Rochester Center for
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The Existence of Competitive Equilibrium Over an Infinite Horizon with
Production and General Consumption Sets

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Abstract. Although many theorems have been proved on the existence of competitive equilibrium in production economies with an infinite set of goods and a finite set of consumers, nearly all suffer from a major defect. Their consumption possibility sets are required to equal the positive orthant. This rules out trade in personal services and it does not allow for substitutions between goods on the subsistence boundary. We show both equilibrium existence and core equivalence for economies with production and general consumption sets.

Our method of attack is similar to that introduced by Peleg and Yaari (1970). We first show our economy has a compact core. We then consider the set of equal treatment allocations in the core of replica economies, which is also non-empty and compact. As the equal treatment cores are nested, they have an intersection. Take an allocation in the intersection and separate the set of weakly preferred trades from zero. We first use prices in ba, and then modify them to obtain a price vector in ℓ^1 that still separates. This is our equilibrium price vector. Our proof also shows that any allocation in the core of every replica economy must be an equilibrium. Standard arguments show that the converse is true, yielding core equivalence.

Department of Economics
University of Rochester
Rochester, NY 14627

1. Introduction

Many theorems have been proved on the existence of competitive equilibrium with an infinite set of goods and a finite set of consumers since the path-breaking papers of Peleg and Yaari (1970) and Bewley (1972). Many of these theorems also allow for production. However, nearly all suffer from a major defect. Their consumption possibility sets are required to equal the positive orthant. See Magill (1981), Aliprantis and Brown (1983), Jones (1984), Mas-Collel (1986), Yannelis and Zame (1986), Zame (1987), and many others. An exception is Back (1988). Back describes his consumption set as lying in the positive orthant. However, all that he really requires is that the consumption set have a finite lower bound. Then moving the origin to this lower bound puts the consumption set in the positive orthant. What must be avoided is the requirement that a lower bound lie in the consumption set. The relationship between our models, which are not equivalent, is discussed in the conclusion. A consumption set which is equal to the positive orthant has two major inadequacies. First, it does not allow trade in personal services, even when production is absent and *a fortiori* it does not allow for the use of labor services in production. Second, it does not allow for substitutions between goods on the subsistence boundary of the possible consumption sets.

The need to consider economies with an infinite set of goods arises in the study of economies which do not have a definite termination date. It may be reasonable to suppose that an infinity of goods for delivery on one date belong to a compact space which can be adequately approximated by a finite subset of goods. However, if we wish to deal with an economy which has an indefinite or infinite horizon, this approximation is not available. Goods for delivery at different dates must be regarded as different goods. It was a major contribution of *Value and Capital* (Hicks, 1939) to provide a full-scale analysis of an economy in which this fact is properly recognized. Then, if the horizon is infinite, the number of goods must be infinite, even when deliveries are scheduled at discrete intervals. The infinity of goods must be dealt with simultaneously since the market is analyzed as though all trades occur at the beginning of time. This is a limiting form of the futures economy of Hicks. When uncertainty is introduced in the manner of Debreu (1959) by also distinguishing goods by the states of the world in which they are delivered, the sets of possible future states in each period must be foreseen. It may be possible to do with only a finite number of states of the world in each period, but to give an infinite horizon a finite approximation is difficult unless an arbitrary truncation of time is used with goods in the terminal period valued in an arbitrary way.

Our method of attack is similar to that used by Peleg and Yaari (1970) for an exchange economy. However, unlike the generalizations of Aliprantis, Brown and Burkinshaw (1987), our results apply not only to the case of an economy with production, but also with trade in a variety of labor services. Their method of proof is based on a theorem of Scarf on the nonemptiness of the core. Scarf's theorem can be used to show the equal treatment core of every replica economy is non-empty. Peleg and Yaari then show that any point in the intersection of these cores is an equilibrium. In contrast to Back (1988), this approach has the advantage of proving a core equivalence property. An allocation is an equilibrium if and only if it is in the equal treatment core of every replica of the economy. By modifying Peleg and Yaari's methods, we are able to avoid interiority or properness conditions, which are not appropriate in our infinite horizon setting.[1]

The proof of existence of equilibrium proceeds in several steps. We first show our economy has a core. Moreover, the core is compact. We then consider equal treatment allocations in the core of replica economies. Each of these is also non-empty and compact. These equal treatment cores are nested, so they have an intersection. The final step is to show that all allocations in the intersection are in fact equilibria. The prices are found by weakly separating the set of weakly preferred trades from zero. We first separate with prices in ba, and then modify these to separate with prices in ℓ^1. This is our candidate equilibrium price vector. The last step of the proof is to show that the candidate price vector yields an

[1] Aliprantis, Brown and Burkinshaw find that our preference assumptions are incompatible with uniform properness (1989, p. 174), while Back (1988) finds that Inada conditions on utility can generate improper preferences on ℓ^∞.

equilibrium.

In the course of proving the existence of equilibrium we have shown that any allocation in the equal treatment core of every replica economy must be an equilibrium. Standard arguments show that the converse is true, yielding core equivalence.

In Section Two, we set up the equilibrium model. Section Three shows that core allocations exist, and in Section Four we show that the intersection of the cores of replica economies is non-empty. Section Five then proves the main results of the paper. Any allocation that is in the equal treatment core of every replica is in fact an equilibrium. The existence of equilibria is an immediate corollary. In fact, an allocation is an equilibrium if and only if it is in the equal treatment core of every replica. Concluding remarks are in Section Six. Basic facts about the space *ba* of bounded finitely additive measures (or charges) are presented in the Appendix.

2. The Model

The commodity space s^n is the Cartesian product $\prod_{t=0}^{\infty} \Re^n(t)$, endowed with the product topology where $\Re^n(t)$ has the norm topology (Berge, 1963, p. 78). If $z \in s^n$ then $z = \prod_{t=0}^{\infty} z_t$, and $z_t \in \Re^n(t)$ represents quantities of goods in period t. We will use $z \geq w$ to mean $z_{it} \geq w_{it}$ for all goods i and times t. The notation $z > w$ means $z \geq w$ and there is a t with $z_{it} > w_{it}$ for all i.

There are a finite number of traders, $1, \ldots, H$. The set of possible net trades for the h^{th} trader in period t is C_t^h. The set of net trades for the h^{th} trader over the infinite horizon is $C^h \subset s^n$ where $C^h = \prod_{t=1}^{\infty} C_t^h$. We do not place any additional restrictions on the growth of conceivable consumption sequences, even though the technology may be bounded. In a decentralized economy the consumer does not take into account aggregate technical possibilities when choosing a consumption sequence, but rather looks at what is affordable. Any bound on growth of the optimal consumption path must come out of the budget constraint.

In our model any endowments of goods are included in the specification of the consumption set. For $w_t^h \in C_t^h$, $w_{it}^h < 0$ means that the quantity $-w_{it}^h$ of the i^{th} good is provided by the h^{th} consumer at time t and $w_{it}^h > 0$ means that the quantity w_{it}^h of the i^{th} good is received by the h^{th} consumer at time t. For C_0^h the h^{th} consumer will be able to provide any initial stocks of goods which he may possess, including produced goods. However, for C_t^h with $t > 0$ the h^{th} consumer offers only labor services and other unproduced goods.

A binary relation Q is said to be *irreflexive* if zQz does not hold for any z. A relation is said to be *antisymmetric* if zQy implies not yQz, and it is *transitive* if xQy and yQz implies xQz. Note that a transitive, irreflexive relation is automatically antisymmetric. There is an irreflexive and transitive relation P^h of *strict preference* defined on C^h and a correspondence, also denoted by P^h, defined on C^h by $P^h(z) = \{w : w \in C^h \text{ and } wP^hz\}$.

The production set is $Y = \sum_{t=1}^{\infty} \bar{Y}_t$ where $\bar{Y}_t \subset \{0\} \times \cdots \times \{0\} \times \Re_-^n(t-1) \times \Re_+^n(t) \times \{0\} \times \cdots$. The set $\bar{Y}_t$ represents the possibilities of producing goods belonging to the t^{th} period with goods belonging to the $(t-1)^{st}$ period. Inputs are negative numbers and outputs are positive numbers. Let Y_t be the projection of $\bar{Y}_t$ into the coordinate subspace $\Re^n(t-1) \times \Re^n(t)$. Then $(u_{t-1}, v_t) \in Y_t$ implies $u_{t-1} \leq 0$ and $v_t \geq 0$. The inputs and outputs of the production sector include the capital stocks. These stocks do not appear in the consumer net trading sets C_t^h except for $t = 0$. In an economy with certainty, the ownership of capital stocks is inessential. Only the value of investment is significant for the consumer and the sequence of investment values over time is implicit in the pattern of consumption.

The economy $\mathcal{E}$ is given by the list $\{Y, C^1, \ldots, C^H, P^1, \ldots, P^H\}$. Let $P^h(w)$ denote the strictly preferred set to w. If $z \in P^h(w)$, the trade w is less desirable to the trader than the trade z. A *lower section* of the correspondence $f : X \to \{\text{subsets of } X\}$, at a point $y \in X$, is the set $\{z : y \in f(z)\}$. The *graph* of f is the set $\{(y,z) : z \in f(y)\}$. We define $C = \sum_{h=1}^{H} C^h$. Let $\ell^\infty = \{z \in s^n : z_t \text{ is bounded over } t\}$ and $\ell^1 = \{z \in s^n : \sum_{t=0}^{\infty} |z_t| \text{ is finite}\}$ where $|\cdot|$ is any norm on $\Re^n$. Define the *preference relation* R^h on C^h by $x^hR^hy^h$ if and only if not $y^hP^hx^h$. We will say that x^h is *indifferent* with y^h, written $x^hI^hy^h$, if $y^hR^hx^h$ and $x^hR^hy^h$.

Recall that x is an *extreme point* of a convex set C if there do not exist distinct $x', x'' \in C$ and α, $0 < \alpha < 1$ with $x = \alpha x' + (1-\alpha)x''$. If I is a subset of $\{1, \ldots, H\}$ define $x_I = \sum_{h \in I} x^h$. The economy is *strongly irreducible* if whenever I_1, I_2 is a non-trivial partition of $\{1, \ldots, H\}$, and $x_{I_1} + x_{I_2} \in Y$, there are $z_{I_1} + z_{I_2} \in Y$ with $z^h P^h x^h$ for $h \in I_1$ and for $h \in I_2$, $z^h \in C^h$ when x^h is not an extreme point of C^h and $z^h \in \alpha C^h$ for some $\alpha > 0$ when x^h is an extreme point of C^h.

This differs from the standard definition of irreducibility, although it is in the same spirit. Unlike the standard definition, this definition requires in some cases that the I_2 consumers be able to improve the trades of the I_1 consumers by actual moves to other trades in the I_2 possible trading sets. This stronger form of irreducibility is needed to establish that allocations to replicates of a consumer in the core of a replicated economy are indifferent. When the trading sets are translates of the positive orthant, both irreducibility and strong irreducibility are implied by strict monotonicity. The assumptions are:

ASSUMPTIONS.

(1) *Y_t is a closed, convex cone with vertex at the origin.*

(2) *$Y_t \subset \Re^n_- \times \Re^n_+$ with $Y_t \cap (\{0\} \times \Re^n_+) = \{(0,0)\}$.*

(3) *$C^h = \prod_{t=0}^{\infty} C^h_t$ is convex, closed, and bounded below. There is an interval $[\bar{z}, \bar{w}]$ such that $\bar{z}, \bar{w} \in \ell^\infty$ and $C^h = (C^h \cap [\bar{z}, \bar{w}]) + s^n_+$.*

(4) *For all h the correspondence P^h is convex valued and, relative to C^h, is open valued and has open lower sections. The preference relation P^h is irreflexive and transitive. The weakly preferred set $R^h(x)$ is the closure of $P^h(x)$ for all $x \in C^h$.*

(5) *Let $x \in C^h$. If $z > x$ then $z \in P^h(x)$.*

(6) *The economy $\mathcal{E}$ is strongly irreducible.*

(7) *There is $\bar{x}^h \in C^h - Y$ with $\bar{x}^h \leq 0$ and $\bar{x}^h_t = \bar{x}^h_s$ for all s and t. Moreover, $\bar{x} = \sum_{h=1}^{H} \bar{x}^h < 0$. For any x^h, let $z^h \in R^h(x^h) - Y$ and $\delta > 0$, then there is a τ_0 such that for each $\tau > \tau_0$, there is an $\alpha > 0$ with $(z^h_0 + \delta e_0, z^h_1, \ldots, z^h_\tau, \alpha \bar{x}^h_{\tau+1}, \ldots) \in R^h(x^h) - Y$.*

The technology exhibits constant returns to scale by Assumption 1. Of course, diminishing returns can be accommodated by introducing artificial entrepreneurial factors. Assumption 2 implies Bewley's Exclusion Assumption. It also insures that inputs precede outputs, and that production requires inputs ($Y \cap s^n_+ = \{0\}$). If y is in Y, $y_0 = u_0 \leq 0$. Moreover, if $y_0 = 0$, $y = 0$ since outputs require inputs. As a result, $Y - s^n_+$ can contain no straight lines. To see this let $z, -z \in Y - s^n_+$. Then there are $y, y' \in Y$ with $z \leq y$ and $-z \leq y'$. But $0 \leq y + y' \in Y$, so $y + y' = 0$. Finally, $y_0 \leq 0$ and $-y_0 \leq 0$, so $y = 0$. As $z, -z \leq 0$, $z = 0$.

In an exchange economy, a lower bound b^h for C^h has often been taken to be the negative of a vector of endowments held by the h^{th} trader which lies in C^h. In a production economy where productive services are traded, the requirement that b^h lie in C^h would be very restrictive. Even in a trading economy it is not satisfactory since it implies that the subsistence level for consumers allows no substitution between goods. Also, if personal services are traded, and $C^h \subset s^n_+$, labor services provided by the consumer must be measured in units of his labor time without distinguishing the type of labor performed. In our model the last part of Assumption 3 makes $b^h \in C^h$ unnecessary.

It is clear that given any C^h that satisfies the first part of Assumption 3, a set satisfying the second part may be derived by intersecting C^h with a properly chosen interval $(\bar{z}, \bar{w})$ and adding s^n_+. However, we may give an explicit example. Consider a consumer who supplies two different types of labor, and consumes an all-purpose consumption good. This consumer has T units of potential labor time. The consumption set at time t is $C_t = \{(c_t, \ell_{1t}, \ell_{2t}) : c_t \geq 0$, $T + \ell_{1t} + \ell_{2t} \geq 0, T + \ell_{1t} \geq 0, T + \ell_{2t} \geq 0\}$. Then $C = \prod_{t=0}^{\infty} C_t$ satisfies Assumption 3 with $\bar{z}_t = (0, -T, -T)$ and $\bar{w}_t = (0, 0, 0)$. Note that the consumer may become satiated in each type of labor when none is being supplied without violating Assumption 5, provided that the all-purpose consumption good is always valuable. This also permits the use of entrepreneurial factors that do not affect the consumer's utility. This example is a prototype of the kind of consumption set we are concerned with. Note the different marginal dis-utilities for the different types of labor would preclude using aggregate labor time (or leisure) as a good here.

Assumption 4 will guarantee that preferences can be represented by a continuous utility function. Note that R^h inherits transitivity from P^h. Assumption 5 (periodwise monotonicity) is fairly straightforward. We have already discussed Assumption 6 (strong irreducibility).

The first part of Assumption 7 provides for a path of production and consumption for the economy as a whole in which all goods are in excess supply in every period. This may be thought of as a kind of Slater condition (see Uzawa, 1958, p. 34). It is also a weakening of Bewley's (1972) Adequacy Assumption. Requiring $\bar{x}^h > 0$ for all h would be equivalent to the Adequacy Assumption. Also, take $w^h \in C^h$ and $y \in Y$ with $w^h - y = \bar{x}^h \leq 0$. By monotonicity, $w^h - \bar{x}^h \in C^h$. Thus $w^h - \bar{x}^h \in C^h \cap Y \neq \emptyset$.

The second part of Assumption 7 may seem a bit strange at first glance. It is important to understand it since it plays a key role. Among other things, it implies for any consumer that we can replace the tail of his stream of net trades after production with 0 and still remain in $R^h(x^h) - Y$, provided inputs are increased in the first period since $R^h(x^h) + s^n_+ \subset R^h(x^h)$ by Assumption 5.

Negative components of vectors in Y represent inputs, as do the positive components of vectors in $-Y$. Thus, Assumption 7 says that if the initial stocks held by a consumer are increased by a positive quantity of all goods, he can supply a uniform amount of all goods in all periods after a time τ, using the production possibilities, and still be as well off. In fact, it is enough that the production sector be able to supply uniform net outputs of produced goods in late periods while still allowing each consumer to subsist and supply unproduced goods. Capital accumulation models will often obey this stronger condition, which is formalized below.

ASSUMPTION. *Suppose that for all $y \in Y$ and $\delta > 0$ there is a $b^h \in C^h$ and τ_1 such that for each $\tau > \tau_1$, there is an $\alpha > 0$ with $y(\tau) = (-\delta e_0 + y_0, y_1, \ldots, y_\tau, b^h_{\tau+1} - \alpha\bar{x}^h_{\tau+1}, \ldots) \in Y$.*

The vector b^h may contain labor services. This assumption can be used to derive the second part of Assumption 7. Suppose $z^h = w^h - y \in R^h(x^h) - Y$ with $w^h \in R^h(x^h)$ and

$y \in Y$. Let τ_1, b^h and $y(\tau)$ be as in the assumption, and take τ_2 with $w^h(\tau) = (\delta e_0 + w_0^h, w_1^h, \ldots, w_\tau^h, b_{\tau+1}^h, \ldots) \in R^h(x^h)$. These vectors are in the consumption set and approach $w^h + (\delta e_0, 0, \ldots)$, which is preferred to w^h. The fact that preferred sets are open in C^h implies that there is τ_2 with $w^h(\tau) \in R^h(x^h)$ for $\tau > \tau_2$. Now take $\tau \geq \tau_1, \tau_2$ and subtract to obtain $w^h(\tau) - y(\tau) = (w_0^h - y_0 + 2\delta e_0, w_1^h - y_1, \ldots, w_\tau^h - y_\tau, \alpha \bar{x}_{\tau+1}^h, \ldots) \in R^h(x^h) - Y$. This implies the second part of Assumption 7.

Now consider a standard growth model with one produced good and using labor and the produced good as factors of production. Output per unit of labor input is given by a concave function f with $f' > 0$. The set Y is given by the production function F where $F(K, L) = Lf(K/L)$ for $K, L > 0$ and $F(K, L) = 0$ otherwise. Let $Y_t = \{(u_{1t}, u_{2t}, v_{1t}, v_{2t}) : 0 \leq v_{1t} \leq F(-u_{1t}, -u_{2t}),\ v_{2t} = 0\}$ where "1" indexes the produced good and "2" indexes labor. There is one consumer with initial endowment k. The consumer can supply up to one unit of labor in each period. Thus $C_0 = \{(c, \ell) : c \geq -k,\ \ell \geq -1\}$ and $C_t = \{(c, \ell) : c \geq 0, \ell \geq -1\}$ for $t = 1, 2, \ldots$. The utility function is $u(w) = v(c_0 + k) + \sum_{t=1}^{\infty} \beta^t v(c_t)$ where v is a bounded continuous function with $0 < \beta < 1$. Recall that the c_t are quantities traded. The actual consumption is $c_0 + k$ for $t = 0$, due to the initial endowment of k, and c_t for $t = 1, \ldots$. For the sake of simplicity we assume that labor does not affect utility. Suppose there is a $\bar{k} \leq k$ with $f(\bar{k}) \geq 2\bar{k}$. Now let $\bar{x}_t = (-\bar{k}/3, -1/3)$ and $b = ((-\bar{k}, -2/3), (0, -2/3), \ldots) \in C$. We may think of $1/3$ unit of labor being devoted to a subsistence activity outside of Y. It is easy to see that $\bar{x}$ is in $C - Y$ since $F(\bar{k}/3, 1/3) = f(\bar{k})/3 \geq 2\bar{k}/3$ allows $v_{1t} = 2\bar{k}/3$ and $u_{1t} = -\bar{k}/3$ yielding $y_{1t} = \bar{k}/3$. Let $y \in Y$. Suppose inputs at time zero are increased by δe_0. Maintain consumption levels at each time, while accumulating capital stocks u_{1t}. These will be higher (more negative) at each time period than on the original path. At any time, we may stop following y and devote this capital stock to maintaining steady output. If the capital stock is at least $\bar{k}$, we can follow $y_t' = b_t - \bar{x}_t = (\bar{k}/3, -1/3)$ henceforth, otherwise we must settle for some fraction α of y_t'. In either case, Assumption 7 will be satisfied.

This example shows why some sort of joint condition on consumption and production is needed. The production sector alone cannot necessarily produce positive outputs of all goods after the first period. Inputs of labor or other non-produced goods may be required. A joint condition on consumption and production is then needed to obtain positive output of all goods.

The set of possible trades with production for the h^{th} consumer is $C^h - Y$. The *set of admissible price vectors* will be $S = \{p \in s_+^n : p\bar{x} < \infty\}$ where $p\bar{x} = \sum_{t=0}^{\infty} p_t \bar{x}_t$. Unlike the admissible price vectors in most models of the competitive economy, the price vectors in S are not all contained in the dual of the commodity space. The lower bound on C^h insures that pw is either finite or $+\infty$ for all $w \in C^h$. We do not use the dual space $(s^n)^*$ as our price space. The elements of $(s^n)^*$ have only a finite number of non-zero components. This means that the price of all goods must be zero at almost all times, hence it is totally inappropriate to consider prices in $(s^n)^*$. For $p \in S$ the *budget set* of the h^{th} trader is $B^h(p) = \{x : x \in C^h$ and $px \leq 0\}$. No bundles with infinite value are contained in the budget set. A *competitive equilibrium* for the economy $\mathcal{E} = \{Y, C^1, \ldots, C^H, P^1, \ldots, P^H\}$ is a list $(p, y, x^1, \ldots, x^H)$ such that p is admissible and the following conditions are met.

COMPETITIVE EQUILIBRIUM.

(I) $px^h \leq 0$ *and* $z \in P^h(x^h)$ *implies* $pz > 0$.

(II) $y \in Y$ *where* $y_0 = u_0$ *and* $y_t = u_t + v_t$ *for* $t \geq 1$. *Also* $p_{t-1}u_{t-1} + p_t v_t = 0$ *for* $t \geq 1$, *and* $z \in Y$, *with* $z_t = u'_t + v'_t$, *implies* $p_{t-1}u'_{t-1} + p_t v'_t \leq 0$ *for all* $t \geq 1$.

(III) $\sum_{h=1}^{H} x^h = y$.

The first condition is the usual demand condition. The second condition is the profit condition. The third condition is the balance condition. Our objective is to prove that an equilibrium exists.

3. The Core is Non-Empty

By an *allocation* of net trades we mean a list $\tilde{x} = (x^1, \ldots, x^H)$ such that $x^h \in C^h$ for all

h. A *feasible allocation* must also satisfy the condition $\sum_{h=1}^{H} x^h \in Y$. Then the set of feasible allocations for the economy is $F = \{(x^1, \dots, x^H) : x^h \in C^h$ for all h, and $\sum_{h=1}^{H} x^h \in Y\}$. Let us say that an allocation $\{z^h\}_{h=1}^{H}$ of net trades admits an *improving coalition* B if there is an allocation $\{w^h\}_{h \in B}$ over the members of B such that $\sum_{h \in B} w^h \in Y$ and $w^h \in P^h(z^h)$ for all $h \in B$. The *core* of the economy $\mathcal{E}$ is the set of feasible allocations which do not admit any improving coalitions.

LEMMA 1. *The set of feasible allocations is non-empty, compact and convex.*

PROOF. By Assumption 7, $(\bar{x}^1, \dots, \bar{x}^H) \in F$, so F is non-empty. The set F is convex and closed since Y and all the C^h are both convex and closed. By Tychonoff's Theorem (Berge, 1963, p. 79), it is sufficient to prove F is bounded in each coordinate.

We first show that bounded inputs at time $t-1$ yield bounded outputs at time t. Suppose not. There is a sequence $(u_{t-1}^s, v_t^s) \in Y_t$ and constant B with $|u_{t-1}^s| \le B$ and $|v_t^s| \to \infty$. Consider $(u_{t-1}^s, v_t^s)/|v_t^s| \in Y_t$. Since this is bounded, it has a convergent subsequence with limit $(0, v)$ and $|v| = 1$. But Y_t is closed, so $(0, v) \in Y_t$, which contradicts Assumption 2.

The second step is to show that bounded outputs at time t imply that inputs at t are bounded. Let $\bar{z}$ be the lower bound on the C^h given by Assumption 3. Let $\tilde{x}$ be a feasible allocation with $\sum_{h=1}^{H} x_t^h = y_t = u_t + v_t$. Since $u_t + v_t \in C$, $H\bar{z}_t \le u_t + v_t$, so $H\bar{z}_t - v_t \le u_t \le 0$. Thus bounded outputs at t imply bounded inputs at t.

Finally, inputs at time zero are bounded since $u_0 \in C$. By induction, both inputs and outputs are bounded at each time t. Thus $x_t = u_t + v_t$ is bounded too. QED

Let F^h be the projection of F into the h^{th} consumer's net trading set C^h. The next proposition uses standard arguments to show that preference order can be represented by a continuous utility function on F^h.

PROPOSITION 1. *There exists a continuous function* $u^h : F^h \to \Re$ *such that* $x^h P^h z^h$ *if and only if* $u^h(x^h) > u^h(z^h)$.

PROOF. Since the preference correspondence is both open-valued and has open lower sections, and F^h is compact, we claim there are best (b) and worst (a) elements of F^h. Suppose there is not a worst element in F^h. For each $y \in F^h$ there is some $x \in F^h$ with $y \in P^h(x)$. It follows that $\{P^h(x) : x \in F^h\}$ is an open cover of F^h. By compactness, it has a finite subcover $\{P^h(x_n)\}_{n=1}^N$. Take a worst element x^* of $\{x_1, \ldots, x_N\}$. By transitivity and irreflexivity, x^* cannot be in any of the $P^h(x_n)$. This contradicts the fact that the $P^h(x_n)$ cover F. It follows that a worst element of F^h exists. A similar argument using open lower sections shows that a best element exists.

Let $J = \{(1-\theta)a + \theta b : 0 \leq \theta \leq 1\}$ and define $u^h((1-\theta)a + \theta b) = \theta$. For arbitrary $x \in F^h$, consider $J \cap P^h(x)$ and $J \cap P^h_-(x)$ where $P^h_-(x) = \{y : xP^h y\}$ is the lower section of P^h at x. Both of these are open. Since J is connected, either one of these sets is empty, or there is a unique θ with x indifferent to $(1-\theta)a + \theta b$. In the latter case, define $u^h(x) = \theta$. If $J \cap P^h_-(x)$ is empty, x must be indifferent to the worst point of F^h and we set $u^h(x) = 0$. If $J \cap P^h(x)$ is empty, x must be indifferent to the best point of F^h and we set $u^h(x) = 1$.

For any $x \in F^h$, $\{y \in F^h : u^h(y) > u^h(x)\} = P^h(x)$ and $\{y \in F^h : u^h(y) < u^h(x)\} = P^h_-(x)$ by transitivity of P^h and R^h. Since both of these sets are open in F^h, the utility function is continuous. QED

Let u^h be a continuous utility function representing P^h on F^h. Let $U(\tilde{x})$ be the vector of utilities $(u^h(x^h))$ and $\tilde{F} = U(F)$. The set $\tilde{F}$ is the utility possibility set of the economy. Note $\tilde{F}$ is compact, hence bounded. For any coalition S define $V(S) = \{z \in \Re^n : z_h \leq u^h(x^h)$ for all $h \in S$ with $x^h \in C^h$ and $\sum_{h \in S} x^h \in Y\}$. This is the set of utility vectors whose projection on the utility subspace of the coalition S lies in or below the utility possibility set of S. Note that $V(S)$ is closed, non-empty, comprehensive ($z \in V(S)$ and $y \leq z$ implies $y \in V(S)$), and bounded above in $\Re^S$. Moreover, if $x \in V(S)$ and $x_h = y_h$ for all $h \in S$, then $y \in V(S)$.

Let $\mathcal{B}$ be a non-empty family of subsets of $\{1, \ldots, H\}$. Define $\mathcal{B}_h = \{S \in \mathcal{B} : h \in S\}$. A family $\mathcal{B}$ is *balanced* if there exist non-negative weights w_S with $\sum_{S \in \mathcal{B}_h} w_S = 1$ for all h. A

V-allocation is an element of $V(1,\ldots,H)$. A coalition S can *improve on* a V-allocation x if there is a $y \in V(S)$ with $y_h > x_h$ for all $h \in S$. The core of V is the set of V-allocations that cannot be improved upon by any coalition. The following theorem is from Scarf (1967).

THEOREM (SCARF). *Suppose* $\bigcap_{S\in\mathcal{B}} V(S) \subset V(1,\ldots,H)$ *whenever* $\mathcal{B}$ *is a balanced family. Then* V *has a non-empty core.*

THEOREM 1. *Under Assumptions 1-4 and 7 the economy* $\mathcal{E}$ *has a non-empty core.*

PROOF. Let $\mathcal{B}$ be a balanced family of sets with balancing weights w_S and let $(z_1,\ldots,z_H) \in \bigcap_{S\in\mathcal{B}} V(S)$. For each coalition S there are $x_S^h \in C^h$ for $h \in S$ with $\sum_{h\in S} x_S^h = y^S \in Y$ and $u^h(x_S^h) \geq z_h$ for all $h \in S$. Now consider $x^h = \sum_{S\in\mathcal{B}_h} w_S x_S^h$. Note that $u^h(x^h) \geq u^h(x_S^h) \geq z_h$ for some $S \in \mathcal{B}_h$ by convexity of preferences (Assumption 4). Also $\sum_{h=1}^H x^h = \sum_{h=1}^H \sum_{S\in\mathcal{B}_h} w_S x_S^h = \sum_{S\in\mathcal{B}} w_S \left(\sum_{h\in S} x_S^h\right) = \sum_{S\in\mathcal{B}} w_S y^S \in Y$. Thus $(z_1,\ldots,z_H)$ is feasible for the entire economy due to the feasiblity of $(x^1,\ldots,x^H)$. Therefore $(z_1,\ldots,z_H) \in V(1,\ldots,H)$. Scarf's theorem now shows the core of V is non-empty.

Now let $\tilde{z} = (z^1,\ldots,z^H)$ be in the core of V and take $\tilde{x} \in F$ with $U(\tilde{x}) \geq \tilde{z}$. It is clear that $\tilde{x}$ must be in the core of $\mathcal{E}$. Therefore the core of $\mathcal{E}$ is non-empty. QED

4. Edgeworth Equilibria

We now replicate the economy $\mathcal{E}$. In the r^{th} replica $\mathcal{E}_r$, there are r identical copies of each trader in $\mathcal{E}$. Each copy has the same trading set and preference correspondence as the original trader. We will use the idea of *equal treatment core*. The equal treatment core K_r is equal to the set of allocations in the core of the replicated economy $\mathcal{E}_r$ such that each trader in $\mathcal{E}_r$ who is a replica of a given trader in $\mathcal{E}$ undertakes the same net trade. Then an allocation in K_r may be represented by $\{x^h\}_r$ where $\{x^h\}$ is the allocation of net trades to the original traders and r is the number of replications. Let K_1 be the core of the economy $\mathcal{E}_1 = \mathcal{E}$. We must first show that the equal treatment core is not empty for any r.

Let $\{x^{hk}\}_r$ represent an arbitrary allocation in the r^{th} replica economy where the k^{th}

replicate of the h^{th} consumer in $\mathcal{E}$ undertakes the trade x^{hk}.

LEMMA 2. *If $\{x^{hk}\}_r$, $h = 1, \dots, H$, and $k = 1, \dots, r$, is an allocation in the core of $\mathcal{E}_r$, then, for h given, $x^{hj} I^h x^{hk}$ holds for all j and k.*

PROOF. Let the allocation $\{x^{hk}\}$ where $h = 1, \dots, H$ and $k = 1, \dots, r$ lie in the core of the economy $\mathcal{E}_r$. Suppose $x^{hj} I^h x^{hk}$ does not hold for some h, j, k. From convexity of $P^h(x)$, it follows that $R^h(x)$ is convex as the closure of $P^h(x)$. Consider a replicate of original consumer h with index $hj(h)$ that satisfies $x^{hk} R^h x^{hj(h)}$ for all $k = 1, \dots, r$. That is, $hj(h)$ has an allocation that is no better, and perhaps poorer, than the allocation of any other replicate of h. Consider the coalition $B = \{1j(1), \dots, Hj(H)\}$ of worst-off replicates, and give $hj(h)$ the net trade $x^h = (1/r) \sum_{k=1}^{r} x^{hk}$.

For each h it follows that $x^h R^h x^{hj(h)}$ by convexity of $R^h(x)$, and, if $x^{ik} P^h x^{ij(i)}$ for some i, k, then $x^i P^i x^{ij(i)}$. Now $\sum_{h=1}^{H} x^h = (1/r) \sum_{k=1}^{r} \sum_{h=1}^{H} x^{hk} \in Y$ since $\{x^{hk}\}$ is feasible and Y is a cone. Thus $\{x^h\}$ is a feasible allocation. Strong irreducibility and convexity allow us to spread the gain received by $ij(i)$ to all $hj(h)$. Let $I_1 = \{h : h \neq i\}$ and $I_2 = \{i\}$. This yields a feasible allocation $\{z^h\}$ with $z^h P^h x^h$ for $h \in I_1$ since x^i is not an extreme point of C^i. Take the convex combination $\{\lambda x^h + (1 - \lambda) z^h\}$ for $0 \leq \lambda \leq 1$. This is a feasible allocation, is preferred by all $h \in I_1$ to $\{x^h\}$, and, for λ sufficiently close to 1, is also preferred by i. It follows that $x^h P^h x^{hj(h)}$ for all $hj(h) \in B$. Thus B is an improving coalition, and $\{x^{hk}\}$ cannot be in the core. This contradiction shows $x^{hj} I^h x^{hk}$ for all h, j, k. QED

LEMMA 3. *The equal treatment core K_r of $\mathcal{E}_r$ is non-empty if the core of $\mathcal{E}_r$ is non-empty.*

PROOF. By Lemma 2, for any core allocation, the allocations received by the replicates of a given h in the original economy are indifferent. Then by the convex valuedness of the relation R^h, the equal treatment allocation in which each replicate of h receives $x^h = (1/r) \sum_{k=1}^{r} x^{hk}$, satisfies $x^h R^h x^{hk}$ for all h, k. Since there is no improving coalition for the allocation $\{x^{hk}\}$, there is certainly not one for $\{x^h\}_r$. Thus $\{x^h\}_r$ is in the core of $\mathcal{E}_r$. QED

Since the core of $\mathcal{E}_r$ is non-empty by Theorem 1, $K_r \neq \emptyset$ is an an immediate corollary of Lemma 3.

COROLLARY. *$K_r \neq \emptyset$ for any $r \geq 1$.*

Our immediate objective is to show there is a common element in all the K_r. Let $K = \bigcap_{r=1}^{\infty} K_r$. That is, $\{x^h\} \in K$ if $\{x^h\}_r \in K_r$ for all r. Any allocation in K is called an *Edgeworth equilibrium* (Aliprantis, Brown and Burkinshaw, 1987).

THEOREM 2. *K is non-empty.*

PROOF. Fix r. We first show K_r is closed. Since $K_r \subset F$ and F is compact, this will imply K_r is compact. Let $\{x^{h,s}\}_r \in K_r$, $s = 1, \ldots,$ be a sequence of allocations in K_r which converge to a limit $\{x^h\}_r$. Suppose $\{x^h\}_r$ is not in K_r. Let w^{hi} be a net trade for the i^{th} copy of the h^{th} original trader. There is an improving coalition B such that $w^{hi} \in P^h(x^h)$ for $hi \in B$ and $\sum_{hi \in B} w^{hi} \in Y$. By the fact that $P^h(x^h)$ has open lower sections, $w^{hi} \in P^h(x^{h,s})$ will hold when s is large. This implies that B is improving for $\{x^{h,s}\}_r$ for large s, and thus $\{x^{h,s}\}_r$ is not in K_r for large s. As this is contradicts the hypothesis, $\{x^h\}_r$ must be in K_r, which is therefore closed and compact.

It is clear that the K_r form a nested sequence of non-empty compact sets. Let $\{x^{h,r}\} \in K_r$. Since this sequence is contained in the compact set K_1, it has a limit point $\{x^h\}$. Moreover, since the tail of the sequence past r is in K_r, and K_r is closed, $\{x^h\} \in K_r$ for each r. Thus $\{x^h\} \in K$. QED

5. The Existence of Competitive Equilibrium

To prove the existence of equilibrium we will show that $(x^1, \ldots, x^H) \in K$ implies that there are p and y such that $(p, y, x^1, \ldots, x^H)$ is a competitive equilibrium. Let G be the convex hull of $\bigcup_{h=1}^{H} R^h(x^h)$. The key is to separate $G - Y$ from zero. The vector that performs this separation will be our equilibrium price vector.

The proof proceeds via a series of lemmas. We first show G is closed, hence $G - Y$

is closed. The next step is to show $G - Y$ does not intersect the negative orthant. We then construct a vector $\bar{c}$ which will be used to normalize prices. This allows us to define ϵ-price sets in ba which approximately separate $G - Y$ from 0. Lemma 6 shows these sets are non-empty.

We then show that the ϵ-price sets form a decreasing sequence of non-empty compact sets, and so have an intersection. We take an arbitrary vector in that intersection and use the Yosida-Hewitt theorem to throw away its purely finitely additive part, leaving us with an ℓ^1 price vector p^*. The normalization by $\bar{c}$ insures that p^* is non-zero.

The next three lemmas show first that truncations of p^* approximately separate $G - Y$ from 0, and then that p^* itself separates $G - Y$ from 0. At this point we only have a quasi-equilibrium. Finally, the point $\bar{x}$ together with strong irreducibility guarantees that the cheaper point condition is satisfied, and that the quasi-equilibrium is an equilibrium.

We start by showing G is closed. The lower bound on consumption plays an important role here.

LEMMA 4. *G is closed in s^n.*

PROOF. By Assumption 3, each C^h is bounded below by $\bar{z}$, and therefore G is bounded below by $\bar{z}$. Suppose $z^s \in G$ and $z^s \to z$. We must show that $z \in G$. Let $z^s = \sum_{h=1}^{H} w^{hs} = \sum_{h=1}^{H} \alpha_{hs} z^{hs}$ where $\alpha_{hs} \geq 0$, $\sum_{h=1}^{H} \alpha_{hs} = 1$ and $z^{hs} \in C^h$.

The α_{hs} are contained in the unit interval, so we may assume that they converge, by passing to a subsequence if necessary. If any of the w^{hs} were unbounded, the fact that each w^{hs} is bounded below would imply z^s is also unbounded, contradicting the convergence of z^s. Therefore each of the w^{hs} is bounded. Then the projection of each factor of the Cartesian product lies in compact set, and the w^{hs} lie in a compact set by Tychonoff's theorem. Thus we can choose a further subsequence where each of the w^{hs} converge, $w^{hs} \to w^h$.

Let $I = \{h : \alpha^h > 0\}$. For $h \in I$, $w^{hs}/\alpha_{hs} = z^{hs} \to w^h/\alpha_h \in C^h$. For $h \notin I$, $\alpha^{hs}\bar{z} \leq w^{hs}$ where $\bar{z}$ is the lower bound on C^h from Assumption 3. Taking the limit shows $0 \leq w^h$.

Now consider $w^h/\alpha_h + \sum_{h \notin I} w^h$, which is in $R(x^h)$ by periodwise monotonicity. Moreover, $\sum_{h \in I} \alpha_h (w^h/\alpha_h + \sum_{h \notin I} w^h) = \sum_{h=1}^{H} w^h = z$. Therefore $z \in G$. QED

We will need the following theorem adapted from Choquet (1962).

THEOREM (CHOQUET). *If Z is a product closed convex set in s^n which contains no straight lines, then for any two product closed subsets X, Y of Z, the sum $X + Y$ is closed.*

COROLLARY. *$G - Y$ is closed in s^n.*

PROOF. Recall $G - Y \subset \bar{z} + s^n_+ - Y$. Both $G - \bar{z}$ and Y are closed and contained in $s^n_+ - Y$. Also, $s^n_+ - Y$ contains no straight lines (Assumption 2). Thus we need only show $s^n_+ - Y$ is closed and apply Choquet's theorem.

Let $z^n \to z$ with $z^n \in Y - s^n_+$. Then there are $y^n \in Y$ with $z^n \leq y^n$. Since z^n converges, the y^n_0 are bounded below. But $y^n_0 \leq 0$, so the y^n_0 are bounded. By the proof of Lemma 1, this implies y^n_t is bounded for each t. Thus y^n has a convergent subsequence with limit $y \in Y$. Since $z^n \leq y^n$, $z \leq y$ and $z \in Y - s^n_+$. Thus $Y - s^n_+$ is closed. QED

LEMMA 5. *If $K \neq \emptyset$ then there is no $z \in G$ and $y \in Y$ such that $z - y < 0$ (i.e., $z - y \leq 0$ and $z_t - y_t < 0$ for some t).*

PROOF. Let $P(\tilde{x})$ be the convex hull of the $P^h(x^h)$. In light of the periodwise monotonicity assumption for preferences, it is sufficient to prove that there is no $z \in P(\tilde{x})$ and $y \in Y$ such that $z - y = 0$. In other words, it is sufficient to show that $Y \cap P(\tilde{x}) = \emptyset$.

Suppose not. There is a set of consumers I and weights α_i such that $\sum_{i \in I} \alpha_i z^i = y \in Y$, $\alpha_i > 0$, $\sum_{i \in I} \alpha_i = 1$, and $z^i \in P^i(x^i)$. An equivalent condition for z^i is that $\sum_{i \in I} \alpha_i (z^i - y) = 0$. For any positive integer k, let a^k_i be the smallest integer greater than or equal to $k\alpha_i$. For each $i \in I$, take $y^i \in C^i \cap Y$ and let $w^i_k = (k\alpha_i / a^k_i)(z^i - y^i) + y^i$.[2] As w^i_k is a convex combination of z^i and y^i, it lies in C^i. Moreover, $w^i_k \to z^i$ as $k \to \infty$. Since the preferred

[2] Except for this step, which has been modified to accommodate our consumption sets, we use the argument of Debreu and Scarf (1963). In their case $0 \in C^i \cap Y$, so they can take $y^i = 0$.

sets are relatively open (Assumption 4), we have $w_k^i \in P^i(x^i)$ for large k, which we now fix.

Also

$$\sum_{i \in I} a_i^k w_k^i = \sum_{i \in I} (k\alpha_i z^i - k\alpha_i y^i + a_i^k y^i) = ky + \sum_{i \in I} (a_i^k - k\alpha_i) y^i.$$

As $0 \leq a_i^k - k\alpha_i \leq 1$, $\sum_{i \in I} a_i^k w_k^i \in Y$. Thus the coalition of a_i^k consumers of type i for $i \in I$ can improve on $\tilde{x}$. The improving coalition can be formed if the original economy has been replicated $\max\{a_i^k\}$ times. This is in contradiction to the hypothesis. Therefore $Y \cap P(\tilde{x}) = \emptyset$. In other words, $\{x^h\}_r$ in the core for all r implies that the production set Y intersected with the convex hull of the the preferred trades of the original consumers, $P^h(x^h)$, is empty. QED

Choose α and τ such that $d^h = (x_0^h + e_0, x_1^h, \dots, x_\tau^h, \alpha \bar{x}_{\tau+1}^h, \dots) \in R^h(x^h) - Y$ by Assumption 7. Let $\bar{d}^h = d^h + (2e_0, 0, \dots)$. By the definition of d^h and monotonicity, $\bar{d}^h \in \ell^\infty \cap (G - Y)$, as is $\bar{c}^h = \bar{d}^h - \alpha \bar{x}^h \geq \bar{d}^h$. Note that $\bar{c}_t^h = 0$ for $t = \tau + 1, \dots$. Let $\bar{c} = (1/H) \sum_{h=1}^H \bar{c}^h = -\alpha \bar{x}/H + (1/H) \sum_{h=1}^H \bar{d}^h$. For $0 < \epsilon < 1$ we define the *price set* $S(\epsilon) = \{p \in ba_+ : p\bar{c} = 1 \text{ and } pw \geq -\epsilon \text{ for all } w \in (G - Y) \cap \ell^\infty\}$.

We take an indirect approach, using price vectors in ba. In contrast, Peleg and Yaari use price sets in s_+^n. We could try to to define $S(\epsilon)$ analogously in s_+^n, with $pw \geq -\epsilon$ for all $w \in G - Y$. However, Peleg and Yaari rely on the fact that a lower bound of the consumption set (or the set of possible net trades) is in the consumption set. This allows them to show their price sets are closed in s_+^n, when boundedness yields compactness. In our case, the lower bound is not in the set of possible net trades. We must replace it with a cheapest vector in the set of possible net trades for any given price vector. To avoid defining a different cheapest vector for every price vector, which causes technical difficulties, we settle on a price vector first. Our $S(\epsilon)$ will be compact and have the finite intersection property. We can then take a vector in the intersection of the $S(\epsilon)$, which can be modified to obtain the desired price vector in ℓ^1. The next lemma shows that $S(\epsilon)$ contains a finitely non-zero vector.

LEMMA 6. *For any ϵ, $0 < \epsilon < 1$, there is $p \in (s^n)^*$ such that $p \in S(\epsilon)$ with $|p_0| > 0$. Moreover, whenever $p \in S(\epsilon)$, $p_{t-1}u_{t-1} + p_t v_t \leq 0$ for all $(u_{t-1}, v_t) \in Y_t$ for all $t = 1, 2, \ldots$.*

PROOF. For $\epsilon > 0$ let $a(\epsilon) = (-\epsilon e_0, 0, 0, \ldots)$ where $e_0 = (1, \ldots, 1)$. By Lemma 5, $a(\epsilon) \notin G - Y$. By the Corollary to Lemma 4, $G - Y$ is closed in the product topology. Also, $\{a(\epsilon)\}$ is compact. By a separation theorem (Berge, 1963, p. 251) there is a continuous linear functional $f \in (s^n)^*$ with $f \neq 0$ such that $f(z) > f(a(\epsilon)) + \delta$ for any $z \in G - Y$ and some $\delta > 0$.

Any such f may be represented by a vector $p \in s^n$ with $p \neq 0$ but $p_t = 0$ for all but finitely many t. Thus

$$f(z) = pz = \sum_{t=0}^{\infty} p_t z_t \geq -\epsilon|p_0| + \delta,$$

for any $z \in G - Y$ and some $\delta > 0$. Periodwise monotonicity and the separation condition imply that $p \geq 0$. Thus we have for some $p \geq 0$, $p \neq 0$,

$$pz > -\epsilon|p_0| \qquad \text{for all } z \in G - Y. \tag{1}$$

Now $x^h \in R^h(x^h)$ for all h and $\sum_{h=1}^{H} x^h = y$ for some $y \in Y$ implies that $0 \in G - Y$. Setting $z = 0$ in (1) shows $|p_0| > 0$.

Since $d^h \in G - Y$, we have $pd^h > -\epsilon|p_0|$ by equation (1). Now $p\bar{c}^h = 2|p_0| + pd^h - \alpha p\bar{x}^h \geq (2 - \epsilon)|p_0|$ by equation (1). Thus $p\bar{c}^h > |p_0|$ for $0 < \epsilon < 1$. Define $\hat{p} = p/p\bar{c}$. Consider $z \in G - Y$. If $\hat{p}z \geq 0$, then $\hat{p}z \geq -\epsilon$. On the other hand, if $\hat{p}z < 0$, $\hat{p}z > pz/|p_0| \geq -\epsilon$. Thus $\hat{p} \in S(\epsilon)$ for $0 < \epsilon < 1$. This is the desired vector in $S(\epsilon)$.

Now let $p \in S(\epsilon)$. Since $Y = Y + Y$ and $0 \in G - Y$, it follows that $-Y \subset G - Y$. Therefore (1) implies $pz < \epsilon$ for all $z \in Y$. Since $\alpha z \in Y$ for any $\alpha > 0$, it follows that $pz \leq 0$ for all $z \in Y$. However, $(0, \ldots, 0, u_{t-1}, v_t, 0, \ldots) \in Y$ for all $t = 1, 2, \ldots$ and $(u_{t-1}, v_t) \in Y_t$. Therefore $p_{t-1}u_{t-1} + p_t v_t \leq 0$ for all $(u_{t-1}, v_t) \in Y_t$. QED

Recall that, for two topological vector spaces E and F, $\sigma(E, F)$ denotes the weakest topology on E such that the map $p \mapsto px$ is continuous on E for each $x \in F$. When E is the dual of a Banach space F, $\sigma(E, F)$ is referred to as the *weak** topology.

LEMMA 7. *The intersection of the price sets, $S = \bigcap_{0<\epsilon<1} S(\epsilon)$ is non-empty.*

PROOF. Lemma 6 implies $S(\epsilon)$ is non-empty. Furthermore, $S(\epsilon)$ is $\sigma(ba, \ell^\infty)$-closed since the inner product is $\sigma(ba, \ell^\infty)$-continuous, and the inequalities that define $S(\epsilon)$ are weak inequalities.

Now let p be an arbitrary element of $S(\epsilon)$. Consider the point $\bar{c} + \alpha\bar{x}/H$, which is in $G - Y$ by construction. Thus $p\bar{c} + \alpha p\bar{x}/H \geq -\epsilon$, or

$$-p\bar{x} \leq H(1+\epsilon)/\alpha. \tag{2}$$

Since $p \geq 0$, and $\bar{x}$ is constant and strictly positive, $S(\epsilon)$ is bounded by (2). By Alaoglu's theorem (see Appendix), weak* closed and bounded sets are weak* compact, so $S(\epsilon)$ is compact.

Finally, the argument of Theorem 2 shows that the intersection of the $S(\epsilon)$ is non-empty.

QED

Let $\bar{p} \in S$. The Yosida-Hewitt theorem (see Appendix) allows us to decompose $\bar{p}$ into the sum of an ℓ^1-vector p^*w and a pure charge. However, p^* makes sense when w is any element of $G - Y$, even if $w \notin \ell^\infty$. Define $w_{it}^- = 0$ when $w_{it} \geq 0$ and $w_{it}^- = w_{it}$ for $w_{it} < 0$. For $w \in C^h$, $\bar{z}^- \leq w^-$, hence $w^- \in \ell^\infty$. Now $p^*w = p^*(w - w^-) + p^*w^-$. The first term is either finite or $+\infty$ while the second term is finite. Thus p^*w is either finite or $+\infty$. Now consider $y \in Y$. We can write $y = \sum_{t=1}^{\infty} y(t)$ with $y(t) \in \bar{Y}_t \subset Y$. Each $y(t) \in \ell^\infty$ and $p^*y(t) \leq 0$ by Lemma 6. Thus p^*y is either finite or $-\infty$. Combining the results shows that p^*z is either finite or $+\infty$ for $z \in G - Y$.

The next step is to show that truncations of p^* approximately separate $G - Y$ (not just $(G - Y) \cap \ell^\infty$) from 0. We then use a limiting argument to show that p^* itself separates $G - Y$ from 0.

LEMMA 8. *Let $\epsilon > 0$. Then $\sum_{t=1}^{\tau} p_t^* z_t \geq -\epsilon$ for τ sufficiently large whenever $z \in G - Y$.*

PROOF. Let $\eta > 0$ and $z \in G - Y$. Write $z = \sum_{h=1}^{H} \alpha_h z^h$ with $z^h \in R^h(x^h) - Y$. For τ large, $\hat{z}^h = (\eta e_0 + z_0^h, z_1^h, \ldots, z_\tau^h, 0, \ldots) \in R^h(x^h) - Y$ by Assumption 7 and monotonicity. Apply $\bar{p} \in S$ to find $\eta |p_0^*| + \sum_{t=1}^{\tau} p_t^* \hat{z}_t = \bar{p}\hat{z} \geq 0$ for τ large. Rearrange and set $\epsilon = \eta |p_0^*|$ to obtain the result. QED

As $p^* \in \ell^1$ and $p^* \geq 0$, $p^* z$ achieves a finite minimum over C^h at some point in the $\sigma(\ell^\infty, \ell^1)$-compact set $\hat{C}^h = [\bar{z}, \bar{w}] \cap C^h$. Let $\hat{z}^h$ be a point in $\hat{C}^h$ where that minimum is achieved. Thus $p^* w^h \geq p^* \hat{z}^h$ for any $w^h \in C^h$. Moreover, since $C^h = \prod_{t=1}^{\infty} C_t^h$, $p_t^* z_t^h \geq p_t^* \hat{z}_t^h$ for any $z^h \in C_t^h$.

LEMMA 9. *The vector p^* satisfies $p^* z \geq 0$, for all $z \in G - Y$.*

PROOF. Let z be an arbitrary element of $G - Y$. Thus $z = w - y$ with $w \in G$ and $y \in Y$. We will show $p^* z \geq 0$ by showing $p^* w \geq 0$ and $p^* y \leq 0$. Now $w = \sum_{h=1}^{H} \alpha_h w^h$ for some $w^h \in R^h(x^h)$ and $\alpha_h \geq 0$ with $\sum_{h=1}^{H} \alpha_h = 1$. Define

$$w_t^h(\tau) = \begin{cases} w_0^h + \delta e_0 & t = 0 \\ w_t^h & t = 1, \ldots, \tau \\ \hat{z}_t^h & t = \tau + 1, \ldots \end{cases}$$

Since $C^h = C_0^h \times C_1^h \times \cdots$, we have $w^h(\tau) \in C^h$. Moreover, the fact that $P^h(x^h)$ is open relative to C^h implies $w^h(\tau) \in P^h(x^h)$ for τ sufficiently large. Define an increasing function $\tau(\delta)$ so that $w^h(\tau(\delta)) \in P^h(x^h)$. Now set $c_t^h = w_t^h - \hat{z}_t^h$ and define $c^h(\tau(\delta))$ by

$$c_t^h(\tau(\delta)) = \begin{cases} c_t^h & t = 0, \ldots, \tau(\delta) \\ 0 & t = \tau(\delta) + 1, \ldots \end{cases}$$

Note that $w^h(\tau(\delta)) = c^h(\tau(\delta)) + (\delta e_0, 0, \ldots) + \hat{z}^h$ and $p_t^* c_t^h(\tau(\delta)) \geq 0$ from the definition of $\hat{z}^h$.

Let $\epsilon > 0$. Now $\sum_{h=1}^{H} \alpha_h w^h(\tau(\delta)) \in G$, so Lemma 8 yields a τ_0 with

$$-\epsilon \leq \sum_{h=1}^{H} \alpha_h \sum_{t=0}^{\tau} p^* w_t^h(\tau(\delta)) = \delta |p_0^*| + \sum_{h=1}^{H} \alpha_h \sum_{t=0}^{\tau} p_t^* c_t^h(\tau(\delta)) + \sum_{h=1}^{H} \alpha_h \sum_{t=0}^{\tau} p_t^* \hat{z}_t^h$$

for $\tau \geq \tau_0$. Thus for $\tau \geq \tau_0, \tau(\delta)$

$$\delta |p_0^*| + \sum_{h=1}^{H} \alpha_h \sum_{t=0}^{\tau(\delta)} p_t^* c_t^h \geq - \sum_{h=1}^{H} \alpha_h \sum_{t=0}^{\tau} p_t^* \hat{z}_t^h - \epsilon.$$

Since $p^* \in \ell^1$ and $\hat{z}^h \in \ell^\infty$, the right hand side approaches $-\sum_{h=1}^{H} \alpha_h p^* \hat{z}^h - \epsilon$ as $\tau \to \infty$. Let $\delta \to 0$ so $\tau(\delta) \to \infty$. Then $\sum_{t=0}^{\tau(\delta)} p_t^* c_t^h \to p^* c^h$ since the sum is monotone increasing. It is possible that $p^* c^h = +\infty$. Now let $\epsilon \to 0$ to obtain $p^* w = \sum_{h=1}^{H} \alpha^h p^* c^h + p^* \sum_{h=1}^{H} \alpha_h \hat{z}^h \geq 0$. Note that $p^* w$ may be $+\infty$.

The argument in Lemma 6 shows $\bar{p}(0, \ldots, 0, u_{t-1}, v_t, 0, \ldots) \leq 0$ whenever $(u_{t-1}, v_t) \in Y_t$. But $\bar{p} = p^*$ on such vectors. Since $Y = \sum_t \bar{Y}_t$, $p^* y \leq 0$ for all $y \in Y$. Combining the two results shows $p^* z \geq 0$. QED

We claim that $(p^*, y, x^1, \ldots, x^H)$, where $y = \sum_{h=1}^{H} x^h$, is a competitive equilibrium for $\mathcal{E}$. By Lemma 9 $p^* x^h \geq 0$ for all h since $x^h \in G \subset G - Y$. But then $0 \leq \sum_{h=1}^{H} p^* x^h = p^* y \leq 0$ as in Lemma 6. Thus $p^* x^h = 0$ for all h and $p^* y = 0$.

To complete the proof that condition I holds we must show that $w^h \in P^h(x^h)$ implies $w^h \notin B^h(p^*) = \{z^h \in C^h : p^* z^h \leq 0\}$. Any point that is preferred by h to x^h must lie outside the budget set. A final result is:

Proposition 2. *If there is a $w^h \in C^h$ such that $pw^h < 0$ and $pz^h \geq 0$ for all $z^h \in P^h(x^h)$, then $pz^h > 0$ for all $z^h \in P^h(x^h)$.*

Proof. Suppose $z^h \in P^h(x^h)$ and $pz^h = 0$. Since $P^h(x^h)$ is open in C^h by Assumption 4, there is a point $y^h = \alpha w^h + (1 - \alpha) z^h$ such that $y^h \in P^h(x^h)$ and $py^h < 0$. This contradicts the hypothesis. Therefore, such a z^h can not exist. QED

THEOREM 3. *Under Assumptions 1-7 the economy $\mathcal{E}$ has a competitive equilibrium with prices in ℓ^1.*

PROOF. From Proposition 2 we see that Condition I will be completed if it can be proved that every consumer has a point z^h in the consumption set such that $p^* z^h < 0$. Consider $\bar{x}^h \in C^h - Y$. Then $\bar{x}^h = w^h - y$ for some $w^h \in C^h$ and $y \in Y$. Then $p^* w^h = p^* \bar{x}^h + p^* y \leq p^* \bar{x}^h$. Since $\bar{x} = \sum_{h=1}^{H} \bar{x}^h < 0$, at least one consumer has $p^* \bar{x}^h < 0$ and hence $p^* w^h < 0$.

Let I_1 be the set of indices h such that there is a $z^h \in C^h$ with $p^* z^h < 0$. Let I_2 be the complementary set of indices. We have just shown that I_1 is non-empty. Suppose that I_2 is non-empty. Use strong irreducibility to obtain $z_{I_2} \in C_{I_2}$ and $y' \in Y$ such that $y' = z_{I_1} + z_{I_2}$, where $z^h \in P^h(x^h)$ for all $h \in I_1$. Taking the inner product of both sides with p^* gives

$$p^* y' = p^* z_{I_1} + p^* z_{I_2}. \tag{3}$$

Also $p^* z_{I_1} > 0$ by Proposition 2 since $z^h \in P^h(x^h)$ for all $h \in I_1$. Now consider the $h \in I_2$. By strong irreducibility, z may be chosen so that $z^h \in \alpha C^h$ for some $\alpha > 0$, which yields $p^* z^h \geq 0$ by the definition of I_2. This means the right hand side of (3) is strictly positive. But $p^* y' \leq 0$ by Lemma 6. This contradiction implies that I_2 is empty. In other words, strong irreducibility of the economy implies that each consumption set has points with negative value if any consumption set has a point with negative value. Then by Proposition 2, $z^h \in P^h(x^h)$ implies $p^* z^h > 0$ for all h. This establishes the second part of Condition I for competitive equilibrium.

Condition II is implied by $p^* y = 0$ and the proof of Lemma 6, and condition III follows from the definition of a feasible trade. Therefore $(p^*, y, x^1, \ldots, x^H)$, where $y = \sum_{h=1}^{H} x^h$, is a competitive equilibrium of $\mathcal{E}$. QED

In fact, we have actually shown that any Edgeworth equilibrium is a competitive equilibrium. Standard arguments show that any competitive equilibrium is an Edgeworth equilibrium.

COROLLARY. *Under Assumptions 1-7 an allocation is an Edgeworth equilibrium of the economy $\mathcal{E}$ if and only if there is a price vector $p^* \in \ell^1$ for which it is a competitive equilibrium.*

6. Conclusion

Our results have been presented in a general model of intertemporal equilibrium. The advantage of this is that we can make assumptions that are economically natural. The intertemporal structure is more specific than we actually require.

There are two directions of generalization — production and consumption. The key properties of production that we used were the compactness of the feasible set, the fact that $Y - s_+^n$ is closed and contains no straight lines, and Bewley's exclusion assumption. Provided these hold, our proofs are still correct.

As for consumption, we have used more continuity than necessary. If we assume that preferences are represented by an upper semicontinuous utility function, Scarf's theorem still applies. Although lower semicontinuity is used in Lemmas 2, 5 and 9, and Proposition 2, all but Lemma 9 only need lower semicontinuity on line segments. Lemma 9 requires lower semicontinuity when the tail of a trading stream is replaced by the tail of $\hat{z}$. Both of these conditions are weaker than lower semicontinuity in the product topology.

These facts suggest that the same proof applies to continuous time capital accumulation models, using the compact-open topology. Becker, Boyd and Sung (1989) contains the relevant details on continuity and compactness in this topology.

As far as existence of equilibrium is concerned (rather than core equivalence), many cases covered by our model are also covered by Back (1988) when feasible paths are uniformly bounded. However, growing economies pose some extra problems in Back's model. The space must be renormed to allow for growth. But then the Adequacy assumption must also be strengthened if Back's proof is to apply. It is not enough to find $\bar{y}$ and $\bar{x}^h$ with $\bar{y} - \sum_{h=1}^{H} \bar{x}^h$ bounded above zero. The bound must grow at the maximal growth rate of the economy. In contrast, our model handles growing economies without modification.

Another possible application of our results is to a model with uncertainty, such as the model of Debreu and Hildenbrand presented in Bewley (1972). As long as there are countably many states of the world, our techniques could be used in such a setting. By weakening the continuity requirements, our methods may also apply when there are uncountably many states of the world.

Appendix: The Space *ba*

The dual of ℓ^∞ (under the norm topology) is the space *ba* of finitely additive measures. A *finitely additive measure* (or *charge*) is a mapping μ from a σ-field $\mathcal{F}$ to the real numbers such that $\|\mu\| = \sup_A |\mu(A)| < \infty$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$. We can define an integral in the usual way.

In our case, $\mathcal{F}$ is the collection of all subsets of the positive integers. The measure corresponding to $p \in (\ell^\infty)'$ is defined by $\mu(A) = p \cdot I_A$ where I_A is the indicator function of A. Note that $\|\mu\| = \|p\|$. When A_n are disjoint, $\sum_{n=1}^N I_{A_n}$ does not converge in ℓ^∞, since $\|\sum_{n=1}^N I_{A_n} - \sum_{n=1}^{N+1} I_{A_n}\| = \|I_{A_{N+1}}\| = 1$. It follows that we cannot conclude that the finitely additive measure μ is countably additive.

An example of a charge that is not countably additive is a Banach limit. Let $c = \{x \in \ell^\infty : \lim x_t \text{ exists}\}$. Let S be the shift operator $(Sx)_t = x_{t+1}$. A *Banach limit* is any linear functional on ℓ^∞ with $\liminf x_t \leq p(x) \leq \limsup x_t$, and $p(x) = p(Sx)$. (See Rudin, 1973, p. 82.) A Banach limit may be obtained by using the Hahn-Banach theorem to extend $p(x) = \lim x_t$ defined on c to all of ℓ^∞. For $x \geq 0$, $px \geq 0$ since $\liminf x_t = 0$, so a Banach limit must be positive. They also have the interesting property that $p(x) = 0$ whenever there is a T with $x_t = 0$ for $t > T$.

A non-negative finitely additive measure such that no non-zero countably additive measure ν obeys $0 \leq \nu \leq \mu$ is called *purely finitely additive* (a *pure charge*). A charge is pure if its positive and negative parts are pure charge. The measure generated by a Banach limit is a pure charge since it assigns zero measure to each point in the positive integers. Yosida and Hewitt (1952, p. 52) prove:

YOSIDA-HEWITT THEOREM. *Any non-negative finitely additive measure μ has the form $\mu = \mu_0 + \mu_\infty$ where μ_0 is non-negative and countably additive and μ_∞ is non-negative and purely finitely additive. Moreover, μ_0 and μ_∞ are unique.*

Since countably additive measures on the positive integers correspond to summable sequences, the Yosida-Hewitt theorem lets us uniquely decompose any element of $(\ell^\infty)'$ into a summable part and a purely finitely additive part.

We also require Alaoglu's Theorem, a form of which may be found in Rudin (1973, p. 66).

ALAOGLU'S THEOREM. *Any weak* closed, bounded set in the dual of a Banach space is weak* compact.*

Since ba is the dual of ℓ^∞, this implies that any $\sigma(ba, \ell^\infty)$-closed and bounded set is $\sigma(ba, \ell^\infty)$-compact. We should also note that such sets are not necessarily metrizable. As a result, sequences cannot characterize the topology. In particular, we cannot conclude that a bounded sequence in ba has a convergent subsequence, only that it has a convergent *subnet*.

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