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Abstract.

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In neoclassical optimal growth models the stability of the paths of capital accumulation depends on the discount parameter. We prove here that, for discount factors small enough, the policy function which describes an optimal path can be of any type. The result is achieved by making use of the notion of α-concavity and it leads to a constructive approach. Given any twice differentiable map we show how to construct an optimal growth problem which produces that map as the optimal policy function. An obvious consequence is that also "chaos" is possible in this class of models. J.Econ.Theory.

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1. INTRODUCTION.

Many dynamic models used in current research assume, either explicitely or implicitely, that the actions taken by the economic agents are very regular and predictable. Also it is often claimed that this "well behaved" behaviors are logical consequences of some hypothesis on the rationality of the agents. In this paper we investigate wheter these assumptions are infact sufficient to ensure such a regularity: we will find that this is not the case. In particular we prove that even very irregular (i.e. chaotic) dynamic behaviors are possible.

Capital theory provides the best established analytical framework in which to carry out this exercise. In this case a reduced form model is considered. An aggregate welfare function is defined in terms of the state variables and its discounted sum is maximized. The resulting optimal choice is a sequence of vectors of capital stocks which are Pareto Optima for the underlying economy. It is then customary to interpret such a sequence as a competitive equilibrium over time for some decentralized economy.

The concept of policy function plays a central role in describing these aggregate accumulation paths. It is defined as the map which associates to any given capital stock the future capital stock which is optimal according to the intertemporal objective. Under standard assumptions, these functions will be continuous from some compact set into itself. Clearly the set of all C^0-functions is very wide: consequently many efforts have been placed in finding economically meaningful assumptions capable of restricting the possible policy functions within some smaller subset. The best results in this area belong to the so-called Turnpike Theory: loosely speaking they provide conditions for the policy function to be a map with a unique and dynamically stable fixed point. Unfortunately the whole problem is dramatically complicated by the presence of discounting: the optimal solution depends on the discount factor, and such dependence, even if continuous, can actually be very complicated. In fact most of the Turnpike Theorems are stated in terms of discount parameters very close to one, i.e., for problems in which discounting is relatively unimportant. On the other hand some well known examples of instability and other new researches on the periodicity of optimal paths have definitely proved that the class of functions with a unique and globally attractive fixed point does not exhaust the set of policy functions. The reader is referred to Benhabib and Nishimura [1] for a recent work on this topic, some earlier literature is also quoted there.

All this leads to a very natural question: "Can an arbitrary continuous function defined on a compact subset K of the positive orthant be an optimal policy function for some value of the discount parameter in a neoclassical optimal accumulation model?".

We provide here a first affirmative answer to this question. It is not a complete solution of the problem above because we have to restrict ourselves to a less general set of maps. Specifically we prove that the claim above is true for the set of twice differentiable functions and not for that of all the continuous functions. The former is actually dense in the latter, but the restriction does not seem to be unessential. In fact it is not clear to us if an extension to the continuous functions is possible.

Let us recall that the results presented here are a continuation and specialization of the research initiated in Boldrin and Montrucchio [2]. They are a continuation because the "Density Theorem" contained there is improved upon here; and a specialization because in the first work we considered a very large class of models of economizing over time, whereas in the present article we restrict ourselves to the neoclassical optimal growth model only. Formally this amounts to adding some monotonicity conditions on the return function, but this apparently innocuous variation requires some new techniques to be handled.

We have organized the article in this way: the next Section contains a formal statement of the problem and some standard results from the precedent literature. Section 3 is devoted entirely to list and explain the notions of α-concavity and concavity-β which are essential in proving the two main Theorems of Section 4. We conclude the paper with a simple economic example and some comments on future research. For reasons of homogeneity and brevity we have omitted any detailed economic applications here: this is left for further work in which special attention will be paid to the two-sector model.

2. STATEMENT OF THE PROBLEM.

The neoclassical model of optimal growth with discounting is completely described by problem (P) and the subsequent assumptions (A.1)-(A.3) below.

$$\begin{aligned} &\text{Max} \sum_{t=0}^{\infty} V(k_t,k_{t+1})\delta^t \\ &\text{subject to } (k_t,k_{t+1})\varepsilon T,\ t=0,1,2,\dots. \\ &k_0 \text{ given in } K \\ &\delta\varepsilon(0,1)\ . \end{aligned} \tag{P}$$

(A.1) The set of feasible capital stocks $K\subset R^n_+$ is convex and compact. The technology set $T\subset KxK$ is also convex and compact and it satisfies: $proj_1(T) = K$. We also assume that K has non-empty interior.

(A.2) The return function $V: KxK\rightarrow R$ is continuous and concave. $V(k,\cdot)$ is strictly concave for every fixed $k\varepsilon K$.

(A.3) $V(k,k')$ is strictly increasing in k and strictly decreasing in k'.

To simplify our notation from now on, we will denote the set of capital stocks which are technically feasible from k with $T(k) = \{k'\varepsilon K$, such that $(k,k')\varepsilon T\}$.

The solution to (P) will be a sequence of vectors in K, say $\{k_0, k_1, k_2,\dots.\}$ with k_0 exogeneously given. It can be shown that under (A.1)-(A.3) such an optimal path exists and it is unique. These same results can be actually obtained under a set of assumptions weaker than (A.1)-(A.3), see for example McKenzie [7].

The optimal solution $\{k_0, k_1, \ldots\}$ can be alternatively represented by iterating a policy function τ_δ . This will be a map from K into K such that: $k_1 = \tau_\delta(k_0)$, $k_2 = \tau_\delta(k_1)$, ..., etc. . The subscript δ is used to underline the dependence of the policy function on the discount parameter: this parameterization turns out to be a crucial step in the development of our central argument.

The relevant features of τ_δ are probably better understood by introducing the techniques of Dynamic Programming. Define the value function associated to (P) as:

$$W_\delta(k_0) = \text{Max} \sum_{t=0}^{\infty} V(k_t, k_{t+1})\delta^t \tag{1}$$

$$\text{subject to } (k_t, k_{t+1}) \varepsilon T$$

$$k_0 \text{ given in K.}$$

It is well known that $W_\delta(k_0)$ turns out to be the (unique) fixed point of a functional equation which is induced by a contraction operator of modulus δ over the space of all the continuous functions (see for example Denardo [5]). This functional equation is the Bellman Equation:

$$W_\delta(k) = \text{Max } \{V(k,k') + \delta W_\delta(k'), \text{ s.t. } k' \varepsilon T(k)\} \tag{2}$$

This allows a "formal" definition of the policy function as:

$$\tau_\delta(k) = \text{Arg max } \{V(k,k') + \delta W_\delta(k'), \text{ s.t. } k' \varepsilon T(k)\} \tag{3}$$

Notice that the iterates of (3) over K define the dynamical system: $k_{t+1} = \tau_\delta(k_t)$, t=0,1,2,.... , which describes the optimal path starting at the initial condition k_o . A more detailed discussion of the relation between (P) and τ_δ can be found in Boldrin and Montrucchio [2].

To place our enquiry within the framework of the contemporary research we report, without proofs, some previously established facts about the behavior of the dynamical system induced by (3).

<u>FACT 1</u>. Under assumptions (A.1)-(A.3) the following properties are true:

a) τ_δ is a continuous map from K into K,

b) the map $\delta \mapsto \tau_\delta$ is continuous from the interval [0,1) into $C^0(K,K)$,

c) $C^0\text{-}\lim_{\delta \to 0^+} \tau_\delta = \theta$,

where θ is the policy function of the short run problem:

$$\text{Max } \{V(k_o,k_1), \text{ s.t. } k_1 \varepsilon T(k_o),\ k_o \text{ given in } K\}$$

and the C^0-lim refers to the topology of the uniform convergence.

<u>Proof</u>. See Boldrin and Montrucchio [2]. Notice that Assumption (A.3) is not needed at all in proving this result.

<u>FACT 2</u>. Assume dim(K) = 1 and that V is of class C^1. These two additional Assumptions, plus (A.1)-(A.3), imply that: there exists a δ^1 such that for any $\delta < \delta^1$ the policy function turns out to be defined as:

$$\tau_\delta(k) = \text{Min } \{k' \varepsilon K, \text{ s.t. } k' \varepsilon T(k)\}.$$

This means that the optimal policy function is trivially defined by the technology for any given initial state vector. In particular: $\tau_\delta(k) = 0$ for all $k \varepsilon K$ if $(k,0) \varepsilon T$ for all k.

<u>Proof</u>. This is just Lemma 1 in Deneckere and Pelikan [6].

Remark 1. Notice that the same result is no longer true in the n-dimensional case when a generic convex and compact T is considered. To be more precise, the situation turns out to be the following: the policy function will take on values only on the lower boundary of T(k) for all δ lower than a certain critical value δ^1 . But in general the boundary of T(k) does not reduce to a single point, hence the only relevant information gained from assuming very small discount factors is that the dynamics induced by the policy function is restricted on a (n-1)-dimensional manifold. If the "putty-putty" hypothesis (or simply: $0 \varepsilon T(k)$ for all k in K) is added, then the n-dimensional analogous of Fact 2 can be replicated.

Remark 2. It is also easily seen (compare for example the proof given in [6]) that the imposition of some "Inada-type" conditions on the behavior of V at the boundary of T will rule out the result for all the values of δ different from zero. This should be taken in account when Fact 2 is intended to be applied to the standard aggregate model of optimal growth for which "Inada-type" assumptions are usually made.

FACT 3. Consider problem (P) under (A.1)-(A.3) plus some technical hypothesis. Then for any given V there exists a δ^2 such that for all $\delta \varepsilon (\delta^2, 1]$ the optimal policy function τ_δ possesses a unique fixed point $k(\delta)$ in the interior of K. Moreover this fixed point is globally attractive.

Proof. The literature proving Turpike Theorems is actually very large. Proofs of statements equivalent to Fact 3 can be found, for example, in McKenzie [7] and Scheinkman [10].

It is apparent that our knowledge of the nature of the policy function (3) is very poor. Using the continuity of the relation between δ and τ_δ (Fact 1) we can characterize the behavior of τ_δ for δ in a neighbourhood of zero for the one-dimensional case (Fact 2), and in a neighbourhood of one for the n-dimensional case (Fact 3). For δ very small in the n-dimensional case we can only say that τ_δ has a qualitative behavior very close to that of θ (Fact 1 again), but nothing more precise. What is worse is that we know almost nothing of τ_δ when the discount parameter is outside these two neighbourhoods. The aim of the present paper is to show that (A.1)-(A.3) are too weak for imposing any other significant restriction on τ_δ when δ is allowed to take on any value on the unit interval. This will imply that, as long as only those assumptions are maintained, problem (P) can hardly provide any falsifiable prediction when the true value of the discount parameter is not known

3. ALPHA-CONCAVITY AND OPTIMIZATION PROBLEMS.

All through this Section we will consider concave and finite maps f and g from X to R, and real valued functions V(x,y) defined on XxY, with X and Y convex subsets of Euclidean spaces. We will denote with $||\cdot||$ the usual Euclidean norm of R^n .

DEFINITION 1. We say that f is α-concave , α>0, if:

$$f(x) + (1/2)\alpha||x||^2 \text{ is concave over X.}$$

DEFINITION 2. The function f is called concave-β if:

$$f(x) + (1/2)\beta||x||^2 \text{ is convex over X.}$$

The notion of α-concavity was used in Rockafellar [9] to provide a Turnpike Theorem for the continuous-time version of problem (P).

The following alternative characterization will be useful later on; the proof follows trivially from the standard definitions of concavity and convexity.

PROPOSITION 1.

(a) f is α-concave if and only if:

$$f[(1-\lambda)x+\lambda y] \geqslant (1-\lambda)f(x) + \lambda f(y) + (1/2)\alpha\lambda(1-\lambda)||x-y||^2,$$

for all x,yεX and every λε[0,1].

(b) f is concave-β if and only if:

$$f[(1-\lambda)x+\lambda y] \leqslant (1-\lambda)f(x)+\lambda f(y)+(1/2)\beta\lambda(1-\lambda)||x-y||^2,$$

for all x,yεX and all λε[0,1].

Now consider the function V(x,y). A natural extension of Definition 1 is given by:

DEFINITION 3. We say that V is (α,β)-concave if:

$$V(x,y) + (1/2)\alpha||x||^2 + (1/2)\beta||y||^2 \text{ is concave over } X\text{x}Y.$$

In the following we need this particular case only:

DEFINITION 4. V is called α_y-concave if:

$$V(x,y) + (1/2)\alpha||y||^2 \text{ is concave over } X\text{x}Y.$$

Notice that the latter corresponds to the Definition of $(0,\alpha)$-concavity for V. The final Definition we need is:

DEFINITION 5. We say that V(x,y) is uniformly concave-β in x if $V(\cdot,y)$ is concave-β for every fixed $y \in Y$.

Remark 3. Some clarification on the meanings of the above Definitions are probably useful for the reader. It is clearly understood (see also Proposition 1), that the notion of α-concavity defines a degree of strong concavity (for $\alpha=0$ we have just the usual notion of concavity). The larger α is the more concave is the function. On the contrary the concavity-β places an upper bound on the degree of concavity.
Although both notions are independent of the differentiability of the function, it happens to be the case that the specific functions we study later on, turn out to be differentiable (actually of class C^2). For this specific class of functions a characterization of α-concavity and concavity-β in terms of second derivatives can be given naturally. Denote with H(x) the Hessian matrix of the function f, then to say that f is α-concave and concave-β is equivalent to: $-\beta||z||^2 \leqslant z'H(x)z \leqslant -\alpha||z||^2$, for every x in X and z in R^n

The sum of two concave functions is always a concave one, whereas this is not the case, in general, for their difference. In the latter case it is intuitively clear that the resulting function will be concave only if the difference in the "degrees of concavity" of the two operands is large enough. As we need to add and subtract concave functions it is worth reminding the reader that if we take two concave functions f and g, such that f is α-concave and g is concave-β, with $\beta \leq \alpha$, then the difference (f-g) is $(\alpha - \beta)$-concave . Consider two real valued functions $W: X \times Y \to R$ and $\Psi: Y \to R$, in this case the above fact can be generalized as follows:

<u>PROPOSITION</u> <u>2</u>. Let $W(x,y)$ be α_y-concave and $\Psi(y)$ concave-β, with $\beta<\alpha$. Then the difference $W(x,y) - \Psi(y)$ turns out to be an $(\alpha-\beta)_y$-concave function on $X \times Y$.

<u>Proof</u>. Indeed $W(x,y) + (1/2)\alpha||y||^2$ is concave over $X \times Y$ by assumption, and $-\Psi(y) - (1/2)\beta||y||^2$ is concave over Y.
Hence their sum: $W(x,y) - \Psi(y) + (1/2)(\alpha-\beta)||y||^2$ is concave. Q.E.D.

At this point we have enough structure to begin our positive analysis. In Boldrin and Montrucchio [2, Theorem 3.1] we used the following family of functions:

$$W(x,y) = -(1/2)||y||^2 + \langle y-\bar{y},\theta(x)\rangle - (1/2)L||x||^2 \qquad (4)$$

with $\theta \varepsilon C^2(X,X)$, $X = Y$ and $\bar{y}$ a fixed value in X, to prove that any C^2-map can be a solution to (P), under (A.1)-(A.2) and $\delta = 0$. The properties of (4) will be exploited here to give an improved version of that Theorem. We proceed by Lemmas.

The critical role played by (4) in the proof of our results will become apparent later on (see especially Lemma 3 and Theorem 3). At this point we want only to single out the following to, intuitive, facts: (a) if (4) is concave then it can be interpreted as a Bellman Equation before the application of the Max operator with respect to y, i.e. we can put: $W(x,y) = V(x,y) + \delta W_\delta(y)$ (b) when (4) is maximized with respect to y it is easily seen that $y = \theta(x)$ is the optimal solutions. These two facts will be exploited at length in this paper.

<u>LEMMA 1</u>. Consider the family (4) for a given map θ: $X \to X$, where $X = \bar{\Omega}$, Ω is open in R^n, X is compact and convex, θ is of class C^2 over Ω and it is continuously extendable with its derivatives over X. Then for any $\alpha \varepsilon [0,1)$ there exists a positive constant L such that the corresponding W is α_y-concave. More precisely it is enough to put:

$$L \geqslant \mu\sigma + 8\gamma^2/(1-\alpha),$$

where: $\gamma = \text{Max}\{||D\theta(x)||,\ x\varepsilon X\}$, $\sigma = \text{Max}\{||D^2\theta(x)||,\ x\varepsilon X\}$, $\mu = \text{diam } X = \text{Max}\{||x_1-x_2||,\ x_1,x_2 \varepsilon X\}$, and D is the derivative operator.

<u>Proof</u>. (<u>see Appendix</u>)

We can now state:

<u>THEOREM 1</u>. Take $X = \bar{\Omega}$, where Ω is an open subset of R^n and X is compact and convex.
Let θ: $X \to X$ be any C^2-map over Ω with derivatives extendable for continuity over X.
Then for every given $\alpha \varepsilon [0,1)$ there exists a C^2-function W: XxX $\to$R such that:

(i) $\text{Max}\{W(x,y),\ \text{s.t. } y \varepsilon X\} = W(x,\theta(x))$

(ii) W(x,y) is α_y-concave over XxX.

<u>Proof</u>. Let W(x,y) be defined by the family (4). Then (i) follows by the first order conditions and (ii) has been proved in Lemma 1. Q.E.D.

Remark 4. It is immediately seen that Theorem 3.1 in Boldrin and Montrucchio [2] refers to the case $\alpha = 0$. The relevance of the α-concavity improvement for the understanding of (P) will be apparent later on.

THEOREM 2. Take a function W from XxY into R and assume it is uniformly concave-β in x, then:

$$\Psi(x) = \text{Max}\ \{W(x,y),\ y\varepsilon Y\},$$

is also concave-β.

Proof. Let x and x' be two points in X, y^* the maximizer of W associated to $\lambda x + (1-\lambda)x'$ for a given $\lambda\varepsilon[0,1]$. Then:

$$\Psi[\lambda x+(1-\lambda)x'] = W[\lambda x+(1-\lambda)x',y^*] \leqslant$$
$$\leqslant \lambda W(x,y^*) + (1-\lambda)W(x',y^*) + (1/2)\beta\lambda(1-\lambda)||x-x'||^2 \leqslant$$
$$\leqslant \lambda\Psi(x) + (1-\lambda)\Psi(x') + (1/2)\beta\lambda(1-\lambda)||x-x'||^2 .$$

The last inequality corresponds to the concavity-β of Ψ by Proposition 1 above. Q.E.D.

Remark 5. We need to stress that Theorem 2 does not hold any longer if a (non-trivial) constraint is added to the maximization problem. In other words if the maximizer y^* has to be selected in a (convex and compact) set dependent on x, say T(x), then the concavity-β of Ψ does not follow in general from the concavity-β of W in x. Since the case we are studying includes such a constraint (see problem (P) and Assumption (A.1)), a straightforward application of Theorem 2 is not possible. This unpleasant complication will be considered further and solved in the next Section.

We complete our technical background with the two following applications of the theory developed so far: they will be both useful in proving Theorem 3.

PROPOSITION 3. For any given positive number L the function W given in (4) is uniformly concave-β in x for every $\beta \geq \beta^* = L + \mu\sigma$.

Proof. We have only to prove that:

$$F(x,y) = -(1/2)||y||^2 + \langle y-\bar{y}, \theta(x)\rangle + (1/2)(\beta-L)||x||^2$$

is convex for any given y when $\beta \geq \beta^*$

Take a fixed y in X and set:

$$f(t) = F(x_0+tx_1, y)$$

where x_0 and (x_0+tx_1) are two points in X to which the same qualifications introduced in the proof of Lemma 1 apply. Then:

$$f''(t) = \langle y-\bar{y}, D^2\theta(x_0+tx_1)(x_1,x_1)\rangle + (\beta-L)||x_1||^2$$

so that:

$$f''(t) \geq (\beta-L)||x_1||^2 - |\langle y-\bar{y}, D^2\theta(x_0+tx_1)(x_1,x_1)\rangle| \geq$$

$$\geq (\beta-L)||x_1||^2 - ||y-\bar{y}||\cdot||D^2\theta(x_0+tx_1)||\cdot||x_1||^2 \geq$$

$$\geq ||x_1||^2[(\beta-L) - \mu\sigma].$$

Hence $\beta \geq \beta^*$ implies $f''(t) \geq 0$. Q.E.D.

COROLLARY 1. If W is the same as in Proposition 3 and β is such that W is uniformly concave-β in x for a given L, then the function:

$$\Psi(x) = \text{Max}\ \{W(x,y),\ \text{s.t.}\ y \in X\} = W(x,\theta(x)).$$

is also concave-β for all $\beta \geq L + \mu\sigma$.

Proof. It follows from Proposition 3 and Theorem 2.

4. THE MAIN THEOREM: INDETERMINACY OF THE POLICY FUNCTIONS.

The mathematical tools we have been exposing and discussing in Section 3 will be applied here to the n-sector neoclassical optimal growth model defined by problem (P) under (A.1)-(A.3). Actually we will develop most of the subsequent analysis considering problem (P) under (A.1)-(A.2) solely. Assumption (A.3) will be introduced in the last step only (Theorem 4). This procedure has been chosen to shorten the proof of the main Theorem (Theorem 3 below) and to keep it consistent with the tools developed in Section 3.

The next Lemma contains a characterization of the policy function τ_δ which turns out to be essential in proving our main Theorem. This Lemma is just a restatement of Theorem 5.4 of Montrucchio [8]. From now on we will return to the notation of Section 2.

<u>LEMMA 3</u>. A map $\theta: K \to K$ is the policy function τ_δ of the optimal growth model (P) under (A.1)-(A.2), for a fixed value of δ in $[0,1)$, if and only if the two following conditions are satisfied:

(i) There exists a real (concave) function $W(k,k')$ defined on KxK such that:

$$\text{Max } \{W(k,k'), \text{ s.t. } k' \varepsilon T(k)\} = W(k,\theta(k)).$$

(ii) Setting $\Psi(k) = W(k,\theta(k))$, the real function $W(k,k') - \delta\Psi(k')$ satisfies the hypothesis (A.2).

<u>Proof</u>.

<u>Necessity</u>. Let the policy function τ_δ be equal to θ for a given return function $V(k,k')$ and a technology set T satisfying (A.1)-(A.2). By the Bellman's Equation (2) we must have:

$$\text{Max } \{V(k,k') + \delta W(k'), \text{ s.t. } k' \varepsilon T(k)\} = W(k) \tag{2'}$$

and:

$$\theta(k) = \text{Arg max } \{V(k,k') + \delta W(k'), \text{ s.t. } k' \varepsilon T(k)\} \qquad (3')$$

Put: $W(k,k') = V(k,k') + \delta W(k')$, to obtain:

$$\text{Max } \{W(k,k'), \text{ s.t. } k' \varepsilon T(k)\} = W(k,\theta(k)),$$

so that (i) is satisfied.

Moreover, if we set: $\Psi(k) = W(k,\theta(k))$, we obtain $\Psi(k) = W(k)$, by (2'),

In this way we have:

$$W(k,k') - \delta\Psi(k') = V(k,k') + \delta W(k') - \delta W(k') = V(k,k')$$

which is concave in (k,k') and strictly concave in k' by (A.2). Hence (ii) is satisfied.

<u>Sufficiency</u>. Define the return function of (P) as: $V(k,k') = W(k,k') - \delta\Psi(k')$, which satisfies (A.2) by (ii). Then:

$$\text{Max } \{V(k,k') + \delta\Psi(k'), \text{ s.t. } k' \varepsilon T(k)\} = \text{Max } \{W(k,k'), \text{ s.t. } k' \varepsilon T(k)\} = \Psi(k)$$

and:

$$\text{Arg max } \{V(k,k') + \delta\Psi(k'), \text{ s.t. } k' \varepsilon T(k)\} =$$

$$= \text{Arg max } \{W(k,k'), \text{ s.t. } k' \varepsilon T(k)\} = \theta(k),$$

by hypothesis (i).

This, in turn, implies that θ is the optimal policy function of a problem (P) where we have: $V(k,k') = W(k,k') - \delta\Psi(k')$, as the return function, the discount parameter δ is given and the value function defined in (2') is exactly $\Psi(k) = \text{Max } \{W(k,k'), \text{ s.t. } k' \varepsilon T(k)\}$. Q.E.D.

Unfortunately we cannot make full use of the last characterization result to obtain our main Theorem. This is due to the fact that Theorem 2 is essential in our strategy of proof (see Theorem 3). But, as we have noted in the Remark 5 above, Theorem 2 cannot be applied when a non trivial constraint is added. To overcome this difficulty we will give a weaker version of Lemma 3, to obtain a result compatible with the unconstrained maximization of

Theorem 2. This is accomplished in the following Corollary. We omit the proof because it is self evident in the light of the proof of Lemma 3.

<u>COROLLARY</u> <u>2</u>. A set of <u>sufficient</u> <u>conditions</u> in order to obtain a map θ: $K \to K$ as the optimal policy function τ_δ of problem (P) under (A.1)-(A.2) for a given $\delta \varepsilon [0,1)$ is:

i) $(k,\theta(k)) \varepsilon T$, for every $k \varepsilon K$.

ii) There exists a concave, real valued function $W(k,k')$ such that:

$$\text{Max } \{W(k,k'), \text{ s.t. } k' \varepsilon K\} = W(k,\theta(k)).$$

iii) Setting $\Psi(k) = W(k,\theta(k))$, the real function $W(k,k') - \delta\Psi(k')$, is concave in (k,k') and strictly concave in k'.

Finally we can prove our central result:

<u>THEOREM</u> <u>3</u>. Take any $\theta \varepsilon C^2(K;K)$, satisfying the assumptions of Theorem 1, and such that: $(k,\theta(k)) \varepsilon T$ for every $k \varepsilon K$.
Then there exists a discount parameter $\delta^* \varepsilon (0,1)$, the value of which depends on θ, such that for every fixed $0 < \delta < \delta^*$ we can construct a return function $V_\delta(k,k')$ satisfying (A.2) and with the following property: The optimal policy function τ_δ solving (P) under (A.1)- (A.2) with $V = V_\delta$, is the map θ. Moreover a lower bound for δ^* can be estimated as:

$$\delta^* > \delta^{**} = \{8\gamma^2 + \mu\sigma - 4\gamma(4\gamma^2+\mu\sigma)^{1/2}\}/(2\mu^2\sigma^2) > 0.$$

<u>Remark</u> <u>6</u>. It is important to stress that the assumption of monotonicity of V given in (A.3) <u>it</u> <u>is</u> <u>not</u> <u>satisfied</u> by the V_δ we will construct. We consider this problem in the Corollary 3.

<u>Proof</u>. Take any $\theta \varepsilon C^2(K;K)$ such that $(k,\theta(k)) \varepsilon T$ for every $k \varepsilon K$.
By Theorem 1 there exists a $W(k,k')$ such that:

$$\text{Max } \{W(k,k'), \text{ s.t. } k'\varepsilon K\} = W(k,\theta(k)).$$

By Lemma 1 we know it is sufficient to take $W(k,k')$ as defined in (4). The same Lemma implies that this function turns out to be $\alpha_{k'}$-concave over KxK, when:

$$L \geqslant 8\gamma^2/(1-\alpha) + \mu\sigma, \qquad \alpha\varepsilon[0,1) \qquad (5)$$

Let $\Psi(k) = \text{Max } \{W(k,k'), \text{ s.t. } k\varepsilon K\}$, then Ψ is concave-β by Corollary 1, when:

$$\beta \geqslant \beta^* = L + \mu\sigma. \qquad (6)$$

Then Corollary 2 will imply that the function: $V_\delta(k,k') = W(k,k') - \delta\Psi(k')$, is our desired return function if we can prove it is strictly concave in k'. Using Proposition 2 we can see that: $W(k,k') - \delta\Psi(k')$ is $(\alpha-\delta\beta)_{k'}$-concave on KxK.

So we need the three parameters to satisfy:

$$\alpha - \delta\beta > 0. \qquad (7)$$

to have $V_\delta(k,k')$ concave in KxK and strictly concave in k'. Summing up: the Theorem is proved if the conditions (5), (6) and (7) are simultaneously satisfied by some values of the parameters L, α, β such that $\delta\varepsilon[0,1)$.

With some algebra (left to the reader) it is easily seen that the set of solutions to the system (5)-(7) is not empty and that the value:

$$\delta^{**} = \text{Max } \{(L+\mu\sigma)^{-1} - (8\gamma^2)(L^2-\mu^2\sigma^2)^{-1}, \text{ s.t. } L \geqslant 8\gamma^2+\mu\sigma.\}$$

is the largest value of the discount parameter compatible with a non empty solution. Further tedious computations will show that the solution of the latter maximization exercise is exactly the value δ^{**} given in the Theorem. Q.E.D.

Some Remarks on this Theorem seem appropriate.

<u>Remark 7</u>. We need two qualifications on the estimates of δ^* and δ^{**} given above.

First: it should be clear that δ^* depends on the chosen θ. That is: for a given θ there exists a $\delta^*>0$ such that the Theorem follows. With the number δ^{**} we provide a lower bound for δ^*, which depends on the first and second derivatives of θ and on the diameter of K. It must be clear that such an estimate is surely biased downward for a generic θ.
Second: notice that we have been working with a particular family of concave functions, i.e. the family (4). This does not exhaust the class of functions satisfying our Theorem. Hence the lower bound we have found (i.e. δ^{**}) can probably be improved upon by using suitable generalizations of (4).
Finally, let us report here that, with some more computations, the upper- and lower- bounds of δ^{**} can be simply estimated as:

$$(32\gamma^2 + 4\ \mu\sigma)^{-1} < \delta^{**} < (32\gamma^2)^{-1}$$

<u>Remark 8</u>. Our second observation concerns the nature of the V_δ we have constructed. Noticed that these V_δ's satisfy a set of assumptions stronger than (A.2), at least for all $\delta < \delta^{**}$. First: they are of class C^2 and they originate value functions which are also C^2. Second, and more important, they are strongly concave for $\delta < \delta^{**}$. This fact is understood in the following way: when δ is smaller than δ^{**} the set of solutions to (5)-(7) is open: this is obtained by plugging $\beta = L + \mu\sigma$ in (7) so that we are left with the inequality (5) plus the inequality $L < \alpha/\delta - \mu\sigma$. When $\delta < \delta^{**}$ is fixed the solutions will be the intersections between the halfplane $L < \alpha/\delta - \mu\sigma$ and the set of (α,L) satisfying (5). Now: take two pairs (α,L) and (α,L') which are interior to this intersection and such that $L' > L$. Denote with W_L and $W_{L'}$ the maps (4) corresponding to L and L' and with Ψ_L, $\Psi_{L'}$, the respective maxima. It is seen immediately that the following hold:

$$W_{L'}(x,y) + (1/2)(L'-L)||x||^2 = W_L(x,y)$$

and

$$\Psi_{L'}(y) + (1/2)(L'-L)||y||^2 = \Psi_L(y).$$

From which:

$W_{L'}(x,y) - \delta\Psi_{L'}(y) + (1/2)(L'-L)||x||^2 - (1/2)\delta(L'-L)||y||^2 = W_L(x,y) - \delta\Psi_L(y)$

Since $W_L - \delta\Psi_L$ is $[\alpha-\delta(L+\mu\sigma)]_y$-concave we conclude that:

$$W_{L'}(x,y) - \delta\Psi_{L'}(y) + (1/2)(L'-L)||x||^2 + (1/2)[\alpha - \delta(L'+\mu\sigma)]||y||^2$$

is concave. Hence:

$$V_\delta(x,y) = W_{L'}(x,y) - \delta\Psi_{L'}(y)$$

turns out to be $\{L'-L, \alpha-\delta(L'+\mu\sigma)\}$-concave (see Definition 3).

Remark 9. Finally, a few more words on the content of Theorem 3. It is, in some sense, a strong improvement on Theorem 3.3 in Boldrin and Montrucchio [2]. The most significant improvements, as we see them, are the following:

1) The result holds for any $\theta \varepsilon C^2(K^\circ;K)$, whereas formerly we gave only a density version of the statement.

2) As $\delta^* > 0$ we can get rid of the $\delta = 0$ approximation. This fact is critical in permitting us to include the neoclassical optimal growth model in our "world of indeterminacy".

3) Last, but not the least, Theorem 3 underlies a constructive technique which permits the exact computation of a problem (P) for any given C^2 policy function. The technique should be clear from the proof, we will briefly illustrate it after Theorem 4.

We can now tackle a problem we have been postponing so far, i.e. the inclusion of the monotonicity conditions (A.3) in the structure of the return function V. Is it possible with a small modification of family (4), to construct a V_δ satisfying also this latter requirement ? A positive answer is given in the following, non trivial, consequence of Theorem 3.

__THEOREM 4.__ Assume $\theta \varepsilon C^2(K,K)$ is as defined in Theorem 3. Then for every $\delta' \varepsilon (0,\delta^*)$, with δ^* given in Theorem 3, there exists a return function $V_{\delta'}$ depending on δ' and satisfying A.2 and A.3, such that θ is the optimal policy function τ_δ of the associated problem (P) __under__ (__A.1__)-(__A.3__) when $\delta = \delta'$.

__Proof.__ After Theorem 3 the only thing we need to prove is that the monotonicity requirements on the return function V_δ can be satisfied for an opportune choice of parameters.

Consider the following modified version of the family (4):

$$W(k,k') = -(1/2)||k'||^2 + \langle k'-\bar{k},\theta(k)\rangle - (L/2)||k||^2 + \langle a,k\rangle, \qquad (4')$$

where a is a strictly positive n-dimensional vector and $\bar{k}$ is a given point in K.

Notice that all the arguments on α-concavity and concavity-β we have been using in Theorem 1 and 3, hold true even by adding the linear term $\langle a,k\rangle$.

Thus, for $\delta<\delta^*$ and retaining the notation of Theorem 3, we know that: $V_\delta(k,k') = W(k,k') - \delta\Psi(k')$ is concave.

Also:

$(\partial/\partial k)[V_\delta(k,k')] = \{D\theta(k)\}^T(k'-k) - Lk + a > 0,$

when the components of the vector a are large enough. Precisely: $a_i > (LN + \gamma\mu)$, all $i=1,..,n$, where $N = \text{Max}\ \{||k||, \text{ s.t. } k\varepsilon K\}$. Also, we indicate the transpose operator with $\{\bullet\}^T$.

The derivative with respect to k' is:

$(\partial/\partial k')[V_\delta(k,k')] =$

$= -(1-\delta L)k' - \delta a + \theta(k) + \delta\{D\theta(k')\}^T(\bar{k} - \theta(k'))$.

Again, we satisfy the monotonicity requirement by simply imposing a componentwise lower bound on the vector a:

$$a_i > LN + \gamma\mu + N/\delta, \text{ all } i=1,...,n.$$

Summarising: it is easily seen that, for any positive $\delta < \delta^*$ there exists a vector a satisfying the requirements. Notice that $\delta > 0$ is essential in finding a real value for a. Hence:

$$\text{Max}\{W(k,k'), \text{ s.t. } k'\varepsilon T(k)\} = \text{Max}\{V_\delta(k,k') + \delta\Psi(k'), \text{ s.t. } k'\varepsilon T(k)\} = W(k),$$

will be the value function of a problem (P) under (A.1)-(A.3). Finally because:

$$(\partial/\partial k')[W(k,k')\} = 0 \text{ implies } k' = \theta(k),$$

by equation (3), we conclude that θ is the policy function τ_δ of such an optimal growth model . Q.E.D.

Remark 10. We think that the theoretical implications of the last result need not be explained to a careful reader.

Rather, we stress here the constructive approach underlying our main Theorem. The step-by-step procedure is as follows: assume a technology set T is given. Then take a generic map $\theta\varepsilon C^2(K;K)$ such that $(k,\theta(k))\varepsilon T$. Using the formulas of Theorem 3 we compute the critical value δ^{**} . The function W can be calculated by using the family (4') given in the proof of Theorem 4.

At this point pick a $0 < \delta' < \delta^{**}$: for such a δ' the return function is computed by: $V_{\delta'}(k,k') = W(k,k') - \delta'\Psi(k')$, as proved in Corollary 2. Next choose an opportune vector a as indicated in Theorem 4. This will complete the construction of the neoclassical optimal growth model exhibiting θ as the optimal policy function $\tau_{\delta'}$.

Further, remember that in the one-dimensional case and/or when "free disposal" and "reversability" is assumed for T, a lower critical value δ^1 of the discount factor can be computed such that for all δ smaller than this the situation depicted in Fact 2 shows up. The reader has to pay attention to the fact that δ^1 depends on $V_{\delta'}$ which, in turn, depends on the previously chosen θ, so that it does not interfere with our procedure.

Informally speaking, the following scenario will appear: the optimal problem

(P) with $V_{\delta'}$ as a return function exhibits the trivial map $\tau_{\delta}(k) = \inf \{k' \varepsilon K,$ s.t. $k' \varepsilon T(k)\}$ for any $\delta \varepsilon [0, \delta^1]$. Instead for the given $\delta' \varepsilon (\delta^1, \delta^*)$ the associate problem (P) produces $\theta = \tau_{\delta'}$. Finally when δ is in the neighbourhood $(\delta^2, 1]$ the Turnpike property holds.

5. CHAOS IN A TWO-SECTOR MODEL.

To give a flavor of the possible applications of the above results we will construct a very simple two-sector, no-joint-production economy exhibiting a particular dynamic behavior for appropriate values of δ. To make life easier, we consider a very simplified world where utility is linear in consumption, a fixed amount of labor is supplied in each period, the capital depreciation factor equals one and both factors are used in each sector. In this case the Production Possibility Frontier (PPF) $c = V(k, k')$ is the solution of:

$$\text{Max } c = f(k^c, \ell^c) \tag{8}$$
$$\text{s.t. } k' \leqslant g(k^k, \ell^k)$$
$$k^c + k^k \leqslant k, \ \ell^c + \ell^k \leqslant 1.$$

where the total amount of work has been fixed at one and all the variables are constrained to non-negative values.The notation should be self-evident. The only assumptions we make on the two production functions f and g are concavity and positive marginal productivity of both factors. The maximization problem then becomes:

$$\text{Max} \sum_{t=0}^{\infty} V(k_t, k_{t+1})\delta^t \tag{9}$$
$$\text{s.t. } 0 \leqslant k_{t+1} \leqslant g(k_t, 1).$$

The reader is referred to Benhabib and Nishimura [1] for the details of the model.

Consider the famous "chaotic" map $4x(1-x)$ from the unit interval into itself (see for example Collet and Eckmann [3]) and assume we wish to have it as the optimal policy function of (9) for some value of the discount parameter. All we have to do is follow a routine computation. Using Theorem 4 we get the following PPF:

$$V(x,y) = -.0144y^4+.0288y^3-.3074y^2-4.03y+4xy(1-x)-115x^2+2191x. \quad (10)$$

Is equation (10) a consistent solution to problem (8) ? The answer is positive: the reader can easily check that by using:

$$f(k^c,\ell^c) = V[k^c+d(1-\ell^c),1-\ell^c] \quad (11)$$

and

$$g(k^k,\ell^k) = \min\ [k^k/d,\ell^k] \quad (12)$$

with $d < .000257$, the PPF (10) is obtained.

As (11) and (12) satisfy all the standard requirements, we have constructed a two-sector economy exhibiting "chaos" when the discount parameter is equal to $\delta^{**} = .0018$. The reader should note that in this example the warning we have given in Remark 6 above applies. The above value of δ^{**} has been calculated routinely by using the formula given in Theorem 3, but, as we noted, it is a downward biased estimation. In fact the same example goes through by using any δ smaller than .01263. This can be verified by direct computation.

6. CONCLUSIONS.

We consider this paper as the opening to a potentially fruitful field of research. Simply, the main result says that every behavior is possible in a neoclassical optimal growth model: hence the problem of excluding some unrealistic and/or undesirable behaviors is totally open. In particular, this means that the usual practice of assuming very simple dynamic patterns for economies with rationally maximizing agents is totally unjustified. Along this line of research more effort should be put in obtaining "Turnpike-like" results. Certainly, to obtain meaningful results, much more structure than that contained in (A.1)-(A.3) should be added to problem (P).

While the above is a primary research objective, a second one, no less interesting, follows directly by using our strategy of proof. It is the study, and possibly the classification, of the various types of return functions V that can originate a specific policy function. Such a knowledge seems important because it can provide information on the technological structure that are, potentially, falsifiable. Moreover, it is not a trivial exercise because the family (4) is not the only suitable class of functions for obtaining our results. It must be understood that many other functions of the type W can be constructed by using concave maps different from the (negative of) the norm.

Before concluding let us note that a by-product of our research is a complete solution of the question of existence of "chaotic" policy functions which represents optimal accumulation paths. The estimates of the critical value δ^* given in Remark 7 also suggests that the conjecture according to which the values of the discount parameter should be very small to obtain "chaos" is probably true. In fact the estimate $\delta^{**} < (32\gamma^2)^{-1}$ implies

$\delta^{**} < (1/32)$ when τ_δ is "chaotic". The conjecture was suggested in Boldrin and Montrucchio [2] and in Deneckere and Pelikan [6]. It is also clear that the negative conjecture made in Dechert [4] is not true in general. It is valid only for the standard one-sector model, because in that case the particular features of V force τ_δ to be monotone.

Finally, a "bifurcational conjecture" should be investigated: given a V which exhibits complex behaviors for δ small, what kind of bifurcation process leads from the Turnpike state to chaos ? In particular: is the so-called "Feigenbaum scenario" the most likely one ? Indeed this seems a fascinating and challenging problem.

APPENDIX.

Proof of Lemma 1. The proof is divided into two parts.

(1st part). Let $\alpha\varepsilon[0,1)$. Then to show that W is α_y-concave amounts to proving that:

$$W^*(x,y) = -(1/2)(1-\alpha)||y||^2 + \langle y-\bar{y},\theta(x)\rangle - (L/2)||x||^2$$

is concave when L satisfies the above inequality.

Concavity of W^* is equivalent to the concavity of the family of functions $f:R\to R$ defined as:

$$f(t) = W^*(x_0+tx_1,y_0+ty_1)$$

x_0, y_0 fixed in X and (x_0+tx_1), $(y_0+ty_1)\varepsilon X$, $t\varepsilon R$.

The case $x_1 = 0$ is trivial, given the fact that $\alpha < 1$. Without any loss of generality we can assume : $||x_1|| = 1$ and $0 < ||y_1||<+\infty$. To maintain both (x_0+tx_1) and (y_0+ty_1) within X we have to impose:

$$|t| = ||tx_1|| \leqslant \mu \quad \text{and} \quad ||ty_1|| \leqslant \mu.$$

It is easily seen that f will be of class C^2 in the interior points by construction. Hence we can compute its second derivative f'' which turns out to be:

$$f''(t) = -L - h_1(t) - h_2(t),$$

where:

$$h_1(t) = (1-\alpha)||y_1||^2 - 2\langle y_1,D\theta(x_0+tx_1)x_1\rangle, \text{ and}$$

$$h_2(t) = \langle\bar{y}-(y_0+ty_1),D^2\theta(x_0+tx_1)(x_1,x_1)\rangle.$$

To complete the proof of Lemma 1 we make use of the following Lemma:

LEMMA 2. There exists a positive constant M such that: for any $||y_1|| \geqslant M$, $h_1(t)$ is non-negative for all vectors (x_0+tx_1), (y_0+ty_1), x_0, $y_0\varepsilon X$, x_1 and y_1 in R^n, with $||x_1|| \leqslant 1$.

Precisely we can put: $M = 2\gamma/(1-\alpha)$.

Proof of Lemma 2. It is easily proved by using the following estimate:

$$|\langle y_1, D\theta(x_0+tx_1)x_1\rangle| \leqslant ||y_1||\circ||x_1||\circ||D\theta(x_0+tx_1)|| \leqslant \gamma||y_1||.$$

Hence:

$$h_1(t) \geqslant (1-\alpha)||y_1||^2 - 2\gamma||y_1|| = ||y_1||\circ\{(1-\alpha)||y_1|| - 2\gamma\},$$

and, by setting $||y_1|| \geqslant M = 2\gamma/(1-\alpha)$, we have $h_1(t) \geqslant 0$ for all t. Q.E.D.

Proof of Lemma 1 (2nd Part). To study the sign of $f''(t)$ we can now argue separately for the two cases: $||y_1|| \geqslant M$ and $||y_1|| < M$.

For the first case, Lemma 2 assures that $h_1(t) \geqslant 0$, so if we are able to prove that $h_2(t)$ is bounded we can conclude that there exists a (finite) positive value of L such that $f''(t) \leqslant 0$ everywhere. This is immediately obtained because $|h_2(t)| \leqslant \mu\sigma$ and then $L \geqslant \mu\sigma$ will guarantee the desired result.

In the second case($||y_1||<M$) it is possible to see that not only $h_2(t)$ is still bounded but that $h_1(t)$ is also so. In fact:

$$|h_1(t)| \leqslant (1-\alpha)||y_1||^2 + 2\gamma||y_1|| < (1-\alpha)M^2 + 2M\gamma.$$

In both cases L can be determined independently from the choice of x_0, x_1, y_0, y_1. In fact we need to set:

$$L \geqslant (1-\alpha)M^2 + 2\gamma M + \mu\sigma = 8\gamma^2/(1-\alpha) + \mu\sigma.$$

Q.E.D.

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