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STOCHASTIC GAMES OF RESOURCE ALLOCATION: EXISTENCE THEOREMS FOR DISCOUNTED AND UNDISCOUNTED MODELS*

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ABSTRACT

The canonical paradigm for the study of dynamic resource allocation in economic theory, the neoclassical growth model under possible production uncertainty, is extended in this paper to a strategic framework by modelling it as a stochastic game. For a very general specification of the problem, we establish the existence of stationary Markov equilibria in pure (non-randomized) strategies for the discounted game. We then show that, under a boundedness condition, as the discount factor tends to unity, the limit of discounted equilibrium strategies is a pure strategy stationary Markovian equilibrium of the undiscounted game with payoffs evaluated according to the long-run average criterion. The last result is of special interest as there are no available existence results for undiscounted stochastic games and in well-known examples, the limit of discounting equilibria are not equilibria in the undiscounted game. Journal of Economic Literature. Classification Numbers 020, 022, 026.

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1. Introduction

The theory of non–cooperative stochastic games finds an especially fertile application in the economic theory of capital accumulation and intertemporal resource allocation under imperfect competition. The standard model involves a single good – the state variable – that may be consumed or invested. In each period of an infinite–horizon, each of several infinitely–lived agents observes the available stock and independently decides on his consumption level for that period. Any amount left over after consumption by all the agents forms the investment for that period and is transformed to next period's available stock through a (stochastic) production function, and the situation repeats itself from the new state. Agents derive utility solely from their own consumptions and attempt to maximize the (expected) discounted sum of one–period utilities over the infinite horizon. However, since any agent's consumption level in a given period has repercussions for all the other agents' current and future rewards through its impact on the investment level for that period, the need for strategic interaction arises. This gives rise to a stochastic game.

Parametrized models of this sort with discounting, have been studied by Lancaster (1973), Levhari and Mirman (1980), Mirman (1981), Easwaran and Lewis (1985), Reinganum and Stokey (1985), and Cave (1987) among many others. Benhabib and Radner (1988) study the case when no functional form restrictions are placed on the (deterministic) production function, but utility functions are restricted to being linear in consumption. Our point of departure in this paper is the recent work of Sundaram (1989) and Majumdar and Sundaram (1988). Both papers establish the existence of pure–strategy stationary Markov equilibrium (PSSE) to these games in which all players employ strategies that are lower–semicontinuous (lsc) functions of the state variable. These results are obtained, without imposing any functional–form restrictions, under convexity and monotonicity assumptions on the game's structure that are standard in the neoclassical economic theory of intertemporal resource allocation; and a special assumption of symmetry (that all players have identical discount factors and payoff functions) that enables the authors to get around the familiar problems that arise when trying to demonstrate existence of equilibrium in pure strategies. The distinction between the papers is that Sundaram (1989) assumes the transition mechanism to be a deterministic function of investment, while Majumdar and Sundaram (1988) posit atomless transition probabilities.

Our contribution in this paper is twofold: first, we extend the results of these papers by demonstrating (Theorem 1) the existence of PSSE in lsc strateiges under a much more general transition mechanism that admits as special cases the deterministic and atomless mechanisms. More specifically, all we require of the transition probabilities (apart from the usual productivity and weak–continuity assumptions) is that they satisfy a certain "strong stochastic dominance" condition. This condition is little more than the requirement that higher investment yield (probabilistically) higher output.

Our second and main result (Theorem 2) combines this existence theorem for discounted stochastic capital accumulation games, with techniques developed in Dutta (1989) where the asymptotic properties of dynamic programming problems as the discount factor tends to unity are studied. We prove under a value–boundedness assumption that there is a PSSE in lsc strategies to the undiscounted stochastic resource allocation game, when payoffs are evaluated according the limit of means. In fact, we demonstrate that as the discount factor tends to unity, the equilibrium strategies of the discounted game converge (in a sense made specific in the paper) to PSSE of the undiscounted game and the discounted average value functions converge to a state–independent constant which turns out to be a long–run average value of the undiscounted game. To the best of our knowledge, this sharp convergence result is the first existence theorem for such undiscounted stochastic games.

A few remarks are in order. Firstly, it is interesting to contrast our results with existence theorems available in the literature on noncooperative stochastic games. The framework employed in the latter is typically considerably more general than ours; however, existence of equilibrium has been established only in mixed (i.e. randomized) strategies (e.g., Nowak (1985), Mertens and Parthasarathy (1987), Duffie, et al (1988)) that may not even be stationary (Mertens–Parthasarathy, and Duffie, et al), and only for discounted stochastic games. On the other hand, convexity and monotonicity assumptions of the form we employ are standard in the economics literature, and provide for a rich analytical structure. Our framework fully exploits these assumptions to obtain stronger results in this more restrictive framework.

The strengthening comes in two directions. In a stationary Markovian environment, a natural first class of strategies within which to look for an equilibrium is that of stationary Markovian strategies. Moreover, as an extensive literature in economics attests, a characterization of the positive and normative properties of pure strategy stationary Markovian equilibria is a task several orders of magnitude easier than a similar exercise for other equlibrium classes.[1] Finally, as has been argued for example in Maskin–Tirole (1988), stationary Markovian equilibria capture the more intuitive notion of "reaction functions" than do history dependent equilibria. For the symmetric resource game, based on the canonical paradigm of dynamic economics, the neoclassical growth model, we supply a very general existence result for PSSE in both discounted and undiscounted games. A second, and perhaps more important, strengthening pertains to the undiscounted game. As Sorin (1986) has shown by way of an example, the limit of the equilibria of discounted non–cooperative stochastic games may well not be an equilibrium in the corresponding undiscounted game; further

[1]For a complete characterization of the properties of stationary Markovian equilibria in resource games, see Dutta–Sundaram (1989).

even the ϵ–equilibria of the undiscounted game may be "far away" in payoffs from this limit. In fact, as Aumann (1987) points out, there is no general existence theorem in any class of strategies for undiscounted stochastic games. Theorem 2 in this paper, therefore, shows that for an interesting and important class of economic problems, we do have positive results on both the existence and the convergence issues.

It should be pointed out that for the asymmetric version of our game, there is no existence result that we know of. This remains an important open question.

This paper is organized as follows. Section 2 sets up the stochastic resource allocation game, lists our assumptions, and defines a stationary equilibrium to the game. Section 3 collects the main results on the existence of PSSE to the discounted and undiscounted games. All proofs are in section 4. Finally, the Appendix provides sufficient conditions under which our value–boundedness condition (used in proving Theorem 2) holds.

2. The Model

2.1 Preliminary Notation and Definitions

The set of reals (resp. non–negative reals, strictly positive reals) is denoted by $\mathbb{R}$ (resp. $\mathbb{R}_+$, $\mathbb{R}_{++}$).

A function $g: D \to \mathbb{R}$ is lower–semicontinuous or lsc at $x \in D$ iff for all sequences $\{x_n\} \subset D$ such that $x_n \to x$, $\liminf_n g(x_n) \geq g(x)$. g is upper–semicontinuous or usc at $x \in D$ iff $-g$ is lsc at x. Lastly, g is everywhere lsc (resp. usc) on D iff g is lsc (resp. usc) at each $x \in D$.

By analogy with probability distribution functions, a sequence of non–decreasing, right–continuous functions $\{F_n\}$ converges weakly to a limit F with the same properties iff $F_n(x) \to F(x)$ at all x where F is continuous. By Helley's selection theorem (Billingsley (1978, p. 290), for every such sequence of functions $\{F_n\}$, there is a subsequence n(k) of n, and a non–decreasing, right–continuous function F such that $F_{n(k)}$ converges weakly to F.

The support of a distribution function F is denoted by supp.(F). Supp(F) is defined as the interval [a, b] where $a = \inf\{x|F(x) > 0\}$ and $b = \inf\{x|F(x) = 1\} = \sup\{x|F(x) < 1\}$.

2.2 The Stochastic Resource Allocation Game

For notational convenience we confine ourselves to the 2–player situation. A generic player will be indexed by i. In all statements pertaining to i, j will denote the other player.

Time is discrete and continues forever. Periods are indexed by $t = 0, 1, 2, \ldots$. In each period t, the two players observe the available stock $y_t \geq 0$ of a good (the 'resource'), and, independently and simultaneously, decide on the amounts they plan to consume that period. Let $a_{it} \in [0, y_t]$ be player i's planned consumption. If plans are feasible ($a_{1t} + a_{2t} \leq y_t$), then they are executed, and player i's actual consumption, denoted by c_{it}, equals a_{it}. If plans are collectively infeasible ($a_{1t} + a_{2t} > y_t$), then actual consumptions are decided through a pair of "allocation functions" (h_1, h_2) as $c_{it} = h_i(y_t, a_{1t}, a_{2t})$. The amount $x_t = y_t - c_{1t} - c_{2t}$ left over after consumption by the players forms the investment (or savings) in period t. This investment then gets transformed to the period–$(t + 1)$ stock y_{t+1} which is realized according to the (conditional) probability distribution $q(\cdot|x_t)$. The situation now repeats itself from the new state y_{t+1}, and so on ad infinitum. Player i's reward in period t is a function only of his own (actual) consumption in period t, and is given by the utility function $u_i(c_{it})$; however, each player's consumption level in any period affects (through q) the distribution of output in the next period, and hence the future consumption possibilities for both players. Since the objective of both players is to maximize their expected total rewards (defined precisely below) over the infinite horizon, this conflict of interests creates the need for strategic behavior and gives rise to a stochastic game.

We now proceed to list our assumptions on the various components of this game, beginning with the transition mechanism q.

A. The Transition Probabilities:

Departing somewhat from standard practice, we define q to be a (conditional) probability distribution function, so that if y is the random variable distributed according to $q(\cdot|x)$, then $q(\bar{y}|x) = \text{Prob.}\ (y \le \bar{y}|x)$.

The first assumption on q has 2 parts: (i) there is no free production, and (ii) strictly positive investment levels today result always in strictly positive realization of output levels tomorrow, with a non–zero lower bound and a finite upper bound on possible realizations. Formally:

Assumption 1: (i) If $x = 0$, then $q(0|x) = 1$.

(ii) For all $x > 0$, there is a compact interval $I(x) \subset \mathbb{R}_{++}$, such that $\text{supp}\ (q(\cdot|x)) \subset I(x)$.

The next 2 assumptions are concerned with reproductivity of the resource. Assumption 2, usually referred to as a "productivity" or "Inada" condition, states that for sufficiently small investment levels, all realizations will exceed the investment level. Assumption 3 requires the existence of a "maximum sustainable stock": any realization of output from an investment level exceeding this stock will be smaller than the investment. For any $x, y \in \mathbb{R}_+$, let $q(y^-|x) := \lim_{z\uparrow y} q(z|x)$ denote the left–limit of $q(\cdot|x)$ at y.

Assumption 2: There is $\eta > 0$ such that if $0 < x < \eta$, then $q(x^-|x) = 0$, i.e., supp $q(\cdot|x) \subset [x, \infty)$.

Assumption 3: There is $\bar{x} > 0$ such that if $x \geq \bar{x}$, then $q(x|x) = 1$, i.e., supp $q(\cdot|x) \subset [0, x]$.

The next assumption is the standard weak–continuity requirement on q:

Assumption 4: If $x_n \to x$, then the sequence of distribution functions $q(\cdot|x_n)$ converges weakly to $q(\cdot|x)$.

Lastly, our "strong stochastic dominance" condition that requires larger investment levels to yield probabilistically higher stock levels:

Assumption 5: If $x < x'$, then for all $y \in \mathbb{R}_+$, $q(y^-|x) \geq q(y|x')$.

Remark 1: Sundaram (1989) assumes the existence of an increasing, continuous function $f:\mathbb{R} \to \mathbb{R}_+$ with (i) $f(0) = 0$, and (ii) $f(x) \geq x$ for $x \leq \bar{x}$, and $f(x) \leq x$ for $x \geq \bar{x}$ for some $\bar{x} > 0$ and sets $q(y|x) = 0$ if $y < f(x)$, $q(y|x) = 1$ if $y \geq f(x)$. Our assumptions clearly cover this as a special case.

Remark 2: Majumdar and Sundaram (1988) require q to satisfy assumptions 1–4 and to be atomless whenever $x > 0$. Note that, under this atomlessness assumption, strong stochastic dominance and their assumption of <u>weak</u> stochastic dominance ($x < x' \Rightarrow q(y|x) \geq q(y|x')$ for all $y \in \mathbb{R}_+$) are equivalent. Hence, our assumptions cover Majumdar–Sundaram (1988) also as a special case.

Assumptions 1–5 suffice for proving Theorem 1. To prove Theorem 2 we need a stronger version of Assumption 2, that is analogous to (identical to if transitions are deterministic) the standard assumption of the infinite slope of the production function at the origin that is employed in neoclassical growth theory. For $y \in \mathbb{R}_+$, let $m(y) = \inf. \{x | \text{supp. } q(\cdot|x) \subset [y, \infty]\}$, i.e., m(y) is the smallest investment level that will regenerate y with probability 1. Using this notation, Assumption 2 requires that, for small values of y, $m(y) \leq y$. For Theorem 2, we will replace Assumption 2 with

Assumption 2′: $[m(y)/y] \to 0$ as $y \to 0$.

B. <u>The State Space:</u>

We take the initial stock level y_0 to lie in some compact interval $[0, \hat{y}] \subset \mathbb{R}_+$. Define $y^* = \max \{\hat{y}, \bar{x}\}$, where $\bar{x}$ is the maximum sustainable stock of Assumption 3. Then, for all $x \in [0, y^*]$, $q(\cdot|x)$ has support contained in $[0, y^*]$. Therefore, there is no loss of generality in confining analysis to $[0, y^*] \equiv S$. We refer to S as the <u>state space</u>. Since this is only a choice of measurement units, we henceforth set $y^* = 1$.

C. <u>The Utility Functions:</u>

Our first assumption on the utility functions u_i requires them to satisfy the standard assumptions of concavity, monotonicity, and the Inada condition of unbounded marginal utility at the origin:

Assumption 6: For i = 1, 2, $u_i: \mathbb{R}_+ \to \mathbb{R}_+$ is strictly concave and strictly increasing on $\mathbb{R}_+$, and is $C^1 \cap \mathbb{R}_{++}$ with $\lim_{x \downarrow 0} u_i'(x) = +\infty$.

Our second assumption, that of 'symmetry' is special. As we explain in section 4, standard techniques, such as obtaining the equilibrium as a fixed–point of a (continuous) best–response map taking a (compact) set of strategies into itself, are inapplicable in this game. Symmetry enables overcoming this problem by enabling us to consider a different map whose fixed–point yields (after some manipulation) a PSSE to the game.

Assumption 7: (Symmetry) $u_1 = u_2$ (= u, say).

We remark that we are not aware of any existence theorem even under deterministic transitions in the presence of possibly non–symmetric payoffs, but counterexamples are also lacking.

D. Allocation functions:

For the sake of definiteness, we specify a symmetric allocation rule:

$$h_1(y, a_1, a_2) = h_2(y, a_1, a_2) = y/2, \; y > 0$$

The "equal split" rule is a natural candidate to resolve infeasible plans in the symmetric game. The results that follow also hold for a larger class of allocation rules with similar qualitative features.[2] Since the establishment of existence results for the broadest set of allocation rules, which are never invoked in equilibrium, is peripheral to our main question, we do not pursue it any further here.

The tuple $\{S, q, u, h_1, h_2\}$ completes the specification of the stochastic resource allocation game. In the next subsection, we define strategies and equilibria to the discounted and undiscounted versions of this game.

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[2]It is clear that some structure on the allocation rules is indispensible. Else, bizarre specifications of allocation rules could force a dynamic resource game into yielding arbitrary outcomes. On this point, see Dutta–Sundaram (1989, footnote 15).

2.3 Stationary Strategies and PSSE

Definition 1: A stationary strategy (for either player) is a measurable function $g:S \to S$ that satisfies $g(y) \in [0, y]$ $\forall$ $y \in S$.

A player's stationary strategy g specifies his planned consumption level at any $y \in S$ as $g(y)$. Let Γ denote the space of all stationary strategies available to the players. (Note that we have already restricted attention to only pure (i.e., non-randomized) strategies.) Henceforth, we denote a stationary strategy for i by g_i.

A pair of stationary strategies (g_1, g_2) induces in the obvious manner through u, q, and the functions h_i, a t-th period expected reward for player i from each initial state $y \in S$. Denote this reward by $u_i^t(g_1, g_2)(y)$.

In the discounted stochastic resource allocation game with discount factor δ (henceforth, δ – SRG for short), both players discount future rewards by the discount factor $\delta \in (0, 1)$. Thus, in the δ-SRG, player i's total reward from the initial state y when the players employ the stationary strategies (g_1, g_2) is given by:

$$W_i^{\delta}(g_1, g_2)(y) = \Sigma_{t=0}^{\infty} \delta^t u_i^t(g_1, g_2)(y).$$

Definition 2: The stationary strategy $g_1^* \in \Gamma$ is a best-response of player 1 to player 2's stationary strategy g_2 in the δ – SRG iff:

$$W_1^{\delta}(g_1^*, g_2)(y) \geq W_1^{\delta}(g_1, g_2)(y) \quad \forall\ y \in S, \forall g_1 \in \Gamma.$$

This definition is standard and requires no elaboration. We do however note that the stipulation that the best-response be chosen from Γ is not a restriction. A well-known argument in dynamic programming establishes that a stationary non-randomized best–response to a stationary strategy is also a best–response in the space of all strategies, i.e., even when the responding player may randomize and/or condition his actions on histories. Similarly, a best–response of player 2 to $g_1 \in \Gamma$ is defined. We are naturally led to:

<u>Definition 3:</u> <u>The pair</u> $(g_1^\delta, g_2^\delta) \in \Gamma \times \Gamma$ <u>is a PSSE to the δ–SRG iff: for</u> i, j = 1, 2, <u>and</u> $i \neq j$, g_i^δ <u>is a best–response in the</u> δ–SRG <u>to</u> g_j^δ.

In the undiscounted stochastic resource allocation game (U–SRG, for short) both players use the <u>limit–of–means</u> (or: <u>long–run average</u> or LRA) criterion to evaluate payoffs from the game. That is, if $W_i(g_1, g_2)(y)$ denotes the total reward to i from y $\in$ S under the LRA, we have:

$$(2.1) \qquad W_i(g_1, g_2)(y) = \liminf_{T\to\infty} \left[\frac{1}{T} \Sigma_{t=0}^{T-1} u_i^t(g_1, g_2)(y) \right]$$

or, if the limit on the right is well–defined:

$$(2.2) \qquad W_i(g_1, g_2)(y) = \lim_{T\to\infty} \left[\frac{1}{T} \Sigma_{t=0}^{T-1} u_i^t (g_1, g_2)(y) \right]$$

We define best–responses in the U–SRG exactly as in definition 2, with the obvious modification. Finally, exactly as in definition 3, a PSSE to the U–SRG is defined by a pair of strategies (g_1, g_2) that are best–responses to each other in the U–SRG.

Some preliminary observations before proceeding to the next section. Under our specification of the model, trivial equilibria always exist, equilibria in which after some history players choose actions that are collectively infeasible. Indeed, it is readily verified that the stationary strategies (g_1, g_2) specified by $g_1(y) = g_2(y) = y \; \forall \; y \in S$

constitute a PSSE to both the δ–SRG and the U–SRG. Such equilibria are uninteresting for obvious reasons and this motivates the definition:

Definition 4: A PSSE (g_1, g_2) (to the discounted or undiscounted game) is "interior" if for all $0 < y \in S$ we have $g_1(y) + g_2(y) < y$.

Our existence theorems in this paper pertain to interior PSSE. Apart from their greater realism, our desire to search for interior PSSE was motivated at least in part by wishing to find a set of conditions under which the stock would not be extinct in finite time. For more on the importance of this and related issues, we refer the reader to Clemhout and Wan (1987).

Finally, some terminology for the U–SRG. If the limit in (2.1) is well–defined and the LHS of (2.2) is a constant for all $0 < y \in S$ (say, $W_i(g_1, g_2)(y) = \nu_i$) then we define ν_i to be the long–run average (LRA) value of the game to player i in the PSSE (g_1, g_2).

3. Results

Our first theorem establishes the existence of PSSE in lsc strategies to the δ–SRG for $\delta \in (0, 1)$:

Theorem 1: (a) For each $\delta \in (0, 1)$, there is a function $g^\delta \in \Gamma$ satisfying

(i) g^δ is everywhere lsc on S

(ii) $0 < 2g^\delta(y) < y$ for all $0 < y \in S$

and (iii) $\frac{g^\delta(y) - g^\delta(y')}{y - y'} \leq 1/2$ for all $y \neq y' \in S$

such that the stationary strategies $g_1^\delta = g_2^\delta = g^\delta$ constitute a PSSE to the δ–SRG

(b) <u>The corresponding payoff functions</u> $W_i^\delta(g^\delta, g^\delta)$ <u>are upper–semicontinuous and non–decreasing on</u> S.

***Remark*:** Theorem 1 asserts a stronger property of the equilibrium strategies than mentioned above; namely, that these strategies have slopes bounded above everywhere by 1/2 (1/n in the n–player case). This last property is crucial in establishing Theorem 2.

Some new notation based on Theorem 1 is needed to state Theorem 2. Let ψ^δ:S → S represent the savings function in the δ–SRG, i.e., $\psi^\delta(y) = y - 2g^\delta(y)$ at all $y \in S$. For ease of notation, let V_1^δ, V_2^δ denote respectively the total payoff functions in equilibrium in the δ–SRG, that is, $V_i^\delta = W_i^\delta(g_1^\delta, g_2^\delta)$. Since $g_1^\delta = g_2^\delta = g^\delta$, by symmetry we must also have $V_1^\delta = V_2^\delta$. Denote this common function by V^δ. The value–boundedness condition that we need to prove Theorem 2 can now be stated formally.

***Assumption 8*:** There is a function $M:S \to \mathbb{R}$ with $M(y) > -\infty$ for all $y \neq 0$, such that for all $0 < y \in S$, $(V^\delta(y) - V^\delta(1)) \geq M(y)$ for all $\delta \in (0, 1)$.

We have been unable to prove that this condition holds in general. On the other hand (see Appendix), we have found several sufficient conditions on the PSSE of the δ–SCG that ensure it; and we have parametrized examples where the PSSE satisfy these sufficient conditions for an interval of values of the parameter. Assumption 8 is neither vacuous nor degenerate.

Observe that, by the properties of g^δ listed in Theorem 1, for each $\delta \in (0, 1)$ ψ^δ is an <u>upper–semicontinuous</u> and <u>non–decreasing</u> function on S. Let Ψ denote the space of all such functions:

$$\Psi = \{\psi: S \to S \mid \psi(y) \in [0, y] \text{ for all } y \in S;\ \psi \text{ is usc and non-decreasing on } S\}$$

Since non–decreasing usc functions are also right–continuous, it follows from Helley's Theorem (e.g., Billingsley (1978, p. 290)) that every sequence in Ψ has a weakly–convergent subsequence. That is, if $\{\psi^n\}$ is a sequence in Ψ, then there is a subsequence n(k) of n and a function $\psi \in \Psi$ such that $\lim_{k\to\infty} \psi^{n(k)}(y) = \psi(y)$ for all continuity points y of ψ. In particular, $\psi^\delta \in \Psi$ for each $\delta \in (0, 1)$; so if $\delta_n < 1$, $\delta_n \to 1$, this property holds for the sequence $\{\psi^{\delta_n}\}$.

Theorem 2: (i) There is a PSSE to the U–SRG.

(ii) Let $\delta_n < 1$, $\delta_n \to 1$. Assume, wlog, that ψ^{δ_n} converges weakly to $\psi \in \Psi$. Then, the stationary strategies (g_1, g_2) defined by $g_1 \equiv g_2 \equiv g$, where

$$g(y) = \frac{1}{2}(y - \psi(y)) \text{ for all } y \in S$$

constitutes a symmetric PSSE to the U–SRG.

(iii) The total payoff functions $W_i(g_1, g_2)$ associatied with this equilibrium are both equal to a constant ν at all $0 < y \in S$, where ν is defined by

$$\nu = \lim_{n\to\infty} (1 - \delta_n)V^{\delta_n}(1).$$

and, is the long–run value of the game to either player.

In words, the pure strategy stationary equilibria to the U–SRG under the long–run average payoff criterion, may be obtained as the weak limit of the PSSE of the δ–SRGs as $\delta \uparrow 1$. Moreover, the PSSE thus obtained has a constant LRA value on S, which is given by the limit (as $\delta \uparrow 1$) of the normalized payoff functions of the δ–SRG.

4. Proofs

4.1 Preliminary Results

The following lemma, a generalization of Fatou's lemma, is critical in proving Theorems 1 and 2. The proof of this lemma follows from Dutta (1989, Theorem 1).

Lemma 1: Let F_n, F be non-positive, non-decreasing, right-continuous functions from S into $\mathbb{R}_+$, and suppose F_n converges weakly to F. Let $\{x_n\} \subset S$, $x_n \to x$. Then,

$$\text{(4.0)} \qquad \limsup_{n\to\infty} \int F_n(\cdot)dq(\cdot|x_n) \leq \int F(\cdot)dq(\cdot|x).$$

Proof: As $x_n \to x$, $q(\cdot|x_n)$ converges weakly to $q(\cdot|x)$ by assumption 4. The lemma now follows from the Generalized Fatou's lemma (Dutta (1989, Theorem 1)).

Remark 1: If $x_n = x$ for all n, then (4.0) is, of course, just Fatou's lemma, since the functions are all non-positive. On the other hand if $F_n = F$ for all n, then (4.0) follows from the definition of weak-convergence of probability measures, since non-decreasing, right-continuous functions are also usc. Lemma 1 generalizes both results by allowing measures and integrands to vary simultaneously.

Remark 2: A simple corollary of lemma 1 is that (4.0) continues to hold if the condition of non-positivity is replaced by the stipulation that the functions F_n, F are all uniformly bounded above on S, provided the other conditions of lemma 1 hold.

4.1 Proof of Theorem 1

The construction of the proof of Theorem 1 is in the spirit of Sundaram (1989) and Majumdar and Sundaram (1988). Consequently, in the early steps we state without detailed proof several preliminary results. The reader is urged to consult the two papers for details. The notation introduced here is important for it is used again in proving Theorem 2.

Step 1: A Generalized Game

Following Debreu (1954) we define a generalized game, as one in which infeasible plans are ruled out by construction. For any stationary strategy g, suppose that a responding player is restricted to picking strategies only from $\Gamma(g) \subset \Gamma$ defined by

$$\Gamma(g) = \{\gamma \in \Gamma \,|\, \gamma(y) \in [0, y - g(y)] \ \forall\ y \in S]$$

When both players are restricted in this manner to picking plans which are feasible relative to the other player's strategy, we have a generalized game. A best response in such a game we will call a "generalized best-response" or GBR. Consider a symmetric equilibrium (g, g) of the generalized game, and call it an interior symmetric equilibrium if $0 < 2g(y) < y$ at all $y > 0$. Given the allocation rule that splits the stock in the event of infeasible plans, it follows that an interior equilibrium in the generalized game is actually an interior equilibrium in the δ–SRG.

Lemma 2: Let (g, g) be an interior symmetric equilibrium in the generalized game. Then, (g, g) is an interior symmetric equilibrium in the δ–SRG.

Proof: Let V^δ represent the players' value function in the equilibrium (g^δ, g^δ) of the generalized game. Then,

$$\begin{aligned} V^\delta(y) &= u(g^\delta(y)) + \delta \int V^\delta(\cdot)dq(\cdot\,|y - 2g^\delta(y)) \qquad \forall\ y \in S \\ &\geq u(y - g^\delta(y)) + \delta\frac{u(0)}{1-\delta} \\ &> u(\tfrac{y}{2}) + \delta\frac{u(0)}{1-\delta} \end{aligned}$$

The last expression is the returns to the choice of an infeasible plan by a responding player in the δ–SRG. The lemma is hence proved.

From hereon, we concentrate on finding an interior equilibrium in the generalized game. However, the usual technique of obtaining an equilibrium as the fixed point of a suitably constructed best–response mapping cannot directly be applied here, since best–responses cannot be guaranteed to belong to the same space (i.e., satisfy the same restrictions) as the corresponding strategies. Specifically, a sufficient condition (see Majumdar–Sundaram (1988, Theorem 3.1)) for a strategy g to admit a GBR is that g be everywhere lsc on S. However, it is trivial to construct strategies that satisfy these conditions, but admit no GBR that is everywhere lsc on S. Similarly the GBR to an everywhere continuous (resp. differentiable) strategy cannot be guaranteed to be everywhere continuous (resp. differentiable). We explain now how symmetry enables us to overcome this problem.

Step 2: A Common Savings Function

Consider the space Ψ defined earlier (section 3) as the space of non–decreasing, usc functions $\psi: S \to S$ that obey $\psi(y) \in [0, y]$ for each $y \in S$. As in Sundaram (1989) and Majumdar–Sundaram (1988), we identify Ψ with a set of potential savings functions in a stationary equilibrium. Each such function ψ is "consistent" with the symmetric consumption functions $(g(\psi), g(\psi))$ defined by

$$(4.1) \qquad g(\psi)(y) = \frac{1}{2}(y - \psi(y)),\ y \in S.$$

Observe that $g(\psi)$ is lsc on S; and further that for all $y, y' \in S$ with $y \neq y'$, we have

$$(4.2) \qquad [(g(\psi)(y) - g(\psi)(y'))/(y - y')] \leq 1/2.$$

We observe, without proof, the following:

Lemma 3: For each $\psi \in \Psi$, $g(\psi)$ <u>admits a GBR</u> $\hat{g}(\psi)$. <u>The total payoff</u> $V_\psi : S \to \mathbb{R}$ <u>obtained by the responding player in a GBR to</u> $g(\psi)$ <u>is usc and non-decreasing on</u> S.

A proof of Lemma 3 may be found in Majumdar–Sundaram (1988, Theorem 3.1).

As indicated above, on the other hand, $\hat{g}(\psi)$ need not be lsc, and certainly therefore, need not be consistent with any savings function $\psi \in \Psi$ in the sense of (4.1). However, one can prove:

<u>Lemma 4:</u> (i) <u>For each</u> $\psi \in \Psi$, <u>there is a unique GBR</u> $\hat{g}(\psi)$ to $g(\psi)$ <u>such that the function</u> $\hat{\psi}(\psi)$ <u>defined by</u>

$$\hat{\psi}(\psi)(y) = y - g(\psi)(y) - \hat{g}(\psi)(y)$$

<u>is in</u> Ψ.

(ii) <u>Moreover,</u> $\hat{\psi}(\psi)$ <u>can be actually be defined as</u>

$$\text{(4.3)} \qquad \hat{\psi}(\psi)(y) = \max \{x \in \xi(\psi)(y)\}$$

$$\text{where } \xi(\psi)(y) = \underset{x \in [0,\ y-g(\psi(y)]}{\text{argmax}} \{u(y-g(\psi)(y)-x) + \delta \textstyle\int V_\psi(\cdot)dq(\cdot|x)\}$$

<u>Sketch of Proof:</u> By the Principle of Optimality, V_ψ satisfies at each $y \in S$:

$$\text{(4.4)} \qquad V_\psi(y) = \max_{x \in [0,\ y-g)\psi)(y)]} \{u(y-g(\psi)(y)-x) + \delta \textstyle\int V_\psi(\cdot)dq(\cdot|x)\}.$$

From the strict concavity of u and (4.2), it immediately follows that the objective in the RHS of (4.4) is a supermodular function of (x, y). Hence, it follows (see, e.g., Majumdar–Sundaram (1988, Lemma 4.3)) that the correspondence of maximizers in (4.4), $\xi(\psi)$, is non–decreasing, i.e., $y' > y$, $x \in \xi(\psi)(y)$, $x' \in \xi(\psi)(y') \Rightarrow x' \geq x$. The right continuity of V_{ψ} (Lemma 3) and the weak continuity of q (Assumption 4) then imply that $\hat{\psi}$, as in (4.3), is usc, non–decreasing and is in fact, the unique usc, non–decreasing selection from $\xi(\psi)$.

Q.E.D.

We henceforth refer to $\hat{\psi}(\psi)$ as the GBR to ψ (rather than as "the savings function in a GBR to $g(\psi)$"). This GBR map defines a map $H:\Psi \rightarrow \Psi$. At a fixed–point ψ^* of this map H, we have $H(\psi^*)(y) = \psi^*(y)$ for all $y \in S$, or:

$$\begin{aligned} y - g(\psi^*)(y) - \hat{g}(\psi^*)(y) &= \psi^*(y) \\ &= y - 2\, g(\psi^*)(y) \end{aligned}$$

so $g(\psi^*)(y) = \hat{g}(\psi^*)(y)$ for all $y \in S$, or $g(\psi^*)$ is a GBR to itself. By symmetry $g(\psi^*)$ is then an equilibrium to the generalized game.

Lemma 5: At a fixed point ψ^* of $H, g(\psi^*)$ is an interior, symmetric equilibrium of the generalized game.

Proof: We only need to show that $g(\psi^*)$ is interior. The proof follows by exploiting the Inada conditions (Assumptions 2 and 6) on u and q. See Lemma 4.9 in Majumdar–Sundaram (1988) for details.

Q.E.D.

The task is now reduced to finding a fixed point of H.

Step 3: An Expansion of the State Space

For technical reasons, we expand the state space $S = [0, 1]$ to a larger space $S^* = [0, s^*]$, $s^* > 1$, to avoid upper–endpoint problems in S. On S^*, define

$$\Psi^* = \{\psi: S^* \to S^* \mid \psi(y) \in [0, y] \ \forall \ y \in S^*; \ \psi \text{ is usc and non-decreasing on } S^*; \ \psi(s^*) = s^*\}.$$

Since supp $q(\cdot|x) \subset S^*$ for $x \in S^*$ by assumption 3, the generalized game with state space S^* is also well–defined. Lemma 3 and 4 are extended to S^* in the obvious manner, and continue to be valid. Finally, it is clear that the restrictions to S of strategies that constitute a PSSE to the generalized game on S^* constitute a PSSE to the generalized game on S.

We first establish the requisite topological conditions for Ψ^* to have the fixed–point property. Observe that $\Psi^* \subset \Pi(S^*)$, where $\Pi(S^*)$ is the space of all distribution functions corresponding to finite, positive measures μ on the Borel sets of S^* satisfying $\mu(S^*) = s^*$. A well–known result (see, e.g., Parthasarathy (1967, Theorem 6.4.1) establishes that this space of measures, hence $\Pi(S^*)$, is compact in the metrizable topology of weak convergence. It is easily seen that Ψ^* is a closed subset of $\Pi(S^*)$ in this topology ; hence, Ψ^* is also compact, and sequential arguments in Ψ^* suffice. Convexity of Ψ^* is immediate. And invoking the Schauder–Tychonoff fixed–point theorem (e.g., Smart (1974)):

Lemma 6: Ψ^* has the fixed–point property, i.e., any continuous map from Ψ^* into itself has a fixed–point.

To ensure that the GBR map H, of step 1, maps Ψ^* into itself, we modify H at the upper–endpoint of S^* to $H(\psi)(s^*) = s^* \ \forall \ \psi \in \Psi^*$. Observe that a fixed–point ψ^* of H still yields a function $g(\psi^*)$ that is a GBR to itself on $[0, s^*)$, hence on $[0, 1]$. (This is true since supp $q(\cdot|x) \subset [0, x] \ \forall \ x \in [1, s^*)$ by assumption 3, so that if the game starts in $[0, s^*)$, it can never get to s^* which is then irrelevant.) All that remains in the proof of Theorem 1 is, therefore,

Step 4: Continuity of the GBR map $H:\Psi \to \Psi^*$[3]

Let $\{\psi_n\} \subset \Psi^*$ be a sequence converging weakly to ψ, and let $\hat{\psi}_n = H(\psi_n)$. Assume wlog that $\hat{\psi}_n$ converges weakly to $\hat{\psi}$. We need to show that $\hat{\psi} = H(\psi)$.

Recall that V_ψ denotes the payoff in a generalized best–response to ψ. For ease of notation denote V_{ψ_n} by V_n. Since V_n is non–decreasing and right–continuous for each n, Helley's selection Theorem (Billingsley (1978, p. 290)) implies that along some subsequence, again denoted n, V_n converges weakly to a right–continuous, non–decreasing function V. Note also that V_n, V are uniformly bounded above and below on S^* by $u(s^*)(1 - \delta)^{-1}$ and $u(0)(1 - \delta)^{-1}$ respectively.

For ease of notation, let k_n, k, denote the residual stock under ψ_n and ψ respectively, i.e., for $y \in S^*$, $k_n(y) = y - g(\psi_n)(y) = \frac{1}{2}(y + \psi_n(y))$, and $k(y) = y - g(\psi)(y) = \frac{1}{2}(y + \psi(y))$. Note that k_n, k are right–continuous, non–decreasing functions and since ψ_n converges weakly to ψ, so k_n converges weakly to k.

For each n, V_n satisfies at each $y \in S^*$:

[3]This step, the continuity of H, is the point where Majumdar–Sundaram (1988) and Sundaram (1989) critically exploit their respective assumptions of atomless and deterministic transitions.

$$(4.5)\qquad V_n(y) = \max_{x\in[0,k_n(y)]} \{u(k_n(y) - x) + \delta\int V_n(\cdot)dq(\cdot|x)\}$$

$$= u(k_n(y) - \hat{\psi}_n(y)) + \delta\int V_n(\cdot)dq(\cdot|\hat{\psi}_n(y)).$$

Let $\bar{S} \subset S^*$ be defined by $\bar{S} = \{y \in S^* | V, \psi \text{ and } \hat{\psi} \text{ are continuous at } y\}$. Note that $\bar{S}$ is dense in S^*. For $y \in \bar{S}$, $\hat{\psi}_n(y) \to \hat{\psi}(y)$ as $n \to \infty$, so $q(\cdot|\hat{\psi}_n(y))$ converges weakly to $q(\cdot|\hat{\psi}(y))$. Since V_n, k_n converge weakly to V and k respectively, taking limits in (4.5) yields for $y \in \bar{S}$ by Lemma 4.1:

$$(4.6)\qquad \begin{aligned} V(y) &\le u(k(y) - \hat{\psi}(y)) + \delta \operatorname{limsup}_n \int V_n(\cdot)dq(\cdot|\hat{\psi}_n(y)) \\ &\le u(k(y) - \hat{\psi}(y)) + \delta\int V(\cdot)dq(\cdot|\hat{\psi}(y)). \end{aligned}$$

Since $\bar{S}$ is dense in S^*, (4.4) holds for all $y \in S^*$. We now prove that the opposite inequality also holds, and that, in fact, $\hat{\psi}(y)$ solves for each $y \in [0, s^*)$:

$$(4.7)\qquad V(y) = \max_{x\in[0\,,\,k(y)]} \{u(k(y) - x) + \delta\int V(\cdot)dq(\cdot|x)\}.$$

This will clearly complete the proof of continuity of H.

To establish the desired inequality, consider any $y \in [0, s^*)$, and an arbitrary $x \in [0, k(y)]$. Since $\bar{S}$ is dense in S^*, there is a sequence $y_m \downarrow y$, $y_m \in \bar{S}$, and an associated feasible investment $x_m \in [0, k(y_m)]$, $x_m > x$. By the strong stochastic dominance assumption (Assumption 5), there are atomless measures μ_m, $m \geq 0$ such that: $q(\cdot|x_m)$ (weakly) stochastically dominates μ_m which (weakly) stochastically dominates $q(\cdot|x)$. Now,

$$V_n(y_m) \geq u(k_n(y_m) - x_m) + \delta\int V_n(\cdot)dq(\cdot|x_m)$$
$$\geq u(k_n(y_m) - x_m) + \delta\int V_n(\cdot)d\mu_m$$

Since μ_m is atomless, noting that $V_n \to V$ μ_ma.e., and appealing to the dominated convergence theorem yields $\int V_n(\cdot)d\mu_m \to \int V(\cdot)d\mu_m$. So, taking limits,

$$V(y_m) \geq u(k(y_m) - x_m) + \delta\int V(\cdot)d\mu_m$$
$$\geq u(k(y_m) - x_m) + \delta\int V(\cdot)dq(\cdot|x)$$

The last inequality follows since μ_m stochastically dominates $q(\cdot|x)$. Letting $y_m \downarrow y$, we have

(4.8) $$V(y) \geq u(k(y) - x) + \delta\int V(\cdot)dq(\cdot|x), \forall\ x \subset [0, k(y)]$$

Clearly, (4.6) and (4.8) together yield (4.7) for $y < s^*$. Hence, we have proved

<u>Lemma 7:</u> <u>The best response function</u> $H: \Psi^* \to \Psi^*$, <u>is continuous in the weak topology</u>.

Combining Lemmas 6 and 7 we have a fixed point ψ^* of H. But then,

$$\psi^*(y) = y - 2g(\psi^*)(y)$$
$$= y - g(\psi^*)(y) = \hat{g}(\psi^*)(y)$$

or $\hat{g}(\psi^*)(y) = g(\psi^*)(y)$ at all $y < s^*$, implying that $g(\psi^*)$ is a GBR to itself on s^*. The proof that this is also an equilibrium in the original game, follows Lemmas 2 and 5.

4.2 Proof of Theorem 2

Let $\delta_n \uparrow 1$, and let V_n denote the value function under a PSSE in the δ_n – SRG with the properties listed in Theorem 1. Write $\phi_n(y) = V_n(y) - V_n(1)$. Note that in proving Theorem 1 we showed that V_n is non–decreasing and usc, and hence $\phi_n \leq 0$. Now, the optimality equation can be re–written as

$$(4.9) \qquad (1 - \delta_n)V_n(1) + \phi_n(y) = u(k_n(y) - \psi_n^*(y)) + \delta_n \int \phi_n(\cdot)dq(\cdot \mid \psi_n^*(y))$$

ϕ_n is bounded below pointwise by the value boundedness assumption (Assumption 8). It is also non–decreasing and usc by Theorem 1. Hence, there is a subsequence (again denoted by n) on which, $\phi_n \Rightarrow \phi$, $\psi_n^* \Rightarrow \psi^*$ and $(1 - \delta_n)V_n(1) \to \nu$, where ν is a subsequential limit of the bounded sequence $(1 - \delta_n)V_n(1)$. Let $\bar{S} = \{y \in S\text{: } y \text{ is a continuity point of } \phi, \psi^*\}$. For $y \in \bar{S}$, Lemma 1 yields,

$$(4.10) \qquad \nu + \phi(y) \leq u(k^*(y) - \psi^*(y)) + \int \phi(y')dq^*(y' \mid \psi^*(y)).$$

Again, exploiting the denseness of $\bar{S}$ in S, (4.10) can be shown to hold for all $y \in S$. Iterating on (4.10) and using the fact that $\phi \leq 0$ now results in:

$$T\nu + \phi(y) \leq \Sigma_{t=0}^{T-1} u^t(g^*)(y) + E[\phi(y_T) \mid y] \leq \Sigma_{t=0}^{T-1} u^t(g^*)(y)$$

where $E[\phi(y_T) \mid y]$ and $u^t(g^*)(y)$ denote respectively the expected T–th period value of ϕ and the t–th period expected reward under g^*, from the initial state y. Dividing by T and letting $T \to \infty$, we get

$$(4.11) \qquad \nu \leq \liminf_{T\to\infty} \left[\frac{1}{T} \Sigma_{t=0}^{T-1} u^t(g^*)(y) \right]$$

or, that using g^* as a response to g^* yields a LRA payoff of at least ν from any $0 < y \in S$.

We now show that there is no response to g^* which yields a LRA payoff greater than ν from the initial state $y = 1$. Since g^* has a slope bounded above everywhere on S by 1/2, it is trivial to see that the LRA payoff against g^* must be non–decreasing on S. Therefore, if ν is an upper bound on the LRA payoff from $y = 1$, it is also an upper bound from any other initial state $y > 0$. Combining this with (4.8), the proof of Theorem 2 is completed. We proceed in several steps.

Step 1: Identical arguments as in the proof of lemma 7 establish that for any fixed $x \in [0, k^*(y)]$:

(4.12) $$\nu + \phi(y) \geq u(k^*(y) - x) + \int \phi(\cdot)dq(\cdot|x)$$

Combining this with (4.10), we now have:

(4.13) $$\nu + \phi(y) = \max_{x \in [0,k^*(y)]} \{u(k^*(y) - x) + \int \phi(\cdot)dq(\cdot|x)\},\ y > 0.$$

Step 2: Let π be any (i.e., not necessarily stationary) strategy[4] for the player (say, 1) responding against g^* in the U–SRG. From hereon fix the initial state $y = 1$. Let $u^t(\pi, g^*)$ denote the expected t–th period reward from using π against g^*. Finally, let

---

[4]Formally, for $t \geq 0$, let $h_{t+1} \in H_{t+1} = S^{3t} \times S$ be a partial history of states and actions by the 2 players up to period t, and the period–(t+1) state. Then, a strategy π is a sequence of measurable maps $\{\pi_t\}$ such that $\pi_t: H_t \to S$ and $\pi_t(h_t) \in [0, y_t]$ for all t, where y_t is the last element of h_t. Observe that a stationary strategy π is one for which π_t depends in a time–constant way only on the last coordinate of h_t for each t, i.e., $\pi_t = g$ for all t where $g: S \to S$.

μ_π^T denote the period–T marginal distribution over S from using π against g^*. Iterating on (4.2) with y = 1,

$$(4.11) \qquad \nu + \frac{1}{T}\,\phi(y) \geq \frac{1}{T}\left[\,\Sigma_{t=0}^{T-1}\, u^t(\pi,\, g^*)(1) + \int \phi(\cdot)d\mu_\pi^T(\cdot)\,\right].$$

Note that if $\left[\frac{1}{T}\int \phi d\mu_\pi^T\right] \to 0$ as $T \to \infty$, then (4.13) implies

$$(4.14) \qquad \nu \geq \text{limsup}_{T\to\infty}\left[\frac{1}{T}\,\Sigma_{t=0}^{T-1}\, u^t(\pi,\, g^*)(1)\right],$$

thus completing the proof of Theorem 2 by the earlier arguments. We now show that this must indeed be the case for some ϵ–optimal BR to g^* in the U–SRG, $\forall\ \epsilon > 0$.

Step 3: Recall that $m(y) = \inf.\{x | q(y^- | x) = 0\}$, i.e., m(y) is the minimum feasible investment level that will reproduce a stock level of at least y almost surely. $m(\cdot)$ is increasing in y. By the Inada condition (Assumption 2′), there is $\eta^* > 0$ such that $m(y) < y/2$ for $y \in (0, \eta^*)$. Note that for all $y \in S$, $k^*(y) = y - g^*(y) \geq y/2$. For small values of $\xi > 0$, let P_ξ denote the following perturbed best–response problem:

$$(P_\xi) \qquad \begin{array}{lll} \underset{\pi}{\text{Max}} & & \text{limsup}_{T\to\infty}\left[\frac{1}{T}\,\Sigma_{t=0}^{T-1}\, u^t(\pi,\, g^*)(1)\right] \\ \text{subject to:} & & x \in [m(\xi),\, k^*(y)],\ y \geq \xi \\ & & x \in [m(y),\, k(y)],\ \ y < \xi. \end{array}$$

where x is the action recommended by π for any history ending in y. By the earlier observations, this problem is well–defined for small values of ξ. Note that $\xi = 0$ corresponds to the original best–response problem from the initial state y = 1, with "limsup" replacing "liminf" in the objective function.

Let λ_ξ and $\overline{\lambda}$ denote respectively the supremum over all π of the objective function in P_ξ when $\xi > 0$ and $\xi = 0$. Clearly $\overline{\lambda} \geq \lambda_\xi$ for all $\xi > 0$, and λ_ξ is non-decreasing as $\xi \downarrow 0$.

Suppose it were true that $\lambda_\xi \to \overline{\lambda}$ as $\xi \downarrow 0$ (we prove this in step 4). Let $\epsilon > 0$ be given. Pick any $\xi > 0$ such that $\overline{\lambda} - \lambda_\xi < \epsilon/2$. Pick any plan π in P_ξ that is $\epsilon/2$–optimal. Then, of course, π is ϵ–optimal in P_0.

By construction, μ_π^T has full support on $[\xi, 1]$ for all $T > 0$. This implies, that since ϕ is non-decreasing and bounded above, so $\int \phi d\mu_\pi^T$ is uniformly bounded. Therefore, $\frac{1}{T} \int \phi d\mu_\pi^T \to 0$ as $T \to \infty$, or from (4.13)

$$(4.16) \qquad \nu \geq \limsup_{T\to\infty} \left[\frac{1}{T} \Sigma_{t=0}^{T-1} u^t(\pi, g^*)(1) \right] \geq \lambda_\xi - \epsilon/2 \geq \overline{\lambda} - \epsilon.$$

But (4.14) holding for all $\epsilon > 0$ implies $\nu \geq \overline{\lambda}$, which combined with (4.10) proves Theorem 2.

Thus, as the last step in the proof we show that $\lambda_\xi \to \overline{\lambda}$ as $\xi \downarrow 0$.

<u>Step 4:</u> Given $\epsilon > 0$ pick ξ so that $[u(m(\xi)) - u(0)]] < \epsilon$. Since $m(\xi) \to 0$, as $\xi \to 0$, and u is continuous on $\mathbb{R}_+$, this is possible. Let π^* be optimal in P_0. (If no optimal best–response strategies exist, pick π^* to be ϵ–optimal.) Modify π^* to require that the constraints in P_ξ be satisfied. The maximum loss from using this modified strategy is clearly $[u(m(\xi)) - u(0)]$ in every period, or, in other words, this strategy implies that $\overline{\lambda} - \lambda_\xi < \epsilon$.

Q.E.D.

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Appendix

We show here a sufficient condition for Assumption 8 to hold. Consider the deterministic transition game, i.e., $q(\cdot|x)$ is specified by

$$
\begin{aligned}
q(y|x) &= 0 \qquad \text{if } y < f(x) \\
&= 1 \qquad \text{if } y \geq f(x)
\end{aligned}
$$

where $f:\mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, increasing function (the "production function") satisfying $f(0) = 0$. (This is essentially the framework of Sundaram, 1989.) Assume also that f satisfies the usual neoclassical conditions: f is strictly concave on $\mathbb{R}_+$, and is C^1 on $\mathbb{R}_{++}$ with $\lim_{x\downarrow 0} f'(x) = \infty$, $\lim_{x\uparrow\infty} f'(x) = 0$. Note that Assumptions 1–5 are satisfied.

Then, there is a symmetric equilibrium pair (g^δ, g^δ) meeting the conditions of Theorem 1. We will now show that whenever condition (*) below is met, then Assumption 8 is satisfied.[5]

(*) g^δ is non–decreasing on S.

Note that Theorem 1 has nothing to say about (*), and indeed, it is an open question whether, in general, g^δ can also be chosen to satisfy (*). However, the parametrized model of Levhari and Mirman (1980) admits such equilibria for all values of the parameter. (Levhari–Mirman assume $f(x) = x^\alpha$ for $\alpha \in (0, 1)$, and $u(c) = \log c$[6]; they

[5]We also have results that show that if a certain productivity condition is met and either (a) g^δ is differentiable on S, or (b) g^δ is continuous on S and satisfies an asymptotic condition on S, then Assumption 8 is satisfied.

[6]This utility specification is not coverd by our assumption that u(0) is finite. However, as in single agent optimization problems, all of the relevant arguments in Theorems 1 and 2 can be extended to this case, using the Inada conditions on u and q.

show that the linear stationary strategies $g^{\delta}(y) = [(1-\alpha\delta)/(2-\alpha\delta)] \cdot y$ constitute a PSSE.)

Proposition: Suppose the PSSE of (g^{δ}, g^{δ}) satisfies (*). Then, Assumption 8 is satisfied.

Proof: It is easily shown using standard methods that the symmetric first–best problem is uniquely solved by a pair of symmetric stationary strategies (h^{δ}, h^{δ}); and that at all $y \in S$, h^{δ} satisfies

(A.1) $u'(h^{\delta}(y)) = \delta u'(h^{\delta}(y')) \, f'(s^{\delta}(y))$
where $s^{\delta}(y) = y - 2h^{\delta}(y)$, and $y' = f(s^{\delta}(y))$.

It is also not too difficult, using (*) in conjunction with methods used in section 5 of Dutta–Sundaram (1989), to show that the corresponding first–order conditions satisfied by (g^{δ}, g^{δ}) at all $y \in S$ are:

(A.2) $u'(g^{\delta}(y)) \leq \delta u'(g^{\delta}(y^*))f'(\psi^{\delta}(y))(1 - D^{+}g^{\delta}(y^*))$

where $y^* = f(\psi^{\delta}(y))$, $\psi^{\delta}(y) = y-2g^{\delta}(y)$, $D^{+}g^{\delta}$ is the right upper Dini derivate of g^{δ}.

(A.1) and (A.2) together can be shown to imply:

Claim 1: $g^{\delta}(y) \geq h^{\delta}(y)$ for all $y \in S$.

Proof: Suppose for some y, $g^{\delta}(y) < h^{\delta}(y)$. Then, $u'(g^{\delta}(y)) > u'(h^{\delta}(y))$, and $\psi^{\delta}(y) > s^{\delta}(y)$, so from (A.1), (A.2), and the strict concavity of u, $g^{\delta}(y^*) < h^{\delta}(y')$. Moreover, since f is increasing, $y^* > y'$, so once again $u'(g^\delta(y^*)) > u'(h^\delta(y'))$ and $\psi^\delta(y^*) > s^\delta(y')$. Thus, if y_t^*, y_t' represent the period–t values of the state from y under (g^δ, g^δ) and (h^δ, h^δ) respectively, iterating this argument reveals $y_t^* > y_t'$ for all t. ψ^δ non–decreasing (by Theorem 1) combined with f increasing implies y_t^* is a monotone sequence. Similarly, y_t' can be shown to be a montone sequence. By the continuity of g^δ which follows from Theorem 1 and (*), $\bar{y}^* = \lim y_t^*$ is a steady–state of the game under (g^δ, g^δ), i.e.,

$$\bar{y}^* = f(\psi^\delta(\bar{y}^*)).$$

Standard arguments imply similarly that $\bar{y}' = \lim y_t'$ is a steady–state under (h^δ, h^δ). From (A.2), $\delta f'(\psi^\delta(\bar{y}^*))\ (1 - D^+g^\delta(\bar{y}^*)) \geq 1$, while from (A.1), $\delta f'(s^\delta(\bar{y}')) = 1$. There are two cases to consider. If $D^+g^\delta(\bar{y}^*) = 0$, then ψ^δ is strictly increasing at $\bar{y}^*$ and hence, $y_t^* \to \bar{y}^*$ asymptotically (i.e., $y_t^* \neq \bar{y}^*$ for any t). Theorem 3.2 in Dutta–Sundaram (1989), then establishes that $\bar{y}^* < \bar{y}'$, a contradiction. If $D^+g^\delta(\bar{y}^*) > 0$, then $\delta f'(\psi^\delta(\bar{y}^*)) > 1$, i.e., $\psi^\delta(\bar{y}^*) < s^\delta(\bar{y}')$, from the concavity of f. This too is a contradiction, since by the claim from previous arguments, $\bar{y}^* > \bar{y}'$, and the fact that these are steady states implies, $\psi^\delta(\bar{y}^*) \geq s^\delta(\bar{y}')$.

To complete the proof of the proposition, it would be simpler notationally to assume the everywhere differentiability of the payoff functions (V^δ, V^δ) associated wih the PSSE (g^δ, g^δ), although this is not necessary.[7] (Observe that Theorem 1, in any event, ensures that V^δ is almost everywhere differentiable.) Sundaram (1989, Lemma

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[7]Strictly speaking, let $U^\delta(y) = V^{\delta\prime}(y)$, if V^δ is differentiable at y, and 0 otherwise. Let $\mu^\delta(y) = \sup U^\delta(y)$ and suppose, for some $\epsilon > 0$, z_y is such that $V^{\delta\prime}(z_y)$ exists and $V^{\delta\prime}(z_y) > \mu^\delta(y) - \epsilon$, $z \in [y,1]$. From the fundamental theorem of calculus, the first equality above can be replaced by, $V^\delta(1) - V^\delta(y) \leq [V^{\delta\prime}(z_y) + \epsilon](1-y)$. The rest of the arguments follow.

III.1) shows that V^δ is differentiable at $y \in S$ iff g^δ is differentiable at y, in which case

$$V^{\delta\prime}(y) = u'(g^\delta(y))(1 - g^{\delta\prime}(y)).$$

Finally, we need the result from Dutta (1986) where it is shown that there is a function $\alpha: S \to S$, $\alpha(y) > 0$ if $y > 0$, such that the symmetric first–best strategies (h^δ, h^δ) satisfy $h^\delta(y) \geq \alpha(y)$ for all $\delta \in (0, 1)$, for all $y \in S$, $y > 0$.

Now note that for some $z_y \in (y, 1)$

$$\begin{aligned} V^\delta(1) - V^\delta(y) &= V^{\delta\prime}(z_y)(1-y), \text{ by the mean value theorem} \\ &= u'(g^\delta(z_y))(1 - g^{\delta\prime}(z_y))(1 - y) \\ &\leq u'(g^\delta(z_y))(1-y), \text{ by } (*) \\ &\leq u'(h^\delta(z_y))(1-y), \text{ by claim} \\ &\leq u'(\alpha(z_y))(1-y). \end{aligned}$$

Defining $M(y) = (y - 1)u'(\alpha(z_y))$, it follows that $V^\delta(y) - V^\delta(1) \geq M(y)$ for all $y \in S$.

Q.E.D.