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1. ***Introduction.*** We examine the problem of fairly allocating an infinitely divisible good among a group of agents whose preferences are single–peaked: Up to some critical level, an increase in the consumption of the good increases an agent's welfare; beyond that level, the opposite holds (the critical level not being necessarily the same for all agents). Our search will be for well–behaved procedures to identify for each such problem one or several satisfactory allocations of the good. Such general procedures will be called ***solutions.***

This model, which was recently considered by Sprumont (1989), can be given several interpretations. Sprumont offers the following two. First is distribution at disequilibrium prices: consider a two–commodity economy in which resources are supposed to be allocated via the Walrasian mechanism. However, prices are in disequilibrium; they may not have had the time to stabilize, or an unanticipated shock may have destroyed some existing equilibrium, or prices may have been kept from adjusting in order to achieve some social objective. Distribution must take place however, so a rationing rule has to be defined. If consumers' preferences are strictly convex, then, when restricted to the budget lines, they are single–peaked.[1]

Another situation for which the model would be appropriate is when a certain quantity of labor has to be supplied by a team in order to complete some task, and the problem is to divide the task among them. If workers are paid an hourly wage and their disutility of labor is concave, then their induced preferences on the labor they supply are single–peaked.

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[1]If preferences were only convex, then their restriction to the budget lines would have a single "plateau". Our analysis would require only small changes to be applicable to that case. Sprumont's model does not actually fit the typical problem of fair rationing in all of its details. See Section 8 for a slightly different model which would be more descriptive of that situation.

Next is the standard problem of distributing a good among agents who become satiated when their consumption reaches a certain level, in situations where the good is not freely disposable.

Imagine that a fixed amount has to be allocated among a group of consumers when each of them cares not only about what he consumes, but also about what others consume. Assuming, as is quite natural, that he cares mainly about himself when his consumption is low, and about others when his consumption is high (theirs is then correspondingly low), then his preferences over his own consumption are single–peaked.

Finally, models with single–peaked preferences have also been extensively analyzed in social choice theory and they are popular in political science.

Sprumont established the existence of a unique efficient and anonymous solution, which he named the "uniform rule", having the property that, for every agent, truthfully announcing his preferences is a dominant strategy in the associated revelation game.

Here we pursue the analysis of this model. Our focus, however, is on equity. We apply several of the equity notions that have been used in the literature on fair allocation. We also define a variety of solutions that make use of the specific features of this model. Then, we evaluate these solutions.

Much progress has been made in the last few years in the understanding of fairness issues based on ordinal concepts, that is, concepts that depend only on agents' preferences, and not on concepts of utility. (For a review of this literature, see Thomson, 1989b). These issues have been studied in a variety of different situations, but mainly in the standard context of allocating infinitely divisible goods among agents with monotonic preferences. We draw on the conceptual apparatus that has been developed in the analysis of such "classical" problems of fair division. In particular, one of the central concepts in our analysis is that of an envy–free allocation.

Our main conclusion is that the uniform rule should be considered to be the most important solution to the problem of allocating an infinitely divisible good among agents with single–peaked preferences. Indeed, it satisfies a greater number of desirable properties than any other solution. First of all, it selects efficient allocations. It is also single–valued, that is, it always makes a very precise recommendation. It responds appropriately to changes in the amount to be distributed. Its recommendation for any economy always "agrees" with its recommendation for associated subeconomies, and it is essentially the only rule for which this is so. It also satisfies a converse of this property. It is the only allocation to survive under certain asymptotic enlargements of the economy. Finally, there are several combinations of these properties that the uniform rule is the only one to satisfy.

2. ***Notation.*** The model, and much of the notation, follow Sprumont (1989). There is an amount $M \in \mathbb{R}_+$ of some infinitely divisible good that has to be allocated among a set $N = \{1,\ldots,n\}$ of agents. An ***economy*** is a list $R = (R_i)_{i \in N}$ of n continuous preference relations defined over $[0,M]$. These preference relations are ***single–peaked:*** for each i, there is $x_i^* \in [0,M]$ such that for all x_i, $x_i' \in [0,M]$, if $x_i' < x_i \leqq x_i^*$, or if $x_i^* \geqq x_i > x_i'$, then $x_i P_i x_i'$ (P_i denotes the strict preference relation associated with R_i, and I_i the indifference relation).[2] Let $\boldsymbol{p(R_i)} \in \mathbb{R}_+$ be the preferred consumption according to R_i. Let $\boldsymbol{p(R)} = (p(R_1),\ldots,p(R_n))$ be the vector of preferred consumptions. Each preference relation R_i can be described in terms of the function $\boldsymbol{e_i}$: $[0,M] \to [0,M]$ defined as follows: Given $x_i \leqq p(R_i)$, $e_i(x_i) \geqq p(R_i)$ and $x_i I_i e_i(x_i)$ if this is possible, and $e_i(x_i) = M$ otherwise; given $x_i \geqq p(R_i)$, $e_i(x_i) \leqq p(R_i)$ and $x_i I_i e_i(x_i)$ if this is possible, and $e_i(x_i) = 0$ otherwise. ($e_i(x_i)$ is the consumption on the other side of agent i's preferred consumption that is indifferent for him to x_i, if there is such a consumption. If there is none, $e_i(x_i)$ is the end–point of the interval $[0,M]$ on the other side of his preferred consumption.)

[2]Our analysis below would apply, with straightforward modification, to the case of preferences with a single "plateau", (footnote 1), i.e. preferences for which the set of maximal elements is a segment, instead of a singleton.

A ***feasible allocation*** is a list $x = (x_i)_{i \in N} \in \mathbb{R}^n_+$ such that $\sum_{i \in N} x_i = M$. Note that free disposal of the good is not assumed.[3] Let X be the set of feasible allocations.

Our objective is to distribute the amount M equitably. A ***solution*** is a mapping φ which associates with every admissible preference profile R a non–empty subset $\varphi(R)$ of the set of feasible allocations. Each of the points in $\varphi(R)$ is interpreted as one possible recommendation.

An example of a solution is the following:

Pareto solution, P: $x \in P(R)$ if $x \in X$ and there is no $x' \in X$ with $x_i' R_i x_i$ for all i, strict preference holding for at least one i.

We will search for well–behaved solutions. In general, we would prefer being able to make precise recommendations and to impose on solutions the requirement of single–valuedness. However, this property is a luxury we can rarely afford in economics. Few are the domains where it can realistically be expected. It is therefore of great interest that the domain under consideration allows for a number of such solutions; we will introduce and discuss a variety of them.

We will often refer by comparison to the domain of "classical" economies, where there is a finite number of infinitely divisible goods and a finite number of agents with monotone preferences.

We will find it convenient in later sections (5, 6 and 7) to assume preferences to be defined over $\mathbb{R}_+$ instead of [0,M]. On such a domain, solutions may be required to depend only on the restriction of each R_i to [0,M], or they may be allowed to depend on the whole of each R_i. The former choice is equivalent to the formulation we have adopted except in situations where M may vary, a possibility that we will also examine. We will come back to this issue latter on. Finally, we could choose preferences to be defined over

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[3]Otherwise, for each i, we would replace R_i by a preference relation that is strictly monotone up to $p(R_i)$ and satiated above $p(R_i)$. The analysis of the resulting allocation problem would be very much like the analysis of standard problems.

some fixed interval $[0,M_0]$, where M_0 is possibly different from M. This would be appropriate in the case of allocating a task among workers, M_0 being the maximal amount of time each worker can work.

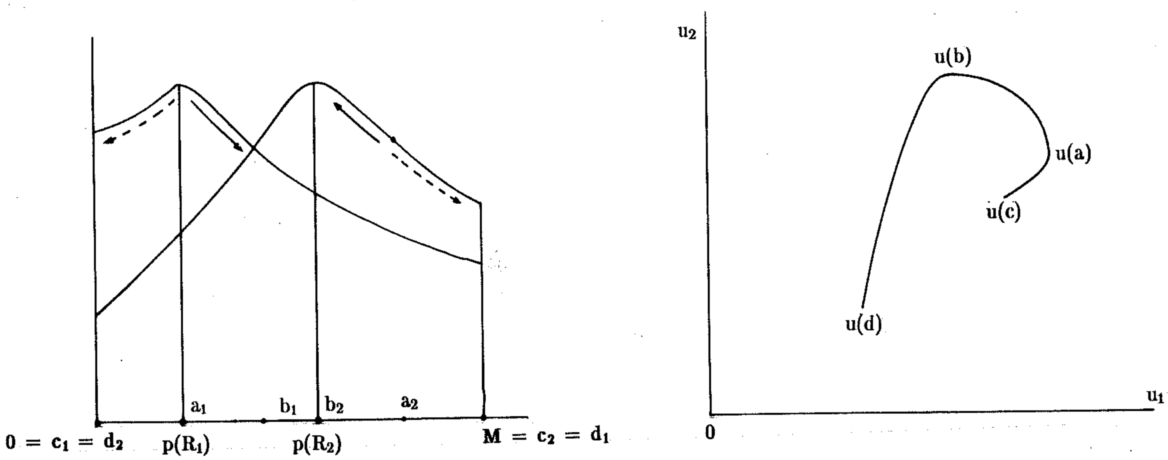
Domain restrictions of interest are when for all i (i) $0 I_i M$. (This restriction would apply in the fix–price interpretation of the model, when indifference curves are asymptotic to the axes), and (ii) R_i admits a concave numerical representation.

The intersection of two solutions φ and φ' is denoted $\varphi\varphi'$.

3. ***Some properties of the Pareto and no–envy solutions.*** First, we establish some elementary properties of the Pareto solution and of the "no–envy" solution, which is undoubtedly the most important ordinal solution to the problem of fair allocation. The set of efficient allocations has a much simpler structure here than in classical economies, but the set of envy–free and efficient allocations has a complicated structure, just as in classical economies.

We start with the Pareto solution. As noted by Sprumont, $x \in P(R)$ if and only if $x \in X$ and (i) when $\Sigma p(R_i) \geqq M$, $x_i \leqq p(R_i)$ for all i, and (ii) when $\Sigma p(R_i) \leqq M$, $x_i \geqq p(R_i)$ for all i. From this characterization, it follows directly that P(R) is convex, a fact that will be useful later on.

For each i, let $u_i:[0,M] \to \mathbb{R}$ be a continuous numerical representation of agent i's preferences, and let $u(X) \equiv \{\bar{u} \in \mathbb{R}^n | \exists x \in X \text{ with } \bar{u}_i = u_i(x_i) \text{ for all } i\}$; u(X) is the image of the feasible set in utility space. The set u(X) need not be comprehensive ($S \subseteq \mathbb{R}^n$ is "comprehensive" if for all $t, t' \in \mathbb{R}^n$, if $t \in S$ and for all i, $\inf\{t''_i | t'' \in S\} \leqq t'_i \leqq t_i$, then $t' \in S$). This is an important implication of the fact that no free–disposability



The set of feasible allocations and its image in utility space.

Figure 1

assumption is made. For example, suppose that $p(R_1) = M$ and $p(R_2) = 0$. Then, $P(R) = \{(M,0)\}$ and $u(X)$ is a monotone path with endpoints $(u_1(0),u_2(M))$ and $(u_1(M),u_2(0))$.

Figure 1 indicates some of the features of $u(X)$ for a typical 2–person economy. Note that for the example $u(c)$ Pareto–dominates $u(d)$ but in general, there need be no Pareto–domination between $u(c)$ and $u(d)$.

The next concept will play a central role in our analysis.

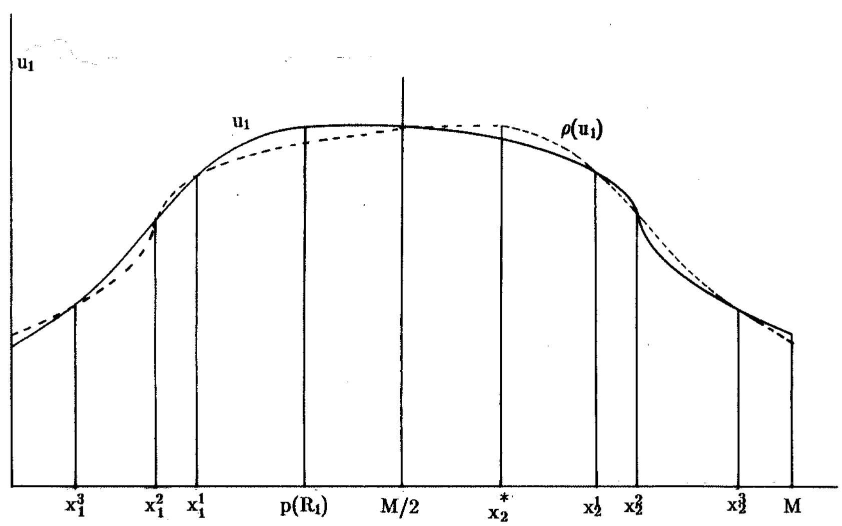
No–envy solution, F (Foley, 1967): $x \in F(R)$ if $x \in X$ and for all i, j, $x_i R_i x_j$.

There always exist envy–free and efficient allocations in our model. Indeed, the "uniform allocation", introduced later, always exists and has both properties.

In classical economies, the set of envy–free and efficient allocations usually does not have a simple structure, even for small numbers of agents and commodities. This remains true here, in spite of the special features of the model. In particular, in contrast with $P(R)$, $FP(R)$ is not convex.

For $n = 2$, $FP(R)$ is a countable union of closed intervals. Here is a procedure to construct $FP(R)$. Suppose, without loss of generality, that $p(R_1) + p(R_2) \geqq M$ and that $p(R_1) \leqq p(R_2)$. This implies that $p(R_2) \geqq M/2$. In Figure 2, the solid line is the graph of a numerical representation of R_1, u_1. The dotted line is its symmetric image with respect to the vertical line of abscissa M/2, i.e. the graph of the function $\rho u_1:[0,M] \to \mathbb{R}$ defined by $\rho(u_1(x_1)) = u_1(M-x_1)$ for all $x_1 \in [0,M]$. If $x = (x_1,x_2) \in X$, then x_1 and x_2 are symmetric of each other with respect to that vertical line. Let $M/2 \geqq x_1^1 > x_1^2 > \ldots$ and $M/2 \leqq x_2^1 < x_2^2 < \ldots$ be the successive points of intersection of the two graphs. Note that $x^1 = (x_1^1,x_2^1)$, $x^2 = (x_1^2,x_2^2)$,... are all feasible allocations. To simplify, we assume that all intersections are transversal.[4] If $x \in P(R)$, then $x_1 \leqq p(R_1)$ and $x_2 \leqq p(R_2)$.

Since $p(R_2) \geqq M/2$, agent 2 does not envy agent 1 at $x \in P(R)$ if and only if $x_1 \leqq M/2$. Then, agent 1 does not envy agent 2 if and only if the graph of u_1 is above that of $\rho(u_1)$ at x_1. If $p(R_1) < M/2$, the best allocation in FP(R) for agent 1 is $(p(R_1),x_2^*)$ for $p(R_2) \geqq x_2^* = M-p(R_1) \geqq p(R_1)$. Starting from this point, we progressively transfer the good from agent 1 to agent 2 and indicate when the resulting allocation is



Determining the set of envy–free and efficient allocations for a two–person example.

Figure 2

[4]This is the generic case. The description of the envy–free and efficient set will only be slightly different if some intersections were not transversal.

envy–free. If $x_1 \in [x_1^1, p(R_1)]$, $(x_1, M-x_1) \in FP(R)$. If $x_1 \in [x_1^2, x_1^1]$, $(x_1, M-x_1) \notin FP(R)$. Then again, if $x_1 \in [x_1^3, x_1^2]$, $(x_1, M-x_1) \in F(R)$. This alternation occurs until x_1 reaches $M - p(R_2)$ (efficiency requires $x_2 = M - x_1 \leqq p(R_2)$).

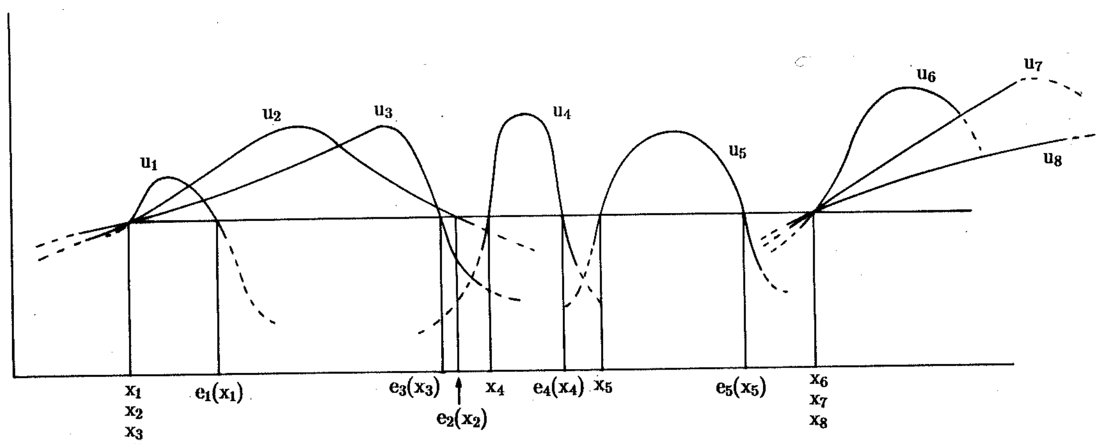
The construction of FP(R) in the case $p(R_1) \geq M/2$ is analogous and we omit it.

We now turn to the n–person case. Again, without loss of generality, suppose that $\Sigma p(R_i) \geqq M$. If agent i does not envy agent j and $x_j \geqq x_i$, then he does not envy any agent k with $x_k \geqq x_j$. As a result, to check whether $x \in FP(R)$ it suffices to check that an agent would not prefer to consume the next greatest amount consumed by anyone else. Then, if $x \in FP(R)$, there is $L \in \mathbb{N}$, a list $0 \leqq a_1 < a_2 \ldots < a_L$, and a partition $\{I_1, I_2, \ldots, I_L\}$ of the set of agents with the following two properties:

(i) for all $\ell = 1,\ldots,L$, and for all $i \in I_\ell$, $x_i = a_\ell \leqq p(R_i)$,

(ii) for all $\ell = 1,\ldots, L-1$ and for all $i \in I_\ell$, $e_i(x_i) \leqq a_{\ell+1}$.

An example with $n = 8$ is represented in Figure 3. There, $L = 4$, $I_1 = \{1,2,3\}$, $I_2 = \{4\}$, $I_3 = \{5\}$, and $I_4 = \{6,7,8\}$.



A typical envy–free and efficient allocation in an 8–person example.[5]

Figure 3

[5]In order not to clutter the Figure, utilities are normalized so that $u_i(x_i) = u_j(x_j)$ for all i, j and their graphs are drawn only in some interval containing their preferred consumptions.

4. ***Other equity notions.***[6] Although the no–envy notion has remained the central concept in the literature on equitable allocation since its introduction, other notions have played important roles. In this section we apply these notions to the present context (subsections b, c), but we also propose others that make use of the specific features of the model under study (subsections d,e,f,g). First, however, we discuss solutions simply defined by operating from equal division two solutions commonly discussed in economics (subsection a).

(a) ***Individual rationality from equal division. Core from equal division.*** We start with the requirements that an allocation Pareto–dominate equal division or that it be in the core from equal division.

Individual–rational solution from equal division, I_{ed}: $x \in I_{ed}(R)$ if $x \in X$ and $x_i R_i (M/n)$ for all i.

Core from equal division, C_{ed}: $x \in C_{ed}(R)$ if $x \in X$ and there is no $S \subseteq N$ and $(x_i')_{i\in S}$ such that $\sum_{i\in S} x_i' = |S|M/n$ and $x_i' R_i x_i$ for all $i \in S$, strict preference holding for at least one $i \in S$.

In classical economies with $n = 2$, any allocation that is individually–rational from equal division is envy–free. However, as soon as $n > 2$, an allocation in the core from equal division may not be envy–free (Feldman and Kirman, 1974). Here, the same conclusions hold, although different proofs are required because of the different structures of the models.

[6]The following definition due to Pazner and Schmeidler (1978), as well as variants and extensions of it, have been very useful in other contexts: The allocation $x \in X$ is ***egalitarian–equivalent for R*** if there exists a reference amount x_0 such that $x_i I_i x_0$ for all i. Let $E^*(R)$ be the set of these allocations. Here, this concept will not be useful, since $E^*P(R)$ will typically be empty. Suppose for instance that there is $x \in X$ such that $x_i = p(R_i)$ for all i. Then, $\{x\} = P(R)$. As a result, if for at least one pair $\{i,j\}$, $p(R_i) \neq p(R_j)$, then $E^*P(R) = \emptyset$.

***Proposition* 1.** Let $x \in I_{ed}(R)$. If $n = 2$, then $x \in F(R)$. Let $x \in C_{ed}(R)$. If $n > 2$, x may not be in $F(R)$.

Proof. Let $N = \{1,2\}$ and let $x \in I_{ed}P(R)$. Without loss of generality, suppose that $x_1 \leqq M/2 \leqq x_2$. Since $x_1 R_1 (M/2)$, $p(R_1) \leqq M/2$, but then $(M/2) R_1 x_2$. Therefore, $x_1 R_1 x_2$ and 1 does not envy 2. Since $x_2 R_2 (M/2)$, $p(R_2) \geqq M/2$ but then $(M/2) R_2 x_1$. Therefore, $x_2 R_2 x_1$ and 2 does not envy 1. This proves the first claim. It is easy to construct examples showing that the inclusion $F \supseteq I_{ed}$ may be strict.

To prove the second claim, let $N = \{1,2,3\}$, $p(R) = (4,11,11)$, $e_2(9) = 12$, and $M = 24$. Let $x = (4,9,11)$. Note that $x \in X$. Since $x_i \leqq p(R_i)$ for all i, $x \in P(R)$. For each i, $x_i R_i (M/3)$. Since agents 1 and 3 receive their preferred consumptions, $\{1,3\}$ cannot improve upon x. If $\{1,2\}$ can improve upon x, it is with (y_1,y_2) such that $y_1 = x_1$ and $x_2 < y_2 < e_2(x_2)$. Then, $y_1 + y_2 < x_1 + e_2(x_2) = 4 + 12 = 16$. Since $2M/3 = 16$, no improvement is possible. If $\{2,3\}$ can improve upon x, it is with (y_2,y_3) such that $x_2 < y_2 < e_2(x_2)$ and $y_3 = x_3$. Then, $9 + 11 = x_2 + x_3 < y_2 + y_3$. Since $2M/3 = 16 < 20$, no improvement is possible. Therefore, $x \in C_{ed}(R)$. Since $x_3 P_2 x_2$, $x \notin F(R)$.

Q.E.D.

Note that $I_{ed}(R)$ is a convex subset of X. Since $P(R)$ has that property too, the solution $I_{ed}P$ is convex-valued.

Finally, we observe that unfortunately the core is not guaranteed to be non-empty.

***Proposition* 2.** There are economies R for which $C_{ed}(R) = \emptyset$.

Proof. Let $N = \{1,2,3\}$, $p(R) = (2,4,4)$, and $M = 9$. Then, $M/3 = 3$. Let $x \in X$. If given access to $2M/3$, agents 1 and 2 can both receive their preferred consumptions. For $\{1,2\}$ not to be able to improve upon x, we need $x_1 = 2$ and $x_2 = 4$. Since agents 2 and 3 have the same preferred consumptions, we also need $x_3 = 4$. But this is impossible since then $x_1 + x_2 + x_3 = 10 > M$.

Q.E.D.

(b) ***Fair treatment of groups.*** Next, we consider concepts designed to evaluate the relative treatment of groups, instead of individuals. First, we compare groups of equal cardinalities.

No–envy solution for groups, G (Schmeidler and Vind, 1972): $x \in G(R)$ if $x \in X$ and for all groups $G, G' \subseteq N$ with $|G| = |G'|$, there is no $(y_i)_{i \in G}$ such that $\sum_{i \in G} y_i = \sum_{j \in G'} x_j$ and $y_i R_i x_i$ for all $i \in G$, strict preference holding for at least one $i \in G$.

Note that if $x \in G(R)$, then $x \in P(R)$ (simply take $G = G' = N$). It might be argued that it is more natural to only compare the welfares of distinct groups, or only those of non–overlapping groups. Then, efficiency would have to be required separately.

Clearly, any group envy–free allocation is envy–free.

The set $G(R)$ can be characterized as follows: If $x \in G(R)$, recall that $x \in P(R)$ and assume without loss of generality that $\Sigma p(R_i) \geqq M$, so that $x_i \leqq p(R_i)$ for all i. Then, x passes the test if and only if for all $G, G' \subseteq N$ with $|G| = |G'|$, $\sum_{i \in G'} x_i \leqq \sum_{i \in G} x_i$ or $\sum_{i \in G} e_i(x_i) \leqq \sum_{i \in G'} x_i$.

The existence of group envy–free allocations will be discussed later on (Corollary 1).

The next definition allows us to compare the welfares of groups of different sizes.

Strong no–envy solution for groups, G^*: $x \in G^*(R)$ if $x \in X$ and for all groups $G, G' \subseteq N$, there is no $(y_i)_{i \in G}$ such that $\sum_{i \in G} y_i = \frac{|G|}{|G'|} \sum_{j \in G'} x_j$ and $y_i R_i x_i$ for all $i \in G$, strict preference holding for at least one $i \in G$.

Clearly $G^*(R) \subseteq G(R)$ and again, if $x \in G^*(R)$, then $x \in P(R)$. Also, supposing without loss of generality that $\Sigma p(R_i) \geqq M$, so that $x_i \leqq p(R_i)$ for all i, x passes the test if and only if for all $G, G' \subseteq N$, $\frac{|G|}{|G'|} \sum_{i \in G'} x_i \leqq \sum_{i \in G} x_i$ or $\sum_{i \in G} e_i(x_i) \leqq \frac{|G|}{|G'|} \sum_{i \in G'} x_i$.

(c) ***The Proportional solution.*** The next definition (mentioned by Sprumont) is based on the principle of proportionality, which underlies much of the theory of allocative fairness

(Young, 1988 quotes Aristotle: "What is just ... is what is proportional and what is unjust is what violates the proportion").

Proportional solution, Pro: $x = \text{Pro}(R)$ if $x \in X$ and there exists $\lambda \in \mathbb{R}_+$ such that $x_i = \lambda p(R_i)$ for all i.

Note that for this rule to be well–defined, the preferred consumption of at least one agent should be positive. Unless $\Sigma p(R_i) = M$, at a proportional allocation, no agent with a positive preferred consumption reaches it. Clearly, a proportional allocation is necessarily efficient, but it need not be envy–free.

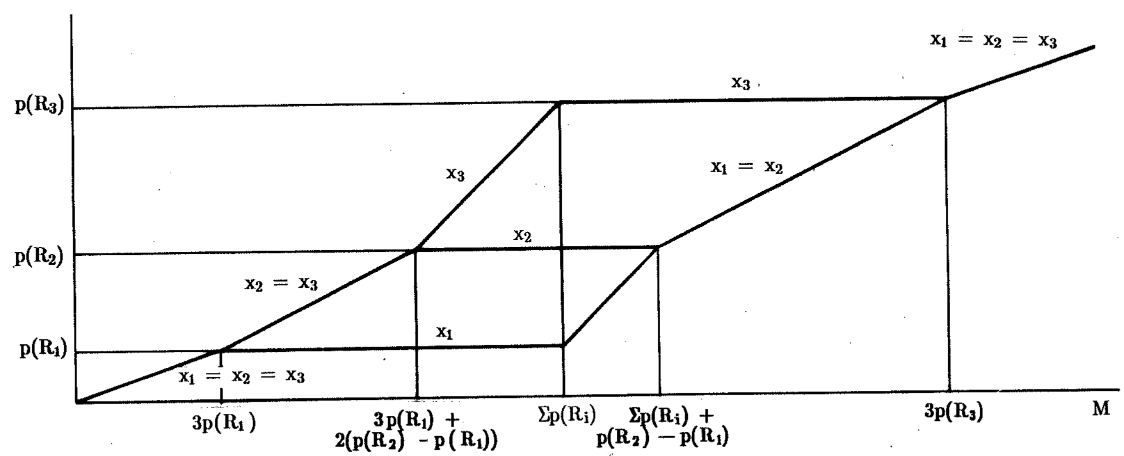
The following variant might be useful. It has the advantage of treating units of the good above or below the preferred consumptions symmetrically, as do all of the other solutions that we will discuss.

Symmetrically proportional solution, Pro*: $x = \text{Pro}^*(R)$ if $x \in X$ and (i) when $\Sigma p(R_i) \geqq M$, there exists $\lambda \in \mathbb{R}_+$ such that $x_i = \lambda p(R_i)$ for all i, and (ii) when $\Sigma p(R_i) \leqq M$, there exists $\lambda \in \mathbb{R}_+$ such that $M-x_i = \lambda[M-p(R_i)]$ for all i.

(d) ***The uniform rule.*** The following selection from the no–envy solution will play an important role below.

Uniform rule, U (Sprumont): $x = U(R)$ if $x \in X$ and (i) when $\Sigma p(R_i) \geqq M$, $x_i = \min\{p(R_i),\lambda(R)\}$ for all i, where $\lambda(R)$ solves $\Sigma\min\{p(R_i),\lambda(R)\} = M$, and (ii) when $\Sigma p(R_i) < M$, $x_i = \max\{p(R_i),\lambda(R)\}$ for all i, where $\lambda(R)$ solves $\Sigma\max\{p(R_i),\lambda(R)\} = M$.

The uniform allocation is obtained by successively making the agents who receive the least as well–off as possible. Here are the payments as a function of M (Figure 3 illustrates the rule for n=3). For M small, all agents receive the same amount; this holds until all have received an amount equal to the smallest preferred consumption. Then, the agent with the smallest preferred consumption does not receive anything for a while. Instead, any increase in M is divided equally among the remaining agents until each of them has received an amount equal to the second smallest preferred consumption. Then, the agent with the second smallest preferred consumption does not receive anything for a while... This process continues until each agent has received his preferred consumption. Any increase beyond $\Sigma p(R_i)$ goes first to the agent with the smallest preferred consumption until he has received an amount equal to the second smallest preferred consumption. A further increase is divided equally among the agents with the two smallest preferred consumption until they have received an amount equal to the third smallest preferred consumption ... This goes on until all agents have reached the largest preferred consumption. Afterwards, they share equally any further increase.[7,8] Figure 3 illustrates the rule for n=3.



An illustration of the uniform rule in the three–agent case

Figure 4

[7]This way of describing the uniform rule is not quite in agreement with the model as formulated so far, where preferences are defined over [0,M] for some fixed M. We have adopted it anyway because it is very convenient and it yields equivalent results. See Sections 5 and 6 for a further discussion of the issue of domains.

[8]The similarity between the uniform rule and the rule proposed in the Talmud for the adjudication of conflicting claims (O'Neill, 1983; Aumann and Maschler, 1985; Young, 1987) should be noted. Indeed, the algorithm describing that solution is identical up to the point where each agent has received his preferred consumption, by replacing the vector of preferred consumptions by the vectors of claims divided by two.

The uniform rule can be criticized on the grounds that it gives full satisfaction (i.e. their preferred consumptions) to some of the agents (those with low preferred consumptions if $\Sigma p(R_i) \geqq M$; those with high preferred consumptions if $\Sigma p(R_i) \leqq M$) at the expense of the others. But, Sprumont showed that it is essentially the only strategy–proof rule. On that basis alone, it should be taken very seriously. In addition, it has a number of other very desirable properties, as we will discover throughout the remainder of this paper. In fact, it satisfies many more of these properties than any other solution. However, we have:

***Proposition* 3.** The uniform rule does not satisfy no–envy for groups.

Proof. Let $N = \{1,2,3,4\}$, $p(R) = (1,2,3.5,5)$, and $M = 10.5$. Let $x = (1,2,3.5,4)$, $G = \{1,4\}$, and $G' = \{2,3\}$. Note that $U(R) = x$. Let $y_1 = 1$ and $y_4 = 4.5$. Then, $y_1+y_4 = x_2+x_3 = 5.5$, $y_1R_1x_1$ and $y_4P_4x_4$. Therefore G envies G′.

Q.E.D.

Proposition 3 confirms that no–envy for groups is indeed stronger than no–envy for individuals, since the uniform rule picks envy–free allocations. The next result relates the uniform rule to two of the criteria seen earlier.

***Proposition* 4.** The uniform allocation Pareto–dominates equal division but it may not be in the core from equal division, even if the latter is not empty.

Proof. Let R be given and $x = U(R)$. Without loss of generality, suppose that $\Sigma p(R_i) \geqq M$. Then, $x_i \leqq p(R_i)$ for all i. For any i such that $x_i = p(R_i)$, then of course $x_iR_i(M/n)$. If $x_i < p(R_i)$ for some i, then $x_j \leqq x_i$ for all j, which implies $M/n \leqq x_i$. But then again $x_iR_i(M/n)$. This proves the first claim.

To prove the second claim, let $N = \{1,2,3\}$, $p(R) = (2,5,6)$, $e_2(4) = 5.5$, and $M = 12$. Then, $U(R) = (2,5,5)$. Note that $x = (2,4,6) \in C_{ed}(R)$. Indeed, $x_iR_i(M/3)$ for all i, so that $x \in I_{ed}(R)$. If given access to $2M/3 = 8$, $\{1,3\}$ cannot improve upon x since both agents receive their preferred consumptions at x. For $\{1,2\}$ to improve upon x with

(y_1,y_2), we need $y_1 = 2$ and $y_2 \in]4,5.5[$ so that $y_1+y_2 < 2+5.5 = 7.5 < 8$. For {2,3} to improve upon x with (y_2,y_3), we need $y_2 \in]4,5.5[$ and $y_3 = 6$ so that $y_2 + y_3 > 4+6 = 10 > 8$. Finally $x \in P(R)$. Therefore, $C_{ed}(R) \neq \emptyset$. However, if given access to $2M/3 = 8$, {1,3} can improve upon (2,5) with (2,6).

Q.E.D.

(e) ***The Equal–distance solution.*** We suggest next to compare distances from preferred consumptions unit for unit as opposed to proportionately (in contrast with c), and to select allocations at which all agents are equally far from their preferred consumptions. It is, of course, not in general possible to ensure that all agents be at the same distance from their preferred consumptions since boundary problems may occur. So, instead, we offer the following definition.

Equal–distance solution, Dis: $x = Dis(R)$ if $x \in X$ and (i) when $\Sigma p(R_i) \geqq M$, there exists $a \geqq 0$ such that $x_i = \max\{0,p(R_i)-a\}$ for all i, and (ii) when $\Sigma p(R_i) \leqq M$, there exists a $\geqq 0$ such that $x_i = p(R_i) + a$ for all i.

The proportional solution (and the symmetrically proportional solution), as well as the equal–distance solution produce efficient, but not necessarily envy–free, allocations.

(f) ***The preferred–consumption–maximizing solutions.*** Since the uniform allocation is defined by giving their preferred consumption to the agents with the smallest preferred consumption, one might think that at that allocation, the number of agents reaching their preferred consumptions is greater than at any other envy–free and efficient allocation. But this is not true: Let N = {1,2}, p(R) = (4,5), $e_1(1) = 5$, and M = 6. Then, U(R) = (3,3). However, at $y = (1,5) \in FP(R)$, agent 2 reaches his preferred consumption while at U(R), no one does. We propose below selection procedures from P and FP based on the number of agents reaching their preferred consumptions.

Given $x \in X$, let $pk(x,R) = |\{i | x_i = p(R_i)\}|$.

Definition. $P^*(R) = \{x \in P(R) | pk(x,R) \geqq pk(y,R)$ for all $y \in P(R)\}$.

Definition. $H^*(R) = \{x \in FP(R) | pk(x,R) \geqq pk(y,R)$ for all $y \in FP(R)\}$.

Neither one of these solutions is single–valued.

(g) ***The Equal–sacrifice solution.*** Finally, we propose evaluating allocations $x \in X$ on the basis of the difference between what agent i receives and what the equivalent bundle is on the other side of his preferred consumption. This is because the number $c_i(x) = |e_i(x_i) - x_i|$ is a measure of the size of agent i's "upper contour set at x_i", that is, the size of his "sacrifice at x". Selecting efficient allocations for which sacrifices are equal across agents would of course be appealing but, such allocations will not exist in general. As second best, we recommend the allocation at which sacrifices are as "equal as possible".

Equal–sacrifice allocation, Sac: $x = Sac(R)$ if (i) when $\Sigma p(R_i) \geqq M$, there exists $\sigma \geqq 0$ such that $e_i(x_i) - x_i \leqq \sigma$ for all i, strict inequality holding only if $x_i = 0$, and (ii) when $\Sigma p(R_i) \leqq M$, there exists $\sigma \geqq 0$ such that $x_i - e_i(x_i) \leqq \sigma$ for all i, strict inequality holding only if $x_i = M$.

It is easy to check that this solution is well–defined. It is not so straightforward however to define a similar selection from the envy–free and efficient correspondence. One possibility is to pick the allocation at which the vector of sacrifices is lexicographically maximal. Selection procedures based on lexicographic operations are standard in game theory and social choice.

Formally, let $\tilde{t}$ be obtained by rewriting the coordinates of $t \in \mathbb{R}^n$ in decreasing order. Given t and $t' \in \mathbb{R}^n$, say that ***t is lexicographically smaller than t'***, written $t <_L t'$, if $[\tilde{t}_1 < \tilde{t}'_1]$, or $[\tilde{t}_1 = \tilde{t}'_1$ and $\tilde{t}_2 < \tilde{t}'_2]$, or ... Also, let $c(x) = (c_1(x),\ldots,c_n(x))$.

Definition. $H^{**}(R) = \{x \in FP(R) | c(x) \leqq_L c(y)$ for all $y \in FP(R)\}$.

In contrast with rules c, d, e, f, the rules defined here have the advantage of not depending only on the preferred consumptions.

5. ***Resource monotonicity.*** In the next sections, we turn to properties of solutions pertaining to changes in the parameters of the problem. First, we consider changes in the amount to be divided, limiting our attention to single–valued solutions. An economy is now denoted by the pair (R,M).

Up to this point, we have always taken preferences to be defined over the interval [0,M], where M is whatever amount is to be divided. This implies that even in situations where the natural domain of definition of preferences would be larger, only their restriction to the set of "realistic" consumptions of each agent was deemed relevant for the evaluation of an allocation. Since the property we study next pertains to variations in resources, for its formulation to be in line with the above interpretation, the domain over which preferences are defined should be allowed to change.

Then we have two choices. Either we require solutions to depend only on the restriction of preferences to their realistic domain, or we allow them to depend on features of preferences outside of their realistic domain. The latter choice enlarges the class of possible solutions. To illustrate, there now exist two natural ways of defining the proportional solution. Given a preference relation R_i defined on $\mathbb{R}$ and given $M \in \mathbb{R}$, let $\boldsymbol{p(R_i)}$ be the preferred consumption of R_i on $\mathbb{R}$ (the "unconstrained" preferred consumption of R_i), and $\boldsymbol{p_M(R_i)}$ be the preferred consumption of R_i on [0,M] (the "constrained" preferred consumption of R_i). Using this notation, our earlier definition of the proportional allocation of (R,M) would be the feasible allocation x such that for some λ, $x_i = \lambda p_M(R_i)$ for all i. If each $p(R_i)$ is finite, we can now give an alternative definition, by choosing the feasible allocation x such that for some λ, $x_i = \lambda p(R_i)$ for all i. Most of the results to be established below do not depend on which formulation is adopted and we will limit ourselves to the first one.

Suppose now that the amount to be divided increases. If agents had monotone preferences, it would be desirable that all gain. This property has been the object of much attention recently in the context of classical economies (Chun and Thomson, 1988;

Moulin and Thomson 1988). This is also what we would like to require here if even after the increase there is not too much of the good, so that each agent is initially to the left of his preferred consumption and remains so, as efficiency requires. If what is to be divided increases so much that agents are forced beyond their preferred consumptions, then, at some point, some of them may have to be negatively affected, and the natural requirement in that case is that if one of them is indeed negatively affected, then they all are. If there is initially "too much" of the good, so that all agents have already passed their preferred consumptions, the opposite would be appealing: all agents should continue losing. In general, the property that we will want to consider is that all agents be affected in the same direction: all lose or all gain.

Resource–monotonicity. For all (R,M), (R,M′), either $\varphi_i(R,M')R_i\varphi_i(R,M)$ for all i, or $\varphi_i(R,M)R_i\varphi_i(R,M')$ for all i. ***Strict resource monotonicity*** holds if, in addition, whenever one of the preferences is strict, then they all are.

We omit the straightforward proof of the following result.

***Proposition* 5.** The uniform rule, the proportional solution, the equal–distance solution, and the equal–sacrifice solution are all resource monotonic.

Note that the uniform rule is not strictly resource–monotonic. Indeed, as a function of the amount to be divided, what each agent i receives increases to $p(R_i)$, remains stationary for a while, then increases again (Figure 4). Correspondingly, his utility increases, then remains fixed, then decreases. Similarly, neither the equal–distance solution nor the equal–sacrifice solution are strictly resource monotonic.

The proportional solution is strictly resource–monotonic if all preferred consumptions are positive.

By redefining the equal–distance and equal–sacrifice solutions by using the whole preference relations instead of their restrictions to the interval [0,M],[9] then we would obtain two strictly resource–monotonic solutions.

Does there exist a strictly resource–monotonic selection from the envy–free and efficient correspondence? Certainly, the answer cannot be yes if no restrictions are made on preferences: If an agent's preferred consumption is 0, then efficiency requires him to consume 0, for any $M \in [0, \Sigma p(R_i)]$. The question in the case of all positive preferred consumptions is open.

6. ***Variable population.*** Next, we formulate properties pertaining to changes in the number of agents. Previous studies of the problem of fair allocation in economies of variable size are Chichilnisky and Thomson (1987), Thomson (1988) and Tadenuma and Thomson (1989a,b). The first two references consider classical economies and the latter two economies with indivisible goods.

To allow for changes in the number of agents, the notation has to be adapted again. There is now an infinite population of "potential agents", indexed by the integers, $\mathbb{N}$. Let $\mathscr{P}$ be the class of finite subsets of $\mathbb{N}$. An ***economy*** is a pair $e = ((R_i)_{i \in Q}, M)$, or simply (R_Q, M), where $Q \in \mathscr{P}$. Let $X(e)$ be the set of feasible allocations of e. As in the previous section, we will assume preferences to be defined over $\mathbb{R}_+$. (However, see the remark following Proposition 10.)

The following property pertains to single–valued solutions. On classical domains where preferences are monotone, it is natural to require that all agents initially present be affected negatively when newcomers arrive with claims as valid as theirs, resources being kept fixed. In the present context, this requirement can also be imposed if in the initial economy, agents have not reached their preferred consumptions. On the other hand, if

[9]In the case of the equal–distance solution, this requires all preferred consumptions to be finite; in the case of the equal–sacrifice solution, this requires that the function e_i be well–defined on $]0,\infty[$.

there is so much of the good that agents are initially beyond their preferred consumptions, the arrival of additional agents may help and it becomes natural to require that if one of the agents initially present gains, then they all do. In general, the requirement that will be useful here is that all agents be affected in the same direction, as Chun (1986) had proposed in a different context.

Population Monotonicity: For all Q, $Q' \in \mathcal{P}$ with $Q' \subset Q$, for all (R_Q,M), either $\varphi_i(R_Q,M) R_i \varphi_i(R_{Q'},M)$ for all $i \in Q'$, or $\varphi_i(R_{Q'},M) R_i \varphi_i(R_Q,M)$ for all $i \in Q'$. ***Strict Population Monotonicity*** holds if in addition, in each of the previous cases, whenever one of the preferences is strict, then they all are.

Our first result is rather negative:

***Proposition* 6.** The uniform rule, the proportional solution, and the equal–distance solution do not satisfy Population Monotonicity.

Proof: To prove the result for *U*, let $Q' = \{1,2\}$, $p(R_{Q'}) = (1,3)$, and $M = 5$. Then, $U(R_{Q'},M) = (2,3)$. Let $Q = \{1,2,3\}$ with $p(R_3) = 2$. Then, $U(R_Q,M) = (1,2,2)$. Agent 1 gains from agent 3's arrival, whereas agent 2 loses.

To prove the result for ***Pro***, let $Q' = \{1,2\}$, $p(R_{Q'}) = (1,2)$, $e_1(.5) = 1.5$, $e_2(1) = 5$ and $M = 6$. Then, $Pro(R_{Q'},M) = (2,4)$. Let $Q = \{1,2,3\}$ with $p(R_3) = 9$. Then, $Pro(R_Q,M) = (.5,1,4.5)$. Agent 1 gains from agent 3's arrival, whereas agent 2 loses.

To prove the result for ***Dis***, $Q' = \{1,2\}$, $p(R_{Q'}) = (1,2)$, $e_1(.5) = 1.5$, $e_2(1.5) = 4$, and $M = 5$. Then, $Dis(R_{Q'},M) = (2,3)$. Let $Q = \{1,2,3\}$ with $p(R_3) = 3.5$. Then, $Dis(R_Q,M) = (.5,1.5,3)$. Agent 1 gains from agent 3's arrival, whereas agent 2 loses.

Q.E.D.

Another open question is whether there exist population monotonic, or perhaps strictly population monotonic, selections from the envy–free and efficient solution? As far as the stronger property is concerned, the answer cannot be an unqualified yes since if an agent's preferred consumption is zero, then he should get 0 at any efficient allocation in any economy such that $\Sigma p(R_i) \leqq M$. The inequality is preserved by the addition of new agents.

It should be noted that population monotonic selections from the Pareto solution exist; the equal–sacrifice solution is one, as formally stated next. We omit the straightforward proof of this result.

***Proposition* 7**: The equal–sacrifice solution satisfies Population Monotonicity.

Next, we consider a property that has played a very important role in some recent literature (reviewed in Thomson, 1989a). Essentially, it says that if x provides a desirable division of M among some group Q, then the restriction of x to any subgroup $Q' \subset Q$ constitutes a desirable division among the members of that subgroup of what they have collectively received at x, namely, $\sum_{i \in Q'} x_i$.

Consistency* (*Cons*)**: For all $Q, Q' \in \mathcal{P}$ with $Q' \subset Q$, for all $e = (R_Q, M)$, for all $x \in \varphi(e)$, $x_{Q'} \in \varphi(t^x_{Q'}(e))$, where $t^x_{Q'}(e) = (R_{Q'}, \sum_{i \in Q'} x_i)$. (The economy $t^x_{Q'}(e)$ will be called a ***reduced economy.) ***Bilateral Consistency* (*B.Cons*)** is the weakening of consistency obtained by requiring $|Q'| = 2$.

The Pareto solution is consistent; so is the no–envy solution and so is their intersection (Consistency is preserved under intersection). The uniform rule is consistent too. The proportional solution is consistent if it is defined by imposing proportionality to the unconstrained preferred consumptions but not if the constrained preferred consumptions are used (see Section 5 for a discussion of these two formulations). None of the individually–rational from equal division and efficient correspondence, the core correspondence from equal division or the preferred–consumption–maximizing solution of section 4(f) are consistent. We only give proofs for one positive result and one negative result.

***Proposition* 8.** The uniform rule satisfies Consistency. The preferred consumption–maximizing selection from FP does not.

Proof. Let $e = (R_Q, M)$ and $x = U(e)$. Without loss of generality, we suppose that $\sum_{i \in Q} p(R_i) \geqq M$. Then, for all $i \in Q$, $U_i(e) = \min\{p(R_i), \lambda(e)\}$ where $\lambda(e)$ solves $\sum_{i \in Q} \min\{p(R_i), \lambda(e)\} = M$. Since $U_i(e) \leqq p(R_i)$ for all $i \in Q$, it follows that given any $Q' \subset Q$, $\sum_{i \in Q'} U_i(e) \leqq \sum_{i \in Q'} p(R_i)$. Therefore, in the reduced economy $e' = (R_{Q'}, M')$ where $M' = \sum_{i \in Q'} x_i$, we have $M' \leqq \sum_{i \in Q'} p(R_i)$. Then, for all $i \in Q'$, $U_i(e') = \min\{p(R_i), \lambda(e')\}$ where $\lambda(e')$ solves $\sum_{i \in Q'} \min\{p(R_i), \lambda(e')\} = M'$. But the solution to this equation is $\lambda(e) = \lambda(e')$. Therefore, $U_i(e') = U_i(e)$ for all $i \in Q'$. U does not satisfy *Cons.*

To prove the claim concerning H^*, let $Q = \{1,2,3,4\}$ and $M = 12$. Suppose that $p(R) = (2,3,4,5)$ and that $e_1(1) = 3$ and $e_2(1) = 4$. Let $x = (2,2,4,4)$. Then $pk(x,R) = 2$. We claim that there is no $y \in FP(R)$ with $pk(y,R) > 2$. There are four cases to consider.

(i) $y_i \neq p(R_i)$ only for $i = 1$. Then, by feasibility, $y_1 = 0$ and 1 envies 2.

(ii) $y_i \neq p(R_i)$ only for $i = 2$. Then, by feasibility, $y_2 = 1$ and 2 envies 1.

(iii) $y_i \neq p(R_i)$ only for $i = 3$. Then, by feasibility, $y_3 = 2$ and 3 envies 2.

(iv) $y_i \neq p(R_i)$ only for $i = 4$. Then, by feasibility, $y_4 = 3$ and 4 envies 3.

Clearly, there is no y with $pk(y,R) = 4$. Therefore, $x \in H^*(R)$.

Now, let $Q' = \{2,4\}$ and $e' = (R_{Q'}, \sum_{i \in Q'} x_i)$. Note that $pk(x_{Q'}, R_{Q'}) = 0$. Also, $H^*(e') = \{(3,3),(1,5)\}$ since at each of these two allocations y, $pk(y, R_{Q'}) = 1$ and there is no $y \in X(e')$ with $pk(y, R_{Q'}) = 2$. Hence $x_{Q'} \notin H^*(e')$. H^* does not satisfy *Cons.*

Q.E.D.

We now ask whether there exist consistent selections from the no–envy and efficient solution other than the uniform rule. The answer is a very limited yes. Indeed, our next result is that essentially any such solution has to contain the uniform rule. The only additional requirement we use is the very mild requirement that small changes in the amount to be distributed never lead to large changes in the desired allocation. A characterization of the uniform rule can then be obtained by demanding the rule to be single–valued.

M–Continuity (M–Cont): Let $\{M^\nu\}$ be a sequence in $\mathbb{R}_+$ converging to M. For each $\nu \in \mathbb{N}$, let $x^\nu \in \varphi(R_Q,M^\nu)$. If $x^\nu \to x$, then $x \in \varphi(R_Q,M)$.

***Proposition* 9.** If a subsolution of the envy–free and efficient solution satisfies Consistency and M–Continuity, then it contains the uniform rule.

***Proof*.** Let $e = (R_Q,M)$ be given and let $x = U(e)$. Without loss of generality, suppose that $\sum_{i\in Q} p(R_i) \geqq M$.

Let $0 < a_1 < a_2 \ldots < a_L$ be the ordered list of distinct non–zero amounts received by the members of Q at x. Since the case $M = 0$ is trivial, we assume that $L \geqq 1$. Let $Q_1 = \{i \in Q | x_i = 0 \text{ or } x_i = a_\ell \text{ for } \ell < L\}$ and $Q_2 = Q\backslash Q_1$ (note that Q_1 could be empty). Since $\sum_{i\in Q} p(R_i) \geqq M$, it follows from the definition of U that $x_i = p(R_i)$ for all $i \in Q_1$ and that $x_i \leqq p(R_i)$ for all $i \in Q_2$.

The economy $e = (R_Q,M)$ is now augmented as follows. Let $\tilde{Q}$ be a group of L new agents indexed by ℓ, $Q' = Q \cup \tilde{Q}$, and $M' = M + \sum_{\ell\in\tilde{Q}} a_\ell$. Let $\tilde{\mathbb{N}} = \{\nu \in \mathbb{N} | 1/\nu \leqq a_1\}$. In what follows, ν will be an arbitrary element of $\tilde{\mathbb{N}}$. For each $i \in Q$, let $R_i^\nu = R_i$. For each $\ell \in \tilde{Q}$, let R_ℓ^ν be such that $p(R_\ell^\nu) = a_\ell$ and $e_\ell(a_\ell - 1/2\nu) = M'$. Let $e^\nu = (R_{Q'}^\nu, M')$. Let $x' \in \mathbb{R}^{Q'}$ be such that $x'_Q = x$ and $x'_\ell = a_\ell$ for all $\ell \in \tilde{Q}$.

Note that $x' \in X(e^\nu)$. Also, since $\sum_{i\in Q} p(R_i) \geqq M$ by hypothesis, $p(R_i^\nu) = p(R_i)$ for all $i \in Q$ by construction, and $\sum_{\ell\in\tilde{Q}} p(R_\ell^\nu) = a_\ell$ also by construction, it follows that

$\sum_{i \in Q'} p(R_i^\nu) \geqq M + \sum_{\ell \in \tilde{Q}} a_\ell = M'$. Therefore, if $\bar{x} \in P(e^\nu)$, then $\bar{x}_i \leqq p(R_i^\nu)$ for all $i \in Q'$. Since x' has that property, $x' \in P(e^\nu)$.

Now, let $y^\nu \in \varphi(e^\nu)$. Since $\varphi \subseteq P$, $y_i^\nu \leqq p(R_i^\nu)$ for all $i \in Q'$, as we just saw. Therefore, (i) $y_i^\nu \leqq x'_i$ for all $i \in Q_1$ and $y_\ell^\nu \leqq x'_\ell$ for all $\ell \in \tilde{Q}$. Obviously, (ii) $y_i^\nu \leqq M'$ for all $i \in Q_2$.

We now claim that (iv) $p(R_\ell^\nu) - 1/\nu \leqq y_\ell^\nu \leqq p(R_\ell^\nu) = a_\ell$ for all $\ell \in \tilde{Q}$. If not, by (i), there is $\ell \in \tilde{Q}$ such that $y_\ell^\nu < p(R_\ell^\nu) - 1/\nu$. Then, for agent ℓ not to envy any $i \in Q_2$ at y^ν, we need $y_i^\nu \leqq y_\ell^\nu$ or $y_i^\nu \geqq e_\ell^\nu(y_\ell^\nu) > M'$. Using (i) and (ii) we then obtain (iii) $y_i^\nu \leqq y_\ell^\nu \leqq x'_\ell \leqq x'_i$ for all $i \in Q_2$. But then (i) and (iii) together imply $\sum_{i \in Q'} y_i^\nu \leqq \sum_{i \in Q} x'_i + \sum_{\ell' \in \tilde{Q} \setminus \{\ell\}} x'_{\ell'} + x'_\ell - 1/\nu = \sum_{i \in Q'} x'_i - 1/\nu = M' - 1/\nu < M'$, in contradiction with $y^\nu \in X(e^\nu)$. This proves (iv).

For each $i \in Q$ with $x_i > 0$, let $\ell \in \tilde{Q}$ be such that $a_\ell = x_i$. From (i), agent i does not envy agent ℓ at y^ν only if (v) $y_i^\nu \geqq y_\ell^\nu$. Together with (i) and (iv), we conclude that (vi) $x'_i - 1/\nu \leqq y_i^\nu \leqq x'_i$ for all $i \in Q_1$. Now (iv), (v), and (vi) together with the feasibility condition $\sum_{i \in Q'} y_i^\nu = M'$ imply (vii) $x'_i - 1/\nu \leqq y_i^\nu \leqq x'_i + (|Q'|-1)/\nu$ for all $i \in Q_2$. Therefore, by (vi) and (viii), $y_Q^\nu \to x$ as $\nu \to \infty$.

By ***Cons***, $y_Q^\nu \in \varphi(t_Q^{y^\nu}(e^\nu))$, where $t_Q^{y^\nu}(e^\nu) = (R_Q, M^\nu)$ and $M^\nu = \sum_{i \in Q} y_i^\nu$ satisfies $|M^\nu - M| \leqq |Q|/\nu$. As $\nu \to \infty$, $y_Q^\nu \to x$ and $M^\nu \to M$. By ***M–Cont***, $x \in \varphi(e)$.

Q.E.D.

Remark. We indicate here how the proof would have to be modified if solutions were required to depend only on the restrictions of preferences to [0,M] where M is whatever amount is to be divided. First of all, in the statement of Consistency the definition of a reduced economy should be changed to $(R_{Q'}|[0,M'], M')$ where $M' = \sum_{i \in Q'} x_i$. Then, for each $i \in Q$, take R_i^ν to be a preference relation on $[0,M']$ whose restriction to $[0,M]$ is R_i

and such that $p(R_i^\nu) = p(R_i)$. (The need to specify the preferred consumption of R_i^ν may arise only for $i \in Q_2$.) For each $\ell \in \tilde{Q}$, take R_ℓ^ν to be a preference relation on $[0,M']$ such that $p(R_\ell^\nu) = a_\ell$ and $e_\ell(a_\ell - 1/2\nu) = M$. In the last step, consider the reduced economy $(R_{Q|[0,M^\nu]}, M^\nu)$.

The next two results are direct consequences of Proposition 9.

***Theorem* 1.** The uniform rule is the only single–valued selection from the envy–free and efficient solution satisfying Consistency and M–Continuity.

***Corollary* 1.** There are economies that have no group envy–free allocations.

Proof. This is a consequence of the fact that the group no–envy solution G satisfies the hypotheses of Proposition 9 and that U is not a subsolution of G, as established in Proposition 3.

Q.E.D.

Consider now the following converse of consistency, which allows us to deduce the desirability of an allocation for some economy from the desirability of its restrictions for the associated reduced economies: Given a feasible allocation x for the large economy, if for every subgroup of two agents, the restriction of x to that subgroup provides an equitable way of allocating between them the sum of what they have received, then x itself is desirable for the whole economy. Many of the models where conditions of this type have been found useful are reviewed in Thomson (1989a).

***Converse Consistency* (*Conv.Cons*):** For all $e = (R_Q, M)$, for all $x \in X(e)$, if for all $Q' \subset Q$ with $|Q'|=2$, $x_{Q'} \in \varphi(R_{Q'}, \sum_{i \in Q'} x_i)$, then $x \in \varphi(e)$.

***Proposition* 10.** The Pareto solution, the no–envy solution, their intersection, and the uniform rule all satisfy Converse Consistency. However, neither the individually rational from equal division and efficient solution nor the core from equal division do.

Proof. Let $e = (R_Q,M)$ and $x \in X(e)$. Suppose $x_{Q'} \in P(R_{Q'},\sum_{i\in Q'} x_i)$ for all $Q' \subset Q$ with $|Q'| = 2$. Using the characterization of P, we conclude that for every such group Q', $x_i \leqq p(R_i)$ for all $i \in Q'$ or $x_i \geqq p(R_i)$ for all $i \in Q'$. This implies that either $x_i \leqq p(R_i)$ for all $i \in Q$, or $x_i \geqq p(R_i)$ for all $i \in Q$. Therefore, $x \in P(e)$. P satisfies ***Conv.Cons.***

The result for F and U is clear, and we omit the proofs. The result for FP follows then from the fact that ***Conv.Cons*** is preserved under intersection.

Now, let $N = \{1,2,3\}$, $p(R) = (1,3,7)$, $e_2(1) < 4$, and $M = 9$. Also, let $x = (1,1,7)$. Note that $x \in P(R)$. Also, since agents 1 and 3 receive their preferred consumptions, their consumptions dominate equal–division in any reduced economy they may belong to. Finally, observe that $x_2 I_2[(x_1+x_2)/2]$ and that $x_2 R_2[(x_2+x_3)/2] = 4$. Therefore, $x_{Q'} \notin I_{ed}(R_{Q'},\sum_{i\in Q'} x_i)$ for all $Q' \subsetneq N$. However, $x \notin I_{ed}(M/3)$ since $M/3 = 3P_2x_2$. Therefore, $I_{ed}P$ does not satisfy ***Conv.Cons.***

Since in the previous example, it so happens that $x_{Q'} \in C_{ed}(R_{Q'},\sum_{i\in Q'} x_i)$ for all $Q' \subsetneq N$, if follows that C_{ed} does not satisfy ***Conv.Cons*** either.

Q.E.D.

Next, we turn to the group no–envy solution. This solution coincides with the no–envy solution in the 2–person case, so we should of course not expect it to satisfy Converse Consistency. In order for the hypotheses of Converse Consistency to have any power, they should say something for subgroups of at least four persons since this is the smallest number of agents needed for a meaningful comparison of the welfares of groups (i.e. groups containing more than one person). In the next Proposition, Converse Consistency should be understood in this weaker sense. It will be referred to as ***Weak Converse Consistency.*** Yet we have:

***Proposition* 11.** The group–no–envy solution does not satisfy Weak Converse Consistency.

Proof. Let $Q = \{1,2,...,6\}$, $R_1 = R_2$, $e_1(0) = .4$, $R_3 = R_4 = R_5$ and $e_3(1) = 1.5$, $e_6(2) = 2.5$, and $M = 5$. Let $e = (R_Q,M)$ and $x = (0,0,1,1,1,2)$.

It is easy to check that at x no "one–person" group envies any other "one–person" group. The laborious proof that no group of cardinality 2 envies any other such group is relegated to the appendix.

We conclude by noting that {1,2,6} envies {3,4,5}. Indeed, let $y_1 = y_2 = .25$ and $y_6 = 2.5$. We have $y_1+y_2+y_6 = x_3+x_4+x_5$. Also, $y_1P_1x_1$, $y_2P_2x_2$ and $y_6R_6x_6$. Therefore, G does not satisfy ***Conv.Cons.***

Q.E.D.

For the next result, we return to Converse Consistency as originally stated.

***Theorem* 2.** The uniform rule is the only single–valued selection from the envy–free and efficient solution satisfying Bilateral Consistency, Converse Consistency, and M–Continuity.

Proof. From the proof of Proposition 9, we deduce that if $\varphi \subseteq FP$ satisfies ***B.Cons*** and ***M–Cont,*** then $\varphi(e) \supseteq U(e)$ for all $e = (R_Q,M)$ with $|Q| = 2$. Since both U and φ satisfy ***Conv.Cons,*** $\varphi(e) \supseteq U(e)$ for all $e = (R_Q,M)$ with $|Q| > 2$. But by single–valuedness of φ, $\varphi = U$.

Q.E.D.

We conclude this section with another characterization of the uniform rule based on individual rationality from equal division instead of no–envy.

***Theorem* 3.** The uniform rule is the only single–valued selection from the individually–rational from equal division and efficient correspondence satisfying M–Continuity and Bilateral Consistency.

Proof. We have already seen that U satisfies all the properties listed in the Theorem. Conversely, let φ be a solution satisfying all the properties. We first show that $\varphi = U$ for all $e = (R_Q,M)$ with $|Q| = 2$. Let $x = U(e)$. Without loss of generality, suppose that $\sum_{i \in Q} p(R_i) \geqq M$, $Q = \{1,2\}$, and $x_1 \leqq x_2$. If $x_1 = x_2$, then $x_1 = x_2 = M/2$ and $I_{ed}(e) = \{x\}$. If $x_1 < x_2$, then by definition of U, $x_1 = p(R_1)$. If $x_1 = 0$, then $P(e) = \{x\}$. If $x_1 > 0$, let $n \in \mathbb{N}$ be such that $0 \leqq y_0 = (x_2 - x_1)/(n-2) < x_1$. Note that $n > 2$. Let $x_0 = x_1 - y_0$. For all $\nu \in \mathbb{N}$ such that $1/\nu \leqq x_0$, let $e^\nu = (R^\nu_{Q'}, M')$ be such that $Q' = \{1,\ldots,n\}$, $M' = nx_1 (= M + (n-2)x_0)$, $R^\nu_1 = R_1$, $R^\nu_2 = R_2$, and for each $i \in \{3,\ldots,n\}$, R^ν_i satisfies $p(R^\nu_i) = x_0$ and $e^\nu_i(x_1) = x_0 - 1/\nu$.

Let $y^\nu = \varphi(e^\nu)$. Note that $\sum_{i \in Q'} p(R^\nu_i) = \sum_{i \in Q} p(R_i) + (n-2)x_0 \geqq M + (n-2)x_0 = M'$, and since $\varphi \subseteq P$, it follows that (i) $y^\nu_i \leqq p(R^\nu_i)$ for all i. Since $M'/n = x_1$, and $\varphi \subseteq I_{ed}$, it follows that (ii) $y^\nu_1 = x_1$ and (iii) for all $i \in \{3,\ldots,n\}$, $y^\nu_i \in [x_0 - 1/\nu, x_1]$. By (i) and (iii), we obtain that (iv) for all $i \in \{3,\ldots,n\}$, $y^\nu_i \in [x_0 - 1/\nu, x_0]$. Then, (ii) and (iv) together yield (v) $y^\nu_2 \in [x_2, x_2 + (n-2)/\nu]$.

By *B.Cons*, $(y^\nu_1, y^\nu_2) = \varphi(R_Q, y^\nu_1 + y^\nu_2)$. By (ii) and (v), $(y^\nu_1, y^\nu_2) \to (x_1, x_2)$ and $y^\nu_1 + y^\nu_2 \to M$. By *M.Cont*, $x = \varphi(e)$. Therefore, φ coincides with U for two–person economies.

Now, let $e = (R_Q, M)$ be an arbitrary economy and let $x = \varphi(e)$. By *B.Cons*, $x_{Q'} = \varphi(R_{Q'}, \sum_{i \in Q'} x_i)$ for all $Q' \subset Q$ with $|Q'| = 2$. By the previous paragraph, $x_{Q'} = U(R_{Q'}, \sum_{i \in Q'} x_i)$. Since U satisfies *Conv.Cons*, $x = U(e)$.

Q.E.D.

7. ***Economies with a large number of agents.*** In classical economies, the case of a large number of consumers is of particular interest. Indeed, when preferences are smooth and sufficiently diverse, the equal–income Walrasian allocations are the only envy–free and efficient allocations (Varian, 1974). In the present context, one solution also plays a special role. Once again, it is the uniform rule!

Here we will assume for convenience that preferences are defined over the interval [0,M] for some fixed M. For the problem to remain interesting, we will increase the amount to be divided along with the number of agents, and assume that the per capita amount to be divided is always equal to M.

One standard way of enlarging an economy is simply by replication. Note first that most of the solutions we discussed are invariant under replication. For example, take the no–envy solution. If e^k is obtained by replicating k times some model economy e, and two agents are of the same type in e^k, then they receive the same amount at any allocation in $F(e^k)$. If $y \in F(e^k)$, then y is the k–replica of some $x \in F(e)$. Conversely, the replica of any $x \in F(e)$ is an element of $F(e^k)$.[10] (Similarly, the individual rational from equal division and efficient solution, the uniform, equal–distance, and equal–sacrifice rules are all invariant under replication.) Here too, it is only when preferences are sufficiently diverse that the set of envy–free allocations will shrink. The sketch of a simple convergence result appears in the appendix.

Can any statement be made about $G^*(e^k)$ as $k \to \infty$? Since G may be empty (Corollary 1) and $G^* \subseteq G$, there is no assurance that $G^*(e^k)$ remains non–empty for all k. However, if it remains non–empty for all k, then the only allocation that can belong to all replicas, when correspondingly replicated, is U(e):

***Theorem* 4.** Let e be some economy and $x \in P(e)$. If for each k, the k–replica of x belongs to $G^*(e^k)$, then $x = U(e)$.

Proof. Without loss of generality, suppose that $\Sigma p(R_i) \geqq M$ so that if $x \in P(e)$, then $x_i \leqq p(R_i)$ for all i. If $y \in P(e^k)$, it remains true that $y_i \leqq p(R_i)$ for all i. Let $x \in P(e)$. Let $a_1 < a_2 < \ldots < a_L$ be the various distinct amounts received at x by the agents in e. For each $\ell < L$, let i be such that $x_i = a_\ell$. Suppose that $x_i < p(R_i)$. Then there are $K, k' \in \mathbb{N}$ such that $x_i < \frac{k'x_i + (K-k')a_{\ell+1}}{K} < p(R_i)$. In the K–replica of e, let G′ be a group composed of k′ agents of type i and K − k′ agents of type j, where j is any agent such that $x_j = a_{\ell+1}$. In e^K, any one–person group composed of one agent of type i

[10]The no–envy solution is also invariant under replication in classical economies.

would prefer the average holdings of any such group G′. Therefore $y \notin P(e^k)$.

Q.E.D.

8. ***The case of different initial endowments.*** A slightly different situation from the one considered up to now is when each agent is initially endowed with some amount of the good and the problem is to reallocate these initial endowments in some fair way. This is more typical of situations where rationing has to take place.

Let $\omega_i \in \mathbb{R}_+$ be agent i's initial endowment and $X_i = [a_i,b_i]$ be his consumption set, satisfying $\omega_i \in X_i$. An economy is now denoted $((R_i)_{i \in N},(\omega_i)_{i \in N},(X_i)_{i \in N})$, or simply (R,ω,X). The set of feasible allocations is the set $\overline{X} = \{(x_i)_{i \in N} \mid \sum_{i \in N} x_i = \sum_{i \in N} \omega_i$ and for each i, $x_i \in X_i\}$. We will still designate by M the total resources of the economy: $M = \sum_{i \in N} \omega_i$.

The traditional concepts, as well as the more specific solutions that we have found useful above, can be adapted with no difficulty to this situation. We just give a few examples: $t \in \mathbb{R}^N$ is an ***envy-free net change for*** (R,ω,X) if $\Sigma t_i = 0$, $t + \omega \in \overline{X}$ and for no i, j, $(\omega_i+t_j)P_i(\omega_i+t_i)$; $x \in \overline{X}$ is an ***individually-rational allocation for*** (R,ω,X) if $x_i R_i \omega_i$ for all i; $t \in \mathbb{R}^N$ is a ***uniform net change for*** (R,ω,X) if $\Sigma t_i = 0$, $t + \omega \in \overline{X}$, and (i) when $\Sigma p(R_i) \geqq M$, $t_i = \max\{p(R_i) - \omega_i, -\lambda(R)\}$ if $p(R_i) \leqq \omega_i$ and $t_i = \min\{p(R_i) - \omega, \lambda(R)\}$ if $p(R_i) \geqq \omega_i$ where $\lambda(R)$ is chosen so that $\Sigma t_i = 0$.)

9. ***Conclusions.*** We have considered the problem of fairly allocating an infinitely divisible good among agents with single-peaked preferences. This specification of the domain has significant implications. The properties of the standard equity notions when applied to this problem are indeed quite different from what they are when these solutions are operated on classical domains. Also, on this domain we could define a variety of appealing solutions that have no counterparts on classical domains.

Our second most important conclusion is that the uniform rule, a solution that had previously been shown to be extremely well–behaved from the viewpoint of incentives, is also the best–behaved from a variety of other viewpoints. Indeed, it is resource–monotonic, consistent, and conversely consistent. It is the only single–valued selection from either the no–envy and efficient solution or from the individually–rational from equal division and efficient solution to be consistent and to respond continuously to changes in the amount to be divided. Finally, it has certain asymptotic properties shared by no others. Rare are the situations where the same method so generally dominates all others. Usually, one finds that some solutions perform well in some situations and others perform well in some other situations. On the basis of what we now know, the uniform rule can wholeheartedly be advocated as the best solution to the problem of fair allocation in economies with single–peaked preferences.

Appendix

***Proof of Proposition* 11.** We supply here the proof that no group of cardinality 2 envies any other group of the same cardinality.

For convenience, we indicate the compositions of groups by their holdings at x; for instance a group denoted (0,1) is any group containing agent 1 or agent 2, together with agent 3 or agent 4 or agent 5. The proof is in several steps:

(a) No group envies another group of the same composition.

(b) No group (0,0) (say {1,2}) envies any group. Indeed, if $y_1 R_1 x_1$ and $y_2 R_2 x_2$, then $y_1 \leqq .4$ and $y_2 \leq .4$, so that $y_1 + y_2 \leq .8$. Since the holdings of any group (0,1), (0,2), (1,1), or (1,2) are at least 1, we are done.

(c) No group (0,1) (say {1,3}) envies (0,0). Indeed, if $y_1 R_1 x_1$ and $y_3 R_3 x_3$, then $y_1 \geqq 0$ and $y_3 \geqq 1$ so that $y_1 + y_3 \geqq 1$. Since the holdings of (0,0) are 0, we are done.

(d) No group (0,1) (say {1,3})envies any group (0,2) or (1,2), or (1,1). Indeed, if $y_1 R_1 x_1$ and $y_3 R_3 x_3$, then $y_1 \leqq .4$ and $y_3 \leqq 1.5$ so that $y_1 + y_3 \leqq 1.9$. Since the holdings of any group (0,2), (1,1) or (1,2) are at least 2, we are done.

(e) No group (0,2) (say {1,6}) envies any group (0,0), (1,1), or (0,1). Indeed, if $y_1 R_1 x_1$ and $y_6 R_6 x_6$, then $y_1 \geqq 0$ and $y_6 \geqq 2$, so that $y_1 + y_6 \geqq 2$. Since the holdings of (0,0), (1,1), or (0,1) are at most 2, we are done.

(f) No group (0,2) (say {1,6}) envies any group (1,2). Indeed, if $y_1 R_1 x_1$ and $y_6 R_6 x_6$, then $y_1 \leqq .4$ and $y_6 \leqq 2.5$, so that $y_1 + y_2 \leqq 2.9$. Since the holdings of (1,2) are 3, we are done.

(g) No group (1,1) (say {3,4}) envies any group (0,0), (0,1), or (0,2). Indeed, if $y_3 R_3 x_3$ and $y_4 R_y x_y$, then $y_3 \geqq 1$ and $y_4 \geqq 1$, so that $y_3 + y_4 \geqq 2$. Since the holdings of (0,0), (0,1), or (0,2) are at most 2, we are done.

(h) No group (1,1) (say {3,4}) envies any group (1,3). Indeed, if $y_3R_3x_3$ and $y_4R_4x_4$, then $y_3 \leqq 1.5$ and $y_4 \leqq 1.5$ so that $y_3 + y_4 \leqq 3$. Since the holdings of (1,2) are 3, we are done.

(i) No group (1,2) (say {3,6}) envies any other group. Indeed, if $y_3R_3x_3$ and $y_6R_6x_6$, then $y_3 \geqq 1$ and $y_6 \geqq 2$. Since the holdings of any other group are at most 3, we are done.

Sketch of a convergence result (Section 7).

Assume preferences to be drawn from a pool with the following characteristics. The support of the distribution of the preferred consumptions is the whole interval [0,M]. For each preferred consumption p in [0,M] and for each $0 \leqq x_i \leqq p$, (resp. $p \leqq x_i \leqq M$), the support of the conditional distribution of $e_i(x_i)$ is the entire subinterval (p,M) (resp. [0,p]). Let $\{e^k\} = \{(u_{N_k}, M_k)\}$ be a sequence of economies of increasing cardinality ($|N^k| \to \infty$) ordered by inclusion ($k' > k$ implies $N_k \subseteq N_{k'}$) and such that $M^k = M|N^k|$ for all k. Then, we claim that the set of envy-free and efficient allocations converges to the uniform allocation, in the following sense: Let $\{x^k\}$ be a sequence such that $x^k \in FP(e_k)$ for all k. Then, for all $\epsilon > 0$, and for all i, there is $\bar{k}$ such that for all $k > \bar{k}$, $|x_i^k - U_i(e^k)| < \epsilon$.

We assume without loss of generality that for k large enough, if $x \in P(e^k)$ then $\Sigma p(R_i) \geqq M|N_k|$. Suppose by way of contradiction that there is $\epsilon > 0$ and some i such that for some $\bar{k} \in \mathbb{N}$, $i \in N^k$ for infinitely many $k \geqq \bar{k}$ and $x_i^k < p(R_i) - \epsilon$ infinitely often. Then in each $e^{k'}$, no agent j can consume any amount in $[p(R_i)-\epsilon, e_i(p(R_i)-\epsilon)]$, (otherwise agent i would envy agent j at $x^{k'}$). Since there is a positive proportion of agents with preferred consumptions in that interval, each of these agents should consume x_i^k (otherwise they would envy agent i at $x^{k'}$). But then, for k large enough, the union of the "forbidden intervals" $[x_j^k, e_j(x_j^k)]$ taken over all j such that $p(R_j) \in [p(R_i) - \epsilon, p(R_i)]$

is essentially the whole of $[p(R_i)-\epsilon, M]$. Therefore, almost all agents consume less than $p(R_i)-\epsilon$. This is in contradiction with feasibility, since the endowment per capita is equal to M.

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