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Abstract

Monetary economists have recently begun a serious study of money supply rules that allow the Fed to adjustably peg the nominal interest rate under rational expectations. These rules vary from procedures that produce stationary nominal magnitudes to those that generate nonstationarities in nominal variables. Our paper investigates the determinacy properties of three representative interest rate rules.

We use Blanchard and Kahn's solution technique as a starting point. It doesn't directly apply, so we first modify their procedure. We then narrow the range of solutions by considering the ARMA solutions of Evans and Honkapohja and the global minimum state variable solution of McCallum. We then examine these solutions in light of the expectational stability notions employed by DeCanio, Bray and Evans.

Two of the three classes of rules yield a unique admissible solution. The exclusion of bubbles usually rules out the general ARMA solutions present in Evans and Honkapohja and leads to unique solutions via a saddlepoint property. Nonetheless, the nonstationary money supply rules we examine do not generally yield a well determined system over all parameter values. We employ the global minimum state variable methodology of McCallum and Evans' expectational stability in an effort to insure uniqueness. Although these methods are usually in agreement, one of the nonstationary rules yields a global minimum state variable solution that is expectationally unstable when the central bank is sensitive to interest rate deviations. Moreover, under these conditions, an alternative (non-global) minimum state variable solution *is* expectationally stable, casting doubt on the applicability of McCallum's global procedure in this context.

1. Introduction

Beginning with McCallum's (1981) article, monetary economists have been able to seriously study the use of an interest rate instrument in an economic environment that incorporates rational expectations. Dotsey and King (1983, 1986) and Canzoneri, Henderson and Rogoff (1983) have examined a variety of money supply specifications that allow the central bank to target or adjustably peg the nominal interest rate. Barro (1989), building on the work of Goodfriend (1987) and McCallum (1986), has used the concept of interest rate smoothing in an attempt to generate nominal variables that have statistical properties that resemble actual time series.

The literature has produced a number of money supply rules that allow the Fed to adjustably peg the nominal interest rate. The rules vary from procedures that produce stationary nominal magnitudes to those that generate nonstationarities in nominal variables. Although determinacy issues are central to much of McCallum's work, the various types of rules have not been subjected to a systematic investigation. Our paper carries out that investigation.

We use Blanchard and Kahn's (1980) procedure as a starting point for studying three representative forms of interest rate rules. Since their procedure doesn't directly apply to our problems, the first order of business is to modify it so that it does apply. We then narrow the range of solutions using the methods of Evans and Honkapohja (1986) and of McCallum (1983). Finally, we examine these solutions in light of the expectational stability notions employed by DeCanio (1979), Bray (1982) and Evans (1985, 1986).[1]

By focusing on solutions that meet the nonexplosiveness criteria of Blanchard and Kahn (1980) we find that two of the three classes of rules yield a unique admissible solution. The exclusion of bubbles usually rules out the general ARMA solutions present in Evans and Honkapohja and leads to unique solutions via a saddlepoint property similar to that used in Blanchard and Kahn (1980).

Nonetheless, the nonstationary money supply rules which are employed in McCallum (1986) and extended by Barro (1989) do not generally yield a well determined system over all parameter values — there are an infinity of solutions. Focusing on solutions involving a minimal set of state variables helps somewhat, but uniqueness may still be a problem. In an effort to insure uniqueness, McCallum (1983) has proposed a subsidary principle, further refining the set of admissible solutions to those minimum state variable solutions that hold

[1] General discussions of expectational stability may be found in Blume, Bray and Easley (1982) and Frydman and Phelps (1983).

globally — for all values of the parameters. Both McCallum and Barro appeal to the global minimum state variable methodology of McCallum (1986).

However, our methodology indicates that only Barro's model has a unique nonexplosive solution when the interest rate is pegged. This occurs because, in his model, the central bank is concerned about the variance of price level surprises. This concern leads to a restriction of the admissible parameter values that the coefficients on the interest rate feedback terms are allowed to have.

When it comes to interest rate rules, we are not as optimistic as they are in applying McCallum's technique. Although Evans (1986) finds that expectational stability often provides support for the subsidiary principle, that is not always the case here. In fact, under one of the nonstationary rules, we find that the global minimum state variable solution is expectationally unstable when the central bank is sensitive to interest rate deviations. Moreover, under these conditions, an alternative (non-global) minimum state variable solution *is* expectationally stable, casting doubt on the applicability of McCallum's subsidiary principle in this context.

The paper proceeds as follows. Section Two gives three variants of the basic model, differing only in the interest rate rule. Section Three contains our modification of Blanchard and Kahn's eigenvalue counting rules. In Section Four, we examine the solutions to our systems and investigate their nominal determinacy properties. Section Five analyzes their expectational stability, while Section Six discusses Barro's model. Section Seven contains concluding remarks.

2. The Model

The basic economic structure is similar to McCallum (1981, 1986), and Goodfriend (1987). The real side of the economy is depicted by a Fisher relationship between the nominal rate of interest, the stochastic real rate of interest, and the expected rate of inflation. Thus

$$R_t = a + E_t p_{t+1} - p_t + r_t \qquad (1)$$

where R_t is the nominal rate of interest, $a+r_t$ is the real rate of interest, a serially independent stochastic process with mean a, p_t is the logarithm of the current price level, and $E_t p_{t+1}$ is the expectation of the log of next period's price level, conditional on current information. The current information set includes observations on all current and past values of the endogenous variables and the stochastic disturbances. The demand for money is given by

$$m_t^d = p_t - cR_t + v_t \qquad (2)$$

where m_t is the logarithm of nominal money balances and v_t is a mean zero white noise disturbance term. The initial money supply m_0 is given. The disturbance term includes the effects of changes in income on the demand for money as well as shifts in taste and transactions technology.

The model is closed by specifying the money supply process. Here we choose to examine three different candidates.

$$m_t^s = b + \lambda(R_t - R^*). \tag{3a}$$

$$m_t^s = m_{t-1} + \lambda(R_t - R^*), \qquad m_0 \text{ given} \tag{3b}$$

$$m_t^s = m_{t-1} + \lambda(R_t - E_{t-1}R_t), \qquad m_0 \text{ given.} \tag{3c}$$

Equations (3a) and (3b) are in the spirit of McCallum (1986), while (3c) is somewhat representative of the specification used in Goodfriend (1987), Dotsey and King (1983, 1986), and Canzoneri, Henderson, and Rogoff (1983). The parameter λ determines how much the monetary authority reacts to changes in the interest rate. In all three specifications we approximate the behavior of a peg by letting $\lambda \to \infty$.

Equation (3a) implies a stationary money supply, while (3b) and (3c) yield nonstationary movements in money. As will be shown below, equation (3b) produces determinacy problems not found in the other two systems.

3. Root Counting Rules

In order to analyze the determinancy properties of our interest rate rules we need a methodology. The methodology that we find most appealing is that of Blanchard and Kahn (1980), since it rules out explosive solutions and is carefully based on a general theory of uniqueness for a system of linear difference equations. They require that the conditional expections of future x do not explode. Specificially, $E_t x_{t+i}$ is polynomially bounded in the sense that for all t there exist random variables $\bar{x}_t$ and integers n_t with $|E_t x_{t+i}| \leq (1+i)^{n_t}\bar{x}_t$ for all i. The nonexplosiveness criteria seems sensible to us since it mimics the role of transversality conditions in well specified optimization models. Also, as an empirical matter, analyzing explosive bubbles does not seem particularly relevant.

To facilitate comparison with Blanchard and Kahn (1980), we refer to variables with $E_t x_{t+1} = x_{t+1}$ as predetermined and those with $E_t x_{t+1} \neq x_{t+1}$ as non-predetermined. Of course, $E_t x_t = x_t$ for all x. Blanchard and Kahn assume that the initial values of the predetermined variables are given, while the initial values of non-predetermined variables are not given. We break camp with them here. Our analysis in this section shows that it is *not* the stochastic properties of the variables which matter, but rather the presence or absence of initial conditions.

Blanchard and Kahn's formulation is not immediately applicable to our set of problems since systems involving (3b) or (3c) fall under their example C due to the presence of terms containing $E_t P_{t+2}$. As they point out, problems in this category can not be addressed by the counting rules in their theorem. We are, therefore, required to modify their methodology. Our method will prove useful for a wide class of models that include past expectations of current and future variables and is of interest in and of itself. Although our method is applicable to a much wider variety of systems, the theorem we present is optimized for the systems set forth in section 2. In addition to meriting investigation in their own right, the interest rate rules we analyze in sections 4 and 5 serve double duty as interesting examples of our methods.

Our starting point is the solution procedure of Blanchard and Kahn (1980). The matrix multiplying the current variables is put into Jordan form J, with the eigenvalues listed in order of increasing modulus. The matrix that performs the diagonalization is denoted S. Blanchard and Kahn's procedure is to partition J into blocks corresponding to the n_{in} eigenvalues inside or on the unit circle, and the n_{out} eigenvalues outside the unit circle. The vector of variables gives the n_{pre} predetermined variables first, followed by the n_{non} non-predetermined variables. Partition S accordingly. It is crucial that the $n_{out} \times n_{non}$ matrix S_{22} be of full rank. Blanchard and Kahn then restrict their attention to the polynomially bounded solutions of the system. They find that there is a unique solution when $n_{out} = n_{non}$, there is an infinity of solutions when $n_{out} < n_{non}$, and, for almost all initial conditions, there is no solution when $n_{out} > n_{non}$.

The importance of their rank condition cannot be underestimated. When the rank condition fails, the ill-behaved parts of the solution cannot be ruled out through polynomial boundedness. The root counting procedure fails. The following simple example illustrates this. Consider the system $x_{t+1} = 2x_t$ and $E_t p_{t+1} = p_t/2$ where x_t is predetermined and p_t is non-predetermined. Since $n_{out} = n_{non} = 1$, root counting would yield a unique polynomially bounded solution for all initial conditions. However, for $x_0 \neq 0$, there is no polynomially bounded solution since $x_t = 2^t x_0$. When $x_0 = 0$, there are many polynomially bounded solutions. To see the indeterminacy, let w_t be an arbitrary martingale, so $E_t w_{t+1} = w_t$. Consider $p_t = \alpha(1/2)^t w_t$ and $x_t = 0$ for α arbitrary. This is a solution since $E_t p_{t+1} = p_t/2$, and is clearly polynomially bounded.[2]

Our problems appear to be only slightly different from Blanchard and Kahn's. Nonetheless, their results do not directly apply to our problems due to the presence of future

[2] Martingale solutions are considered in Pesaran (1981) and Gourieroux, Laffont and Monfort (1982). See Pesaran's (1987) book for a complete exposition.

expectations. Fortunately, similar results do apply. [3]

Let P_t be the $m-$vector of variables, Ω_t an $m-$vector of disturbance terms, F an m^*-vector, Q an $m^* \times 2m$ matrix and let A, B, C and D be $m \times m$ matrices. The equations we are interested in will have the form

$$P_{t+1} = AP_t + BE_tP_{t+1} + CE_tP_{t+2} + DE_{t+1}P_{t+2} + \Omega_{t+1}, \qquad \text{s.t. } Q\begin{bmatrix} P_0 \\ E_0P_1 \end{bmatrix} = F \quad (4)$$

where $X_0 = E_0P_1$. All three of our cases fit into this format, as does Blanchard and Kahn's second example C, $P_{t+1} = \alpha(E_tP_{t+2} - E_tP_{t+1}) + \epsilon_t$. The initial condition on m_t can be written in terms of P_0 and E_0P_1 by using (1) and (2). This yields the initial condition $(1+c)p_0 - cE_0p_1 = m_0 + ca + cr_0 - v_0$.

Start by applying E_t to (4). This yields

$$E_tP_{t+1} = AP_t + BE_tP_{t+1} + (C+D)E_t(E_{t+1}P_{t+2}) + E_t\Omega_{t+1}. \quad (5)$$

Provided $C+D$ is invertible, the substitution $X_t = E_tP_{t+1}$ puts this into the Blanchard and Kahn format

$$E_t\begin{bmatrix} P_{t+1} \\ X_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -(C+D)^{-1}A & (C+D)^{-1}(I-B) \end{bmatrix}\begin{bmatrix} P_t \\ X_t \end{bmatrix} + \begin{bmatrix} 0 \\ E_t\Omega_{t+1} \end{bmatrix}.$$

To study uniqueness, we consider the case with $\Omega_t = 0$ and $F = 0$. Equation (4) becomes

$$[I, -D]\begin{bmatrix} P_{t+1} \\ X_{t+1} \end{bmatrix} = [A, B]\begin{bmatrix} P_t \\ X_t \end{bmatrix} + [0, C]E_t\begin{bmatrix} P_{t+1} \\ X_{t+1} \end{bmatrix} \quad (6)$$

while equation (5) becomes

$$E_t\begin{bmatrix} P_{t+1} \\ X_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -(C+D)^{-1}A & (C+D)^{-1}(I-B) \end{bmatrix}\begin{bmatrix} P_t \\ X_t \end{bmatrix}. \quad (7)$$

If there are two solutions to (4), their difference must solve (6).

We cannot blindly apply the Blanchard and Kahn results to equation (7). Some of its solutions may not satisfy (6). Our method is to first solve (7), and then substitute into (6) to see if it imposes any additional restrictions.

Let

$$K = \begin{bmatrix} 0 & I \\ -(C+D)^{-1}A & (C+D)^{-1}(I-B) \end{bmatrix}.$$

Let J be its Jordan form with eigenvalues arranged in order of increasing modulus and let S satisfy $SJ = KS$. Partition both J and S into four blocks each so that J_{11} and S_{11} are $m_{in} \times m_{in}$ and J_{22} and S_{22} are $m_{out} \times m_{out}$.

[3] Our technique also applies to the recalcitrant example C of Blanchard and Kahn.

THEOREM. *Suppose that K is invertible and QS has full rank. If $m_{in} > m^*$, there are infinitely many polynomially bounded solutions, if $m_{in} < m^*$ most initial conditions do not admit any polynomially bounded solutions, while if $m_{in} = m^* \leq m$ with $S_{11} - DS_{21}$ invertible, all initial conditions yield a unique polynomially bounded solution.*

PROOF. First consider the case $m_{in} < m^*$. Take the expectation at time zero in equation (4) and multiply by S^{-1}. Setting $R_t = S^{-1}\begin{bmatrix} E_0P_t \\ E_0P_{t+1} \end{bmatrix}$, this yields $R_{t+1} = JR_t + E_0S^{-1}\begin{bmatrix} 0 \\ \Omega_t \end{bmatrix}$ with initial condition $QSR_0 = F$. If there is a solution to (4), R_t will be polynomially bounded and deterministic. By subtracting a particular solution, we may assume $\Omega_t = 0$ without loss of generality. Then $R_t = J^tR_0$, and polynomial boundedness requires that the last m_{out} entries in R_0 be 0. The inital condition $QSR_0 = F$ then gives us m^* equations in m_{in} unknowns. Since QS has full rank, it will usually have no solutions. It follows that (4) will usually have no solutions.

Now suppose $m_{in} > m^*$. It is tedious but straightforward to use Blanchard and Kahn's procedure to show that at least one solution to (4) exists. Again consider the homogeneous equation obtained by taking expectations at time zero and multiplying by S^{-1}. Any polynomially bounded solution to this with initial condition $QSR_0 = 0$ can be added to a solution to (4) to obtain another solution. As long as the last m_{out} entries in R_0 are 0, R_t will be polynomially bounded. Since there are $m^* < m_{in}$ equations in m_{in} unknowns, there are many such R_0, and the solution is not unique.

Finally, suppose $m_{in} = m^* \leq m$. Again, there is at least one solution. We restrict our attention to the associated homogeneous stochastic difference equation (6). Pesaran (1987) shows that the general solution to (6) is

$$\begin{bmatrix} P_t \\ X_t \end{bmatrix} = SJ^t\mathbf{M}_t = K^tS\mathbf{M}_t$$

where $\mathbf{M}_t$ is an arbitrary martingale. It is easily verified that $[I, -D]K = [A, B] + [0, C]K$. Substituting in (6) we obtain $[I, -D]SJ^{t+1}(\mathbf{M}_t - \mathbf{M}_{t+1}) = 0$.

Again, the last m_{out} entries of $\mathbf{M}_t$ must be zero in view of polynomial boundedness. Since $m_{in} \leq m$, we have $[I, -D]SJ^{t+1}(\mathbf{M}_t - \mathbf{M}_{t+1}) = (S_{11} - DS_{21})J_{11}^{t+1}(\mathbf{M}_{1,t+1} - \mathbf{M}_{1,t}) = 0$. Then the invertibility of $S_{11} - DS_{21}$ and J_{11} implies $\mathbf{M}_{1t} = \mathbf{M}_{1,t+1}$. Finally, plugging in the initial conditions, we see that $\mathbf{M}_t = 0$, and the solution is unique. QED

One interesting aspect of this type of equation is that the distinction between predetermined and non-predetermined variables is unimportant in the solution procedure. We can only discover which variables are which type by solving the equations. What was important was the presence or absence of intitial conditions. The predetermined/non-predetermined distinction is important to Blanchard and Kahn only through the initial conditions. If there are predetermined variable without initial conditions, trying to use the distinction causes problems.

A one-sector deterministic optimal growth problem illustrates this problem. Linearizing the Euler equations, we obtain a second-order difference equation in the capital stock k_t. Introducing $x_t = k_{t+1}$ as a variable gives a first-order system of the type considered by Blanchard and Kahn. The saddlepoint property will typically hold, and there will be one root inside the unit circle and one root outside it. However, since everything is deterministic, all variables are predetermined and root-counting implies that there are (usually) no polynomially bounded solutions since there are two predetermined variables, k_t and x_t. Yet it is well-known that the linearized system will have a unique polynomially bounded solution. The problem is that the Blanchard and Kahn framework requires us to impose an initial condition on the future capital stock x_t. Of course, only one such initial condition is consistent with optimality. Any other initial condition will not yield a solution. Our framework handles the optimal growth problem correctly by only specifying one initial condition, for current capital. Our root counting implies that there is a unique solution — the correct result.

4. The Solutions

Now that we are armed with an appropriately modified Blanchard-Kahn result, we are ready to investigate the relations between the three types of solutions to our systems — the Blanchard-Kahn type, the Evans-Honkapohja ARMA solutions and McCallum's minimum state variable solutions.

(a) The stationary money supply rule

Equations (1), (2), and (3a) yield a reduced form expression for prices of

$$p_t = \frac{1}{1+\lambda+c}[b - \lambda R^* + (\lambda+c)E_t p_{t+1} + (\lambda+c)(a+r_t) - v_t]. \tag{8}$$

This expression is easily put in the framework above, yielding

$$E_t p_{t+1} = \frac{1}{\lambda+c}[(1+\lambda+c)p_t - w_t]$$

where $w_t = b - \lambda R^* + (\lambda+c)(a+r_t) - v_t$.

The root of this equation is $(1+\lambda+c)/(\lambda+c)$. For large λ, this root is outside the unit circle. Since there is precisely one non-predetermined endogenous variable at time t, there is a unique nonexplosive solution. It is

$$p_t = (\lambda+c)a + b - \lambda R^* + \frac{(\lambda+c)r_t - v_t}{1+\lambda+c}. \tag{9}$$

This is also the global minimum state variable solution derived in McCallum (1983, 1986). When $R^* = a$, this solution will approach a limit as $\lambda \to \infty$. This limit approximates the solution for large λ. It is not to be confused with the solution to the limit of equation (8), which is without obvious economic meaning. The money supply rule (3a) does not directly incorporate a trend rate of money growth. It makes intuitive sense that the nominal interest rate target is the expected real rate of interest. In this case the nominal interest rate is $R_t = R^* + (r_t + v_t)/(1 + \lambda + c)$ and will fluctuate randomly around its target. Even though b is not its target, the money stock $m_t = b + \lambda(r_t + v_t)/(1 + \lambda + c)$ likewise fluctuates randomly about b.

The limiting value of the price level as $\lambda \to \infty$, $p_t(\infty)$, is well-defined for $R^* = a$ and equal to

$$p_t(\infty) = ac + b + r_t.$$

As in other interest rate pegging literature, the price level is unaffected by the money demand disturbance. The limiting behavior of the price level is also seen to be equal to the price level of a money supply rule in which the central bank buys and sells bonds at a nominal interest rate that is expected to produce a money supply of b. That is, if the monetary authority chooses to buy and sell bonds at a nominal interest given by

$$R_t = \frac{1}{c}(E_{t-1}p_t - b),$$

the price level $p_t(\infty)$ and interest rate $a = R^*$ emerge as the unique polynomially bounded solution when $c > 0$. The interpretation of the limiting value of (9) as a peg is thus well motivated.

While (9) represents the unique nonexplosive solution to (8), there are infinitely many explosive solutions. The class of these with finite degree ARMA representations can be found by employing the method used in Evans and Honkapohja (1986). First eliminate the constant term with the transformation $q_t = p_t - (ac + b)$. The finite degree ARMA solutions are then found by substituting $q_t = \sum_{i=1}^{k} a_i q_{t-i} + \sum_{i=0}^{l}(b_i v_{t-i} + c_i r_{t-i})$. The general ARMA solution is given by

$$p_t = \frac{1}{\lambda + c}[(1 + \lambda + c)p_{t-1} - (b - \lambda R^*) + v_{t-1}] - (a + r_{t-1}) + b_0 v_t + c_0 r_t \tag{10}$$

where b_0 and c_0 are arbitrary. The nonexplosive solution (9) solves this for $b_0 = -1/(1+\lambda+c)$ and $c_0 = (\lambda + c)/(1 + \lambda + c)$. In other cases, it is easy to see that the difference equation for p_t is explosive for positive λ. Arbitarily limiting the solution to bounded price levels, one could solve (8) forward as in Sargent (1979) and end up with equation (9). Alternatively, restricting the solutions to be expectationally stable in the sense of Evans (1985) rules out solutions other than (9).

(b) Nonstationary money supply rules

We next analyze rule (3b). The reduced form equation for prices derived from (1), (2), and (3b) is

$$p_t = \frac{1}{1+\lambda+c}[\lambda(a-R^*) + (1+c)p_{t-1} - cE_{t-1}p_t + (\lambda+c)E_t p_{t+1} \\ - cr_{t-1} + v_{t-1} + (\lambda+c)r_t - v_t]. \quad (11)$$

In this case, it's simpler to walk through the steps of the theorem rather than invoke it directly.

Update, take expectations conditional on date t information, substitute $x_t = E_t p_{t+1}$, and rearrange to obtain the matrix form

$$\begin{bmatrix} E_t p_{t+1} \\ E_t x_{t+1} \end{bmatrix} = \frac{1}{\lambda+c}\begin{bmatrix} 0 & \lambda+c \\ -(1+c) & 1+\lambda+2c \end{bmatrix}\begin{bmatrix} p_t \\ x_t \end{bmatrix} + \frac{1}{\lambda+c}\begin{bmatrix} 0 \\ \lambda(R^*-a)+cr_t-v_t \end{bmatrix}. \quad (12)$$

This new system is in our framework and has eigenvalues $\mu = 1$ and $\nu = (1+c)/(\lambda+c)$. Since at most one eigenvalue will be outside the unit circle, there is a temptation to use Blanchard and Kahn's proposition 3 to conclude there are an infinity of nonexplosive solution. This would be an error. The transition from (11) to (12) has introduced some extraneous solutions which solve (12) but not (11).

Let p_t be any solution to (12) which also solves (11). As before, any other solution to (12) can be obtained by adding $\mu^t \mathbf{m}_{1t} + \nu^t \mathbf{m}_{2t}$ where $\mathbf{m}_{1t}$ and $\mathbf{m}_{2t}$ are martingales and μ and ν are the eigenvalues. Consider first the case when $|\nu| > 1$. The polynomial boundedness requirement implies $\mathbf{m}_{2t} = 0$. Now substitute in (11), and use the fact that p_t solves (11) to obtain $\mathbf{m}_{1,t+1} = \mathbf{m}_{1t}$. It follows that (11) has a unique solution, provided we know the initial price level. Provided we know the initial price level, the extra constraint imposed by (11) has the same effect on determinacy as reducing the number of non-predetermined variables by one. Roughly speaking, (11) turns p_t into a predetermined variable.

Now consider the case where $|\nu| < 1$. Again substituting in (11), and using the fact that $\nu = (1+c)/(\lambda+c)$, we obtain $\lambda\nu^{t+1}\mathbf{m}_{2,t+1} + \mathbf{m}_{1,t+1} = \lambda\nu^{t+1}\mathbf{m}_{2t} + \mathbf{m}_{1t}$. Even given an initial price level, there are an infinity of solutions to (11). In particular, taking the limit as $\lambda \to \infty$ will not produce a unique solution. The solution to an interest rate peg with this underlying money supply rule will suffer from nominal indeterminacy.

The myriad of nonexplosive finite-degree ARMA solutions can be found by extending the methodology of Evans and Honkapohja (1986) so that it encompasses equation (11). First remove the constant term with the transformation $q_t = p_t - \lambda(R^* - a)t/(\lambda - 1)$ (the time dependence arises from the unit root). Then set $q_t = \sum_{i=1}^{k} a_i q_{t-i} + \sum_{i=0}^{l}(b_i v_{t-i} + c_i r_{t-i})$. It is easily verified that there are at most two non-zero lags. There are three solutions, depending on whether $a_1 = (1+\lambda+2c)/(\lambda+c)$ or a_1 solves $(\lambda+c)a_1^2 - (1+\lambda+2c)a_1 + 1 + c = 0$. The first is the most general form and is given by

$$q_t = \frac{1}{\lambda+c}[(1+\lambda+2c)q_{t-1} - (1+c)q_{t-2} + cr_{t-2} - v_{t-2} + v_{t-1} - (\lambda+c)r_{t-1}] \\ + b_0[v_t - cv_{t-1}/(\lambda+c)] + c_0[r_t - cr_{t-1}/(\lambda+c)] \tag{13}$$

where b_0 and c_0 are arbitrary. The roots of this difference equation are 1 and $(1+c)/(\lambda+c)$. When $|1+c| < |\lambda+c|$, this solution is nonexplosive.

The other two solutions are actually specializations of (13). The second solution has $a_1 = 1$, and is the solution picked by McCallum. This solution obeys

$$q_t = q_{t-1} + \frac{1}{1+c}[(\lambda-1)v_t + (\lambda+c)r_t + v_{t-1} - cr_{t-1}]. \tag{14}$$

This solution occurs when $b_0 = (\lambda-1)/(1+c)$ and $c_0 = (\lambda+c)/(1+c)$. Choosing these values implies that (13) can be obtained from (14) by multiplying (14) by $I - (1+c)L/(\lambda+c)$, where I and L are the identity and lag operators, respectively.[4]

The third solution has $a_1 = (1+c)/(\lambda+c)$ and obeys

$$q_t = \frac{1+c}{\lambda+c}q_{t-1} + \frac{1}{\lambda+c}[v_{t-1} - cr_{t-1}] + r_t. \tag{15}$$

When $|1+c| < |\lambda+c|$, this is also a minimum state variable solution. However, since it fails to be admissible for some parameter values ($\lambda < 1$), it is not the *global* minimum state variable solution of McCallum. Similarly, choosing $b_0 = 0$ and $c_0 = 1$ implies that (13) can be obtained from (15) by multiplying it by $(I - L)$. Note that adding a constant term to (15) also gives an equation that solves (13), since the constant disappears upon application of $I - L$. Further note that this would fail for (15′) since the constant term in (13′) is zero.

Transforming back to p_t yields, respectively

$$p_t = \frac{1}{\lambda+c}[\lambda(R^* - a) + (1+\lambda+2c)p_{t-1} - (1+c)p_{t-2} + cr_{t-2} - v_{t-2} + v_{t-1} - (\lambda+c)r_{t-1}]$$

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[4] To further see that McCallum's solution is not the unique nonexplosive solution when $\lambda < 1$, one can add the linear combination of martingales $\mu^t m_{1t} + \nu^t m_{2t}$ to equation (14) where $m_{1t} = \sum_{j=1}^{t} \alpha r_j$ and $m_{2t} = \sum_{j=1}^{t} \beta \nu_j$ with α and β arbitrary. Since $|\mu|, |\nu| \leq 1$, this is also a nonexplosive solution. Also, a little algebra confirms that the solution with arbitrary martingales can be transformed into equation (13).

$$+ b_0[v_t - cv_{t-1}/(\lambda + c)] + c_0[r_t - cr_{t-1}/(\lambda + c)]. \tag{13'}$$

$$p_t = \frac{\lambda}{\lambda - 1}(R^* - a) + p_{t-1} + \frac{1}{1+c}[(\lambda - 1)v_t + (\lambda + c)r_t + v_{t-1} - cr_{t-1}]. \tag{14'}$$

$$p_t = \frac{1}{\lambda + c}[\frac{\lambda}{\lambda - 1}(1 + c)(R^* - a) + (1 + c)p_{t-1} + v_{t-1} - cr_{t-1}] + r_t. \tag{15'}$$

Perhaps a more intuitive way of seeing the underlying indeterminacy is to iterate (12) forward n periods and to examine the system. Let $\Omega_t = \begin{bmatrix} 0 \\ cr_t - v_t \end{bmatrix}$ and $Z = \begin{bmatrix} 0 \\ \lambda(R^* - a) \end{bmatrix}$ for notational simplicity. Then (12) becomes

$$E_t \begin{bmatrix} p_{t+n} \\ x_{t+n} \end{bmatrix} = S^{-1} J^n S \begin{bmatrix} p_t \\ x_t \end{bmatrix} + \frac{1}{\lambda + c}[S^{-1}(\sum_{i=0}^{n-1} J^i) SZ + S^{-1} J^{n-1} S \Omega_t] \tag{16}$$

where J is the diagonal matrix composed of the eigenvalues of (12) and S^{-1} represents the matrix of corresponding eigenvectors. They are given by

$$J = \frac{1}{\lambda + c} \begin{bmatrix} 1+c & 0 \\ 0 & \lambda + c \end{bmatrix} \qquad \text{and} \qquad S^{-1} = \begin{bmatrix} \lambda + c & 1 \\ 1 + c & 1 \end{bmatrix}.$$

Examining the first row of (16) yields

$$\begin{aligned} E_t p_{t+n} &= (\lambda + c)(\alpha^n - 1) E_t p_{t+1} + [(1 + c) - (\lambda + c)\alpha^n] p_t \\ &\quad + \lambda \left[\frac{1 - \alpha^n}{1 - \alpha} - n \right] (R^* - a) + (\alpha^{n-1} - 1)(cr_t - v_t) \end{aligned} \tag{17}$$

where $\alpha = (1 + c)/(\lambda + c)$ and we have substituted $x_t = E_t p_{t+1}$. Since $c > 0$, we observe that $|\alpha| > 1$ for $-1 - 2c < \lambda < 1$. Dividing (17) by α^n and letting $n \to \infty$, we obtain

$$E_t p_{t+1} = p_t + \frac{\lambda}{\lambda - 1}(R^* - a) + \frac{cr_t - v_t}{1 + c},$$

which is the global minimum state variable solution in equation (14′).

The other case of interest is when $\lambda < -1-2c$ or $\lambda > 1$, which implies $|\alpha| < 1$. Requiring $R^* = a$ and letting $n \to \infty$, equation (17) becomes

$$E_t p_{t+1} = \frac{1}{\lambda + c}[(1 + c)p_t - (cr_t - v_t) + (\lambda - 1)E_t p(\infty)].$$

Recalling that $R^* = a$, we see that this agrees with (15′) when $E_t p(\infty) = 0$. However, there exists an infinity of solutions for $E_t p_{t+1}$, one for each arbitrary choice of $E_t p(\infty)$. Interestingly, if one constrains oneself to the choice $E_t p(\infty) = E_t p_{t+1}$, the global minimum state variable solution (14′) is obtained. Essentially, McCallum's (1983) procedure picks this solution.

The global minimum state variable procedure for the above model works in the following manner. The procedure works by perturbing the model and examining a more general specification of the money supply rule

$$m_t^s = hm_{t-1} + \lambda(R_t - R^*).$$

The procedure then picks the solution associated with the eigenvalue that does not go to zero when h is zero. When $h = 0$, we know the nonexplosive solution is unique. The procedure indicates that the solution associated with the positive root of the difference equation for the price level is the correct one and implies that the coefficient on p_{t-1} in the solution for p_t is one. As mentioned, this means that $E_t p_{t+j} = E_t p_t$ for all j and hence $E_t p(\infty) = E_t p_{t+1}$.

In many instances the global minimum state variable methodology has intuitive appeal. A casual look out the window doesn't seem to indicate that indeterminacy is an important aspect of the economic environment. Adopting a solution that is robust for various values of preference or technology parameters seems sensible since unique solutions often characterize well-defined optimization problems. In the case of interest rate (money supply) rules the parameters in question, h and λ, are just arbitrarily chosen by the modeler or policymaker. In a standard optimizing model with money in the utility function choosing $h = 1$ does not violate any transversality conditions, yet leads to a nominal indeterminacy. An alternative interpretation to McCallum (1986) is that a money supply rule given by (3b) is not well specified, rather than his interpretation that his particular solution should be chosen. Inferences drawn from the solution when $h = 0$ may not be relevant for a solution when $h = 1$ as economic systems that are stationary are quite different from those that are not.

(c) An alternative nonstationary money supply rule

If instead of (3b) we close our economic system with equation (3c), the results are strikingly different. The reduced form becomes

$$\begin{aligned} p_t = [&(1+c)p_{t-1} + (\lambda + c)E_t p_{t+1} + (\lambda - c)E_{t-1}p_t \\ &- \lambda E_{t-1}p_{t+1} - cr_{t-1} + v_{t-1} + (\lambda + c)r_t - v_t]/(1 + \lambda + c). \end{aligned} \tag{18}$$

Updating puts this in our form with $A = (1+c)/(1+\lambda+c)$, $B = (\lambda - c)/(1+\lambda+c)$, $C = -\lambda/(1+\lambda+c)$ and $D = (\lambda+c)/(1+\lambda+c)$. Taking expectations and using $x_t \equiv E_t p_{t+1}$ yields

$$\begin{bmatrix} E_t p_{t+1} \\ E_t x_{t+1} \end{bmatrix} = \frac{1}{c} \begin{bmatrix} 0 & c \\ -(1+c) & 1+2c \end{bmatrix} \begin{bmatrix} p_t \\ x_t \end{bmatrix} + \begin{bmatrix} 0 \\ r_t - v_t/c \end{bmatrix}.$$

The eigenvectors for this system are $\mu = 1$ and $\nu = (1+c)/c$. As both are non-zero, the matrix K is non-singular. It is easily verified that

$$S = \begin{bmatrix} 1 & c \\ 1 & 1+c \end{bmatrix}$$

transforms K into Jordan form. Since $c > 0$, we have one eigenvector outside the unit circle. Since $S - DS = 1/(1+\lambda+c) \neq 0$ and the first column of QS is $1+c-c=1$, the hypotheses of the theorem hold. Since we have one initial condition and one root outside the unit circle, there is a unique polynomially bounded solution from a given initial money stock. [This could also be easily verified by looking for additional solutions of the form $p_t + \mu^t \mathbf{m}_{1t} + \nu^t \mathbf{m}_{2t}$ where p_t is a solution to (18). The polynomial boundedness implies $\mathbf{m}_{2t} = 0$. Substituting in (18) shows that $\mathbf{m}_{1,t+1} = \mathbf{m}_{1t}$.]

An extension of the Evans and Honkapohja (1986) technique yields the following second-order solution

$$\begin{aligned} p_t = {} & [(1+2c)p_{t-1} - (1+c)p_{t-2} - v_{t-2}]/c + v_{t-1}/(\lambda+c) + r_{t-2} - r_{t-1} \\ & + b_0[v_t - (\lambda+\lambda c+c^2)v_{t-1}/c(\lambda+c)] + c_0[r_t - (\lambda+\lambda c+c^2)r_{t-1}/c(\lambda+c)] \end{aligned} \tag{19}$$

where b_0 and c_0 are arbitrary.

As in the previous section, there are two first-order equations solving (18). The first is the global minimum state variable solution of McCallum,

$$p_t = p_{t-1} + \frac{1}{1+c}[(\lambda-1)v_t + (\lambda+c)r_t + v_{t-1} - cr_{t-1}] \tag{20}$$

and the second is the explosive solution

$$p_t = \frac{1+c}{c}p_{t-1} - v_t + \frac{v_{t-1}}{c} - r_t. \tag{21}$$

As before, (19) may be obtained from (20) by setting $b_0 = (\lambda-1)/(1+c)$ and $c_0 = (\lambda + c)/(1+c)$ and multiplying (20) by $I - (1+c)L/c$, while (19) is obtained from (21) by setting $b_0 = -1$ and $c_0 = 0$ and multiplying by $I - L$. However, unlike the previous section, (21) always yields an explosive solution.

Although (20) is a well-defined nonexplosive solution for finite but arbitrarily large λ, the variance of the price level goes to infinity as $\lambda \to \infty$. As McCallum (1986) notes, this result is model specific. An augmented model that includes an aggregate supply term containing $p_t - E_{t-1}p_t$ would yield a price level with finite variance no matter how large λ became. As before, letting $\lambda \to \infty$ can be interpreted as having the central bank buy and sell bonds at a nominal interest rate that will yield a money supply equal to last period's money supply. When the aggregate supply term is included, buying and selling bonds at

$$R_t = \frac{1}{c}[E_{t-1}p_t - m_{t-1}]$$

yields an equation similar to (20) as the limiting solution of this augmented model.

(d) A comparison of all three rules

From an intuitive standpoint our results are a little puzzling. Responding to interest rate deviations from a preset value is a well-defined procedure when the underlying money supply rule is stationary, yet yields indeterminacy when the money supply is nonstationary. This indeterminacy occurs even when $R^* = a$ the expected real interest rate. Yet a nonstationary money supply rule that responds to deviations of the nominal interest rates from last period's expectation of the current nominal interest rate yields unique nonexplosive solutions with the property that $E_{t-1}R_t = a$ in equilibrium.

With respect to the latter puzzle our conjecture is that specifying the nominal interest rate target at $E_{t-1}R_t$ places restrictions on beliefs that are not present with money supply rule (3b). For example today's expectations of future money supplies, m_{t+1}, are equal to m_t with money supply rule (3c).

With specification (3b), $E_t m_{t+n} = m_t + \lambda \sum_{j=1}^{n}(E_t R_{t+j} - a)$. Deviations of $E_t R_{t+j}$ from a cumulate in expectations of future money. With $\lambda > 1$ this leads to an entire family of price level paths that are consistent with various departure of $E_t R_{t+j}$ from a. Essentially, money supply rule (3b) does not pin down the future and hence does not uniquely define current nominal quantities when $\lambda > 1$.

This inability to uniquely define expectations of future money is potentially a problem for money supply rule (3a). However, deviations of $E_t R_{t+j}$ from $R^* = a$, do not cumulate. That is, it is impossible for $E_t m_{t+j}$ to stray too far from the value b. Apparently this is sufficient to guarantee uniqueness and that $E_{t-1}R_t = a$ (when $R^* = a$).

An alternative way of examining the difference between rules (3b) and (3c) would be to posit a hybrid money supply rule

$$m_t^s = m_{t-1} + \lambda_1(R_t - E_{t-1}R_t) + \lambda_2(E_{t-1}R_t - a), \qquad m_0 \text{ given.}$$

With this rule the eigenvalues are $(1+c)/(\lambda_2 + c)$ and 1 implying that $-1 - 2c < \lambda_2 < 1$ is needed for uniqueness of the nonexplosive solution. Considering only positive values of λ_2, deviations of $E_t R_{t+n}$ from a can potentially accumulate in the expectations of m_{t+n}. However with $\lambda_2 < 1$ these deviations are insufficient to generate nonuniqueness. Hence with $\lambda_2 < 1$, $E_t R_{t+n} = a$.

From our discussion one might also draw the conclusion that observing nonstationary money is inconsistent with the central bank responding to nominal interest rate deviations around some arbitrary value. This is not the case, since one could append a nonstationary money control error, $x_t = x_{t-1} + e_t$, to equation (3a) without influencing the discussion concerning uniqueness. Interestingly, the rule (3a) with a random control error is identical to

$$m_t^s = m_{t-1} + \lambda(R_t - R^*) - \lambda(R_{t-1} - R^*) + e_t, m_0 \qquad \text{given.}$$

The eigenvalues when this rule is used are $(1+\lambda+c)/(\lambda+c)$ and 1, which implies a unique nonexplosive solution.

5. Barro's Model

In a recent article, Barro (1989) builds on the work of Goodfriend (1987) and McCallum (1986) in an attempt to generate a money supply rule that produces time series properties of nominal variables that are consistent with actual observations. He also employs McCallum's (1983) procedure to isolate a unique global minimum state variable solution. We have seen that this procedure can not always be relied upon to guarantee a unique nonexplosive solution and, as we will show in the next section, does not always produce expectationally stable solutions. It is, therefore, worth investigating Barro's extension with our methodology.

For our purposes, a simplified version of Barro's money supply rule can be written as

$$m_t^s = m_{t-1} + \lambda_1(R_t - R^*) - \lambda_3(R_{t-1} - R^*), \qquad m_0 \text{ given}$$

where we have altered the model by holding the interest rate target constant. The eigenvalues for the system using this money supply rule and (1) and (2) are $(1+\lambda_3+c)/(\lambda_2+c)$ and 1. Clearly not all values of λ_3 and λ_1 will produce a unique nonexplosive solution.

In Barro's model, however, the central bank is concerned about two objectives. One is to smooth nominal interest rates by minimizing $E(R_t - R^*)^2$. The other is to minimize the variance of unexpected price level movements, i.e., $E(p_t - E_{t-1}p_t)^2$. The first concern implies the λ_3 should be set arbitrarily large, while the second implies that $\lambda_1 = (1+\lambda_3)\theta - c(1-\theta)$ with $\theta = \sigma_v^2/(\sigma_v^2 + \sigma_r^2)$. As λ_3 gets arbitrarily large, the limiting value of $\lambda_1/\lambda_3 = \theta$. The restriction placed on parameter values by a concern for price level surprises guarantees uniqueness for this model. Hence, in searching for money supply rules that yield nominal determinacy, one can employ reasonable loss functions that generate the appropriate parameter restrictions.

6. Expectational Stability

Evans (1985) proposes an alternative to McCallum's (1983) procedure for choosing a particular solution from a class of solutions. He requires that any solution be expectationally stable. Take a small deviation from a rational expectations solution. Use this to solve your system, and then update expectations. This process must converge to the same rational expectations solution. The time t expectation of p_{t+1} at the N−th stage will be denoted $E_t^N p_{t+1}$. We will restrict our attention to linear expectations functions having the form

$$E_t^N p_{t+1} = \alpha_N + \beta_N p_t + \gamma_N e_t + \delta_N u_t \tag{22}$$

where $e_t = [(\lambda + c)r_t - v_t]/(1 + \lambda + c)$ and $u_t = (v_t - cr_t)/(1 + \lambda + c)$. Updating, taking expectations at time t, and substituting in (22), we obtain

$$E_t^N p_{t+2} = (1 + \beta_N)\alpha_N + \beta_N^2 p_t + \beta_N \gamma_N e_t + \beta_N \delta_N u_t. \tag{23}$$

(a) Trend stationary money supply

Equation (8) with $R^* = a$ can be written as

$$p_t = A_0 + A_1 E_t p_{t+1} + e_t \tag{24}$$

where $A_0 = (b+ca)/(1+\lambda+c)$ and $A_1 = (\lambda+c)/(1+\lambda+c)$. We can easily check to see if (10) is expectationally stable as $\lambda \to \infty$. Since u_t does not appear in (24), we may omit it from the expectations function without loss of generality, leaving $E_t^N p_{t+1} = \alpha_N + \beta_N p_t + \gamma_N e_t$. Update (24), take expectations at time t and use (23) with $\delta_N = 0$ to get $E_t^{N+1} p_{t+1}$. This yields the recursive relationships

$$\begin{aligned} \alpha_{N+1} &= A_0 + A_1(1 + \beta_N)\alpha_N \\ \beta_{N+1} &= A_1 \beta_N^2 \\ \gamma_{N+1} &= A_1 \beta_N \gamma_N. \end{aligned} \tag{25}$$

This system has two types of steady state solution. The first is $\alpha = b + ca$, $\beta = 0$ and $\gamma = 0$. Obviously, this is implied by equation (10). The second has $\alpha = -A_0/A_1$, $\beta = 1/A_1$ and γ arbitrary. Linearizing (25), we see that the three roots for the first steady state are A_1, 0 and 0. Thus for $|A_1| < 1$, α_N, β_N and γ_N converge to their steady state values as $N \to \infty$. Since $|A_1| < 1$ for arbitrarily large λ, the interest rate rule (3a) is expectationally stable. Similarly, linearizing about the second steady state yields roots $1 + A_1$, 2 and 1. This second steady state is unstable since one of the roots is larger than 1.

(b) Non-stationary money supply

We next check to see if the minimal state variable solution of (11) is expectationally stable. When $R^* = a$, (11) has the form

$$p_t = B_1 E_{t-1} p_t + B_2 E_t p_{t+1} + B_3 p_{t-1} + u_{t-1} + e_t \tag{26}$$

where $B_1 = -c/(1+\lambda+c)$, $B_2 = (\lambda+c)/(1+\lambda+c)$, $B_3 = (1+c)/(1+\lambda+c)$, and e_t and u_t are as before. Updating (26), taking the expectation at time t and using (22) and (23) yields the recursive relationships

$$\begin{aligned}
\alpha_{N+1} &= [B_1 + B_2(1+\beta_N)]\alpha_N \\
\beta_{N+1} &= B_1\beta_N + B_2\beta_N^2 + B_3 \\
\gamma_{N+1} &= (B_1 + B_2\beta_N)\gamma_N \\
\delta_{N+1} &= (B_1 + B_2\beta_N)\delta_N + 1.
\end{aligned}$$

This iterative system has two types of stationary solutions: The first has $\alpha = 0$, $\beta = 1$, $\gamma = 0$ and $\delta = (1+\lambda+c)/(1+c)$ (giving (14′)); the second has α arbitrary, $\beta = (1+c)/(\lambda+c)$, $\gamma = 0$ and $\delta = (1+\lambda+c)/(\lambda+c)$ (giving (15′)). Linearizing around the first steady state we obtain eigenvalues $B_1 + 2B_2$ (twice) and $B_1 + B_2$ (twice). These eigenvalues are $(2\lambda+c)/(1+\lambda+c)$ and $\lambda/(1+\lambda+c)$. When $-(1+c)/2 < \lambda < 1$, this solution is expectationally stable while for $\lambda > 1$ or $\lambda < -(1+c)/2$, it is expectationally unstable.

The second type of solution is not unique, but rather a continuum. Because of this, we only ask for stability in the sense that small deviations from a member of the solution set lead to convergence to another member of that set. These solutions have eigenvalues 1, $(2+c)/(1+\lambda+c)$ and $1/(1+\lambda+c)$ (twice). This system is expectationally unstable for $-(2+c) < \lambda < 1$. When $\lambda < -(2+c)$ or $\lambda > 1$, three of the eigenvalues have modulus less than 1. Only stability of α_N is still in doubt because of the unit eigenvalue.[5] However, given initial α_0 we find that

$$\alpha_N = \alpha_0 \prod_{i=0}^{N-1} [B_1 + B_2(1+\beta_i)].$$

Thus

$$\log[\alpha_N/\alpha_0] = \sum_{i=0}^{N-1} \log[B_1 + B_2(1+\beta_i)].$$

We now apply the ratio test to the sum. Since $\beta_{i+1} = B_1\beta_i + B_2\beta_i^2 + B_3$, the relevant ratio is $(\log[B_1 + B_2(1 + B_1\beta_i + B_2\beta_i^2 + B_3)])/(\log[B_1 + B_2(1+\beta_i)])$. Using L'Hôpital's rule and the facts that $\beta_i \to \beta = (1+c)/(\lambda+c)$ and $\beta = B_1\beta + B_2\beta^2 + B_3$ allows us to obtain the limit $B_1 + 2B_2\beta = (2+c)/(1+\lambda+c)$. Since this has modulus less than one when $\lambda > 1$, the series, and hence the product, converges. It follows that the solution set is stable. In particular,

[5] It pays to be careful here. Evans (1985) claims stability with a unit root in his Proposition 2, part II. However, actual calculation of the solution reveals that the equilibrium set is not stable. Rather, his system converges to the other steady state if a is below its steady state value, while his system blows up if a is above its steady state value.

(15′) gives the expectationally stable solution for large λ, while the minimum state variable solution (14′) is expectationally unstable.

(c) Alternative non-stationary money supply

Using (3c) implies a different outcome. Equation (18) can be written as

$$p_t = C_1 E_{t-1} p_t + C_2 E_{t-1} p_{t+1} + C_3 E_t p_{t+1} + C_4 p_{t-1} + u_{t-1} + e_t,$$

$C_1 = (\lambda - c)/(1+\lambda+c)$, $C_2 = -\lambda/(1+\lambda+c)$, $C_3 = (\lambda+c)/(1+\lambda+c)$ and $C_4 = (1+c)/(1+\lambda+c)$ with u_{t-1} and e_t as before. The presence of the constant term creates additional difficulties since this equation has a unit root. To handle it, we include a trend term in the expectation function. Thus $E_t^N p_{t+1} = \alpha_N + \beta_N p_t + \gamma_N e_t + \delta_N u_t + \eta_N t$. Proceeding as before, we obtain the following recursive relationships:

$$\begin{aligned}
\alpha_{N+1} &= [C_1 + (C_2 + C_3)(1 + \beta_N)]\alpha_N + \eta_N \\
\beta_{N+1} &= C_1 \beta_N + (C_2 + C_3)\beta_N^2 + C_4 \\
\gamma_{N+1} &= [C_1 + (C_2 + C_3)\beta_N]\gamma_N \\
\delta_{N+1} &= [C_1 + (C_2 + C_3)\beta_N]\delta_N + 1 \\
\eta_{N+1} &= [C_1 + (C_2 + C_3)(1 + \beta_N)]\eta_N.
\end{aligned}$$

Again, there are two steady state solutions corresponding to (20) and (21). The first has $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\delta = 1/C_4$ and $\eta = 0$ while the second has α arbitrary, $\beta = (1 + c)/c$, $\gamma = 0$, $\delta = (1+\lambda+c)/c$ and $\eta = 0$. Linearizing around the minimum state variable solutions in (20), the conditions for expectational stability are $|C_1 + 2(C_2 + C_3)| = |(\lambda+c)/(1+\lambda+c)| < 1$ and $|C_1 + C_2 + C_3| = |\lambda/(1 + \lambda + c)| < 1$. These conditions are satisfied for positive values of λ. The solution corresponding to (21) requires that $|(\lambda + c)/(1 + \lambda + c)| < 1$ and $|(2 + \lambda + c)/(1 + \lambda + c)| < 1$. These cannot be simultaneously satisfied.

7. Summary and Conclusions

This paper provides a detailed examination of the nominal determinacy properties of various interest rate rules. The class of rules examined is broad enough to essentially span most of the literature on this subject. Our analysis indicates that care should be taken when specifying policies in which the central bank responds to nominal interest rates. This is especially true when the underlying money supply rule displays nonstationarity, since determinacy issues are sensitive to specification of the interest rate feedback term. In this case the preferred model includes responses to deviations from last periods expectation of the current nominal interest rate rather than responses to some arbitrary target. If for other reasons, as in Barro (1989), it is desirable to explore how monetary responses to an exogenous interest rate target affect economic outcomes then the investigator should be careful to restrict the admissable parameter values that are assigned to feedback coefficients.

In examining this class of policies we have drawn from a wide range of literature that deals with the solutions to rational expectations models. This literature ranges from the undetermined coefficient approach of Lucas to the martingale method of Pesaran (1987) and the general ARMA solutions of Evans and Honkapohja (1986). We have also looked at the expectational stability properties of the models and find that there is a one to one correspondence between unique nonexplosive solutions and expectationally stable solutions. When uniqueness is a problem we find that under a peg the general class of ARMA solutions are expectationally stable, but that McCallum's solution is not.

To rigorously examine the question of uniqueness we have extended the counting rule methodology of Blanchard and Kahn (1980) to models that include past expectations of current and future variables. Since rational expectations models with this attribute are fairly common, our methods used to establish our theorem should be useful in a variety of other contexts.

Appendix: Blanchard and Kahn Revisited

Although Blanchard and Kahn's result is not implied by our result, or vice-versa, our method does apply to their system. A close reading of the Blanchard and Kahn paper reveals that the predetermined/non-predetermined distinction is only required when the number of predetermined variables is equal to the number of roots inside the unit circle. However, our method shows that predeterminedness is not required here either.

Suppose there are m' variables X_t with initial conditions and m variables without initial conditions P_t. These correspond to the predetermined and non-predetermined variables, respectively. The system considered by Blanchard and Kahn is

$$\begin{bmatrix} I_{m'} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_{t+1} \\ P_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} E_t X_{t+1} \\ E_t P_{t+1} \end{bmatrix} + \Omega_t.$$

For uniqueness, we may consider the homogeneous case ($\Omega_t = 0$) without loss of generality. Apply E_t and set $Q_t = \begin{bmatrix} X_t \\ P_t \end{bmatrix}$ to obtain $E_t Q_{t+1} = AQ_t$. The Jordan form of A is $J = S^{-1}AS$ and the general solution to the homogeneous equation is $Q_t = SJ^t\mathbf{M}_t$ where $\mathbf{M}_t$ is an arbitrary martingale. If $m_{out} = m$, the last m entries of $\mathbf{M}_t$ are zero. Substituting back in the original equation, we obtain $S_{11}J^{t+1}\mathbf{M}_{1,t+1} = [ASJ^t\mathbf{M}_{1t}]_1 = S_{11}J^{t+1}\mathbf{M}_{1t}$. When S_{11} is invertible, this implies $\mathbf{M}_{1,t+1} = \mathbf{M}_{1t}$. The solution is deterministic and is determined by the m'-vector $\mathbf{M}_{10}$. The m' initial conditions imply $\mathbf{M}_{1t} = \mathbf{M}_{10} = 0$. It follows that the inhomogeneous system has a unique solution.

Our method also applies to their first example C. It is $Y_t = aE_{t-1}Y_t - Z_t$. For simplicity, suppose the forcing process Z_t obeys $E_{t-1}Z_t = 0$. There are no initial conditions. Applying E_{t-1} to the equation yields $0 = (1-a)E_{t-1}Y_t$. Unless $a = 1$, $E_{t-1}Y_t = 0$. Substituting back in the homogeneous equation yields $Y_t = -Z_t$ as the unique solution.

The second example C, $P_{t+1} = \alpha(E_tP_{t+2} - E_tP_{t+1}) + \epsilon_t$ doesn't quite fit the statement of our theorem since zero is a root of K. Nonetheless, the same techniques apply and yield a unique solution provided P_0 is given.

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