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ABSTRACT

Consider a heterogeneous but divisible commodity, bundles of which are represented by the (measurable) subsets of the good. One such commodity might be land. The mathematics literature has considered agents with utilities that are nonatomic measures over the commodity (and hence are additive). The existence of "α–fair" allocations, in which each agent receives a utility proportional to his utility of the endowment of the entire economy, was demonstrated there. Here we extend these existence results to α–fair efficient allocations, envy–free allocations, envy–free efficient allocations, group envy–free and nicely shaped allocations of these types. We examine utilities that are not additive and relate the mathematics literature to the economics literature. We find sufficient conditions for the existence of egalitarian–equivalent efficient allocations. Finally, we consider the problem of allocating a time interval (uses of a facility). Existence of an envy–free allocation had been demonstrated in earlier literature. We show that any envy–free allocation is efficient as well as group envy–free. This last result is extended to a more general setting.

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I. ***Introduction.*** A farmer dies leaving instructions to divide his land fairly among his sons. A land reform law stipulates that each latifundia is to be divided fairly among all the farmers in a village. Several communities undertake a drainage of swamps and face the problem of dividing the reclaimed land among themselves. How should such divisions be carried out?

Somewhat more generally, consider a "heterogeneous good" that can be divided in a variety of ways among a group of agents with equal claims on it. Imagine each agent to be endowed with a preference relation over the various admissible subsets. How can the equity of a division be defined and can equitable divisions be achieved?

This problem has been addressed in the mathematics literature, in the following abstract formulation which we describe using standard economic terminology. The good to be divided is modeled as a measure space. There are n agents whose preferences are defined over the measurable subsets. Each agent's preference relation is representable by a function that is a non–atomic measure. A vector of distributional coefficients α in the (n–1)–dimensional simplex is given. The search is for a partition such that, using these representations, the utility of each agent i is at least α_i times the utility he would enjoy from consuming the whole amount available. The existence of such allocations is demonstrated.

This literature will be our point of departure. In spite of its great mathematical generality and elegance, it suffers from several limitations. First, it does not address at all the issue of efficiency, seriously limiting its relevance to economists. This requirement will be imposed throughout in this paper.

Second, it does not attempt to allow for general preferences of the kind that are standard in economics. Instead, preferences are required to have additive numerical representations. We will explain later that allowing arbitrary preferences would be unproductive, but one of our objectives here is to investigate how far one can depart from this case. We will first work with preferences that do have additive representations but we will also present some results on the non-additive case. These require that preferences admit representations that exhibit some form of "decreasing marginal utility", a condition that should be particularly appealing to economists.

Finally, the mathematics literature is unduly narrow in the specification of the equity criteria that it considers. There now exists in economics a well-developed literature devoted to the formulation and the analysis of equity concepts. The concept that has played the central role is that of an envy-free allocation, that is, an allocation such that nobody would prefer what someone else received to what he received. We will analyze the question of existence of envy-free and efficient allocations in the present model. However, other concepts have been found useful and we will consider several of them as well. Indeed, we believe that the application of these concepts to the problem of dividing a heterogeneous commodity is long overdue.

In Section II, we specify the model and state the basic definitions. Most of the focus of this paper is on the no-envy concept and variants of it. In Section III, we consider countably additive utilities and in Section IV, more general ones. In Section V, we investigate the existence of partitions satisfying the property of egalitarian-equivalence, one of the main competing notions. In Section VI, we give conditions on the model sufficient for all envy-free allocations to be efficient. In Section VII, we conclude.

II. *The Model.* We consider the division of a plot of land among a group of agents with equal claims on it. We model this plot of land as a measurable subset L of the Euclidean space $\mathbb{R}^k$. Let m be Lebesgue measure on $\mathbb{R}^k$. $\mathscr{B}$ is the σ-algebra of measurable subsets of L, representing the possible parcels into which it can be divided. Capital letters denote elements of $\mathscr{B}$. Given $A \in \mathscr{B}$, the topological closure of A in $\mathbb{R}^k$ is denoted by $\bar{A}$, and its topological boundary by ∂A. $A, B \in \mathscr{B}$ are called ***adjacent*** if

$\partial A \cap \partial B$ contains a homeomorphic image of $(0,1)^{k-1}$.[1]

There are $n \in \mathbb{N}$ agents. Each agent i, for i=1,...,n, is endowed with a preference relation R_i over $\mathscr{B}$ Let I_i denote the indifference relation associated with R_i and P_i the strict preference relation. Let $R = (R_1,...,R_n)$ be the list of preference relations. Let Π^n be the set of n–element measurable partitions of L, or simply ***partitions***. A partition $B = (B_1,...,B_n) \in \Pi^n$ is (Pareto–)***efficient for R*** if there is no other partition $A = (A_1,...,A_n) \in \Pi^n$ such that $A_i R_i B_i$ for all i, with strict preference holding for at least one i. Assume that each R_i can be represented by a "utility function" $u_i: \mathscr{B} \to \mathbb{R}$. Let $u = (u_1,...,u_n)$ be a list of such utility representations. A sequence of partitions $(B^t)_{t=1}^{\infty} \in \Pi^n$ is ***limit efficient for R*** if for every $\epsilon > 0$ there exists $t^* \in \mathbb{N}$ such that for all $t > t^*$, there does not exist $A \in \Pi^n$ such that for all i, $u_i(A_i) - u_i(B_i^t) > \epsilon$. Note that this definition indeed does not depend on the utility representations.

We now present our main equity notion.

Definition. A partition $B \in \Pi^n$ is ***envy–free for R*** if for all i and j, $B_i R_i B_j$.

Thus, a partition is envy–free if no agent would prefer someone else's parcel to his own. This concept is the central one in the economics literature (to our knowledge, it has not been used at all in the mathematics literature). It was proposed initially by Foley (1967), and later developed by Kolm (1972), Varian (1974), and many others. For a recent survey of this literature, see Thomson (1989). A useful weakening of the concept is the following.

Definition. A sequence of partitions $(B^t)_{t=1}^{\infty} \in \Pi^n$ is ***limit–envy–free for R*** if for every $\epsilon > 0$ there exists $t^* \in \mathbb{N}$ such that for all $t > t^*$ and for all i and j, $u_i(B_j^t) - u_i(B_i^t) < \epsilon$.

We will also consider the following utility–based notion.[2]

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[1]Much of this notation is taken from Hill (1983).

[2]It is known under the name "fair", in the mathematics literature, but we avoid this

Let $\Delta^{n-1} = \{\alpha \in \mathbb{R}^n_+ | \Sigma\alpha_i = 1\}$ be the (n–1)-dimensional simplex.

Definition. Given $\alpha \in \Delta^{n-1}$, a partition $B \in \Pi^n$ is ***α-fair for u*** if for all i, $u_i(B_i) \geq \alpha_i \cdot u_i(L)$.

Here, each agent is required to receive at least a given fraction of the utility he would derive from consuming the whole amount available, the fractions being required to sum to one. This notion was developed by Borsuk (1933), Stone and Tukey (1942), Steinhaus (1949), Dubins and Spanier (1961), and Hill (1983). It seems to be the normative standard in the mathematics literature.

Two more criteria will be used below. Note that they depend only on preferences.

Definition. A partition $B \in \Pi^n$ is ***group envy-free for R*** if for every pair of groups of agents C_1, C_2 with $|C_1| = |C_2|$ there is no partition $\{A_i\}_{i \in C_1}$ of $\bigcup_{i \in C_2} B_j$, such that $A_i R_i B_i$ for all $i \in C_1$ with at least one strict preference.

This definition is adapted from Schmeidler and Vind (1972). If an allocation is group envy-free, it is of course envy-free and efficient (take C_1 and C_2 of cardinality one to establish the first property, and take $C_1 = C_2 = N$ to establish the second one). If the reader finds it more natural to only compare the welfare of distinct groups, or perhaps of non-overlapping groups, then efficiency should be required separately.

Definition. A partition $B \in \Pi^n$ is ***egalitarian-equivalent for R*** if there is some measurable "reference" parcel E such that $B_i I_i E$ for all i.

This definition is adapted from Pazner and Schmeidler (1978). An egalitarian-equivalent partition is such that each agent is indifferent between his parcel and some fixed reference parcel.

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term, which has been given other formal meanings.

III. ***Countably Additive Utility.*** We will open our discussion by noting that the existence of envy-free and efficient allocations cannot be expected if no restrictions are imposed on preferences. Indeed, imagine L to be divided into two measurable subsets, L_1 and L_2, with $m(L_1) = m(L_2)$. Let $n = 2$ and let $u_1(B) = v_1(m(B \cap L_1), m(B \cap L_2))$, $u_2(B) = v_2(m(B \cap L_1), m(B \cap L_2))$ for some functions $v_1, v_2: \mathbb{R}^2_+ \to \mathbb{R}$. This economy is analogous to a 2×2 Edgeworth box economy with (possibly) nonconvex preferences represented by the utility functions v_1, v_2. Thus, the examples of Edgeworth box economies where no envy-free and efficient allocation exist (Varian, 1974) also apply to our heterogeneous commodity. Notice also that this interpretation of the model implies that the commodity space is a generalization of standard exchange models with homogeneous commodities. However, we shall impose different assumptions concerning preferences.

We will start by assuming preferences to have representations that are non-atomic measures.[3] All of the results in the mathematics literature quoted earlier rely on the well-known Lyapunov theorem, which states that the range of any real-valued, nonatomic vector measure is compact and convex (see Rudin (1973, p. 114) for an elegant proof).

The first result is straightforward: if for each i, agent i's preferences can be represented by a nonatomic measure, then efficient partitions[4] exist. This follows immediately from the compactness part of the Lyapunov Theorem. For further discussion, we refer to section II.2 of Dubins and Spanier (1961).

The next result, due to Dubins and Spanier (1961, Corollary 1.1), addresses

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[3]Many of the papers cited here, such as Hill (1983, p.441), point out that the theorems in this literature can fail if atoms are allowed in the utility measures. For example, every utility could be a probability measure that assigns probability one to the same point $x \in L$.

[4]We could similarly establish the existence of "utilitarian" partitions (partitions maximizing the sum of utilities) or "Rawlsian" partitions (partitions whose associated vector of utilities is lexicographically maximal).

the issue of existence of an α–fair partition.

***Theorem* 0.** Suppose that for each i, agent i's preferences can be represented by a nonatomic probability measure u_i. Then, given $\alpha \in \Delta^{n-1}$, there exists a partition $B \in \Pi^n$ such that $u_i(B_j) = \alpha_j$ for all i and j.

Hence, if each u_i is a nonatomic probability measure, then there exists an α–fair partition (by taking i=j in the result).

The following results are easy consequences of Theorem 0: first, together with the compactness part of the Lyapunov Theorem, it implies the existence of an α–fair and efficient partition. Also, setting $\alpha_i = 1/n$ for all i and noting that the last sentence of the Theorem holds for all $j \neq i$, the existence of an envy–free partition follows.

Showing that an envy–free and efficient partition exists is more difficult. In standard exchange economies, such existence theorems generally involve a fixed–point theorem (or a tool equivalent to a fixed–point theorem). The typical approach is to divide the economy's endowment equally among all traders, establish existence of an equilibrium relative to these endowments, and show that any resulting equilibrium allocation is envy–free and efficient. In the case of a heterogeneous commodity, there is no a priori (that is, independent of preferences) way of dividing the total endowment so that all traders necessarily have the same budget in equilibrium. However, we have the following existence result which requires preferences to be monotonic:

Definition*:** A preference relation R_i is ***monotonic if for all $B, B' \in \mathcal{B}$, $B \subsetneq B'$, $m(B) < m(B')$ implies $B' P_i B$. If u_i represents a monotonic preference relation, we will also say that ***u_i is monotonic***.

***Theorem* 1.** If for each i, R_i can be represented by a monotonic function u_i which is a measure absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^k$, then there exists a group envy–free and efficient partition.

The proof uses the following result of Berliant's (1985).

Lemma **1.** If $p^* \in L^1$ is an equilibrium price and $(a_1^*,...,a_n^*) \in (L^\infty)^n$ is an extreme point of the associated set of equilibrium allocations, then a_i^* is an indicator function for a set in $\mathscr{B}$ for each i.

Proof of Theorem **1.** Since each u_i is monotonic and absolutely continuous, using the Radon – Nikodym Theorem (see Rudin (1974, p. 130)) we can write $u_i(B) = \int_B h_i(x)$ dm(x), where $h_i > 0$ a.s. Extend this utility to $\beta \in L^\infty$ via $\bar{u}_i(\beta) = \int \beta(x) \cdot h_i(x)$ dm(x). Let each trader's initial endowment be $(1/n) \cdot 1_L$, where $1_L \in L^\infty$ is the indicator function for L. Apply Bewley's (1972) existence theorem for the commodity space L^∞, and fix one equilibrium price system $p^* \in L^1$. It is easy to see that the set of allocations that are equilibrium allocations relative to the given prices is a convex set that is also compact in the weak* topology on $(L^\infty)^n$. By the Krein–Milman theorem, this set has an extreme point.

By Lemma 1, the extreme equilibrium allocation is a vector of indicator functions of sets in $\mathscr{B}$; call it $(1_{B_1},...,1_{B_n})$. By standard arguments, this allocation is efficient. To see that it is envy–free, notice that all traders have the same budget, so it must be that $u_i(B_i) = \bar{u}_i(1_{B_i}) \geq \bar{u}_i(1_{B_j}) = u_i(B_j)$. To see that it is in fact group envy–free, suppose the existence of two groups C_1 and C_2 with $|C_1| = |C_2|$ and a measurable partition $\{A_i\}_{i \in C_1}$ of $\bigcup_{i \in C_2} B_j$, such that $A_i R_i B_i$ for all $i \in C_1$, with at least one strict preference. Then the value of $\bigcup_{i \in C_1} A_i$ at prices p^* exceeds the value of $\bigcup_{j \in C_2} B_j$ at those prices. This contradicts the fact that the incomes of all agents are the same at prices p^*.

Q.E.D.

The next result, due to Hill (1983, Theorem 2), extends the Dubins and Spanier result to obtain nicely shaped parcels.

***Theorem* 2.** Let $k \geq 2$. Suppose that L, $A_1,\ldots,A_n$ are open, connected subsets of $\mathbb{R}^k$ with A_i adjacent to L for each i. Suppose also that for each i, agent i's preferences can be represented by a nonatomic probability measure on L. Then, given $\alpha \in \Delta^{n-1}$, there exist disjoint, open, connected subsets $B_1,\ldots,B_n$ of L with (i) B_i adjacent to A_i for each i, (ii) $u_i(B_i) \geq \alpha_i$ for each i, and (iii) $\overline{\bigcup_{i=1}^{n} B_i} = L$.

Here L is the land to be divided, while the A_i are intended to represent the extant land holdings of n property owners or countries (the A_i are not subject to reallocation). We say that an allocation B is ***nicely shaped*** if for each i, B_i is open, connected, and adjacent to A_i. From Theorem 2, one can conclude that there exists a nicely shaped α–fair allocation. Hill (1983, p. 442) remarks that Theorem 2 can easily be extended in two directions.

First, nicely shaped limit–efficient partitions exist, as do nicely shaped similarly defined limit–utilitarian and limit–Rawlsian optima. It is also clear that a sequence of nicely shaped α–fair and limit–efficient partitions exist.

Second, the result can be extended so that for any $\epsilon > 0$, there is a partition $B \in \Pi^n$ satisfying the conclusions of the theorem with "$u_i(B_i) \geq \alpha_i$ for all i" replaced by "$|u_i(B_j) - \alpha_j| < \epsilon$ for all i and j". Again taking $\alpha_j = 1/n$ for all j, we conclude that there exists a sequence of nicely shaped limit envy–free allocations. Finally, using Theorem 1 in conjunction with the proof of the Hill result, there exists a sequence of limit–envy–free and limit–efficient partitions that are nicely shaped (open, connected, and adjacent to A_i). Analogous results hold for the concept of a limit group envy–free partition (which we have not defined formally).

We refer to Dunz (1987) and Berliant (1985) for facts about the core of this model. We simply remark here that under the assumptions of Theorem 1, the core is nonempty. Combining this fact with the proof of the Hill result, nicely shaped ϵ–core partitions exist.

IV. ***Nonlinear Utility.*** Here we introduce preferences that cannot be represented by measures. As noted earlier, we cannot prove much about general set functions, so we make an assumption that is stronger than subadditivity of utility (see the lemma below), and is intuitively related to decreasing marginal utility of a point as sets become larger through set containment.

Definitions. A utility function u_i is ***concave*** if $u_i \equiv \int_B h_i(x,B)\, dm(x)$, where for all B, $B' \in \mathscr{B}$ and for all x such that $x \in B' \subseteq B$, $h_i(x,B') \geq h_i(x,B)$.

Since this assumption will be used in the context of α–fair partitions, this assumption is highly cardinal. Notice that it is a generalization of the assumption that u_i is a nonatomic probability measure. An example of a concave representation is $h(x,B) = f(x)/[m(B)+1]$, where f is some positive density on L. Another example is $h(x,B) = f(x)/[rad(x,B)+1]$, where $rad(x,B) \equiv \sup\{\epsilon \geq 0 \mid B_\epsilon(x) \subseteq B\}$ and where $B_\epsilon(x) \equiv \{y \in L \mid \|x - y\| \leq \epsilon\}$. Next we use our new assumption.

***Theorem* 3.** Let $k \geq 2$. Suppose that $L, A_1, ..., A_n$ are open, connected subsets of $\mathbb{R}^k$ with A_i adjacent to L for each i. Assume that for each i, u_i is concave, with $u_i(B) = \int_B h_i(x,B)dm(x)$. For each $1 \leq i \leq n$, let $\alpha_i \geq 0$ be such that $\sum_{i=1}^{n} \alpha_i \leq 1$. Then, there exist disjoint, open, connected subsets $B_1, ..., B_n$ of L with B_i adjacent to A_i for each i such that $u_i(B_i) \geq \alpha_i \cdot u_i(L)$ for each i.

Proof. For each i, let $\bar{u}_i(B) \equiv \int_B h_i(x,L)\, dm(x) \,/\, u_i(L)$. Let $\alpha_{n+1} = 1 - \sum_{i=1}^{n} \alpha_i$, so that $\sum_{i=1}^{n+1} \alpha_i = 1$ and let $\bar{u}_{n+1}(B) \equiv m(B)/m(L)$. Then for each i, $\bar{u}_i$ is a nonatomic probability measure on L, so by Theorem 2 of Hill (1983), there exist disjoint subsets $(B_1,\ldots,B_{n+1})$ such that $\bar{u}_i(B_i) \geq \alpha_i$ for all i. Hence for i = 1,2,...,n we have $u_i(B) = \int_B h_i(x,B)\, dm(x) \geq \int_B h_i(x,L)\, dm(x) \geq \alpha_i \cdot u_i(L)$.

Q.E.D.

One can conclude from this result that there exists a nicely shaped α–fair partition. The same result but without nicely shaped parcels holds for all dimensions (even k = 1) if Theorem 0 is used in place of the Hill theorem. Next we examine a second concept of decreasing marginal utility.

Corollary 1. In the Theorem above, the hypothesis that for each i, u_i is concave can be replaced by the weaker hypothesis that $u_i(B) = \int_B h_i(x,B) dm(x)$ and for all B, B′ ∈ $\mathscr{B}$ with $B' \subsetneq B$, $\int_{B'} h_i(x,B')\, dm(x) \geq \int_{B'} h_i(x,B)\, dm(x)$.

Next we want to discuss how subadditivity is related to our assumptions. A set function $u: \mathscr{B} \to \mathbb{R}$ is ***subadditive*** if for every A, B ∈ $\mathscr{B}$ with $A \cap B = \emptyset$, $u(A \cup B) \leq u(A) + u(B)$.

Lemma. If $u(B) \equiv \int_B h(x,B)\, dm(x)$ and if for all B, B′ ∈ $\mathscr{B}$ with $B' \subsetneq B$, $\int_{B'} h(x,B')\, dm(x) \geq \int_{B'} h(x,B)\, dm(x)$, then u is subadditive.

Proof. For any disjoint A, B ∈ $\mathscr{B}$, $u(A) + u(B) = \int_A h(x,A)\, dm(x) + \int_B h(x,B)\, dm(x) \geq \int_A h(x,A\cup B)\, dm(x) + \int_B h(x,A\cup B)\, dm(x) = \int_{A\cup B} h(x,A\cup B)\, dm(x) = u(A\cup B)$.

Q.E.D.

In order to discuss compactness of the set of efficient partitions when preferences cannot be represented by additive set functions, it is necessary to impose a topology on $\mathscr{B}$ The topology we employ is given in Berliant and Dunz (1989, Appendix), and is closely related to the topology used in Berliant and ten Raa (1988). Since we only need to know that the set of measurable partitions is compact in this topology, there is no need to reproduce the details of the topology. That the set of measurable partitions is compact in the topology is proved in Berliant and Dunz (1989). We call a set function ***continuous*** if it is continuous in this topology. It is immediately apparent that if each u_i is continuous, then efficient partitions exist[5]. It is also apparent that α–fair efficient partitions exist if utilities are continuous and satisfy one of the above decreasing marginal utility conditions.

V. *Egalitarian–Equivalent and Efficient Partitions.* Existence of egalitarian–equivalent and efficient partitions requires less structure and weaker assumptions than those used for our previous results. What is needed is that the preferences be representable by utility functions that are continuous with respect to a compact topology on $\mathscr{B}$ such that sequences of nested sets converge. We employ any one of the topologies discussed above.

Theorem 4. Suppose that L is a compact, connected, k–dimensional manifold (with boundary). If for each i, agent i's preferences can be represented by a function u_i that is continuous and monotonic, then there exist egalitarian–equivalent and efficient allocations.

Proof. First we rescale the utility functions. Without loss of generality, suppose that $u_i(\emptyset) = 0$ for all i. Let y be an arbitrary point in L, and let $B_r(y)$ be the closed ball in $\mathbb{R}^k$ of radius r with center y. Define $v_i(r) \equiv u_i(B_r(y) \cap L)$. By definition of the topology and continuity of u_i, $v_i(0) = 0$ for all i. Let $\bar{r} = \inf \{r \geq 0 | L \subseteq B_r(y)\} < \infty$. Given the assumptions on L and monotonicity of u_i, v_i is continuous and monotone

[5]Again, this is true of utilitarian and Rawlsian partitions. See footnote 4 for definitions.

increasing on $[0,\bar{r}]$. Let $\bar{u}_i = u_i(L)$. Then, the function r defined by $r_i(u) \equiv v_i^{-1}(u)$ for all u is a well-defined, continuous and monotonic function from $[0,\bar{u}_i]$ to $[0,\bar{r}]$. Let $w_i(B) \equiv r_i(u_i(B))$. Then w_i is simply another representation of u_i. Let the utility possibility set be $W = \{(\overline{w}_1,\ldots,\overline{w}_n) \in \mathbb{R}^n \mid \exists\ B \in \Pi^n \text{ s.t. } \forall i,\ w_i(B_i) = \overline{w}_i\}$. Notice that W is comprehensive, i.e. if $w = (w_1,\ldots,w_n) \in W$ and $0 \leq w_i' \leq w_i$ for each i, then $w' = (w_1',\ldots,w_n') \in W$. Also notice that since $\mathscr{B}$ is compact, W is compact. Let 1 be the vector of n 1's, $(1,\ldots,1)$. Since $(0,\ldots,0) \in W$ and W is compact, max $\{t \cdot 1 \mid t \geq 0,\ t \cdot 1 \in W\}$ is finite and attained. Let t^* be the value of t attaining the maximum. Let $B^* = (B_1^*,\ldots,B_n^*)$ be the partition associated with utility level t^*, $w_i(B_i^*) = t^*$ for all i. Then, B^* is efficient, and $w_i(B_i^*) = t^* = r_i(u_i(B_t{}^*(y)))$, so $u_i(B_i^*) = u_i(B_t{}^*(y))$, and B^* is egalitarian-equivalent.

Q.E.D.

VI. ***When does No-Envy Imply Efficiency?*** Here we examine the general question of when envy-free allocations are group envy-free and therefore efficient. We present a general model and sufficient conditions for this implication to hold.

Our result will hold for the model presented above as well as on another domain. Consider the problem of allocating a set of indivisible objects among a group of consumers of equal or greater cardinality (tasks among a number of workers), each consumer receiving at most one object and all the objects being allocated. In that context, an envy-free allocation is also efficient and group envy-free, as was first noted by Svensson (1983). This result will be a special case of our theorem below.

Let F be the consumption set of every agent (F need not be a subset of a linear space or even have a topological structure). Let $\mathscr{F} \subseteq \prod_{i=1}^{n} F$ be the set of feasible allocations. As before, each agent has a complete preference preorder R_i.

To simplify notation, we restate the definition of a group envy-free allocation.

An allocation $x = (x_1,..,x_n) \in \mathcal{F}$ is ***group envy-free for R*** $= (R_1,...,R_n)$ if for every group of agents C, for every injection π: C $\rightarrow$ {1,2,...,n}, and for every $y \in \mathcal{F}$ with $y_i = x_i \ \forall\ i \notin \pi(C)$, $x_j R_j y_{\pi(j)}$ for all $j \in C$ or $x_j P_j y_{\pi(j)}$ for some $j \in C$. Note that this is just the usual definition of group envy-free where C and $\pi(C)$ are groups of agents of equal cardinality, y is a feasible reallocation of the consumptions (given by x) of the agents in $\pi(C)$, and π is an assignment of this reallocation to the agents of C.

Now we state and prove the main theorem of this section.

Theorem 5. Suppose that

(1) there exists a partial order $\succ$ on F such that for all i, $x_i \succ y_i$ implies $x_i P_i y_i$; and

(2) if $x, y \in \mathcal{F}$ are such that there does not exist a bijection π: {1,...,n} $\rightarrow$ {1,...,n} with $x_i = y_{\pi(i)}$ for all i then there exist i and j such that $x_j \succ y_i$.

If $x \in \mathcal{F}$ is envy-free then it is also group envy-free and therefore efficient.

Proof. Let $x \in \mathcal{F}$ be envy-free, let C be a group of agents, let π: C $\rightarrow$ {1,...,n} be an injection, and let $y \in \mathcal{F}$ with $y_i = x_i$ for all $i \notin \pi(C)$. Suppose π': {1,...,n} $\rightarrow$ {1,...,n} is a bijection such that $y_{\pi'(i)} = x_i$ for all i. Then for every $i \in C$, $y_{\pi(i)} = x_j$ for $j = \pi'^{-1}(\pi(i))$. No-envy implies $x_i R_i x_j = y_{\pi(i)}$ for all $i \in C$. So we can assume the hypothesis of condition (2) holds. Therefore, $x_j \succ y_i$ for some i,j. If $i \notin \pi(C)$ then $y_i = x_i$ and (1) implies $x_j P_i x_i$, which contradicts that x is envy-free. So $i \in \pi(C)$. Let $\pi^{-1}(i) = i' \in C$. Then x envy-free implies $x_{i'} R_{i'} x_j$ and (1) implies $x_j P_{i'} y_i = y_{\pi(i')}$. So $x_{i'} P_{i'} y_{\pi(i')}$ and x is group envy-free.

Q.E.D.

Notice that (1) is a (strict) monotonicity assumption. Condition (2) implies that the situation is one of "pure division" in the sense that giving more to some agent means another agent gets less. This property is always true when two agents are dividing a single commodity. Theorem 5 can be strengthened by replacing (1) and

(2) with:

if x, y $\in \mathcal{S}$ are such that there does not exist a bijection π: $\{1,...,n\} \to \{1,...,n\}$ with $x_i = y_{\pi(i)}$ for all i then there exist i and j such that for all agents a, x_j P_a y_i.

Clearly (1) and (2) together imply this condition.

Next we consider several applications of this result. A particularly interesting version of the model presented in the previous sections is obtained by supposing that there is a one–dimensional continuum which has to be divided into intervals. The most natural example here is time.

It is easy to think of situations where the availability of a facility or a service is beneficial only in intervals. For of a variety of reasons (transportation to and from the facility, preparation, coordination with partners), splitting up the time available into small intervals with each agent receiving several of them, mutually disjoint, would be inefficient. We will then assume that preferences are defined over intervals. (If so desired, and without loss of generality when efficiency is insisted upon, we could extend preferences to unions of intervals by identifying each such union with the most preferred of the maximal intervals it contains.)

The existence of envy–free allocations for that model has already been established by Stromquist (1980) and Woodall (1980). The equity notion they use is no–envy, but they make no mention of efficiency. Our contribution here is to show that in this model and under a natural monotonicity condition, any envy–free allocation is necessarily efficient, and in fact group envy–free (according to our definition above, group no–envy implies both no–envy and efficiency). This is quite surprising since there is no reason in general why the normative concept of no–envy (or for that matter, any equity concept) should have any implication concerning efficiency.

Corollary 2. Let $I \equiv [a,b]$ be a connected interval in $\mathbb{R}$. Assume that each agent has preferences over intervals such that $A \supset B$ implies $A\ P_i\ B$. Then every envy–free partition of I into n intervals is group envy–free.

Proof. Here $\mathcal{F}$ is the collection of all partitions of I into n intervals. Define the partial order in condition (1) by $\supset$, so condition (1) holds. Hence only condition (2) needs to be verified. We can identify an interval in such a partition by its right endpoint. So z represents the partition $\{[a,z_1], [z_1,z_2],\ldots, [z_{n-1},z_n]\}$ where $z_n = b$.[6] We will order feasible allocations by position in this manner instead of by agents.

Let w and z be distinct partitions of I. If $w_1 < z_1$ then we are done since $[a,z_1] \supset [a,w_1]$. So we can assume $w_1 \geq z_1$. Suppose that $w_i \geq z_i$ for all $i \leq j-1$. If $w_j < z_j$ then we are done since $[z_{j-1},z_j] \supset [w_{j-1},w_j]$. So it must be that $w_i \geq z_i$ for all i. But, since w and z are distinct there must be some j for which $w_j > z_j$. Therefore $w_{j+1} > z_{j+1}$, for otherwise $w_{j+1} \leq z_{j+1}$ and $[z_j,z_{j+1}] \supset [w_j,w_{j+1}]$ so we are done. Again by induction it must be that $w_i > z_i$ for all $i > j$ since otherwise $w_i \leq z_i$ while $w_{i-1} > z_{i-1}$, so $[z_{i-1},z_i] \supset [w_{i-1},w_i]$, and we are done. But $w_i > z_i$ for all $i > j$ is impossible, since $w_n = z_n = b$. Therefore (2) is verified.

Apply Theorem 5 to obtain the desired result.

Q.E.D.

As mentioned previously, Stromquist (1980) and Woodall (1980) have shown that an envy–free partition exists in this one–dimensional model provided that preferences have a utility representation that is continuous with respect to interval endpoints. (This type of continuity is the same as the more general form of continuity given in Berliant and ten Raa (1988) or Berliant and Dunz (1989) when specialized to this one–dimensional model.) Hence envy–free and efficient partitions exist provided that preferences admit a continuous utility representation and are monotonic in the

[6]For standard measure – theoretic reasons, two intervals that differ on a set of measure zero are considered equivalent, so elements of a partition are allowed to overlap at a common endpoint.

sense of Corollary 2.

An interesting application of Corollary 2 is to models of urban location, such as Alonso (1964). Each agent's utility depends on the interval of land he receives and how close it is to the central business district given by a. Distance to the city center will be measured from the beginning of the agent's interval of land.

Corollary 3. Given the structure of Corollary 2, suppose agent i's preferences are represented by a utility function u_i where for all $e > 0$ $u_i([c-e,d-e]) > u_i([c,d])$, and $[c,d] \supset [c',d']$ implies $u_i([c,d]) > u_i([c',d'])$. Then all envy-free allocations are group envy-free. If each u_i is also continuous, then there exists an envy-free and efficient partition.

The proof of Corollary 3 is an easy consequence of Corollary 2. Corollary 3 depends crucially on measuring the distance to the central business district from the beginning of a parcel. If this distance were given by a "weighted average" of distances from each point in the parcel to the city center (e.g. distance from the middle of the parcel to the city center), then Theorem 5 might not apply. In this case, condition (1) need not be satisfied.

Corollaries 2 and 3 can easily be extended to cover models in which I is a finite union of disjoint intervals in the real line. This can be accomplished by identifying the endpoints of consecutive intervals.

Finally, we demonstrate how a result of Svensson (1983) is captured by Theorem 5.

Corollary 4. Consider the problem of allocating J indivisible commodities and an amount $M \geq 0$ of a divisible commodity among n ($\geq$ J) agents so that each agent receives at most one of the indivisible commodities. If agents strictly prefer more of the divisible commodity to less, then envy-free allocations are group envy-free.

Proof. If $n > J$, "null" indivisible commodities are added to the model so that the numbers of agents and indivisible commodities are the same. As in the proof of

Corollary 2, we index the vectors representing feasible allocations by the indivisible commodity instead of the agent. That is, if $x \in \mathcal{F}$, then x_i gives the quantity of the divisible commodity consumed by the agent receiving the i^{th} indivisible commodity. Feasibility requires that $x_i \geq 0$ for all i and that $\sum_{i=1}^{n} x_i = M$.

First we consider the case $M = 0$. So there are only indivisible goods in this case. Since a commodity bundle never contains money, a bundle consists of only an indivisible commodity. Hence, every envy–free allocation is clearly group envy–free.

Let $M > 0$. We define the partial order $\succ$ required by condition (1) as follows: $x_i \succ y_j$ iff $i = j$ and $x_i > y_j$. This implies that condition (1) holds. Now let x and y be two distinct feasible allocations. So there must be an i such that $x_i > y_i$, otherwise x and y are not distinct or all of the divisible commodity is not allocated. Hence, $x_i \succ y_i$. This shows that (2) holds and Theorem 5 can be applied.

Q.E.D.

VII. *Concluding comment.* We have examined the existence of allocations satisfying various equity criteria in economies in which a heterogeneous good has to be allocated. Beyond existence, there are a number of important issues that should be tackled next pertaining, in particular, to the existence of selections from the no–envy solution satisfying additional properties. Examples are monotonicity with respect to the amount to be divided (all should benefit from such an increase), and with respect to changes in the number of claimants (all claimants initially present lose in such circumstances). We hope that our existence results will contribute to setting the stage for a thorough investigation of these issues.

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