Rochester Center for Economic Research

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Working Paper No. 242
September 1990

University of Rochester

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May 1990
Revised: September 1990

Working Paper No. 242

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*We are indebted to Tapan Mitra for his detailed suggestions on an earlier draft. We would also like to thank Rabah Amir, Jeffrey Banks, Andrew Caplin, Ken Judd, Andreu MasColell, Leonard Mirman, and participants at a seminar at the Summer Conference on Game Theory in Stony Brook (Summer 1990) for their comments, and Bonnie Huck for considerable technical assistance. The first author acknowledges financial support from the Columbia Council for the Social Sciences. The usual disclaimer applies.

ABSTRACT

This paper examines the extent to which the implications of strategic behavior can differ from those of non–strategic behavior, even when attention is restricted to examining memoriless Markovian solutions in either case. The framework employed is that of the dominant paradigm of dynamic economics, the neoclasical aggregative growth model. This model has been widely utilized in the literature; yet investigation has been conducted almost exclusively in an (implicitly competitive) representative–agent/social planner framework, with analysis focussing on the Markovian solutions to the resulting dynamic programming problem. As is well known, such solutions may be guaranteed to exist under mild continuity conditions, and to possess strong regularity (e.g., monotonicity and turnpike) properties under convexity restrictions.

We compare these Markovian solutions to their game–theoretic analog (Markov–Perfect equilibria, or MPE), which arise when agents in the model behave strategically, rather than competitively. A series of robust examples and general propositions show that these solutions may differ remarkably. ***First***, even under the continuity and convexity assumptions that guarantee existence of well–behaved Markovian solutions to the planner's problem, MPE may fail to exist in pure or mixed strategies (Example I). ***Second***, despite the fact that attention is confined to memoriless game equilibria – so that threat–sustained behavior is ruled out – the properties of the two solutions may differ dramatically: ***all*** Markovian solutions may generate regular dynamics, but the unique differentiable MPE could be chaotic or cyclical (Example II). ***Finally***, while all Markovian solutions of the undiscounted neoclassical model converge to the so–called "golden–rule," there are always MPE in the corresponding game which result in multiple steady–states, with at least one steady–state being ***larger*** than the golden–rule stock (Propositions 5.1 and 5.2). The last set of results show, in particular, that not only is a "tragedy of the commons" not inevitable even in memoriless equilibria, but that ***under***consumption and ***over***accumulation are actually robust possibilities.

1. Introduction and Summary

The representative–agent aggregative growth model constitutes what is undoubtedly the single most dominant paradigm in dynamic economic analysis. This framework – and its heterogeneous–agent equivalent where a social planner maximizes a weighted sum of utilities subject to technological constraints – has been found useful for examining a number of issues in many fields including macroeconomics, monetary economics, and finance. At least a partial reason for the widespread acceptance of this framework[1] has been its analytic tractability. Existence of a Markovian optimal policy to the first–best problem is assured under minimal continuity and compactness assumptions. Under added convexity restrictions, this solution exhibits strong qualitative properties: the implied state trajectory is monotone, and converges to a steady–state from any initial state. Under only slightly stronger assumptions, this steady–state may be shown to be unique.

In this paper, we examine the continued validity (in a qualitative sense) of these properties when agents in the model behave ***strategically***, rather than competitively as they are implicitly assumed to do in the representative–agent/social planner framework. More precisely, we consider the strategic analog of the aggregative growth paradigm obtained by modelling it as a ***dynamic game***[2] along the lines initiated by Lancaster

[1]We refer to this framework alternatively as the representative–agent, social planning, or first–best framework.

[2]Our use of the term "dynamic game" is somewhat narrow and refers specifically to the deterministic analog of a stochastic game (cf. Parthasarathy, 1973).

(1973), and Levhari and Mirman (1980),[3] and examine answers to the following questions:

(i) Are the continuity and convexity conditions that guarantee the existence of well-behaved Markovian solutions to the first-best problem, also sufficient to ensure the existence of ***Markov-Perfect Equilibria*** (MPE)[4] to the corresponding dynamic game?

(ii) When MPE do exist, do they possess qualitatively similar properties (e.g., monotonicity of the state trajectory) to Markovian solutions of the first-best problem?

(iii) What, if anything, can one say about limit points of the state trajectory in an MPE, even if monotonicity does obtain? In particular, does the externality in the strategic formulation resulting from the presence of many players inevitably result in a "tragedy of the commons," i.e., overexhaustion (or underaccumulation) relative to the first-best?

We believe these questions are important for a number of reasons. The aggregative growth model is the canonical paradigm of dynamic economics, and analysis of this model in various contexts has focussed almost exclusively on its Markovian solution. It is of natural interest to inquire as to how different the positive properties of the first-best problem and the corresponding game can be under the same informational structure (in this case, Markovian).[5] But more importantly, the implicit

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[3]Similar parametrized models have also been studied by, among others, Mirman (1979), Easwaran and Lewis (1985), Reinganum and Stokey (1985), and Cave (1987). More general analyses, without functional-form restrictions, may be found in Benhabib and Radner (1988), Dutta and Sundaram (1989, 1990), and Sundaram (1989).

[4]Markov-Perfect Equilibria are (subgame-perfect) equilibria in Markovian strategies. These equilibria are the exact analogs of the Markovian first-best solutions. A precise definition, in the context of our model, is given in section 2. The study of Markov-Perfect equilibria in various settings has recently been the focus of a number of studies. See, e.g., Bernheim and Ray (1987), Hellwig and Leininger (1988), Maskin and Tirole (1988), or Dutta and Sundaram (1989), or chapter 5 of Fudenberg and Tirole (1990).

[5]We remark that the neoclassical growth paradigm offers the ideal framework for examining this question combining as it does a genuinely dynamical structure with its

assumption of competitive behavior underlying the first–best problem may simply be inappropriate in some contexts where there are a few large agents. A substantial degree of divergence in the positive properties of the respective solutions would, in such cases, weaken the basis for using the social–planner framework.[6]

We develop a series of robust examples and an associated set of general propositions to examine each of the questions listed above. Our main conclusion is that a strategic formulation makes for a remarkable difference on each count from assuming a social–planner framework.

(i) Existence of MPE. Although MPE constitute a natural set of equilibria to first analyze in dynamic games, and indeed have been the only class of equilibria studied in many applications, there are no known general set of sufficient conditions which guarantee their existence. The most complete results along these lines have been recently obtained by Duffie, et al (1989), and Mertens and Parthasarathy (1987). Unfortunately, neither paper ensures that the equilibrium demonstrated to exist will be Markovian although the basic model ***is*** Markovian in both cases; moreover, both require transition probabilities to be norm–continuous in players' actions, which precludes applications to deterministic settings such as ours.

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position as a widely used framework. The former property is important if MPE are not to be completely uninteresting as regards equilibrium behavior, as they would be, for example, if one studied a repeated game.

[6]We are ***not*** contending that the first–best solution may not be a good approximation when the number of agents is "large," i.e., that it will not arise as the limit of the MPE as the number of players goes to infinity. While an important open question, this issue is peripheral to the objectives of the paper where we wish to compare the two Markovian solutions when the number of players is a given finite number. More generally, our analysis may be interpreted as inquiring into the distinction between "price–taking" and strategic behavior when agents may have non–trivial market power.

In Section 3, Example 3.1, we present a finite horizon version of the aggregate growth model satisfying the usual convexity and continuity properties that ensure existence of a well–behaved Markovian first–best solution. Yet, we prove that there are ***no*** non–trivial MPE in this game.[7] Indeed, non–trivial MPE fail to exist even in ***mixed*** strategies

While our example is quite robust, we think that such non–existence issues are much more generally true than it suggests. A main reason is that non–convexities appear naturally in a best–response environment, even if the primitive model possesses all of the desired convexity and regularity properties.[8] Within the context of our model, for example, the consequence of a current consumption choice by a player is not the gross output tomorrow, but rather this output net of the other players' planned consumption choices tomorrow. Unless the latter are convex functions of tomorrow's stock, the environment faced by a player seeking a best–response will not be convex.

It should be noted that the heterogeneity in agents' preferences is crucial to our counterexample. Elsewhere (see Dutta and Sundaram (1990), and Sundaram (1989)), we have shown that under strict convexity assumptions and Inada conditions, the ***symmetric*** dynamic game, in which all players have identical payoff structures, always admits an MPE. This assumption of symmetry plays a critical role in circumventing the problems arising from the observations of the previous paragraph. Unfortunately, the methods of those papers are quite special and do not extend to the non–symmetric case.

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[7]For the definition of non–trivial, see subsection 2.5. As noted there, trivial equilibria can easily be ruled out, by imposing appropriate conditions.

[8]Such nonconvexities cause problems in demonstrating existence in models other than ours also. See, for example, Leininger (1986), or Bernheim and Ray (1983, 1987), for a discussion of the difficulties this creates in models of altruistic growth.

Finally, it is important to distinguish our example from those in the literature on the non–existence of consistent plans (for instance, Peleg and Yaari (1973), or Hellwig and Leininger (1988); see also Fudenberg and Tirole (1990, ch. 5)). The latter class of examples deals with finite–horizon games of ***perfect*** information. Thus, backward induction reveals the (unique) MPE if one exists. Non–existence is typically obtained by constructing the game in such a way that the optimization problem resulting from backward induction facing the first player (or one of the earlier players) fails to possess a maximum. In contrast, ours is a ***simultaneous***–move game, so that proving non–existence requires us to show the absence of a ***fixed–point*** of the best–response map. It is all the more important to emphasize, therefore, that this failure to possess a fixed–point is obtained in a ***convex*** primitive model with time–separable preferences.

(ii) State Dynamics. One of the most striking conclusions of the neoclassical growth model is the extreme regularity of stock behavior over time, with monotonic growth or decline occurring from every initial state. In Dutta and Sundaram (1989), we demonstrated that if all players in the game had strictly concave payoff functions that moreover depended only on own consumption levels (we call this a "strict neoclassical payoff structure"), a weaker version of this property would continue to hold in any MPE: namely, the sequence of stock levels from any initial state, would ***eventually*** (i.e., within a finite number of periods) become monotone. In section 4, we reexamine this issue under a more general class of preferences. Specifically, in our example, players' utility functions depend not only on (own) current consumption but also incorporate a kind of "wealth effect" through dependence on the size of the current stock.[9] This example has the feature that for each parameter value (in an interval of

[9]Strictly speaking, this assumption represents a departure from the "classical" traidition that we adhere to in the other examples, where payoffs depend only on own consumption. (We note, however, that many papers in the literature on multisectional growth models have utilized payoffs that are defined directly on the space of stocks rather than on consumption. See, e.g., Benhabib and Nishimura, 1985.) Within the context of our model, several justifications may be given for such an assumption.

values under consideration), ***all*** first–best solutions display very regular behavior: the stock sequence from any initial state is monotone and converges to a unique steady–state independent of the value of the initial state. Moreover, for each parameter value the corresponding dynamic game possesses a differentiable MPE that is essentially (in a sense made clear in Appendix II) the unique differentiable MPE. Yet, as the parameter ranges over the interval of feasible values, the dynamical system determining the state trajectory in this MPE displays an astonishing range of possible behavior, including cycles and chaos.

It is worth distinguishing the content of this example from the (considerably more general) studies of competitive cycles or chaos in multisectoral optimal growth models that have been undertaken in the literature (e.g., Benhabib and Nishimura (1985), or Boldrin and Montrucchio (1986)). In those models, it is the Markovian first–best solutions themselves that exhibit erratic state trajectories. In sharp contrast, first–best dynamics in our example are very regular; indeed, the assumptions that typically drive erratic state dynamics in those models, such as appropriate substitutability conditions across sectors, are all absent in our example. Evidently, then, it is the fact of ***strategic*** interaction – or, more generally, the fact that best–response environments inherit properties in equilibrium that are not possessed by the primitive model – that drives our results.

(iii) Asymptotics. It is commonly believed that in models such as the ones we consider, the externality resulting in the strategic formulation from the presence of many agents will lead to eventual underaccumulation relative to the first–best solutions, i.e., will result in a "tragedy of the commons." Parametrized models studied in the literature (e.g., Lancaster (1973), Levhari and Mirman (1980)) have appeared to

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Think of the natural–resource–extraction interpretation of our framework, and suppose that it is "easier" to extract from a larger stock.

vindicate this view. In Dutta and Sundaram (1989), we examined the validity of this claim in a general convex model with a strictly neoclassical payoff structure. We proved that a "tragedy" would always obtain under apparently weak conditions on the MPE; yet, as we showed through an example, it was possible these conditions could be violated, and indeed the reverse phenomenon of ***under***–consumption and ***over***–accumulation could actually occur under an MPE. The important point to emphasize here, as in the sequel, is, of course, that these results are obtained for ***memoriless*** strategies. It is quite straightforward to obtain perfect equilibria in which "high" steady–states are forcibly sustained through the use of punishment deterrents, but such conditioning on history is precluded in an MPE.

In section 5, we strengthen this result considerably by demonstrating over accumulation (and underconsumption) to be a robust possibility in the ***undiscounted*** game under strict convexity, but with no other restrictions. Specifically, we show that the undiscounted symmetric game ***always*** possesses an MPE which results in overaccumulation from at least some initial states.[10] Secondly, under a different (in fact, complementary) set of conditions, we show that there are MPE under which the ***sole*** non–zero steady–state is one resulting in overaccumulation with all intitial states not leading to this steady–state converging monotonically to zero.

Some final remarks before proceeding to the body of the paper. Firstly, it appears important to emphasize that our results in this paper are not being driven on account of indeterminateness of game equilibria. They are, especially, not restatements of the

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[10]We confine attention to the ***symmetric*** undiscounted game partly because it is the only case for which ***existence*** of MPE is known (cf. Dutta and Sundaram, 1990), and partly to simplify notation. Similar techniques to those we employ can be used to show that if the general undiscounted game possesses MPE in a particular class of strategies, it possesses one resulting in overaccumulation.

principle[11] that "anything can happen" in a game if players are sufficiently patient. Secondly, while our examples are obviously specialized, we believe each example is reflective of more general principles (notably, the non–convexities that arise naturally in the best–response environment of a convex primitive model) that distinguishes strategic equilibria from non–strategic solutions in a fundamental sense. Thirdly, and lastly, in restricting attention to Markovian game–equilibria alone, we only strengthen our case; for, admitting more complex, history–dependent strategies, would merely enlarge the scope of strategic outcomes.

Section 2 discusses the basic framework and collects definitions. Sections 3, 4, and 5 deal with, respectively, existence, dynamics, and asymptotics of MPE. The Appendices contain proofs omitted in the main body of the paper.

2. The Framework of Analysis

2.1 The Basic Model[12]

Since all our examples deal with the 2–player situation, we will, for notational convenience, confine ourselves to describing that case. A generic player will be indexed by i. In all statements pertaining to i, j will denote the other player.

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[11]As in the Folk–Theorems of Fudenberg and Maskin (1986) for repeated games, or Abreu and Dutta (1989) for stochastic games.

[12]Each of the following three sections deals with a variant of the model described here. The required modifications of definitions and concepts to cover these variants is usually immeidate, and is detailed wherever necessary. Results on existence and characterization of MPE in the basic framework itself, under ***strict*** convexity assumptions, may be found in our earlier work (Dutta and Sundaram (1989, 1990), and Sundaram (1989)).

There is a single good which may be consumed or invested. Conversion of investment to output takes one period, and is accomplished through a ***production function*** f: $\mathbb{R}_+ \to \mathbb{R}_+$. We make the standard neoclassical continuity and convexity assumptions on f:

Assumption 1. f is continuous and increasing on $\mathbb{R}_+$ with $f(0) = 0$, and $f(x) > 0$ for $x > 0$. Further, there is $\bar{x} > 0$ such that $f(x) \gtrless x$ as $x \lessgtr \bar{x}$.

Assumption 2. f is concave on $\mathbb{R}_+$, differentiable on $\mathbb{R}_{++}$, and satisfies $\lim_{x \downarrow 0} f'(x) > 1/\delta$.

(Here $\delta \in (0, 1)$ refers to the common discount factor of the 2 players. See below.) Observe that f maps $[0, x]$ into itself for any $x \geq \bar{x}$. Since consumption levels will not be allowed to take on negative values, we may, wlog, restrict attention to initial states in $[0, \bar{x}]$. We define $S \equiv [0, \bar{x}]$, and frequently refer to S as the ***state space***.

Time is discrete and continues forever. Periods are indexed by $t = 0, 1, 2, \ldots$. In each period t of this infinite horizon, the stock y_t available at the beginning of the period is allocated between the players' consumptions (c_{1t}, c_{2t}), and period–t investment x_t. (See subsection 2.2 below for details.) This investment is then converted to the stock y_{t+1} available at the beginning of the next period as $y_{t+1} = f(x_t)$, and the process repeats itself.

For the most part, we assume that players' utility depends only on their own consumption levels.[13] Thus, player i's utility from consuming c_{it} in period t is $u_i(c_{it})$,

[13]In Section 4 alone, we use a formulation where the utility a player obtains depends on his own consumption, and also on the stock level. That is, the period–t utiity of player i is $u_i(c_{it}, y_t)$. Under the separability assumption we utilize there, it is not too difficult to show that the monotonicity properties of the first–best solution obtained under Assumptions 1 to 3 extend to cover this case also. We do not present this alternative formulation here, in order to avoid cluttering notation.

where u_i: $\mathbb{R}_+ \to \mathbb{R}$ is player i's ***utility function***. The usual neoclassical restrictions are imposed on the functions u_i:

Assumption 3. For each i, u_i is a continuous, increasing, concave function on $\mathbb{R}_+$.

Finally, we assume that both players discount the future by the same factor $\delta \in (0, 1)$. The tuple $\{S, f, u_1, u_2, \delta\}$ completes the description of the basic model.

2.2 Histories, Strategies, and Payoffs

In the interest of presenting a unified framework for defining strategies and payoffs in both the game and the first-best framework, it is worthwhile explaining, at this point, how the available stock y_t at the beginning of a period gets allocated between the consumption vector (c_{1t}, c_{2t}) and investment $x_t = y_t - c_{1t} - c_{2t}$. In the first-best framework, this is achieved essentially by ***diktat***: a social planner arrives at this division through solving for a first-best outcome as detailed in subsection 2.3. In the game, on the other hand, players pick strategies independently and simultaneously that specify for each player his desired (or ***planned***) consumption, denoted (say) a_{it} for player i, out of the available stock y_t at each t. [The criteria used by players in arriving at these strategy choices is detailed in subsection 2.4]. If these plans are collectively feasible $(a_{1t} + a_{2t} < y_t)$, then each player receives exactly his planned amount, so player i's consumption c_{it} equals a_{it}. If plans are collectively infeasible

$(a_{1t} + a_{2t} > y_t)$, then we simply assume[14] that each player simply receives half the available stock (so $c_{it} = y_t/2$, in this case).

2.2.1 Histories. A generic ***history*** upto period–t (a t–history, for short), denoted h_t, is a list of stock levels and (planned and actual) consumption levels for the 2 players in each period upto (t –1), and the period–t opening stock. Let H_t denote the set of all possible t–histories h_t.

2.2.2 Strategies.[15] A ***strategy*** π_i, for player i, specifies a planned consumption level for i at each date t, given the history h_t upto that point. Of special interest from the point of view of this paper are ***Markovian*** strategies, which prescribe the same consumption level at a state regardlesss of the history by which that state was reached:

Definition. A (stationary) Markovian strategy for i is a function g_i: $S \to S$ that satisfies $g_i(y) \in [0, y]$ at all $y \in S$.

Observe that the set of all strategies (resp. Markovian strategies) is the same for either player. Denote this common set by Π (resp. Γ).

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[14]It is important to emphasize that the apparent discontinuity of this allocation rule plays no role in the sequel. A variety of other rules could be used without changing our results, including, for instance, continuous ones such as splits proportional to plans under infeasibility, or ones under which collective infeasibility is never an *equilibrium* possibility (see footnote 17 for an example of such a rule). For reasons explained in subsection 2.5, the precise choice of allocation rules is irrelevant to the purpose of this paper.

[15]The reader will note that the definition does not allow for ***mixed***–strategies. This is of no relevance for sections 4 and 5 where pure–strategy MPE possessing the desired properties are proved to exist. However, it ***is*** an important issue for our non–existence result of section 3. Since this is the only point in the paper which mixed strategies are used, the definition of mixed strategies and other relevant issues, is postponed to Appendix I.2, where non–existence in mixed–strategies in this example is established.

Two pieces of terminology relating to Markovian strategy profiles will be especially useful in the sequel. The ***savings function*** ψ associated with a particular Markovian profile (g_1, g_2) is defined by $\psi(y) = \max\{0, y - g_1(y) - g_2(y)\}$ for $y \in S$. Secondly, a Markovian profile (g_1, g_2) will be called ***interior*** iff at all $y \geq 0$, $g_1(y) + g_2(y) \in [0, y]$. An important observation stemming from these definitions and the monotonicity of f is the following: from any initial state $y \in S$, the strategy profile (g_1, g_2) will result in a ***monotone*** sequence of states whenever the associated savings function is non-decreasing on S.

2.2.3 Payoffs. Let $\delta \in (0, 1)$ be given. A strategy profile (π_1, π_2) induces in the obvious maner, for each integer $t \geq 0$ and from any initial state $y \in S$, a t-th period actual consumption level for i (= 1, 2), and, hence, a t-th period utility level for i, denoted $u_i^t(\pi_1, \pi_2)(y)$. The total discounted reward $W_i^\delta(\pi_1, \pi_2)(y)$ for i, from the intitial state y, under the profile (π_1, π_2) and the discount factor δ, is then given by

$$W_i^\delta(\pi_1, \pi_2)(y) = \Sigma_{t=0}^\infty \delta^t u_i^t(\pi_1, \pi_2)(y).$$

2.3 The First-Best Solutions

Given $\delta \in (0, 1)$, the set of all (Pareto-) optimal payoffs possible from an arbitrary initial state $y \in S$, are precisely those arising from solutions to the following problem[16] as α ranges over [0, 1]:

$$\text{(2.1)} \qquad \operatorname*{Max}_{(\pi_1,\pi_2)} [\alpha W_1^\delta(\pi_1, \pi_2)(y) + (1 - \alpha)W_2^\delta(\pi_1, \pi_2)(y)]$$

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[16]Observe that in the specification of the problem we allow the social planner to pick any pair $(\pi_1, \pi_2) \in \Pi \times \Pi$, i.e., including those that might specify collectively infeasible plans at some states after some histories. This is clearly unnecessary, but, writing the problem in this form enables notational simplicity.

The following characteristics of the first-best problem are well-known and are offered here without proof. Fix $\alpha \in [0, 1]$. Then:

Fact 1. Under Assumptions 1 and 3, there is always at least one ***Markovian*** strategy profile (h_1^α, h_2^α) that solves (2.1) from any $y \in S$.

Fact 2. Under Assumptions 1, 2, and 3:

(i) Any Markovian solution (h_1^α, h_2^α) is ***interior***, with, in fact, $h_1^\alpha(y) + h_2^\alpha(y) < y$ at all $y > 0$.

(ii) There is at least one Markovian solution whose associated savings function is non-decreasing on S.

(iii) If u_i is strictly concave, the problem admits exactly one Markovian solution, which, by (ii), always generates a monotone state trajectory. In this case, moreover, the sequence of states resulting from any non-zero initial state converges to a "golden-rule" stock y_δ^*, defined by $y_\delta^* = f(x_\delta^*)$, where $\delta f'(x_\delta^*) = 1$.

Summing up, Markovian first-best solutions are guaranteed to exist under fairly weak conditions; to exhibit well-behaved dynamics under minimal convexity restrictions; and to always converge to a golden-rule stock, independent of the value of $\alpha \in [0, 1]$.

2.4 The Dynamic Game and MPE

In the dynamic game-theoretic formulation of the model, each player takes the strategies of the other players as given and attempts to maximize his own discounted total reward over the infinite horizon. Agents' utilities in this formulation may continue to depend on own consumption levels alone; nonetheless, since any player's consumption level in a period affects – via the level of investment that period – future stock levels, and hence future consumption and payoff possibilities of all agents, the need for strategic behavior arises, giving rise to a ***dynamic game***. Indeed, it is precisely this explicit recognition of the production externalities of consumption decisions, that differentiates players in the game–theoretic formulation from those in the competitive model.

To ease notation, we suppress explicit dependence on the discount factor δ in describing equilibria of the game. Also, since our ultimate purpose is to compare Marovian equilibria of the game with Markovian first–best solutions, we avoid spurious generality in the definitions, by focussing on Markovian strategies alone. It is important to note (see Remark 1 below) that this causes no loss of strategic flexibility in picking best–responses. Recall that Γ denotes the set of all Markovian strategies available to either player.

Definition. $g_1^* \in \Gamma$ is a ***best–response*** (BR) of player 1 to $g_2^* \in \Gamma$ if

$$W_1(g_1^*, g_2^*)(y) \geq W_1(g_1, g_2^*)(y) \ \forall\ y \in S, \forall\ g_1 \in \Gamma.$$

A best–response of player 2 to $g_1^* \in \Gamma$ is similarly defined.

Remark 1. The specification that the BR to a Markovian strategy itself be chosen from Γ is ***not*** a restriction. A standard argument from dynamic programming theory shows that a Markovian best–response to a Markovian strategy continues to a best–response in the space of all strategies, i.e., even if the responding player may condition his actions on histories (indeed, even if the responding player may also randomize).

Definition. A pair of Markovian strategies (g_1, g_2) is a ***Nash equilibrium*** (NE) to the dynamic game if for i, j = 1, 2, and $i \neq j$, g_i is a BR to g_j.

Remark 2. By remark 1, a NE in Markovian strategies is a NE in the space of ***all*** strategies. Indeed, even more is true. The memoriless character of Markovian strategies implies that they are actually ***subgame–perfect*** in a strong sense. Hence, the phrase ***Markov–Perfect Equilibrium*** (MPE).

2.5 The Existence Problem Defined: Interior MPE

The alert reader will have observed that under our specification of the allocation rule under infeasibility, and, indeed, under any rule in which the amount alloted a player is an increasing function of his own bid, "trivial" equilibria always exist, i.e., equilibria which result in collectively infeasible plans at some state(s). [For instance, it is immediate that the pair (g_1, g_2) specified by $g_1(y) = g_2(y) = y$ at all $y \in S$ is an MPE under all such rules.] On the other hand, it is always possible to pick allocation rules under which infeasible plans at any state are ***never*** an equilibrium possibility.[17] This excessive dependence of non–interior equilibria on specification of the allocation rule, a rule which is ***not*** a part of the basic neoclassical model, renders them suspect candidates for comparison with first–best solutions. An even stronger reason for restricting attention to interior MPE is the implication of Fact 2 of subsection 2.3 that any first–best solution (h_1^α, h_2^α) is ***strictly*** interior (i.e., satisfies $h_1^\alpha(y) + h_2^\alpha(y) < y$ at all $y > 0$). The question naturally arises whether interior MPE also exist under the same conditions, and it is this we refer to as the existence problem.

However, even strategies that constitute an interior MPE under one allocation rule, may no longer remain so under another. Consequently, in proving non–existence, it

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[17]The following affords an example: whenever $a_1 + a_2 > y$, then player i receives $[a_j y/(a_1 + a_2)]$, $j \neq i$. We note that a related rule has been employed in a different context by Moulin (1984). See also Van Damme (1987, ch. 7.7).

becomes necessary to show that no interior MPE exist for ***any*** choice of infeasibility allocation rules. It is this strong result that we obtain in the parametrized model of Section 3.[18] A few words on the procedure may be useful. If (g_1, g_2) is an interior MPE to the game (under any allocation rule), it is clear that g_i remains player i's BR to g_j, $j \neq i$, even when i is restricted to choosing a BR from $\Gamma(g_j) \subset \Gamma$ defined by

$$\Gamma(g_j) = \{g: S \to S \mid g(y) \in [0, y - g_j(y)] \ \forall\ y \in S\}.$$

Restricting players' BR choices in this manner yields a "generalized game" in the sense of Debreu (1954). It now follows that if the generalized game fails to possess an MPE, then there are no interior MPE in the original game regardless of the choice of allocation rule.

Finally, a matter of terminology. Since all the remaining sections of this paper deal only with interior MPE, we drop the adjective "interior," and refer to these simply as MPE. Thus, phrases such as "unique MPE" should be understood as "unique interior MPE," or, more precisely, "unique MPE of the generalized game."

3. Non-Existence of MPE: Example I

In this section we examine the first of the 3 questions posed in the Introduction: namely, the existence of (interior) MPE under convexity conditions guaranteeing well-behaved Markovian first-best solutions. By the observation made above, it suffices to demonstrate lack of interior MPE to the generalized game, and this is precisely the route we take. In subsection 3.1, we present a ***finite***-horizon example which admits no

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[18] In particular, if an allocation rule such as the one in the previous footnote is used, this will result in non-existence of ***any*** MPE, interior or otherwise.

interior MPE. In Appendix I.2, this result is extended to show non-existence in mixed-strategies also.

3.1 A Finite Horizon Game

3.1.0 Model Specification

(a) The Production Technology. We assume f: $\mathbb{R}_+ \to \mathbb{R}_+$ is given by

$$f(x) = Kx, \qquad x \in [0, K^{-1}]$$
$$= 1, \qquad \text{otherwise.}$$

where K satisfies $\delta K > 2$. Although f does not meet all our assumptions, these violations are inessential. Observe that we may take the state space S to be the unit interval [0, 1].

b) The Utility Functions. We assume u_1 is given by

$$u_1(c) = c, \; c \geq 0,$$

and that $u_2 = u$, where u is any strictly concave function satisfying Assumption 3 and the Inada condition, $u'(0) = +\infty$.

The horizon of the model we consider T = 3. Periods are indexed by t = 0, 1, 2. Since the horizon is finite, there is a need to consider non-stationary Markovian strategies. Such a strategy, say for player i, is a sequence of functions $g^i = \{g_{it}\}_{t=0}^{T-1}$ such that $g_{it}(y) \in [0, y]$ for all $y \in S$. The interpretation is that $g_{it}(y)$ is the specification of player i's planned consumption at y at date t. The definitions of first–best solutions and MPE are amended in the obvious mannner, and are not detailed here. We note that under Assumptions 1–3, Markovian strategies suffice to obtain any desired first–best payoff vector.

For simplicity, we will assume that in the last period ($t = 2$), players split equally any available stock.[19] The presentation of the example proceeds in 2 steps. First, we show that the 2–period game has an (essentially) unique MPE. Clearly, this MPE ***must*** form the Markovian strategies of the last 2 periods in ***any*** MPE of the 3–period game. As the second step completing the example, we show that there is no MPE of the 3–period game that takes these strategies as continuations.

3.1.1 The 2–period Game

The solution to the following 1–person problem involving player 2 plays an important role in developing MPE of the 2–period game:

$$(3.3) \qquad \max_{c_2 \in [0,y]} \{u(c_2) + \delta u(\tfrac{1}{2}f(y - c_2))\}$$

Standard techniques show that there is a unique solution $h: S \to S$ for this problem; that the associated savings function ψ, defined by $\psi(y) = y - h(y)$ for $y \in S$, is continuous and increasing on S, with $\psi(y) > 0$ at all $y > 0$; and that there is a unique $y^* \in (0, 1]$ such that $\psi(y) = K^{-1}$ for all $y \geq y^*$.

Now consider the 2–period game ($t = 1, 2$). It is clear that the Markovian strategies (g_{11}, g_{21}) constitute an MPE to this game if, and only if:

[19] This assumption is consistent with our infeasibility allocation rule. However, almost any other rule which exhausted terminal stock in some prespecified ratios would also work as well, with appropriate modifications.

$$(3.4) \qquad g_{i1}(y) \in \underset{c_i \in [0\,,\, y - g_{j1}(y)]}{\operatorname{argmax}} \{u_i(c_i) + \delta u_i(\tfrac{1}{2}f(y - g_{j1}(y) - c_i))\}$$

for $i = 1, 2$, for all $y \in S$. On the other hand, given the linearity of u_1, it is not too difficult to see that any best-response of player 1 to g_{21} in the generalized 2-period game must be of the bang-bang variety:

$$\begin{aligned} g_{11}(y) &= 0, \text{ if } y - g_{21}(y) \leq K^{-1} \\ &= y - g_{21}(y) - K^{-1}, \text{ otherwise.} \end{aligned}$$

This enables proving the following lemma (formal proof in Appendix I.1):

Lemma 3.1. The unique[20] Markovian equilibrium of the two period game is given by:

$$(3.4)' \qquad g_{11}(y) = \begin{cases} 0 \ , & y \leq y^* \\ y - y^* & y > y^* \end{cases}$$

$$g_{21}(y) = \begin{cases} h(y) & y \leq y^* \\ h(y^*) & y > y^* \end{cases}$$

Further, the associated equilibrium investment is $\psi(y)$ on $y \leq y^*$ and a constant $\psi(y^*)$, $y > y^*$. Finally, the implied continuation values are

[20]Strictly speaking the equilibria are unique on $[0, y^*]$, i.e. all Markovian equilibria have the properties (3.4) on $[0, y^*]$. Since all of the subsequent analysis is on this subset of the state space, this weaker result suffices for our purposes. The piecework linearity of f plays no role in this uniqueness, which obtains under any f satisfying Assumptions 1 and 2.

$$(3.5) \qquad V_1(y) = \begin{cases} \frac{1}{2}\,\delta K\psi(y) & , \ y \leq y \\ (y-y^*) + \frac{1}{2}\delta K\psi(y^*), & y > y^* \end{cases}$$

$$\begin{aligned} V_2(y) &= u(h(y)) + \delta u[\,\tfrac{1}{2}K\psi(y)], \ y \leq y^* \\ &= u(h(y^*)) + \delta u[\,\tfrac{1}{2}K\psi(y^*)], \ y > y^* \end{aligned}$$

Remark. That these strategies form an MPE is quite clear from (3.3) – (3.4), and the form of 1's best response. That they are the (essentially) unique MPE strategies is less clear, and is the content of the proof in Appendix I.

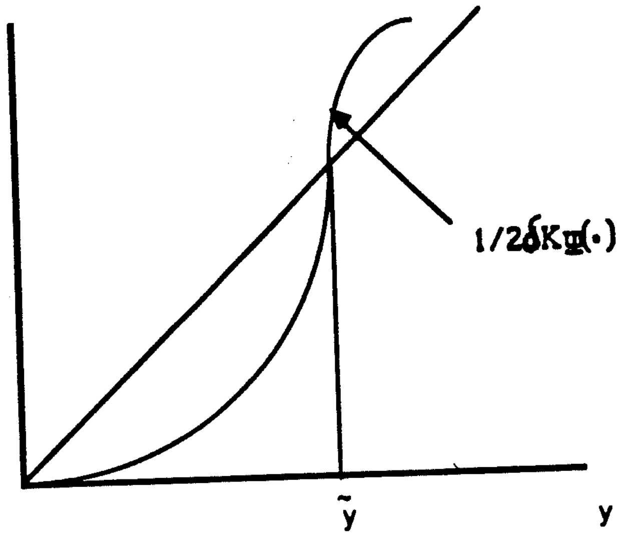
The important point to note is that the convexity of the model has ***no implications for the shape of*** the optimal response, h (and hence the implied investment ψ which may well be non–convex). We now exploit this observation.

3.1.2 The Three Period Game

It is readily seen that g_{i0}, $i = 1, 2$ is part of an MPE for the game iff

$$(3.6) \qquad g_{i0}(y) \in \underset{x \in [0, y-g_j(y)]}{\operatorname{argmax}} \{u_i(y - g_j(y) - x) + \delta V_i(f(x))\}$$

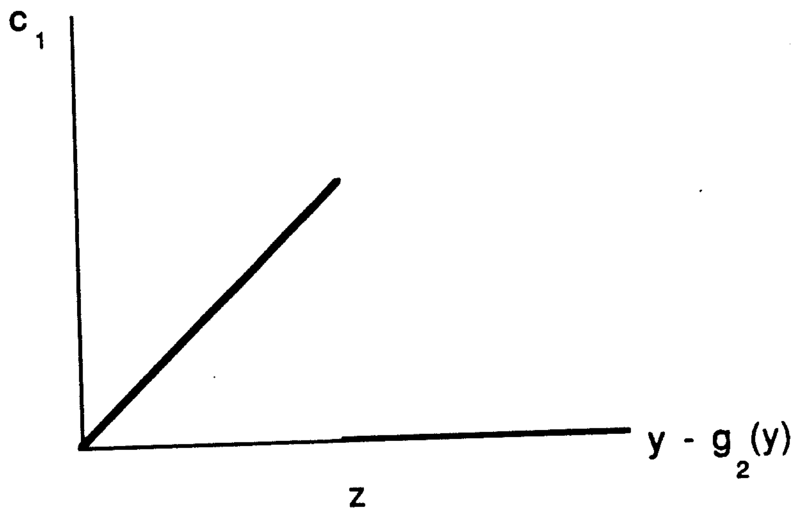
It is immediate in this example from (3.5) that the continuation value V_1 for player 1 has exactly the same shape as ψ on [0, y*]. By suitably picking a non–convex ψ, the resulting non–convexity in V_1 will enable us to demonstrate our non–existence result. Consider a ψ as pictured in Figure 1. (In Appendix I, we give an example of a u, meting our assumptions, that results in such a savings function.)



If such a ψ obtains, it can be shown that:

Lemma 3.2. There do not exist functions g_{i0}, $i = 1, 2$, satisfying (3.6). Therefore, the 3–period generalized game admits no MPE.

Proof. It is evident by the linearity of u_1 that player 1's optimal period–0 action as a function of the ***net*** stock $(y - g_2(y))$ has the following form:



Denote $f^{-1}(\tilde{y})$ by $\hat{z} > 0$. Now pick some $y > 0$ and consider player 2's period–0 optimization problem ***if*** $c_1 = 0$:

$$\underset{x \in [0,y]}{\text{Max}} \{u(y - x) + \delta V_2(f(x))\}$$

Standard arguments using the strict concavity of u show that the correspondence of maximizers (denoted ϕ) to this problem is non-decreasing on S: y, y′ $\in$ S and $y > y'$ $\Rightarrow x \geq x'$ for any $x \in \phi(y)$, $x' \in \phi(y')$. Moreover, $u'(0) = \infty$ implies that $0 < x < y$ for all $x \in \phi(y)$. Let $S^* = \{y \in S | \phi(y) < \hat{z} < y\}$. It is routine to check that $S^* \neq \phi$. We now show that there are no MPE strategies from $y \in S^*$.

Consider any such y. It cannot be the case in an MPE that player 1's period–0 consumption c_1 at y is 0. For then, player 2's period–0 consumption is $c_2 = y - x$ for some $x \in \phi(y)$. But this implies $y - c_2 = x < \hat{z}$, so player 1's optimal action is $c_1 = x > 0$, a contradiction. Similarly c_2 cannot be 0, for this is possible only if $c_1 = y$ which is an inoptimal response against $c_2 = 0$ since $y > \hat{z}$. So $c_1, c_2 > 0$. But $c_1 > 0$ is possible in a best–response iff $c_1 = y - c_2$. In that case, $c_2 > 0$ and $u'(0) = \infty$ implies that in a best–response against $c_1 < y$, player 2's action would satisfy $c_1 + c_2 < y$, a contradiction again. Thus, $c_1, c_2 > 0$ is also impossible, proving the lemma.

Q.E.D.

<u>Remark</u>. It should be emphasized that although the linearity of player 1's returns make the computations very easy, that restriction is in no way essential to the argument. In general, continuation values V_i are going to be non–concave functions. Then, in period 0, for a fixed y, we look for (c_1^*, c_2^*) s.t.

$$c_i^* \in \underset{c_i \in [0,\ y - c_j^*]}{\text{argmax}} \left\{ u_i(c_i) + \delta V_i[k(y - c_j^* - c_i] \right\} \tag{3.8}$$

Equation (3.8) defines a finite-dimensional fixed point problem, but with non-concave objective functions. Hence, the non-existence problem.

4. Irregular dynamics in the MPE: Example II

We now turn to the second object of comparison between Markovian first-best solutions and MPE of the corresponding dynamic game, viz. the extent of regularity exhibited by the state trajectory in either case. The example we present here has 2 symmetric agents whose utility functions depend on (own) consumption and also the size of the current stock, i.e., incorporate a kind of "wealth effect." This example has the characteristic that for each possible value of the parameter (in an interval of values under consideration), the state trajectory in ***any*** first-best solution is extremely well behaved: from any non-zero initial state the sequence of stocks converges monotonically to a unique steady state. Further, for each value of the parameter, the corresponding game has a differentiable MPE that is also essentially the unique differentiable MPE in strictly interior[21] strategies. Yet, in sharp contrast to the regularity of first-best state dynamics, this MPE results in highly complex state trajectories (including chaos) for an interval of parameter values.

4.1 Model Specification

We adopt the following specifications for the various functions and parameters involved:

(a) Utility functions. The agents are presumed symmetric, i.e., to possess identical utility functions and discount factors. We consider a family of utility functions u_δ

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[21]By "strictly" interior, we mean here that the savings function ψ satisfies $\psi(y) \in (0, y)$ for all $0 < y \in S$.

indexed by the common discount factor δ. (u_δ is to be interpreted as the utility function under consideration when the discount factor is δ.) Each agent's instantaneous depends on his own current consumption c, and the size of the current stock y, and is given by

$$u_\delta(c, y) = c + \lambda_\delta(y)$$

where for each δ, $\lambda_\delta\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing C^1 function. In order to compute the MPE of the game, an exact specification of $\lambda_\delta(\cdot)$ will be provided shortly. We note that first–best solutions will be independent of the form of this specification.

(b) Discount factors. The common discount factor δ is allowed to take any value in the interval [1/4, 1/2).

(c) The Production Function. The technology $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ for converting investments into future stocks is given by:

$$f(x) = \begin{cases} 4x & , \quad x \in [0, 1/4] \\ 1 & , \quad x \geq 1/4. \end{cases}$$

Since f maps $[0, \bar{x}]$ into itself for any $\bar{x} \geq 1$, we may, wlog, take the ***state space*** S to be the unit interval [0, 1].

4.2 The First–Best Solutions

It is quite straightforward to see that in any first–best solution, the consumption policy functions (h_1^δ, h_2^δ) must satisfy

$$h_1^\delta(y) + h_2^\delta(y) = \begin{cases} 0 & , \quad y \in [0, 1/4] \\ y-1/4 & , \quad y > 1/4. \end{cases}$$

The argument establishing this runs as follows. Suppose the policies prescribed a strictly positive total consumption at some $y \leq 1/4$. Consider transferring a "small" amount (say, $\epsilon > 0$) of current total consumption to current investment. This results in next period's stock being 4ϵ larger, so that total consumption next period can be increased by 4ϵ, while maintaining the same continuation policy. Since u_δ is linear in c and increasing in y, and since $\delta \geq 1/4$, this alternative policy results in a strict improvement of the objective from the initial state y. Therefore, total consumption cannot be positive from any $y \leq 1/4$. Exactly the same argument establishes that investment from any $y > 1/4$ must be at least 1/4, and, therefore, equal to 1/4. The claim follows.

Observe that in any first-best solution, the sequence of stocks grows from any $y > 0$ until it hits $y^* = 1$, and then remains there forever, establishing regularity of first-best dynamics.

4.3 Dynamics under the MPE

We now adopt a specific form for the functions $\lambda_\delta(\cdot)$ to demonstrate our claim that MPE state dynamics can differ drastically from first-best dynamics. It should be clear, however, that this is being done for the purposes of illustration, and that the validity of the reasoning goes beyond the example. Indeed, even within the example, a wide variety of other dynamical systems can be obtained as MPE, as explained in Appendix II.

For now, let $Q_\delta = (1/2\delta - 1)$, and let $\lambda_\delta(\cdot)$ be specified by:

$$\lambda_\delta(y) = \frac{1}{2} Q_\delta y^2 \ , \ y \in S, \ \delta \in [1/4, 1/2).$$

Note that $Q_\delta > 0$ for all $\delta \in [1/4, 1/2)$, so $\lambda_\delta(\cdot)$ is indeed increasing on S for each δ.

In Appendix II, we show that for each $\delta \in [1/4, 1/2)$, the dynamic game with discount factor δ admits a differentiable MPE in interior strategies. This MPE is, moreover, essentially unique (in a sense explained in Appendix II) for $\delta \in [1/4, 1/3]$ and is almost so for $\delta \in (1/3, 1/2)$. In addition, it is symmetric: the 2 players have the same consumption policy function g_δ, given by

$$g_\delta(y) = (1 - 1/4\delta)y + \lambda_\delta(y) \ , \ y \in S$$

Consequently, the savings function ψ_δ in this MPE is:

$$\begin{aligned} \psi_\delta(y) &= y - 2g_\delta(y) \\ &= (1/2\delta - 1)\ y - 2\lambda_\delta(y) \\ &= Q_\delta y(1 - y). \end{aligned}$$

Since $y(1-y) \leq 1/4$ for all $y \in S$, and $\delta \geq 1/4$, so $\psi_\delta(y) \leq 1/4$ for all $y \in S$. Therefore, from any initial state $y_0 > 0$, the sequence of stocks $\{y_t\}$ evolves according to

$$y_{t+1} = 4Q_\delta y_t(1 - y_t).$$

i.e., y_t is just the t–th iterate of the famous quadratic (or logistic) map $F_\mu(y) = \mu y(1-y)$, where $\mu = 4Q_\delta$. As is well known, this map exhibits a surprising range of dynamic behavior for values of μ near 4 (see, for instance, Devaney, 1989) including chaos for many values of μ (e.g., $\mu = 4$ which corresponds to $\delta = 1/4$).

5. Asymptotics in the MPE

We now turn to the third and last of the questions posed in the Introduction: how inevitable is a "tragedy of the commons" in an MPE? As in Dutta and Sundaram (1989), we use the strict neoclassical framework to address this issue. In subsection 5.3, we show that if $f'(0) > 2$, and the utility functions obey the Inada conditions, the symmetric undiscounted game ***always*** possesses an MPE that does ***not*** result in a "tragedy." Subsection 5.4 adds to this result by showing that if $f'(0) \leq 2$, a somewhat stronger conclusion may be obtained, even without symmetry. Namely, that there are MPE in which from all initial states, the state path either converges to 0, or is overaccumulative.

5.1 Model Specification

(a) Utility Functions. The utility functions u_1 and u_2 are allowed to be any pair of strictly concave functions meeting Assumption 3.

(b) The Production Function. The technology f is any strictly concave function satisfying Assumptions 1 and 2.

Remark. The assumption of ***strict*** concavity of f is only a notation-simplifying device. It ensures uniqueness of the "undiscounted golden-rule" (UGR), but plays no other role in the sequel.

Also for notational simplicity, we normalize the upper-endpoint of S to unity, so $S = [0, 1]$. Our focus in this section is on the ***undiscounted*** model ($\delta = 1$). Payoffs for either player will be evaluated according to the ***long-run average*** (LRA) criterion.[22]

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[22]Although the LRA criterion had well-known weaknesses compared with other undiscounted criteria, such as the overtaking-criterion, it has been widely used in the growth literature, and especially in game theory. This, combined with the simplicity of the LRA as compared to other criteria, are the main reasons we retain it here.

That is, if, in the notation of section 2, $u_i^t(g_1, g_2)(y)$ is player i's period–t payoff under the strategies (g_1, g_2) from the initial state y, the LRA payoff from y under (g_1, g_2) for player i, denoted $W_i(g_1, g_2)(y)$, is given by

$$W_i(g_1, g_2)(y) = \text{liminf}_{T\to\infty} \left[\frac{1}{T} \Sigma_{t=0}^{T-1} u_i^t(g_1, g_2)(y) \right]$$

All relevant definitions from section 2 are modified in the obvious manner to cover this case, and are not detailed here.

5.2 The First–Best Solutions

Under this specification, it is well–known that the state trajectory in any first–best solution, converges, from any non–zero initial state, to the UGR y^* defined by $y^* = f(x^*)$ where $f'(x^*) = 1$. Assumption 2 ensures such an x^* exists; the strict concavity of f implies its uniqueness.

5.3 Underconsumption In the Symmetric Game: A General Proposition

In this subsection, we assume player–symmetric preferences ($u_1 = u_2$), where the common utility function, denoted u, satisfies $u'(0) = +\infty$. We also assume $f'(0) > 2$. Since our purpose in this subsection and the next is to examine the universal validity of a "tragedy" in an MPE, we begin with a definition. Let $y_t(y)$ represent the t–th period value of the state from the initial state y under a given strategy pair (g_1, g_2). We say that (g_1, g_2) leads to a "tragedy" from y if $y_t(y)$ is bounded away from the UGR y^* at a level strictly below it beyond some point in time; and that (g_1, g_2) leads to a "tragedy" if it leads to a "tragedy" from all $y > 0$. Formally:

Definition. The MPE (g_1, g_2) leads to a "tragedy" from y if

(5.1) $$\limsup_{t\to\infty} y_t(y) < y^*.$$

If (5.1) holds from all $y > 0$, then we simply say that (g_1, g_2) leads to a "tragedy."

WE now show that in the symmetric game, there is always at least one MPE that does not lead to a "tragedy" from at least some initial states y:

Proposition 5.1. The symmetric game possesses at least one MPE (g_1, g_2) such that the resulting sequence of states $y_t(y)$ from at least some initial states $y \in S$ satisfies.

(5.2) $$\liminf_{t\to\infty} y_t(y) \geq y^*.$$

Remark. The bulk of the proof is devoted to showing that in the strategies we construct as a candidate MPE, the responding player may, wlog, confine attention to finding a BR from those strategies that always result in a monotone state path. But for this complication, the proof is almost trivial.

Proof. In Dutta and Sundaram (1990), we show that the symmetric game possesses at least one symmetric MPE (g, g).[23] This MPE possesses the following characteristics: (i) the associated savings function ψ is monotone non-decreasing on S, and (ii) the LRA value to either player from y, denoted $\nu(y)$, is a constant ($= \nu$, say) at all $y > 0$.

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[23]Some additional assumptions are also required. See Dutta-Sundaram (1990) for details.

Property (i) implies that for y, y′ $\in$ S, if $y > y'$, then $y_t(y) \geq y_t(y')$ for all t. Let $z = \lim_{t\to\infty} y_t(1)$. Then, $z \geq \lim_{t\to\infty} y_t(y)$ for all y $\in$ S. Define $\alpha: S \to S$ by $f(y - 2\alpha(y)) = y$ for all y $\in$ S. It follows easily from the definition of the LRA payoff that $\nu = u(\alpha(z))$.

There are 2 cases to consider: (a) $z \geq y^*$, in which case we are done, or (b) $z < y^*$. In case (b), we will show that there exists an alternative symmetric MPE (g^*, g^*) such that (g^*, g^*) does not lead to a "tragedy" from an interval of initial states. To this end, define $\hat{y}$ by $\hat{y} > y^*$ and $u(\alpha(\hat{y})) = \nu$. By the concavity of f such a $\hat{y}$ exists. Now define g^* by

$$g^*(y) = g(y),\ y \neq \hat{y}$$

$$= \alpha(\hat{y}),\ y = \hat{y}$$

Let $y_t^*(y)$ be the period–t value of the state from the initial state y under the Markovian profile (g^*, g^*). By construction, $y_t^*(\hat{y}) = \hat{y}$ for all t. Hence, (g^*, g^*) leads to overaccumulation from $\hat{y}$, and indeed from any y $\in$ S for which $\lim_{t\to\infty} y_t^*(y) = \hat{y}$. [Observe that ($g^*$, g^*) gives rise to monotone state paths from each y, so this limit is well–defined.] To complete the proof of the proposition, we now show that (g^*, g^*) is an MPE.

Recall that ν is the constant LRA payoff from any $y > 0$ in the MPE (g, g). It is easy to see that using g^* against g^* also yields an LRA payoff of ν from any $y > 0$. Since g and g^* differ only at $\hat{y}$, it evidently suffices to show now that there is no strategy for the responding player (say, player 1) against g^* that will yield him an LRA payoff strictly greater that ν <u>from the initial state</u> $\hat{y}$.

The crucial step in establising this is the following. Each (not necessarily Markovian) strategy π for player 1 in the generalized game results in a particular sequence of states $\{y_t\}$ from $\hat{y}$. We show that there is no loss of generality in restricting player 1 to using only those strategies that result in this sequence being monotone. That there is no "monotone" strategy that can do better against g* than g* is simple to establish, completing the proof.

Since f is strictly increasing, and g* is a stationary Markovian strategy, we may define the feasible set of LRA payoffs that player 1 may obtain from $\hat{y}$ directly on the set of feasible state paths from $\hat{y}$. Let Ψ denote the set of all such paths, with generic element Y:

$$\Psi = \{Y = \{y_t\} | \forall\ t \geq 0: \ y_{t+1} = f(y_t - g^*(y_t) - c_t), \ c_t \in [0,\ y_t - g^*(y_t)],\ y_0 = \hat{y}\}.$$

Let $\nu(Y)$ denote the LRA payoff to player 1 from (the unique consumption path corresponding to) $Y \in \Psi$. By definition, then, player 1 can obtain no more than ν^* in an LRA BR to g* from $\hat{y}$, where

$$\nu^* = \sup\{\nu(Y) | Y \in \Psi\}.$$

Now let Ψ^R, Ψ^T, and Ψ^M denote the respective subsets of Ψ in which $\hat{y}$ recurs infinitely often (i.o.); in which $\{y_t\}$ follows a T–period cycle for T a positive integer; and in which $\{y_t\}$ is monotone.

$$\Psi^R = \{Y \in \Psi | y_t = \hat{y} \ \ \text{i.o.}\}$$

$$\Psi^T = \{Y \in \Psi | y_{kT} = \hat{y},\ k = 0,\ 1,\ 2,\ \ldots\ \}$$

$$\Psi^M = \{Y \in \Psi | y_{t+1} \geq y_t \text{ for all } t, \text{ or}$$
$$y_{t+1} \leq y_t \text{ for all } t\}$$

Finally, let ν^R, ν^T, ν^M denote respectively the supremum over Ψ^R, Ψ^T, Ψ^M of $\nu(\cdot)$.

The following lemmata are easy to establish:

<u>Lemma 5.1</u>. $\nu^* = \nu^R$.

<u>Lemma 5.2</u>. $\nu^* = \sup_T \nu^T$.

<u>Lemma 5.3</u>. $\nu^M = u(\alpha(\hat{y})) = \nu$.

Lemmata 5.1 and 5.3 are immediate consequences of the facts that g and g* differ only at $\hat{y}$, and (g, g) is an MPE. (For lemma 5.3, recall also that both (g*, g*) and (g, g) generate monotone state paths.) To see lemma 5.2's validity, note that for any $\epsilon > 0$, there must exist $\{y_t\} = Y \in \Psi^R$, and integers m and n, m > n, such that (i) $\nu(Y) \geq \nu^R - \epsilon/2$, (ii) $y_n = y_{m+1} = \hat{y}$, but $y_t \neq \hat{y}$ for $n < t \leq m$, and (iii) the average payoff to player 1 between periods n and m (inclusive) is at least $\nu(Y) - \epsilon/2$. Defining $T = m - n$ now, we have $\nu^T \geq \nu^R - \epsilon$. Since $\epsilon > 0$ is arbitrary, so $\nu^R \leq \sup \nu^T$. The reverse inequality is, of course, immediate since $\Psi^T \subset \Psi^R$ for all T. This completes the proof by lemma 5.1.

These observations imply that to establish (g*, g*) is an MPE, it suffices to show that ν, the LRA value obtained by using g* against g*, satisfies $\nu \geq \nu^T$ for any T. We prove this last step now. Simplify notation first by defining

$$\xi(y) = y - g^*(y) \text{ at all } y \in S,$$

i.e., $\xi(y)$ is simply the net stock facing player 1 at y. Pick any $T \geq 2$, and any $Y \in \Psi^T$. We shall show that $\nu \geq \nu(Y)$.

Since $Y \in \Psi^T$, so Y involves a T–period cycle of stocks $\{y_0, \ldots, y_{T-1}\}$. Let $\{c_0, \ldots, c_{T-1}\}$ denote the corresponding sequence of consumption levels (for the responding player 1) in this cycle. By definition, $y_0 = \hat{y}$, and for $t = 0, 1, \ldots, T-1$,

$$c_t = \xi(y_t) - f^{-1}(y_{t+1}),$$

where, of course, $y_T = \hat{y}$. Therefore, the total consumption over the cycle is

$$\Sigma_{t=0}^{T-1} c_t = \Sigma_{t=0}^{T-1}[\xi(y_t) - f^{-1}(y_{t+1})]$$

$$= \xi(\hat{y}) - f^{-1}(\hat{y}) + \Sigma_{t=1}^{T-1}[\xi(y_t) - f^{-1}(y_t)],$$

where, for the last equality, we have used $y_0 = \hat{y} = y_T$.

Now note that, by construction, $[\xi(y) - f^{-1}(y)]$ is maximized at $y = \hat{y}$, since $\hat{y}$ guarantees the largest steady–state consumption against g^*. This now implies

$$\Sigma_{t=0}^{T-1} c_t \leq T \cdot [\xi(\hat{y}) - f^{-1}(\hat{y})] = Tg^*(\hat{y}).$$

By the strict concavity of u, this translates to

$$\Sigma_{t=0}^{T-1} u(c_t) \leq Tu(g^*(y)) = T\nu.$$

But the LHS is just $T\nu(Y)$, so $\nu \geq \nu(Y)$ as desired. Since T and $Y \in \Psi^T$ were arbitrary, the proposition is proved.

Q.E.D.

5.4 A Second Proposition on Underconsumption

In this subsection, we replace the assumption that $f'(0) > 2$ with the complementary assumption that $f'(0) \in (1, 2]$. The assumption that $f'(0) > 1$ is, of course, made to ensure that the golden-rule y^* is non-zero. We also drop the requirement that $u_1 = u_2$. [Indeed, we do not even require that u_1 or u_2 be concave, but merely that they are continuous and increasing on $\mathbb{R}_+$.] Under these conditions, we show that there are MPE under which the sole non-zero limit point of state paths from any initial state is one resulting in overaccumulation.

Proposition 5.2. Under the conditions states above, there is an MPE (g_1, g_2) of the undiscounted game, and a $\bar{y} \in S$, $\bar{y} > y^*$, such that the resulting state path $\{y_t(y)\}$ from any $y \in S$ converges either to 0 or to $\bar{y}$.

Proof. Define $z = \max \{f(1/2), y^*\}$. (Recall $S = [0, 1]$). Let $\bar{y}$ be any point in $(z, 1)$. Define the strategies (g_1, g_2) by

$$g_1(y) = g_2(y) = y/2, \; y \neq \bar{y}$$
$$= \alpha(\bar{y}), \; y = \bar{y}$$

where, as in subsection 5.3, $\alpha(\bar{y})$ satisfies $f(\bar{y} - 2\alpha(\bar{y})) = \bar{y}$. That (g_1, g_2) satisfies the desired conditions is immediate since $f(0) = 0$. We show (g_1, g_2) is an MPE. Denote by g the common function g_1 and g_2.

First note that since f is concave and $f'(0) \leq 2$, so $f(y/2) < y$ for all $y \in S$, $y \neq 0$. Further, since $\bar{y} > z \geq f(1/2) \geq f(y/2)$ for any $y \in S$, it now easily follows that no matter what strategy is employed against g from any initial state $y \neq \bar{y}$, the state path converges to 0 monotonically. Consequently consumption by the player responding to g also converges to 0, so that the only LRA payoff obtainable by the player responding to g from any initial state $y \neq \bar{y}$, is $u_i(0)$. Since u_i is increasing, it is also the case that $u_i(\alpha(\bar{y})) > u_i(0)$, while deviating from the proposed strategies at $\bar{y}$ yields an LRA payoff of $u_i(0)$ by the above arguments. Thus, (g_1, g_2) is an MPE.

Q.E.D.

Appendix I.1

Proof of Lemma 3.1. Suppose $y \in [0, y^*]$. It is easy to check that the strategies given by (3.4)′ form an equilibrium. Now suppose $(\hat{g}_1, \hat{g}_2)$ is any Markovian equilibrium. Pick an arbitrary $y \geq 0$, and consider

$$\hat{\psi}(y) = y - \hat{g}_1(y) - \hat{g}_2(y)$$

It is clear that $\hat{\psi}(y) \leq K^{-1}$. Suppose in fact that $\hat{\psi}(y) < K^{-1}$. Then, it must be the case that $y - \hat{g}_2(y) < K^{-1}$ and $\hat{g}_1(y) = 0$. But that in turn implies that $\hat{\psi}(y) = \psi(y)$.

On the other hand, if $\hat{\psi}(y) = K^{-1}$, then (with $g_1(y) \geq 0$), player 2 solves the problem

$$\max_{x \in [0,\ y-c_2]} \{u(y - c_2 - x) + \delta u(\tfrac{1}{2}f(x))\}$$

The solution is $\psi(y - c_2) < \psi(y) < K^{-1}$. But by definition of equilibrium, $\psi(y - c_2) = \hat{\psi}(y)$. Hence, we have a contradiction.

We now show that Fig. 2 was "correct."

Proposition A.1. There is a strictly concave function $u: \mathbb{R}_+ \to \mathbb{R}$ satisfying (A3) and $u'(0) = \infty$, such that

(A.1) $$(\tfrac{1}{2}K\delta^2)\psi(y) \lesseqgtr \tfrac{1}{K}y,\ y \lesseqgtr \tilde{y}.$$

(A.2) $$\text{where } \psi(y) = \underset{x\in[0,y]\cap[0,K^{-1}]}{\text{argmax}} \{u(y-x) + \delta u(\tfrac{1}{2}Kx)\}$$

Proof. Consider e.g. $u_\alpha(c) = c^\alpha$ for $\alpha \in (0, 1)$. It is easy to check that if $u = u_\alpha$, then $\psi(y) = \Theta(\alpha)\cdot y$, where $\Theta(\alpha) = [1 + \frac{1}{2}K(\delta K)^{1-\alpha}]^{-1}$. It then follows that $\Theta(\alpha)$ increases in α. So, if u_α was the utility function, as α increases the first inequality is satisfied for $\alpha \in (o, \hat{\alpha})$ and the second for $\alpha \in (\hat{\alpha}, 1)$. To see this is so, note that we need to show

(A.3) $$\tfrac{1}{2}(K\delta)^2 < 1 + \tfrac{1}{2}K(\delta K)^{1-\alpha},\ \alpha \in (0, \hat{\alpha}).$$

At $\alpha = 0$, this inequality is satisfied if $\frac{1}{2}\delta K^2(\delta-1) < 1$, which of course holds for all $\delta \in (0, 1)$. On the other hand, the claim is that

(A.4) $$\tfrac{1}{2}(K\delta)^2 > 1 + \tfrac{1}{2}K(\delta K)^{1-\alpha},\ \alpha \in (\hat{\alpha}, 1)$$

At $\alpha = 1$, this inequality is satisfied if $(K\delta)^2 - K > 2$. Recall that the productivity condition is $K\delta > 2$. So suppose instead that K satisfies this stronger condition, $K\delta > \sqrt{2+K}$. Then, (A.4) is established.

Now suppose $\alpha' \in (0, \hat{\alpha})$, $\alpha'' \in (\hat{\alpha}, 1)$ and

(A.5) $$u(c) = \begin{cases} c^{\alpha'} & ,\ c \in [0, \hat{c}] \\ c^{\alpha''} + \beta & c \in [\hat{c}, \epsilon) \end{cases}$$

Claim 1. There is $\xi > 0$, s.t. for every $\bar{c} \in (0, \xi)$, u is C^0, C^1 except at $\bar{c}$ and strictly concave, with $u'(0) = \infty$, for an appropriate choice of β.

Proof. For every choice of $\bar{c}$, there is clearly a β such that u is C^0. Clearly, it is C^1 except at $\bar{c}$, and $u'(c) = \infty$. Note that the ratio of the slopes of u_α, $\frac{\alpha'}{\alpha''} c^{\alpha'-\alpha''} \to \infty$, as $c \downarrow 0$. So, there is $\xi > 0$ such that $\alpha' c^{\alpha'-1} > \alpha'' c^{\alpha''-1}$, $c \in [0, \xi]$. Pick any $\bar{c} \in (0, \xi)$. The claim is proved.

Consider such a utility function. As $\bar{c} \downarrow 0$, clearly its optimal investment policy uniformly approaches $\Theta(\alpha'')y$. But for any $\bar{c} > 0$, clearly on an initial segment, $\psi(y) = \Theta(\alpha')y$.

For an appropriate "small" choice of $\bar{c}$, the associated ψ is initially flatter that $f^{-1}(y)$ and eventually sharper. It is also that the intersection of the two functions is unique. The proposition is proved.

Appendix I.2

We show here that Markovian equilibria in randomized strategies also fail to exist in the example of section 3. We begin with some definitions and notation. Since we are only concerned here with the example, our definitions are tailored specifically for this case.

For any measurable set $X \subset \mathbb{R}$, let $P(X)$ denote the set of all probability measures on (the Borel subsets of) X. For any $\mu \in P(X)$, supp. μ will denote the support of μ.

Definition 1: A Markovian mixed-strategy for player i is a pair (π_i^0, π_i^1) such that for each $t = 0, 1$, π_i^t: $S \to P(S)$ is a measurable function satisfying supp. $\pi_i^t(y) \subset [0, y]$ for all $y \in S$.

Let $\pi_i = (\pi_i^0, \pi_i^1)$ denote a generic Markovian strategy for player i.

Definition 2: A pair of Markovian strategies (π_1, π_2) is said to be ***interior*** if for $t = 0, 1$, and all $y \in S$, $a_1 + a_2 \leq y$ for all $a_i \in$ supp. $\pi_i^t(y)$, $i = 1, 2$.

That is, a pair of Markovian strateiges is interior if the realized planned actions are collectively feasible at y wp 1, at all y and for all t.

Players are assumed to use the expected utility criterion to evaluate payoffs from mixed strategies. Definitions of MPE and interior MPE are immediate and not detailed here. We are now ready for:

Proposition I.2: There exists no interior mixed-strategy MPE in the game of section 3.

Proof: We will show that any interior mixed-strategy MPE is also a pure strategy MPE, i.e., in any mixed-strategy MPE it must be the case that at all y and t, $\pi_i^t(y)$ has 1-point support for both i. Since section 3 has established the impossibility of a pure strategy MPE, we will be done. For expositional ease we drop the word "interior" from now on.

So let (π_1, π_2) be an MPE. Consider period 1. For each $y \in S$, define $a_i(y)$ by $a_i(y) = \inf \{a | \text{supp. } \pi_i^1(y) \subset [0, a]\}$. It is immediate from the definition of interiority that

$$\text{(I.6)} \qquad a_1(y) + a_2(y) \leq y \quad \text{for all } y \in S.$$

Since (π_1, π_2) is an MPE, so π_2^1 is a BR to π_1^1 from each $y \in S$ in the 2–period game beginning at period 1. Therefore, at any $y \in S$, each $c_2 \in$ supp. $\pi_1^2(y)$ solves:

$$\text{(I.7)} \qquad \underset{c_2 \in [0, y - a_1(y)]}{\text{Max}} \left\{ u(c_2) + \delta \int u(\tfrac{1}{2}f(y - c_1 - c_2))\ \pi_1^1(y; dc_1) \right\}$$

Since u is strictly concave, and f is concave and increasing, the RHS of (I.7) is easily seen to be strictly concave in c_2, and hence to possess a unique maximum. But this implies $\pi_2^1(y)$ has 1–point support for each $y \in S$, and is therefore, equivalent to a pure strategy, say g_2^1: $S \to S$. From subsection 3.1.1, player 2 has a unique BR to each pure strategy of player 2's in this subgame. Therefore, $\pi_1^1(y)$ also has 1–point support at each $y \in S$, so that, in fact, (π_1^1, π_2^1) is simply the MPE (g_1^1, g_2^1) of subsection 3.1.1.

Thus, in the MPE (π_1, π_2), continuation values after period 0 are simply those given by equations 3.5 of lemma 3.1. Now note that player 2's value in this continuation is concave (in fact, strictly concave on $[0, y^*]$), so that an identical argument to that above now establishes that π_2^0 is equivalent to a pure–strategy, say g_2^0: $S \to S$. Once again from each $y \in S$, each element $c_1 \in$ supp. $\pi_1^0(y)$ must be a pure strategy maximizer for player 1, given that continuation values follow the equations (3.5). But for any y such that the ***net*** stock $y - g_2^0(y) \neq \hat{z}$, player 1 has a unique pure–strategy maximizer from figure 3.2. Thus, at all such y, $\pi_i^0(y)$ has 1–point support. It remains only to show that for y such that $y - g_2^0(y) = \hat{z}$ also, player 1 must be using a pure strategy.

Suppose this were not the case for some $\bar{y}$. Note that player 1's BR to ***net*** stocks below $\hat{z}$ is to consume everything, while the unique BR for net stocks above $\hat{z}$ is to consume nothing. At net stocks of $\hat{z}$, either action is optimal, and infact constitute the only optimal actions (cf. figure 3.2). Therefore, if $\pi_1^0(\bar{y})$ does ***not*** have 1–point support, it must be the case that $\pi_1^0(\bar{y}) = 0$ wp p and $\pi_1^0(\bar{y}) = \hat{z}$ wp $(1 - p)$ for some $p \in (0, 1)$. Thus from $\bar{y}$ in the MPE (π_1, π_2), player 2's value is:

$$u_2(g_2^0(\bar{y})) + \delta[pV_2(f(\hat{z})) + (1 - p)V_2(0)] \tag{I.8}$$

Given u_2 strictly concave and $u_2'(0) = +\infty$, it is now immediate that (π_1, π_2) cannot be an MPE since player 2 would prefer to give up a little consumption now rather than accept a continuation value of 0 with positive probability. For suppose, player 2 consumed $g_2^0(\bar{y}) - \epsilon$ for some small $\epsilon > 0$ at $\bar{y}$. Player 1's BR to the new net stock $\bar{y} - g_2^0(\bar{y}) + \epsilon$ ($= \bar{y}_\epsilon$, say) is to consume nothing, so that this strategy has worth

$$u_2(g_2^0(\bar{y}) - \epsilon) + \delta V_2(f(\bar{y}_\epsilon)). \tag{I.9}$$

By the Inada conditions and the strict concavity of u_2 and V_2 it is easily seen that for sufficiently small $\epsilon > 0$, (I.9) strictly dominates (I.8).

Thus, (π_1, π_2) must be a pure strategy equilibrium. By section 3, no such MPE exists.

Appendix II

Let $\delta \in [1/4, 1/3]$ be given. In what follows, we suppress dependence on δ to ease notation.

Let (g_1, g_2) be a differentiable MPE of the dynamic game in strictly interior strategies. Let ψ and ψ^* denote respectively the saving functions under (g_1, g_2) and (g, g). We will show the following:

(i) On the set $S_\delta^* \equiv [0, 1/2\delta - 1]$, we must have $g_1 = g_2 = g$.

(ii) For all $y \in S$, the savings functions ψ and ψ^* satisfy $f(\psi(y)) \equiv 4\psi(y) \in S_\delta^*$, $f(\psi^*(y)) \equiv 4\psi^*(y) \in S_\delta^*$.

By (ii), S_δ^* is clearly the relevant set for dynamic analysis. Therefore, by (i), the essential uniqueness of (g, g) is established. Note that $S_\delta^* \to [0, 1]$ as $\delta \downarrow 1/4$, so that in the limit full uniqueness of (g, g) is obtained.

[For $\delta \in (1/3, 1/2)$, a weaker version of the uniqueness result holds. Let Z be the image of S under $f\text{o}\psi$ in the MPE (g_1, g_2). Identical arguments that we use to prove (i) will establish that $g_1 = g_2 = g$ on Z.]

Finally, we will show that (g, g) is indeed an MPE of the game, completing the proof.

The proof of (i) and (ii) is in 2 steps. First we show that on Z, the image of $f\text{o}\psi$, we must have $g_1 = g_2 = g$, so that $\psi = \psi^*$ on Z. Then we show that for $\delta \in [1/4, 1/3]$, $f(\psi(y)) \in Z$ for all $y \in S$ only if $S_\delta^* \subset Z$. Since ψ and ψ^* agree on Z, this completes the proof.

Pick any $\hat{y} \in S$, and let (i) $\hat{x} = \psi(y)$, and (ii) $\bar{x} = \psi(f(\hat{x}))$. Clearly, for $j = 1, 2$, $\hat{x}$ solves

$$\max_{x \in [0, \hat{y} - g_j(\hat{y})]} \left\{ \hat{y} - g_j(\hat{y}) - x + \delta[f(x) - g_j(f(x)) + \lambda(f(x)) - \bar{x}] \right\}$$

so that, by interiority of the solution, $\hat{x}$ satisfies:

(A.2.1) $$-1 + 4\delta[1 - g_j'(f(\hat{x})) + \lambda'(f(\hat{x}))] = 0$$

or

(A.2.2) $$g_j'(f(\hat{x})) = (1 - 1/4\delta) + \lambda'(f(\hat{x})).$$

Notice that since ψ is continuous and $\psi(0) = 0$, so Z is an interval [0, a]. By (A.2.2), and since $g_j(0) = 0$, the fundamental theorem of calculus yields:

(A.2.3) $$g_j(z) = (1 - 1/4\delta)z + \lambda(z), \; z \in Z,$$

i.e., $g_1 = g_2 = g$ on Z. Therefore, $\psi = \psi^*$ on Z, or $\psi(z) = (\frac{1}{2\delta} - 1)z(1 - z)$, for $z \in Z$.

Next, we claim that $Z \supset S_\delta^*$. To see this note first that we must have $1/2 \in Z$. For suppose $Z = [0, a]$ with $a < 1/2$. Then, $f(\psi(a)) = 4a(1 - a)(\frac{1}{2\delta} - 1) > a$ since $\delta \leq 1/3$, or $f(\psi(a)) \notin Z$, which contradicts the definition of Z. But if $1/2 \in Z$, then $f(\psi(1/2)) \in Z$, or $4(\frac{1}{2\delta} - 1)1/4 = (\frac{1}{2\delta} - 1) \in Z$, so $Z \supset S_\delta^*$ as claimed.

Since (g_1, g_2) coincides with (g, g) on Z, so ψ and ψ^* coincide on Z (hence, on S_δ^*), establishing (ii).

Finally, to prove that (g, g) is actually an MPE, it is easy to show through direct calculation that terminating the game at any point, deviating, or continuing along g all yield the player making a best-response to g, exactly the same value which at a state y is $\frac{1}{4\delta}y$.

As a last point, we note that a variety of other MPE dynamics could have been generated by using alternative specifications for $\lambda(\cdot)$. To wit, any C^1 increasing function $\lambda: \mathbb{R}_+ \to \mathbb{R}_+$ that satisfies

(i) $[2(1 - 1/4\delta)y + 2\lambda(y)] \in (0, y) \quad \forall\, y > 0$

and (ii) $[(1/2\delta - 1)y - 2\lambda(y)] \in (0, 1/4) \quad \forall\, y > 0$

generates a differentiable MPE (g, g) via

$$g(y) = (1 - 1/4\delta)y + \lambda(y),\ y \in S.$$

This result may be proved by direct calculation.

References

Abreu, D., and P. K. Dutta (1989), A Folk–Theorem for Stochastic Games, mimeo, Columbia University.

Benhabib, J., and K. Nishimura (1985), Competitive Equilibrium Cycles, Journal of Economic Theory 35, 284–306.

Benhabib, J., and R. Radner (1988), Joint Exploitation of a Productive Asset: A Game–Theoretic Analysis, mimeo, New York University.

Bernheim, B. D., and D. Ray (1983), Altruistic Growth Economies, Technical Report No. 419, IMSSS, Stanford University.

Bernheim, B. D. and D. Ray (1987), Economic Growth with Intergenerational Altruism, Review of Economic Studies 54, 227–242.

Boldrin, M., and L. Montrucchio (1986), On the Indeterminacy of Capital Accumulation Paths, Journal of Economic Theory 40(1), 26–39.

Cave, J. (1988), The Cold Fish War: Long Term Competition in a Dynamic Game, Rand Journal of Economics (18), Autumn, 596–610.

Debreu, G. (1954), A Social Equilibrium Existence Theorem, Proceedings of the National Academy of Science (38).

Duffie, D; J. Geanakapolos, A. Mas–Colell, and A. McLennan (1989), Stationary Markov Equilibria, mimeo, University of Minnesota.

Dutta, P. K., and R. K. Sundaram (1989), The Tragedy of the Commons? A Complete Characterization of Stationary Equilibria of Dynamic Resource Games, mimeo, University of Rochester.

Dutta, P. K., and R. K. Sundaram (1990), Stochastic Games of Resource Allocation: Existence Theorems for Discounted and Undiscounted Models, mimeo, University of Rochester.

Easwaran, M., and T. Lewis (1985), Appropriability and the Extraction of a Common Property Resource, Economica 51, 393–400.

Fudenberg, D., and E. Maskin (1986), The Folk–Theorem in Repeated Games with Discounting and with Incomplete Information, Econometrica (54).

Fudenberg, D., and J. Tirole (1990), A Course in Game Theory, manuscript, Department of Economics, MIT.

Hellwig, M., and W. Leininger (1988), Markov–Perfect Equilibrium in Games of Perfect Information, DP A–183, University of Bonn.

Lancaster, K. (1973), The Dynamic Inefficiency of Capitalism, Journal of Political Economy 81.

Leininger, W. (1986), The Existence of Nash Equilibria in a Model of Growth with Altruism Between Generations, Review of Economic Studies, 53.

Levhari, D., and L. Mirman (1980), The Great Fish War: An Example Using a Dynamic Cournot–Nash Solution, Bell Journal of Economics 11(1), 322–334.

Maskin, E., and J. Tirole (1988), A Theory of Dynamic Oligopoly I: Overview and Quantity Competition, Econometrica 56(3).

Mertens, J.–F., and T. Parthasarathy (1987), Stochastic Games, CORE Working Paper 8750, Louvain–an–Neuve.

Mirman, L. (1979), Dynamic Models of Fishing, in P. T. Liu and J. G. Sutinen (Eds.) Control Theory in Mathematical Economics, Decker, New York.

Moulin, H. (1984), Implementing the Kalai–Smordinsky Bargaining Solution, Journal of Economic Theory 33, 32–45.

Parthasarathy, T. (1973), Discounted, Positive, and Noncooperative Stochastic Games, International Journal of Game Theory (2).

Reinganum, J., and N. Stokey (1985), Oligopoly Extraction of A Common Property Resource: Importance of the Period of Commitment in Dymic Games, International Economic Review 26(1).

Sundaram, R. K. (1989), Perfect Equilibrium in a Class of Symmetric Dynamic Games, Journal of Economic Theory 47(1), 153–177.

Van Damme, E. (1987), Stability and Perfection of Nash Equilibrium, Springer–Verlag, Berlin.