Rochester Center for
Economic Research

A Theory of Denominations for Government Liabilities

Cooley, Thomas F. and Bruce D. Smith

Working Paper No. 251
October 1990

University of
Rochester

A Theory of Optimal Denominations for Government Liabilities

Thomas F. Cooley and Bruce D. Smith

Rochester Center for Economic Research
Working Paper No. 251

A THEORY OF OPTIMAL DENOMINATIONS FOR GOVERNMENT LIABILITIES

Thomas F. Cooley
University of Rochester

Bruce D. Smith
Cornell University and FRB of Minneapolis

Working Paper No. 251

October 24, 1990

Research supported in part by NSF Grant SES-8921346. WE have benefited from conversations with Stan Engerman, Neil Wallace and Warren Weber.

Introduction

Throughout history, an important policy question has concerned the optimal size of government liabilities; both monetary and non-monetary. For instance, in the 1780s, the United States had a free choice of a currency system. It essentially chose between two proposals, one submitted by Robert Morris, the other by Thomas Jefferson. Morris proposed a system with a basic unit of 1/1440 th of a Spanish dollar, with this selection based on legal valuations of the Spanish dollar then in effect in the various states. Jefferson proposed a basic unit of 1/100th of a dollar. While the system ultimately adopted was essentially Jefferson's, the one area that required compromise was the choice of a smallest currency unit.[1]

Concern about minimum denominations both for money and bonds has, of course, not been confined to this era of American history. There have been many instances in which currency was issued only in relatively large minimum denominations, causing more or less acute "small change shortages."[2] Responses of governments either to alleviate, or to prevent private alleviation of these shortages have often been important policy issues. For instance, in the Jacksonian era when there were many small change shortages, it was common for states to prohibit the issuance of small denomination bank notes.[3] Moreover, small change shortages have been of concern to policymakers even into this century in the U.S. and Canada, as documented by Kemmerer (1910) and Ross (1922).

---

1 For a detailed discussion see Sumner (1891) and Carothers (1930). Parenthetically, according to Carothers, Jefferson, Morris, and Alexander Hamilton thought that smaller minimum denominations led to lower prices. In our analysis, well chosen (in a welfare sense) positive minimum denominations support steady state equilibria with a lower initial price level and a lower inflation rate than can be achieved with a perfectly divisible currency. Thus our results are not immediately supportive of the Jefferson-Hamilton-Morris belief. However, this belief can be correct if the minimum denomination is not chosen optimally. See Cooley and Smith (1990).

2 Carothers (1930), Hanson (1979,1980), Rolnick and Weber (1986), and Glassman and Redish (1988) document episodes of such shortages.

3 In fact Thomas Jefferson was concerned that banknotes be issued only in large denomination and in 1814 urged the Virginia legislature to require banks to eliminate \$5 and \$10 notes, leaving a minimum denomination of \$50 banknotes. See Carothers (1930) and White (1990) for a discussion.

And, the appropriate minimum denomination for government liabilities has been thought to vary with economic conditions. As an example, government bonds that are normally issued only in relatively large denominations were issued in very small denominations in the First and Second World Wars. Currently, there is discussion of whether the smallest currency unit-the penny-should be eliminated.

While Jefferson, Morris, and others received little guidance from the economics of their time on the question of an optimal minimum denomination, it is not clear that they could expect greater guidance today. Discussions by policymakers of whether the penny should be retained typically focus on whether minting pennies is profitable, an argument that is clearly not employed for other monetary policy questions. The literature on outside assets of limited divisibility is small, consisting (to our knowledge) of Bryant and Wallace (1984), Marimon and Wallace (1987), Villamil (1988), and Cooley and Smith (1990).[4] Furthermore, of these only Bryant-Wallace and Villamil discuss the optimal choice of denominations for outside assets.

Bryant-Wallace and Villamil consider two period lived, overlapping generations economies where the government must finance a persistent deficit of given size. In Bryant-Wallace each generation is of finite size, and all young agents are identical. Bryant-Wallace show in this context that, if the deficit is not too large, a set of stationary, Pareto optimal allocations can be supported as steady state equilibria by choosing an appropriate minimum denomination for a single outside asset. Requiring savers to acquire at least some minimum amount allows the government effectively to employ a non-linear tax schedule, which can be chosen to support certain Pareto optimal allocations. Villamil extends this analysis to allow for within-generation heterogeneity (and private information about type), and shows that a stationary Pareto optimal allocation can be supported by issuing as many different government liabilities as there are agent types, with each liability earning a different rate of return.

---

[4] Farmer (1984) formally considers indivisible consumption goods, but his analysis applies equally to indivisible outside assets.

While we draw heavily on the insights of the papers just cited, we view their analyses has having some shortcomings. First, they have the implication that the desired equilibrium can be attained by having each agent purchase one unit of an asset with a positive minimum denomination. This is clearly contrary to observation. Second, these analyses have no discussion of dynamic equilibria, the stability properties of the equilibria of interest, or the possibility of other steady state equilibria. In fact, as we demonstrate below, when each generation contains a large number (a continuum) of agents, the Bryant-Wallace scheme for supporting a Pareto optimal steady state equilibrium also supports a second steady state equilibrium. This latter equilibrium minimizes the utility of young agents (subject to a voluntary participation constraint), and is also poor from the standpoint of initial old agents. Moreover, the optimal steady state equilibrium is unstable, while the poor equilibrium is locally stable. Finally, and in our view most importantly, we demonstrate that there is a severe indeterminacy associated with the optimal equilibrium supported by the Bryant-Wallace scheme. In particular, even if the economy "starts at" the optimal steady state equilibrium, there are no economic forces to keep it there. Or, in other words, starting from the optimal steady state equilibrium, there is a continuum of non-stationary equilibria, all of which converge to the poor steady state equilibrium. All such equilibria minimize young agent utility (subject to voluntary participation) after two periods.

In this paper we consider a setting identical to that of Bryant and Wallace except that each generation contains a continuum of identical agents. In particular, each young generation is two-period lived, there is a single non-storable good at each date, and the government must finance a persistent (real) deficit of given size. We describe the complete set of (non-stochastic) equilibria (dynamical as well as steady state equilibria) when government liabilities are issued in positive minimum denominations. We then show that (as in Bryant-Wallace) the government can always choose a minimum denomination to support a Pareto optimal steady state equilibrium. Such denominations may be chosen so that young agents hold more than one unit of the government liability, however, Nonetheless, generically the same indeterminacy problem arises as in Bryant-Wallace.

We then consider positive minimum denominations that support steady state equilibria that avoid this indeterminacy problem, and that are Pareto superior to all equilibria existing when government liabilities are perfectly divisible. We show that positive minimum denominations always exist that accomplish this. We are also able to establish the existence of an optimal minimum denomination among a restricted set that supports steady state equilibria with particular dynamic (stability) properties. We demonstrate further that, if a more global optimal minimum denomination with the desired properties exists, it belongs to this class.

The paper proceeds as follows. Section I describes the model, and its equilibria for fixed minimum denominations on the single outside asset. Section II considers choosing the minimum denomination to support a Pareto optimal steady state equilibrium, and shows that generically the indeterminacy problem alluded to above must arise. Section III discusses minimum denominations that support steady state equilibria that are Pareto superior to any equilibrium of the divisible asset economy, and that avoid the indeterminacy problem described in Section II. Section III also establishes the existence of an optimal (for young agents) minimum denomination among the set of denominations that support steady state equilibria with particular stability properties. Section IV shows that if an optimal minimum denomination exists among those that avoid the indeterminacy problem of section II, it must belong to this set. Section V concludes by discussing some possible extensions of the analysis.

I. The Model: Preliminaries

We consider an economy consisting of an infinite sequence of two-period lived, overlapping generations, and an initial old generation. Each generation is identical in size, containing a continuum of identical agents with measure one. Time is discrete, and indexed by $t=1,2....$

At each date agents are endowed with a single non-storable consumption good. We let w_j denote the age j endowment of the good ($j=1,2$), and assume $w_j>0$ for each j. In addition, c_j denotes age j consumption by a representative agent. Each young agent has preferences described by the additively separable utility function $u(c_1) + v(c_2)$. The functions u and v are assumed to be strictly increasing, concave, and twice continuously differentiable. In addition, we assume that:

(A.1) $u'(w_1) < v'(w_2)$

and that

(A.2) $0 > cv''(c)/v'(c) \geq -1$,

$\forall\, c \in R_+$ (the consumption set).

There is a single asset which agents in this economy can hold. The asset is issued by the government in indivisible units with a real value of x. Thus, agents can hold assets only in integer multiples of x. Intermediation that permits agents to share assets is assumed to be prohibited. Throughout we let $n=0,1,...$denote an integer. We assume that $x \leq w_1$ and that

(A.3) $u(w_1 - x) + v(w_2 + x) > u(w_1) + v(w_2)$.

The indivisible asset has various interpretations. One is that it is a treasury liability issued in a minimum denomination. Another is that the asset is a currency (possible a specie currency) issued in minimum denominations.[5]

We let the real quantity (per capita) of outstanding assets at t be denoted B_t, and each unit of the asset held earns the gross real return r_t between t and t+1. There is a government that has per capita real expenditures of $g>0$ at each date, which are financed by issuing debt.[6] Thus the government budget constraint is

(1) $g = B_{t+1} - r_t B_t;$ for $t>1$,

and is

(2) $g = B_1 - B_0$,

for t=1. B_0 is the real value of the initial debt held by old agents at t=1, which is endogenous.[7]

A. Equilibrium with Divisible Assets

As a benchmark, we begin by considering the case where the asset is freely divisible. Let

$$f(r_t) = \text{argmax}\,[u(w_1 - s) + v(w_2 + r_t s)],$$

---

5 If this is a specie currency, specie is costlessly coined (or recoined), and has no alternative uses. This interpretation follows Marimon and Wallace (1987).

6 g can also be interpreted as a deficit, and w_j can be interpreted as the age j net of tax endowment. Of course under this interpretation taxes are held fixed throughout.

7 Note that no agents other than the initial old have any endowment of government debt.

so that f is a standard savings function. Assumption (A.1) implies that $f(1) > 0$, and (A.2) implies that $f'(r_t)>0$. In addition we assume that $f^{-1}(0) \geq 0$, and that $f^{-1}(x)$ exists. (This will follow if x is chosen optimally). For future reference, it will be useful to have a notation that incorporates the dependence of savings on endowments. When we wish to do so, we will denote the savings function by $\tilde{f}(w_1,w_2,r_t)$. Note that $\tilde{f}_1 \geq 0 > \tilde{f}_2$ hold.

An equilibrium for the divisible asset economy is a pair of sequences $\{r_t\}_{t=1}^{\infty}$ and $\{B_t\}_{t=0}^{\infty}$ satisfying (1), (2), and

(3) $B_t = f(r_t)$.

Then, for any value B_0, B_1 is given by (2), and from (1) and (3), subsequent values of B_t evolve according to

(4) $B_{t+1} = B_t f^{-1}(B_t) + g$.

The equilibrium law of motion in (4) is depicted in Figure 1. If g is sufficiently small there will be a steady state equilibrium satisfying

(5) $r_t = f^{-1}(B) = (B-g)/B$.

In general there will be more than one value of B that satisfies (5). Let the largest value of B satisfying (5) be denoted $\bar{B}$. As depicted in Figure 1, the steady state equilibrium with $B_t = \bar{B}\ \forall\ t$

is unstable. Also, it is straightforward to show that the steady state equilibrium with $B_t = \bar{B}\ \forall\ t$ is Pareto superior to all other equilibria for this economy. It is not Pareto optimal, of course.

B. Equilibrium with Indivisible Assets

1. Optimal Savings Behavior

We now let the minimum denomination $x>0$ be given, and assume that agents can purchase assets only in integer multiples of x. We begin by characterizing optimal savings behavior, which is done through a sequence of lemmas. Proofs appear in the appendix.

We first show that savings is non-decreasing in the rate of return.

Lemma 1. Let n_1 and n_2 be integers (with $w_1 \geq n_1x$, $w_1 \geq n_2x$), and suppose that at the interest rate r_t' the savings level n_2x is (weakly) preferred to n_1x. If $n_2 > n_1$, then n_2x is strictly preferred to n_1x $\forall\ r_t > r_t'$. Similarly, if $n_2 < n_1$, then n_2x is strictly preferred to n_1x $\forall\ r_t < r_t'$.

Next, let n^* be the largest integer such that $n^*x \leq w_1$. Then, for $0 < n < n^*$, define r(n,x) to be the interest rate that makes young agents indifferent between saving nx and (n-1)x. r(n,x) need not exist for arbitrary n and x, but if it exists it is defined by

$$(6) \qquad u(w_1 - nx) + v[w_2 + r(n,x)nx] \equiv u[w_1 - (n-1)x] + v[w_2 + r(n,x)(n-1)x].$$

We now state conditions under which r(n,x) exists, and describe some properties of this function. (Note in particular that, by Lemma 1, if r(n,x) exists it is unique).

Lemma 2. (a) Suppose that $f^{-1}(nx)$ exists. Then r(n,x) exists and satisfies $f^{-1}[(n-1)x] < r(n,x) < f^{-1}(nx)$. (b) If r(n,x) exists then $f^{-1}[(n-1)x]$ exists and satisfies $f^{-1}[(n-1)x] < r(n,x)$.

It follows immediately from Lemma 2 that if $r(n,x)$ exists for $n \geq 2$, $r(n-1,x)$ exists and satisfies $r(n-1,x) < r(n,x)$. Thus $r(n,x)$ is increasing in its first argument. It is also the case that it is increasing in its second argument.

Lemma 3. $r(n,x)$ is differentiable with respect to x. This derivative (denoted r_2) satisfies $r_2(n,x) > 0$.

When $r_t = r(n,x)$, we have the following result about optimal savings:

Lemma 4. If $r_t = r(n,x)$, then $(n-1)x$ and nx are preferred to any other savings levels.

It is now possible to characterize optimal savings behavior completely.

Proposition 1. If $r_t = r(n,x)$ for some n, savings of $(n-1)x$ and nx are optimal. If $r_t \in (r(n-1,x), r(n,x))$; $n > 1$, then $(n-1)x$ is the (unique) optimal savings level. If $r_t < r(1,x)$, zero savings is optimal.

Proposition 1 follows immediately from Lemmas 1, 2, and 4.

2. Equilibrium

It is now possible to state equilibrium conditions for the indivisible asset economy. Define $n(B_t)$ to be the smallest integer such that $n(B_t)x \geq B_t$. Then if $B_t = n(B_t)x$, it is an equilibrium for all agents to save $n(B_t)x$ iff $r_t \in [r[n(B_t),x], r[n(B_t)+1,x]]$. However, if $B_t \neq n(B_t)x$, then clearly not all agents can hold the same portfolio in equilibrium. It follows from proposition 1 that some agents must save $n(B_t)x$, while others save $[n(B_t)-1]x$. For this to be equilibrium behavior, $r_t = r[n(B_t),x]$

must hold (so in particular, $r[n(B_t),x]$ must exist). Let μ_t denote the fraction of young agents saving $n(B_t)x$. Then if $B_t < n(B_t)x$, market clearing will require that $r_t = r[n(B_t), x]$ and

$$\mu_t n(B_t)x + (1-\mu_t)\,[n(B_t) - 1]x = B_t. \tag{7}$$

Then an equilibrium is a set of non-negative sequences $\{r_t\}_{t=1}^{\infty}$, $\{B_t\}_{t=0}^{\infty}$, and $\{\mu_t\}_{t=1}^{\infty}$ such that (1), (2), (7), and,

$$\begin{aligned} & r_t = r[n(B_t),x]\ ; \quad n(B_t)x > B_t, \\ (8) \qquad & r_t \in [r[n(B_t),x],\, r[n(B_t) + 1,x]]; \qquad n(B_t)x = B_t, \end{aligned}$$

are satisfied.

An equilibrium law of motion for B_t here satisfies

$$B_{t+1} = r[n(B_t),x]B_t + g; \quad n(B_t)x > B_t, \tag{9}$$

and has $B_{t+1} = r_t B_t + g$ if $n(B_t)x = B_t$, with r_t being any value in the interval $[r[n(B_t),x], r[n(B_t) + 1,x]]$. Such a law of motion is depicted in Figure 2.

In general an equilibrium may or may not exist. Moreover, the set of equilibrium sequences for the divisible asset economy and for the indivisible asset economy bear no particular relation to each other. It is possible that steady state equilibria exist for the divisible but not the indivisible asset economy and conversely. Also, the divisible asset economy can have more or fewer steady state equilibria than the indivisible asset economy.

Notice that, if a steady state equilibrium exists, there will always be a set of non-stationary equilibria as well (as shown in Figure 2). Furthermore, if a steady state equilibrium occurs on a vertical portion of the equilibrium law of motion (as at x or 2x in Figure 2), there is a continuum of non-stationary equilibria even if B_1 is given the appropriate steady state value. In fact, in Figure 2, setting $B_t = x$ (or $B_t = 2x$) $\forall\ t \leq T$, and then following a non-stationary path also constitutes an equilibrium. Thus indivisible asset economies give rise to potential indeterminacies even if initial debt levels (values of B_1) are given. We pursue this point further when we turn our attention to the choice of the minimum denomination x.

II. Pareto Optima

A. Optimal Equilibria

Assuming g is not too large, it is straightforward to derive a minimum denomination of $x > 0$ that supports a Pareto optimal equilibrium (and other equilibria as well). This minimum denomination is analogous to that derived by Bryant and Wallace (1984), who consider the problem of choosing a stationary allocation to maximize the utility of young agents:

$$\text{(P.O.)} \qquad \max\ u(c_1) + v(c_2)$$

subject to

$$(10) \qquad c_1 + c_2 \leq w_1 + w_2 - g \ .$$

The solution to this problem, denoted (c_1^*, c_2^*), satisfies (10) with equality and

$$(11) \qquad u'(c_1^*) = v'(c_2^*) \ .$$

Assuming that

$$\text{(A.4)}\quad u(c_1^*) + v(c_2^*) > u(w_1) + v(w_2) \ ,$$

we now construct a minimum denomination that supports this allocation as an equilibrium. Set x = F, with F given by

$$\text{(12)}\quad u'(w_1 - F) = v'(w_2 + F - g) \ .$$

Further, set $B_t = F$, $\forall\, t \geq 1$, and set $B_0 = g - F > 0$. Finally, set $r_t = R$, with R given by $R = (F-g)/F$. We claim that this constitutes an equilibrium with $n(B_t) = n(F) = 1$. To see this, notice that by construction (1) and (2) are satisfied, as is (7) if $\mu_t = 1$, $\forall$ t. Then all that remains to be verified is that $r_t = R \in [r(1,F), r(2,F)]$, if r(2,F) exists. If not, we need only show that $R > r(1,F)$.

First note that r(1,F) exists. This follows from the definition of r(1,F), and the fact that

$$\text{(13)}\quad u(w_1 - F) + v(w_2 + RF) = u(c_1^*) + v(c_2^*) > u(w_1) + v(w_2) \equiv u(w_1 - F) + v[w_2 + r(1,F)F] \ .$$

Moreover, as is apparent from (13), $R > r(1,F)$. Then, if r(2,F) does not exist, we are done. If r(2,F) does exist, it follows from Lemma 2 that $f^{-1}(F)$ exists, and that $r(2,F) > f^{-1}(F)$. In this case our proof is complete if we show that $R < f^{-1}(F)$. But from (12), $F = \tilde{f}(w_1, w_2 - g, 1) > \tilde{f}(w_1, w_2, 1) > \tilde{f}(w_1, w_2, R) = f(R)$. Thus $f^{-1}(F) > R$, as desired.

The equilibrium law of motion for this economy with x = F is depicted in Figure 3. The steady state equilibrium with $B_t = F$ occurs on a vertical portion of the equilibrium law of motion, as $R \in (r(1,F), f^{-1}(F))$. Notice that there is also necessarily a second steady state equilibrium, with

$B_t = B'$, $\forall$ $t \geq 1$, and $r_t = r(1,F)$. By definition $u(w_1 - F) + v[w_2 + r(1,F)F] = u(w_1) + v(w_2)$, so this equilibrium minimizes the utility of young agents (so long as these agents are free to choose a zero savings level). Moreover, the Pareto optimal equilibrium is unstable, while the steady state equilibrium with $B_t = B'$, $\forall$ t, is locally stable.

B. Determinacy

The fact that the optimal equilibrium is unstable while the other is locally stable is an important drawback to the above as a theory of denominational structure. In our view there is a second, more serious, problem as well. In particular, suppose $B_1 = F$. It is still the case that we can choose any value $B_2 \in [r(1,F)F, B_1)$, and construct a non-stationary equilibrium. We can also let $B_t = F$; $t \leq T$, and choose $B_{T+1} \in [r(1,F)F, F]$ for any finite T. Consequently, there is a serious indeterminacy problem even if $B_1 = F$: there are no economic forces operating to keep the economy at the optimal steady state. Moreover, all non-stationary equilibria constructed as described converge to $B_t = B'$, which is a steady state equilibrium with poor welfare properties. Accordingly, it seems natural to attempt to construct a denomination structure for which this indeterminacy problem does not arise. We consider this issue below.

C. A Digression on Bryant and Wallace (1984)

Bryant and Wallace (1984) consider an economy in which each generation is of finite size, so that it is not apparent that they encounter the problem just described. Abstracting from this, however, Bryant-Wallace also require agents to purchase assets with a minimum real value of F. Once in excess of this amount, agents are free to purchase arbitrary incremental asset quantities. This can be captured in our framework quite easily. In Figure 3, for $B_t > F$, the equilibrium law of motion simply coincides with that for the divisible asset economy. Then it is apparent that there are exactly two steady state equilibria, corresponding to $B_t = B'$, $\forall$ t and to $B_t = F$, $\forall$ t.

D. On the Generic Impossibility of a Determinate, Pareto Optimal Steady State Equilibrium.

Any Pareto optimum must have marginal rates of substitution equated for all young agents at each date. Then, in the absence of direct taxation, all young agents must save the same amount at each date. This requires that a Pareto optimal steady state have $B^* = B_t = n(B_t)x$, $\forall$ t. Finally, to avoid the indeterminacy problem described above, the interval $[r[n(B^*), x]B^*, B^*]$ must consist of a single point, as in Figure 4, so that $B^*\{1 - r[n(B^*), x]\} = g$. We now show that it is generically impossible to choose a denomination structure that has $B_t = F$ and $r_t = R$, $\forall$ t, and that has these properties.

Suppose that we attempt to choose a minimum denomination x satisfying the following conditions: (i) $F = (n+1)x$ for some integer n (by the argument of section B, $n \geq 1$ must hold); and (ii) $R = (F-g)/F = r(n+1,x)$. Property (ii) implies that $F[1-r(n+1,x)] = g$, as required, and it is apparent that if x satisfies (i) and (ii), there will exist a steady state equilibrium with $B_t = B^* = F$ as depicted in Figure 4. If such an x can be chosen, we will have $nx = [n/(n+1)]\,(n+1)x = [n/(n+1)]F$, and by property (ii) and the definition of $r(n+1,x)$,

$$(14)\qquad u(w_1-F)+v(w_2+RF) = u(w_1-F)+v(w_2+F-g) = u\left[w_1 - \left(\frac{n}{n+1}\right)F\right]+ v\left[w_2+R\left(\frac{n}{n+1}\right)F\right] = u\left[w_1-\left(\frac{n}{n+1}\right)F\right]+v\left[w_2+\left(\frac{n}{n+1}\right)(F-g)\right]$$

Now define the function H(k,B) by

(15) $H(k,B) \equiv u(w_1 - B) + v(w_2 + B\text{-}g) - u(w_1 - kB) - v(w_2 + kB - kg)$.

Then (14) reduces to $H[n/(n+1),F] = 0$ for some finite integer n.

It will now be useful to collect some facts about the function H. Recall that $\bar{B}$ is the largest steady state value of B_t for the divisible asset economy, and let $H_1(k,B)$and $H_{11}(k,B)$ denote the partial derivatives of H with respect to its first argument. Then we have the following:

Lemma 5. (a) H is continuous, as are H_1 and H_{11}. $H_{11} > 0$ holds, $\forall B \leq w_1$. (b) $H(1,B) = 0, \forall B$. (c) $H(0,B) > 0, \forall B \in [\bar{B},F]$. (d) $H(k,B)$ attains a minimum for some $k \in (0,1), \forall B \in (\bar{B}, w_1)$. The minimum occurs when

(16) $k = Q(B) \equiv f[(B\text{-}g)/B]/B$.

In particular, $Q(B) < 1 \; \forall B \in (\bar{B},w_1)$.

It follows from the lemma that $H(0,F) > 0 > H[Q(F),F]$. Hence, by the intermediate value theorem, there exists a value $k \in (0,Q(F))$ such that $H(k,F) = 0$. Moreover, there is only one such a value in $[0,Q(F)]$, since $H_1(k,F) < 0, \forall k \in (0,Q(F))$. Finally, since $H_1(k,F) > 0, \forall k \in (Q(F), 1)$, and since $H(1,F) = 0$, there is only one value $k < 1$ that satisfies $H(k,F) = 0$ (see Figure 5).

Now suppose that $H[n/(n+1),F] = 0$ holds for some finite integer n. Then we can replace w_1 with $\tilde{w}_1$ (or w_2 with $\tilde{w}_2$) in (15), producing a new function

$\tilde{H}(k,B) = u(\tilde{w}_1 - B) + v(w_2 + B - g) - u(\tilde{w}_1 - kB) - v(w_2 + kB - kg)$, with the property that $\tilde{H}[n/(n+1),F] > 0 > \tilde{H}[(n+1)/(n+2),F]$. Moreover, this can be done with $\tilde{w}_1$ arbitrarily close to w_1 (or $\tilde{w}_2$ arbitrarily close to w_2). Then apparently it is generically impossible for $H[n/(n+1),F] = 0$ to hold for some finite integer n. Or, in other words, it is generically impossible to produce a minimum denomination x satisfying properties (i) and (ii).

If we seek a minimum denomination that (a) supports a steady state equilibrium that is Pareto superior to all steady state equilibria of the divisible asset economy, and (b) avoids the indeterminacy problem discussed in section B, then apparently it generically cannot maximize the (steady state) utility of young agents. Moreover, (b) requires either that the steady state have $B_t = B^* = n(B_t)x$ and $B^*\{1 - r[n(B^*),x]\} = g$, or $B_t = B^* < n(B_t)x$. The next section focuses on denomination structures that support steady state equilibria of the first type.[8]

III. The Existence of a "Second-Best" Minimum Denomination

In this section we focus on minimum denominations that support steady state equilibria such as B^* in Figure 4; that is that have equilibria with $B_t \to B^*$ if $B_1 \in (B^* - \epsilon, B^*)$ for some $\epsilon > 0$, but do not have $B_t \to B^*$ if $B_1 > B^*$. Such steady state equilibria avoid the indeterminacy problems discussed in Section II.B. However, we do not insist that $B^* \geq F$ hold, but only that $B^* > \bar{B}$. Then the steady state of interest is Pareto superior to the best (in a Pareto sense) steady state equilibrium of the divisible asset economy, but need not be Pareto optimal.

We first show that minimum denominations supporting a steady state such as B^* with the desired properties exist. We then show that an optimal minimum denomination exists among those supporting steady states such as B^*; i.e., there is a unique minimum denomination that gives rise to

8 While the argument just given applies to the steady state equilibrium (and Pareto optimum) that maximizes the utility of young agents, it applies equally to any other Pareto optimal steady state equilibrium.

a steady state with the desired property, and that maximizes the welfare of young agents across steady state equilibria having this property.

A. Pareto Improvements from Positive Minimum Denominations

We now want to choose a minimum denomination $x > 0$, and find a (finite) positive integer n as well as values B^* and B_0 such that $B_0 = B^* - g$,

$$(17) \quad (n+1)x = B^*$$

$$(18) \quad r(n+1,x) = (B^* - g)/B^*$$

$$(19) \quad \bar{B} < B^* \leq F.$$

If such values can be found, then setting $B_t = B^*$ and $r_t = r(n+1,x)$, $\forall\, t \geq 1$ constitutes a steady state equilibrium. In particular, (18) and the choice of B_0 imply that (1) and (2) are satisfied. (18) implies that (8) is satisfied, and (7) is satisfied if $\mu_t = 1$, $\forall\, t$. Note also that, by (17), $n(B^*) = n+1$, so that $n(B^*)x = B^*$. Thus the interval $[r(n+1,x)(n+1)x, B^*]$ collapses to the single point $r(n+1,x)(n+1)x$, and the indeterminacy problem of section II.B is avoided.

We observe that if values x, n, and B^* satisfying (17) - (19) can be found, it will be feasible for young agents to save $(n+1)x = B^*$, since $B^* \leq F \leq w_1$. It is also the case that any equilibrium with $B_t = B^*$ and $r_t = r(n+1,x) = (B^* - g)/B^*$, $\forall\, t \geq 1$ will be Pareto superior to any equilibrium of the divisible asset economy. For the initial old generation this is immediate from $B_0 = B_1 - g$ and $B^* > \bar{B}$. For young agents, utility at any date for any equilibrium of the divisible asset economy cannot exceed $u(w_1 - \bar{B}) + v[w_2 + \bar{B}f^{-1}(\bar{B})]$. For a steady state equilibrium of the divisible asset economy as specified, the utility of young agents is $u[w_1 - (n+1)x] + v[w_2 + r(n+1,x)(n+1)x] = u(w_1 - B^*) + v(w_2 + B^* - g)$. $u(w_1 - B) + v(w_2 + B - g)$ is maximized at $B = F$, and is increasing, $\forall B < F$. Thus, $u(w_1 - B^*) + v(w_2 + B^* - g) > u(w_1 - \bar{B}) + v[w_2 + \bar{B} - g] = u(w_1 - \bar{B}) + v[w_2 + \bar{B}f^{-1}(\bar{B})]$ by (19). Therefore, a steady state equilibrium of the type sought will have the configuration depicted in Figure 4, and will constitute a Pareto improvement relative to any equilibrium of the divisible asset economy.

We now show that it is possible to find an integer n, and values B^* and x, satisfying (17) - (19).

Proposition 2. There exists an integer n, a value $x > 0$, and a value B^* such that (17) - (19) hold.

Proof. $r(n+1,x)$ is defined by (6). Substituting (17), (18), and $nx = [n/(n+1)](n+1)x = [n/(n+1)]B^*$ into (6) gives

$$(20) \qquad u(w_1 - B^*) + v(w_2 + B^* - g) = u\left[w_1 - \left(\frac{n}{n+1}\right)B^*\right] + v\left[w_2 + \left(\frac{n}{n+1}\right)(B^* - g)\right],$$

or

$$(20') \quad H[n/(n+1), B^*] = 0.$$

We now show that a positive integer n and a value $B^* \in (\bar{B},F]$ can be chosen so that (20′) holds. Then setting $x = B^*/(n+1)$ satisfies (17).

To establish the result, there are two possibilities to be considered.

<u>Case 1</u>. $H[n/(n+1),F] = 0$ holds for some (finite) integer n. Then we can choose $B^* = F$.

Case 2. $H(k,F) = 0$ holds for some $k \in (0,1)$ but $k \neq n/(n+1)$ for any integer n. In this case, there exists a finite positive integer $\hat{n}$ such that $\hat{n}/(\hat{n} + 1) > k$. Then (see Figure 5), $H[\hat{n}/(\hat{n}+1),F] < 0$. Moreover, we claim that $H(k,\bar{B}) > 0$, $\forall\, k \in [0,1]$. This follows from the fact that $H_1(1,\bar{B}) = 0$, $H_{11}(1,\bar{B}) < 0$, and $H(1,\bar{B}) = 0$. Thus $H[\hat{n}/(\hat{n}+1), \bar{B}] > 0$. Therefore, by the intermediate value theorem, there exists a value $B^* \in (\bar{B},F)$ such that $H[\hat{n}/(\hat{n}+1), B^*] = 0$. But this gives us suitable values B^* and $\hat{n}$, establishing the result.

We have established that a minimum denomination can be chosen that supports a steady state equilibrium satisfying $B_t = B^*$ and $r_t = r(n+1,x)$, $\forall\, t \geq 1$, and $B^* = (n+1)x$. Moreover, it can be chosen so that B^*,n, and x satisfy (17) - (19). In fact, defining k(F) implicitly by $H[k(F),F] = 0$, for every finite integer n with $n/(n+1) > k(F)$, we can find an associated pair (B^*,x). We now show that there is an optimal triple (B^*,n,x) that satisfies (17) - (19).

B. A "Second-Best" Denomination Structure

We now demonstrate the existence of values n (a finite positive integer), $x > 0$, and B^* that yield a maximum level of utility to young agents among all steady state equilibria that have $B_t = B^* = n(B^*)x$ [i.e., satisfy (17)] and $r_t = r(n+1,x)$, $\forall\, t \geq 1$, with B^*,n, and x satisfying (18). [We do not necessarily impose (19).] We thus establish the existence of a best (for young agents) steady state equilibrium of the type depicted in Figure 4. The argument proceeds in two parts. We first establish the existence of an optimal steady state with the properties described, and with B^*,n, and x satisfying (17) - (19). We then relax (19) to obtain an optimum.

We begin by observing that if $B^* = (n+1)x$ and $r_t = r(n+1,x)$, and if (18) holds, then young agent utility at t is $u(w_1 - B^*) + v(w_2 + B^* - g)$. This expression is increasing in B^* $\forall\, B^* \in [0,F)$, decreasing in B^* $\forall\, B^* \in (F,w_1]$, and is maximized at $B^* = F$. Second, define k(B) implicitly by

$H[k(B),B] = 0$ and $k(B) < 1$. Then it is easy to show that, $\forall\, B \in (\bar{B},w_1]$, $k(B)$ exists and is unique.[9] Moreover, $0 < k(B) < Q(B) < 1$, with $Q(B)$ defined as in (16). Then, from Lemma 5, $H_1[k(B), B] < 0$ (see Figure 5). Thus, by the implicit function theorem, $k(B)$ is a continuous function on $(\bar{B},w_1)$.

We first show that B^*,n, and x can be chosen optimally among values satisfying (17) - (19). We have two cases to consider. In the first, $k(F) = n/(n+1)$ for some finite integer n. Then we can set $B^* = F$, $n = k(F)/[1-k(F)]$, and $x = B^*/(n+1)$ and obtain a first-best optimum. Of course generically $k(F) \neq n/(n+1)$ for any integer n. We then construct the desired optimum as follows.

Let $\hat{n}$ be a finite positive integer with $\hat{n}/(\hat{n}+1) > k(F)$. Then for $N \geq \hat{n}$ consider the following problem:

(P.N) max B subject to

(21) $k(B) \in \{k\colon k = n/(n+1);\ n \text{ a positive integer};\ n \leq N\}$,

(22) $B \in [\bar{B},F]$.

Since $N/(N+1) > k(F)$ the constraint set is non-empty (by Proposition 2) and it is clearly compact. Thus (P.N) has a solution, which we denote B_N.

Obviously, $\{B_N\}$ is a non-decreasing sequence which is bounded above (by F). Then $\{B_N\}$ is convergent; say $\lim B_N = B^*$. We now show that $B_{N*} = B^*$ for some finite integer N^*. Then $B_{N*} = B^*$ is the largest value of B (in the interval $[\bar{B},F]$) that can be supported as a steady state

---

9 Existence follows from $H(0,B) > 0 > H[Q(B),B]$. Uniqueness follows from $H_1(k,B) < 0$, $\forall\, k < Q(B)$, and from $H(k,B) < 0$, $\forall\, k \in [Q(B),1)$.

equilibrium satisfying the desired properties. $x = B^*/(N^*+1)$ is then the (second-best) optimal minimum denomination satisfying (17) - (19).

Proposition 3. There exists a finite integer N^* with $B_{N*} = B^*$.

Proof. The proof is by contradiction. If the proposition is false, then $B_N < B^*$ for all finite N. We then construct a subsequence $\{B_{Nj}\}$ as follows. Let N_1 be a finite integer with $N_1/(N_1+1) > k(F)$. By hypothesis, $B_{N1} < B^*$. Moreover, there exists an integer $N_2 > N_1$ such that $N_2/(N_2+1) > 2^{-(1/N1)}$ and $B_{N2} \in (B_{N1},B^*)$. Similarly, there exists an integer $N_3 > N_2$ such that $N_3/(N_3+1) > 2^{-(1/N2)}$ and $B_{N3} \in (B_{N2},B^*)$. Proceeding in this way we obtain a sequence of integers $\{N_j\}$ with $N_{j+1} > N_j$ and $B_{Nj} < B_{Nj+1} < B^*$, $\forall$ j. Moreover, $k(B_{Nj+1}) > k(B_{Nj})$, $\forall$ j [since otherwise B_{Nj+1} would be a feasible solution to the problem $(P.N_j)$].

By construction,

(23) $$k(B_{Nj+1}) > N_j/(N_j+1)$$

$\forall$ j [since otherwise B_{Nj+1} would again be a feasible solution to the problem $(P.N_j)$]. Also $N_j/(N_j+1) \rightarrow 1$, so $\lim k(B_{Nj}) \rightarrow 1$ [since $k(B_{Nj})$ is an increasing sequence bounded above by 1].

Now recall that the function Q(B) defined by (16) is continuous, and $Q(B) < 1$, $\forall\, B > \bar{B}$. Thus $Q(B^*) < 1$, and there exists a value $\epsilon > 0$ such that $Q(B^*) + \epsilon < 1 - \epsilon$. Also, since $\{B_{Nj}\}$ converges to B^*, $\{Q(B_{Nj})\}$ converges to $Q(B^*)$. There therefore exists a value J such that $Q(B_{Nj}) < Q(B^*) + \epsilon$ and $k(B_{Nj}) > 1 - \epsilon$ $\forall\, j > J$. But $k(B_{Nj}) < Q(B_{Nj})$, $\forall$ j, giving the desired contradiction.

Thus, there exists a finite integer N^* with $B_{N*} = B^*$, and a (second-best) optimal minimum denomination can be constructed as described that supports a steady state equilibrium such as B^* in Figure 4, with $B^* \leq F$.

If $B^* < F$ holds, the possibility exists that there exists a value $\hat{B} > F$, a finite positive integer n, and a value $x > 0$ such that

$$(17') \quad (n+1)x = \hat{B}$$

$$(18') \quad r(n+1,x) = (\hat{B}-g)/\hat{B}$$

$$(19') \quad F < \hat{B} \leq w_1$$

and $u(w_1 - \hat{B}) + v(w_2 + \hat{B} - g) > u(w_1 - B^*) + v(w_2 + B^* - g)$. We now investigate this possibility.

For values of $\hat{B}$ satisfying (19), the utility of young agents is decreasing in $\hat{B}$. We thus consider the problem

$$(P'.N) \qquad \min B \text{ subject to (21) and}$$

$$(24) \quad B \in [F,w_1] \ ,$$

where $N/(N+1) > k(F)$. For any N the constraint set for this problem may be empty. If it is empty for all integer N then B^* as constructed above is a (second-best) optimum. If it is non-empty, (say for $N \geq \hat{N}$), then the constraint set is compact, so (P'.N) has a solution. Denote this solution by $\hat{B}_N$.

Obviously $\{\hat{B}_N\}$ is a non-increasing sequence, and is bounded below (by F). Therefore it converges, say to $\hat{B}$. Moreover, the same argument as in the proof of proposition 3 establishes that $\hat{B}_{N'} = \hat{B}$ for some finite integer N'. Then if $u(w_1 - B^*) + v(w_2 + B^* - g) > u(w_1 - \hat{B}) + v(w_2 + \hat{B} - g)$, B^*, N^*, and $x = B^*/(N^* + 1)$ constitute the (second best) optimal set of values sought. If $u(w_1 - \hat{B}) + v(w_2 + \hat{B} - g) \geq u(w_1 - B^*) + v(w_2 + B^* - g)$, then $\hat{B}$, N', and $x = \hat{B}/(N' + 1)$ are the optimal set of values.[10]

We have now established the existence of a best (for young agents) steady state equilibrium that has the configuration depicted in Figure 4. We now consider steady state equilibria that have $B_t = B^*$ and $r_t = r[n(B_t),x]$, $\forall\ t \geq 1$, but that have $B^* < n(B^*)x$, i.e., steady state equilibria that are locally stable.

IV. The Non-existence of an Optimal, Locally Stable, Steady State Equilibrium

Two classes of steady state equilibria avoid the indeterminacy problem discussed in section II.B; steady state equilibria of the type discussed in section III, and locally stable steady state equilibria. Locally stable steady state equilibria that are Pareto superior to any equilibrium of the divisible asset economy have $B_t = \tilde{B}$ and $r_t = r[n(\tilde{B}),x]$ $\forall\ t \geq 1$, with $\tilde{B} < n(\tilde{B})x$, $r[n(\tilde{B}),x] = (\tilde{B} - g)/\tilde{B}$, and $\tilde{B} > \bar{B}$.[11]

In this section, we demonstrate that there is no optimal (for young agents), locally stable steady state equilibrium. This is now stated as

---

[10] If $u(w_1 - B^*) + v(w_2 + B^* - g) = u(w_1 - \hat{B}) + v(w_2 + \hat{B} - g)$, $\hat{B}$, N', and $x = \hat{B}/(N' + 1)$ constitute a Pareto superior set of values to B^*, N^*, and $x = B^*/(N^*+1)$, since $\hat{B} > B^*$. Note also that, if $\hat{B}$ exists, the choices $\hat{B}$, N', and $x = \hat{B}/(N'+1)$ result in a Pareto optimal steady state equilibrium, since $\hat{B} > F$.

[11] Local stability follows from the fact that $B_{t+1} = r[n(\tilde{B}),x]B_t + g$, $\forall\ B_t$ in a neighborhood of $\bar{B}$.

Proposition 4. Let $x > 0$ be given, and suppose that a steady state equilibrium exists satisfying

$$(25) \quad B_t = B(x) < n[B(x)]x, \quad \forall\, t \geq 1,$$

and

$$(26) \quad r\{n[B(x)],x\} = [B(x) - g]/B(x) = r_t, \quad \forall\, t \geq 1,$$

where $B(x)$ denotes the (a) steady state value of B_t when the minimum denomination is x. Then there exists a value $x' > x$ such that

$$(27) \quad B(x') > B(x)$$

$$(28) \quad n[B(x')] = n[B(x)]$$

$$(29) \quad B_t = B(x') < n[B(x')]x', \quad \forall\, t \geq 1,$$

and

$$(30) \quad r_t = r\{n[B(x')],x'\} = [B(x') - g]/B(x'), \ \forall\, t \geq 1.$$

Moreover, young agents prefer the second steady state equilibrium to the first.

Equations (25) and (26) assert that $B(x)$ constitutes a steady state equilibrium for the minimum denomination x, with some agents saving $n[B(x)]x$ and some agents saving $\{n[B(x)] - 1\}x$. (29) and (30) assert the same thing for the minimum denomination x', as well as implying that the new steady state equilibrium is locally stable. (28) implies that $n[B(x)]x'$ and $\{n[B(x)] - 1\}x'$ are the optimal savings levels under the new equilibrium (so young agents continue to save the same integer multiples of the new minimum denomination as they did previously). (27) implies that raising the minimum denomination increases the welfare of the initial old, so that a Pareto improvement is achieved by this increase.

Proof. For any $x > 0$, if $B(x)$ is a locally stable state equilibrium value for B_t, then $B(x)$ satisfies

$$(31) \qquad B(x) = g/\{1 - r[n(B(x)),x]\},$$

with $r\{n[B(x)],x\}$ given by (6). Then if $B(x) < n[B(x)]x$, $B(x)$ is differentiable, and $B'(x) = gr_2(n,x)/[1 - r(n,x)]^2 > 0$ (where the inequality follows from Lemma 3).[12] Thus increasing x locally to x' results in $B(x') > B(x)$. Moreover, this increase can be accomplished while satisfying (28), and hence (29). And, since (31) is equivalent to (30), (30) obviously holds.

Thus we can raise x to x' locally, have $B_t = B(x')$ and $r_t = r\{n[B(x')],x'\}$, $\forall\ t \geq 1$, constitute a locally stable steady state equilibrium, and retain $n[B(x)]$ and $n[B(x)] - 1$ as the optimal integer multiples of the minimum denomination for young agents to save. It only remains, then, to

[12] In particular, $n[B(x)]$ is not changed by changing x locally.

show that local increases in x raise the utility of young agents. The utility of young agents who save nx at the interest rate r(n,x) is [13]

(32) $$W(n,x) = u(w_1 - nx) + v[w_2 + r(n,x)nx] \ .$$

Then, since the optimal value of n is left unchanged by small changes in x, the change in utility associated with a local change in x is given by

$$W_2(n,x) = -u'(w_1 - nx)n + v'[w_2 + r(n,x)nx]\,[nr(n,x) + nxr_2(n,x)] \ .$$

Clearly, then, $W_2(n,x) > 0$ iff

(33) $$r_2(n,x) > -r(n,x)/x + u'(w_1 - nx)/xv'[w_2 + r(n,x)nx].$$

However (33) follows directly from (a.4) in the Appendix.

Therefore local increases in x increase the utility of young agents, establishing the result. We have now shown that any locally stable steady state equilibrium can be dominated in a Pareto sense by an alternate, locally stable steady state equilibrium supported by a larger minimum denomination. There is therefore no optimal, locally stable steady state equilibrium.

---

[13] By definition young agents are indifferent between saving nx and (n-1)x, so either can be used on the right-hand side of (32).

V. Conclusions

In view of the results of Bryant and Wallace (1984), it is not surprising that a minimum denomination for government liabilities can be chosen to support any stationary Pareto optimal allocation as a steady state equilibrium. However, for any specified allocation, the associated equilibrium will generically be unstable. And, even if the economy reaches such an unstable steady state, no forces operate to keep the economy there. Furthermore, if the economy departs from an optimal steady state equilibrium that is unstable, it will converge to a locally stable steady state, and therefore (by Proposition 4) to an allocation that can be improved upon in a Pareto sense.

We have demonstrated that minimum denominations exist that support steady state equilibria that avoid this problem, and that are Pareto superior to all equilibria that exist when outside assets are perfectly divisible. Moreover, we have shown that an optimal minimum denomination (in this class) exists in a restricted sense: an optimal minimum denomination exists among the set of denominations that support steady state equilibria such as B^* in Figure 4. Finally, we have established that, if an optimal minimum denomination exists at all, it must support a steady state equilibrium of this type. We have not shown, however, that the (second-best) optimal minimum denomination derived in section III cannot be dominated by a minimum denomination that supports a locally stable steady state equilibrium. At this point this must remain a topic for future investigation.

Another topic for future investigation would concern extrinsic uncertainty in this context. There are at least two interesting sources of extrinsic uncertainty that merit investigation. One would involve having the returns on government liabilities be determined randomly via a lottery mechanism. Such lotteries on government liabilities issued in relatively large minimum denominations have been common historically,[14] and may be useful here in supporting steady state equilibria that have desired welfare properties but avoid the indeterminacy problems discussed in section II.B. A second would involve sunspot equilibria of the type described by Shell (1977), Azariadis (1981), and Cass and Shell (1983). Both the potential stability of steady state equilibria and the relationship between indivisibilities and sunspot equilibria examined by Shell and Wright (1989) suggest the possibility of sunspot equilibria in worlds with indivisible assets. Developing the apparatus to examine sunspot equilibria in the presence of assets of limited divisibility would be an interesting area for further research.

[14] "Lottery bonds" were issued during the American Revolution, and by the French government at various times during the 18th century. For a discussion see Anderson ( ) and Weir and Velde (1989). The use of tontines by governments was also common in the 18th century, which is obviously another device for confronting a bond-holder with a random return stream. The use of tontines is described by Weir (1989).

APPENDIX

A. Proof of Lemma 1.

By hypothesis,

$$u(w_1 - n_2x) + v(w_2 + r_t'n_2x) \geq u(w_1 - n_1x) + v(w_2 + r_t'n_1x).$$

Define $\Psi(n_1, n_2, r)$ by

$$\Psi(n_1,n_2,r) \equiv u(w_1 - n_2x) + v(w_2 + rn_2x) - u(w_1 - n_1x) - v(w_2 + rn_1x).$$

Then $\Psi(n_1, n_2, r_t') \geq 0$, and

$$\Psi_3 = v'(w_2 + rn_2x)n_2x - v'(w_2 + rn_1x)n_1x.$$

Since (A.2) implies that $v'(w_2 + z)z$ is an increasing function of z, we have that $(n_2 - n_1)\ \Psi_3 > 0$, establishing the result.

B. Proof of Lemma 2.

(a) By definition of $f^{-1}(nx)$ and the strict concavity of agents' utility functions,

(a.1) $\quad u(w_1 - nx) + v[w_2 + nxf^{-1}(nx)] > u[w_1 - (n\text{-}1)x] + v[w_2 + (n\text{-}1)x\, f^{-1}(nx)].$

Similarly

(a.2) $u[w_1 - (n-1)x] + v\{w_2 + (n-1)xf^{-1}[(n-1)x]\} > u(w_1 - nx) + v\{w_2 + nxf^{-1}[(n-1)x]\}$.

Then by the intermediate value theorem there exists a value $r(n,x) \in (f^{-1}[(n-1)x], f^{-1}(nx))$ satisfying (6).

(b) We suppose that $r(n,x)$ exists but $f^{-1}[(n-1)x]$ does not, and derive a contradiction. If $r(n,x)$ exists but $f^{-1}[(n-1)x]$ does not, then $f[r(n,x)] < (n-1)x$, and of course $(n-1)x < nx$. Now define the function $G(z,y)$ by

$$G(z,y) \equiv u(w_1 - z) + v[w_2 + r(n,x)z] - u(w_1 - y) + v[w_2 + r(n,x)y].$$

Clearly $G(y,y) = 0$, and, $\forall\, z > f[r(n,x)]$,

$$G_1(z,y) = -u'(w_1 - z) + r(n,x)v'[w_2 + r(n,x)z] < 0.$$

Therefore $G[nx, (n-1)x] < G[(n-1)x, (n-1)x] = 0$. However, the assumption that $r(n,x)$ exists means (by definition) that $G[nx, (n-1)x] = 0$, giving the desired contradiction.

It follows from Lemma 1 that $f^{-1}[(n-1)x] < r(n,x)$.

C. <u>Proof of Lemma 3</u>

Differentiating (6) with respect to x gives

$$\text{(a.3)} \qquad r_2(n,x) = -r(n,x)/x + \frac{u'(w_1 - nx)n - u'[w_1 - (n-1)x](n-1)}{v'[w_2 + r(n,x)nx]nx - v'[w_2 + r(n,x)(n-1)x](n-1)x}$$

However, it is easy to check that

$$u'(w_1 - nx)/v'[w_2 + r(n,x)nx] \leq \frac{u'(w_1 - nx)n - u'[w_1 - (n-1)x](n-1)}{v'[w_2 + r(n,x)nx]n - v'[w_2 + r(n,x)(n-1)x](n-1)}$$

It then follows that

$$\text{(a.4)} \qquad r_2(n,x) \geq -r(n,x)/x + u'(w_1 - nx)/xv'[w_2 + r(n,x)nx]$$

However, since $nx > f[r(n,x)]$ in order for (6) to hold,

$$\text{(a.5)} \qquad u'(w_1 - nx) > r(n,x)v'[w_2 + r(n,x)nx].$$

Thus $r_2(n,x) > 0$.

D. Proof of Lemma 4.

Suppose $\hat{n}x$, with $\hat{n} \neq (n-1), n$, is a (weakly) preferred savings level, and suppose $\hat{n} > n$. Then

$$\text{(a.6)} \qquad u(w_1 - \hat{n}x) + v[w_2 + r(n,x)\hat{n}x] \geq u[w_1 - (n-1)x] + v[w_2 + r(n,x)(n-1)x].$$

Moreover, there exists a value $\lambda \in (0,1)$ such that $\lambda\hat{n} + (1-\lambda)(n-1) = n$. Therefore, since v is strictly concave, $u(w_1 - nx) + v[w_2 + r(n,x)nx] > u[w_1 - (n-1)x] + v[w_2 + r(n,x)(n-1)x]$. But this contradicts the definition of r(n,x). A similar contradiction results if $0 \leq \hat{n} \leq n-1$.

E. Proof of Lemma 5.

(a) Continuity is obvious. Also,

$$\text{(a.7)} \quad H_1(k,B) = u'(w_1 - kB)B - v'(w_2 + kB - kg)\,(B-g)$$

and

$$\text{(a.8)} \quad H_{11}(k,B) = -u''(w_1 - kB)B^2 - v''(w_2 + kB - kg)\,(B - g)^2 > 0,$$

establishing the remainder of (a).

(b) Obvious.

(c) H(0,B) > 0 iff

$$\text{(a.9)} \quad u(w_1 - B) + v(w_2 + B - g) > u(w_1) + v(w_2).$$

For B < F, $u(w_1 - B) + v(w_2 + B - g)$ is an increasing function of B. Moreover, since $\bar{B}$ is voluntarily held at the interest rate $f^{-1}(\bar{B})$ in the divisible asset economy,

$$u(w_1 - B) + v(w_2 + B - g) \geq u(w_1 - \bar{B}) + v(w_2 + \bar{B} - g) =$$
$$u(w_1 - \bar{B}) + v[w_2 + \bar{B}f^{-1}(\bar{B})] > u(w_1) + v(w_2),$$

$\forall\, B \in [\bar{B},F]$, establishing the result.

(d) By part (a), $H(k,B)$ attains a minimum for $k \in (0,1)$ iff

$$\text{(a.10)} \qquad u'(w_1 - kB) = \left[\frac{B-g}{B}\right] v'(w_2 + kB - kg)$$

holds for some $k \in (0,1)$. (a.10) is equivalent to $kB = f[(B-g)/B]$, or to

$$\text{(a.11)} \qquad k = f[(B-g)/B]/B \equiv Q(B)$$

for some $k \in (0,1)$.

Now $f[(\bar{B}-g)/\bar{B}] = f[f^{-1}(\bar{B})] = \bar{B}$, so $Q(\bar{B}) = 1$. We now demonstrate that $Q(B) < 1$, $\forall\, B > \bar{B}$. To see this, note that by definition, $\bar{B}$ is the largest solution to

$$\text{(a.12)} \qquad B[1 - f^{-1}(B)] = g.$$

(a.12) can be rewritten as

$$\text{(a.12}'\text{)} \qquad f\left[\frac{B-g}{B}\right] = B,$$

so $\bar{B}$ is the largest solution to (a.12$'$). It then follows that

$$f\left(\frac{B-g}{B}\right) < B \tag{a.13}$$

for all $B > \bar{B}$, or that $Q(B) < 1$, $\forall\ B > \bar{B}$. This establishes the result.

REFERENCES

Anderson, William G., *The Price of Liberty: The Public Debt of the American Revolution*, University of Virginia Press, Charlottesville.

Azariadis, Costas, "Self-Fulfilling Prophecies," *Journal of Economic Theory* (25), 1981, pp. 380-96.

Bryant, John and Neil Wallace, "A Price Discrimination Analysis of Monetary Policy," *Review of Economic Studies* (51), 1984, pp. 279-88.

Carothers, Neil, *Fractional Money*, Wiley and Sons, New York, 1930.

Cass, David and Karl Shell, "Do Sunspots Matter?" *Journal of Political Economy* (91), 1983, pp. 193-227.

Cooley, Thomas F. and Bruce D. Smith, "Indivisible Assets, Equilibrium, and the Value of Intermediary Output," manuscript, 1990.

Farmer, Roger, E.A., "Bursting Bubbles: on the Rationality of Hyperinflations in Optimizing Models," *Journal of Monetary Economics* (14), 1984, pp. 29-35.

Glassman, Debra and Angela Redish, "Currency Depreciation in Early Modern England and France," *Explorations in Economic History* (25), 1988, pp. 75-97.

Hanson, John R. II, "Money in the Colonial American Economy: an Extension," *Economic Inquiry* (17), 1979, pp. 281-6.

Hanson, John R. II, "Small Notes in the American Colonies," *Explorations in Economic History* (17), 1980, pp. 411-20.

Kemmerer, Edwin W., *Seasonal Variations in the Relative Demand for Money and Capital in the United States*, National Monetary Commission, Washington, 1910.

Marimon, Ramon and Neil Wallace, "Trade Using Assets Divisible at a Cost," *Journal of Economic Theory* (43), 1987, pp. 223-51.

Rolnick, Arthur J. and Warren E. Weber, "Gresham's Law or Gresham's Fallacy," *Journal of Political Economy* (94), 1986, pp. 185-99.

Ross, Victor, *A History of the Canadian Bank of Commerce*, Vol. II, Oxford University Press, Toronto, 1922.

Shell, Karl, "Monnaie et Allocation Intertemporelle," manuscript, 1977.

Shell, Karl and Randall Wright, "Indivisibilities, Lotteries and Sunspot Equilibria," manuscript, 1989.

Sumner, William Graham, *The Financier and the Finances of the American Revolution*, Augustus M. Kelley, New York, 1891.

Villamil, Anne P., "Price Discriminating Monopoly: a Non-uniform Pricing Approach," *Journal of Public Economics* (53), 1988, pp. 385-92.

Weir, David R., "Tontines, Public Finance, and Revolution in France and England, 1688-1789," *Journal of Economic History* (49), 1989, pp. 95-124.

Weir, David, R. and Francois Velde, "The Financial Market and Government Debt Policy in France, 1750 - 1793," manuscript, 1989.

White, Eugene, "Free Banking, Denominational Restrictions, and Liability Insurance," Manuscript, 1990.

Figure 1

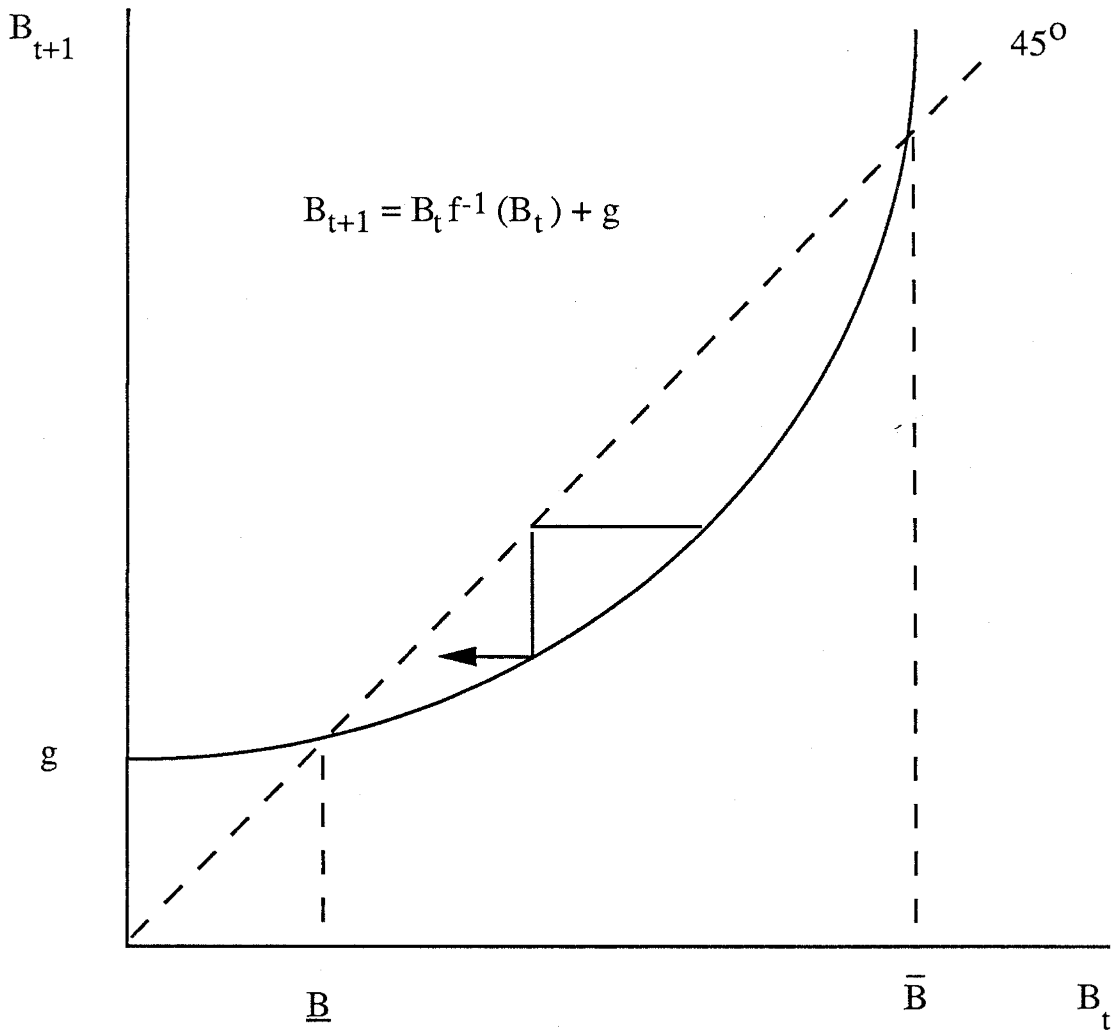

Figure 2

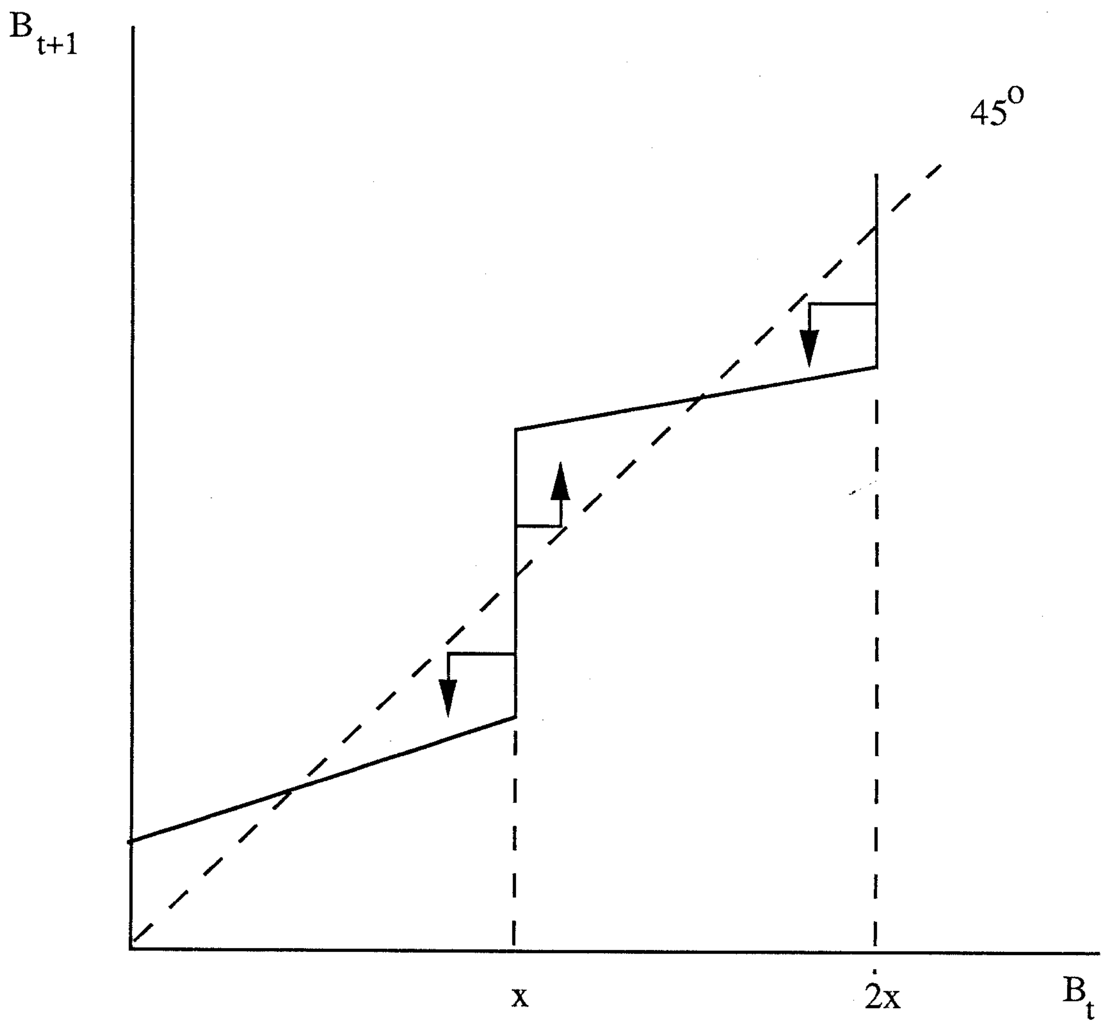

Figure 3

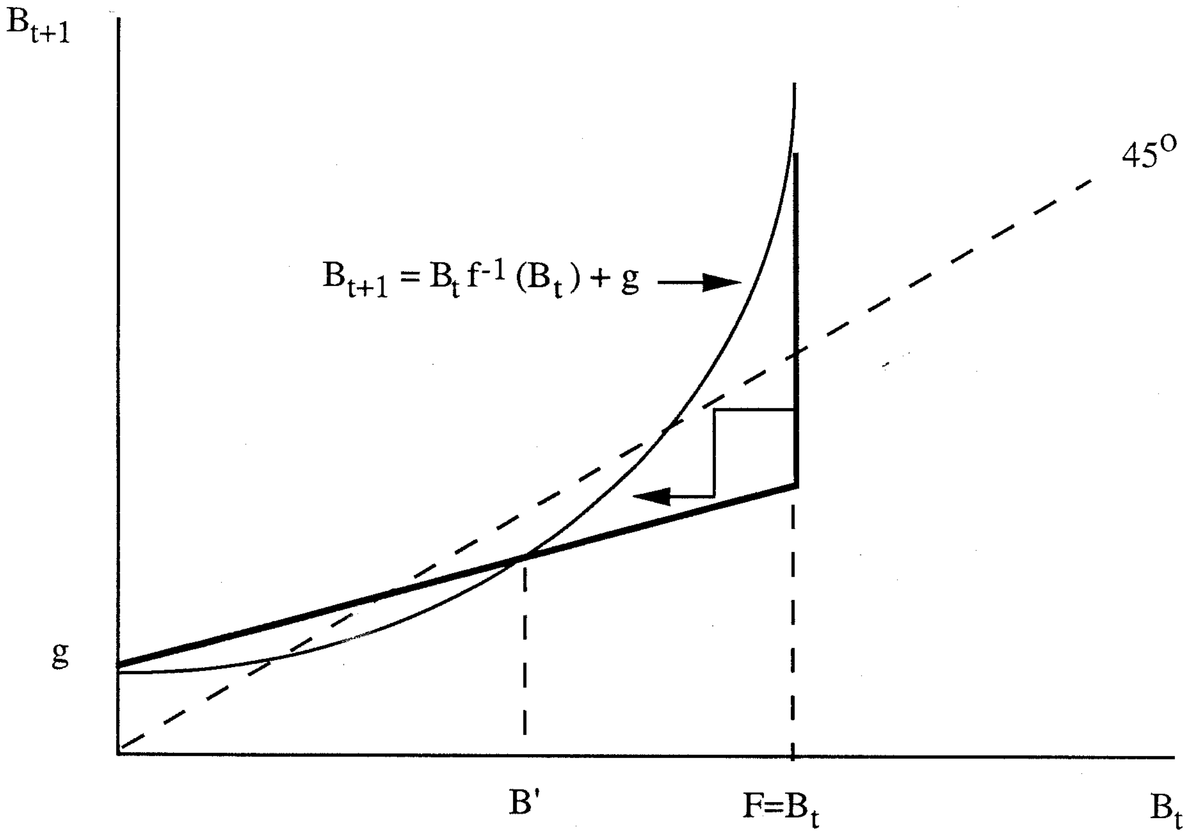

Figure 4

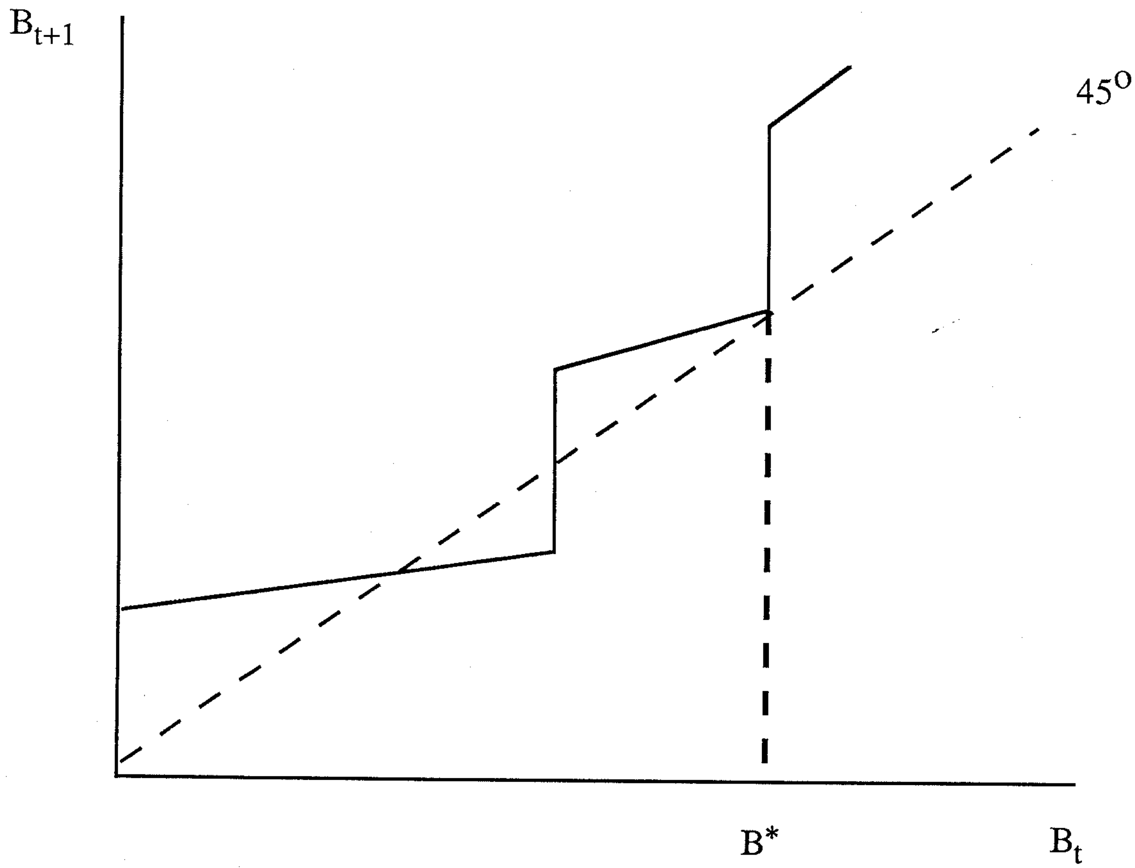

Figure 5