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1. ***Introduction.*** We consider the problem of allocating a single indivisible good to one of several agents. Agents are assumed to have equal rights on this good and some amount of an infinitely divisible good is also available in order to compensate the agents that do not receive it. Alternatively, whoever receives the object may be asked to provide financial compensations to the others. As an example, consider an estate consisting of a house and some liquid and divisible assets such as cash and securities. The estate should be divided up so as to ensure that all heirs receive a fair share. How can this be done?[1]

A ***solution*** to the problem of fair allocation is a correspondence associating with each problem in some admissible class a non-empty subset of its set of feasible allocations, each of the points of which is interpreted as a recommendation for that problem. We investigate the existence of solutions satisfying several properties. These properties are of two types. Some are invariance properties; one of them states the invariance of the solution outcome when some of the agents leave the scene with their payoffs: after their departure with their alloted bundles, consider the problem of fairly allocating among the remaining agents the resources that they have just collectively received. The requirement is that the solution recommends that each of them receive the same bundle as the one that it had initially assigned to him. The other invariance property is the converse of the above. It allows us to deduce the desirability of an allocation on the basis of the desirability of its restrictions to subgroups of agents.

Other properties are monotonicity properties. One has to do with changes in the amount of the divisible good available: all agents should benefit if there

[1]For a general survey of the literature on the problem of fair allocation, see Thomson (1989).

is more of it. The other concerns changes in the number of agents: all agents initially present should be affected in the same direction if additional claimants enter the scene, resources being kept fixed.

The primary normative concept on which our analysis is based is that of an ***envy–free allocation***, that is, an allocation such that no agent would prefer somebody else's bundle to his own. We refer to the correspondence that associates with each economy its set of envy–free allocations as the "no–envy solution". Depending upon the exact specification of the model, including preferences, the amount of the divisible good available, and whether consumption sets are bounded below, there may or may not be envy–free allocations. Several alternative sets of conditions guaranteeing existence have been identified by Svensson (1983), Maskin (1987), and Alkan, Demange and Gale (1988). These results are applicable here. However, we will give a direct and elementary proof under the convenient assumptions imposed by Alkan, Demange and Gale (see Section 2 for details).

Our first task is to establish some elementary facts concerning the structure of the set of envy–free allocations. Svensson had already pointed out that if the number of agents is equal to the number of goods, an envy–free allocation is necessarily efficient; this result easily extends to the case of a number of agents greater than the number of goods, as examined here. A result that is not true in general but holds here is that at any two envy–free allocations, the same agent receives the object (excluding some degenerate situations).

When envy–free allocations exist, there typically is a continuum of them and the question of selection arises. We then look for selections satisfying the consistency conditions described above. Our main finding is that there is a unique best selection: it is the solution that systematically picks the envy–free allocation that is the worst for the winner of the object.[2] It can also be described thus: imagine preferences to be represented by the following utility functions: the utility of each bundle not containing the object is set equal to the amount of money it contains; the utility of each bundle containing the object is set equal to the amount of money that by itself would constitute an indifferent bundle. Then, using these representations, the solution we identified selects the unique efficient allocation at which the utilities of all agents are equal.[3] Note, however, that the solution depends only on the ***preferences*** of the agents.

This solution also satisfies the above monotonicity properties pertaining to changes in the amount of money available and in the number of agents. In fact, it is the only subsolution of the no–envy solution to be population monotonic.

Our analysis parallels that of an earlier paper (Tadenuma and Thomson, 1989), which dealt with the assignment of n indivisible objects among n agents. However, the conclusions reached here are surprisingly different. Most importantly, we had established there the non–existence of non–trivial consistent selections from the no–envy solution. Here, consistent selections exist. Only one is single–valued,[4] and that selection is very well–behaved from a number of other viewpoints.

2. ***The model.*** We consider economies with two goods. One is an indivisible good or "object", such as a job, a house, a contract, ... which is to be

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[2]In some degenerate situations, there could be several Pareto–indifferent allocations with that property.

[3]See footnote 1.

[4]Again, see footnote 1.

attributed to exactly one person. The other is an infinitely divisible good, called "money", which can be used to compensate the others for not receiving the object. We will refer to the person who receives the object as "the winner" and to the others as "the losers". The losers will be said to receive the "null" object.

There is an infinite set of "potential agents", indexed by the integers $\mathbb{N}$. For each $i \in \mathbb{N}$, the consumption space of agent i is the set of pairs $(\delta_i,m_i) \in \{0,1\}\times\mathbb{R}$: $(0,m_i)$ is the bundle containing only m_i units of money and $(1,m_i)$ is the bundle made up of the object together with m_i units of money. Note that no restrictions in sign are imposed on the consumptions of money. For each $i \in \mathbb{N}$, agent i's preferences are assumed to admit a numerical representation u_i: $\{0,1\}\times\mathbb{R} \to \mathbb{R}$ that is continuous and strictly increasing in its second argument, and such that for all $\delta \in \{0,1\}$, $\lim_{m_i\to\infty} u_i(\delta,m_i) = \infty$. An ***economy*** is a list $e = (Q,M,u_Q)$ of (i) a finite set of agents $Q \subset \mathbb{N}$ with $|Q| \geqq 2$, (ii) resources comprising the object and some amount of money $M \in \mathbb{R}$, and (iii) a list $u_Q = (u_i)_{i\in Q}$ of utility functions, one for each of the members of Q. ***A feasible allocation for $e = (Q,M,u_Q)$*** is a pair $(\delta,m) \in [\{0,1\}\times\mathbb{R}]^{|Q|}$ such that $\sum_Q(\delta_i,m_i) = (1,M)$, agent i's bundle being (δ_i,m_i): $\delta_i = 1$ if agent i is the winner and $\delta_i = 0$ otherwise; m_i is the amount of money he receives. The relation $\sum_Q \delta_i = 1$ tells us that exactly one agent receives the object and the relation $\sum_Q m_i = M$ that all of the money available, and no more, is distributed. We also use the notation z for allocations, z_i being the i^{th} component of z. Given a domain of economies $\mathscr{E}$, ***a solution on*** $\mathscr{E}$ associates with every $e \in \mathscr{E}$ a non-empty subset of $Z(e)$, its set of feasible allocations.

Let P be the ***Pareto solution***: Given $e = (Q,M,u_Q)$, $P(e) \equiv \{z \in Z(e) | \nexists\ z' \in Z(e)$ with $[\forall\ i \in Q,\ u_i(z_i') \geqq u_i(z_i)$ and $\exists i \in Q$ s.t. $u_i(z_i') > u_i(z_i)]\}$.

This model has been considered by Luce and Raiffa (1957), Kolm (1972), Crawford and Heller (1979), van Damme (1987), and Moulin (1990). A related one is obtained by assuming that there are arbitrary numbers of indivisible goods and agents. That model, which we will refer to as "the general model", was analyzed by Svensson (1983, 1988), Maskin (1987), Alkan, Demange and Gale (1988), and Tadenuma and Thomson (1989). The purpose of this paper is to pursue the analysis of Tadenuma and Thomson (henceforth TT) in the situation just described. As we will see, many of the conclusions we had reached for the general model do not hold here.

We are interested in achieving allocations that are appealing from the viewpoint of fairness. We take as a primitive concept that of an envy-free allocation, that is, an allocation such that no agent would prefer someone else's bundle to his own:

Definition (Foley, 1967). Given $e = (Q,M,u_Q)$, the allocation $z \in Z(e)$ is ***envy-free for e*** if for no $i, j \in Q$, $u_i(z_j) > u_i(z_i)$. Let $N(e)$ be the set of envy-free allocations of e.

Under the assumptions on preferences made above, $N(e) \neq \emptyset$, as shown by Alkan, Demange, and Gale (1988). Instead of appealing to their general result, we can however give a direct and elementary proof. This proof has the added advantage of identifying the envy-free allocation that will play the central role in the analysis to follow.[5]

Theorem 1. The set of envy-free allocations is non-empty.

[5]We owe this proof to D. Gale. It is of interest that the solution to which we had been led by following an axiomatic approach can also be used to provide a simple proof of the existence of envy-free allocations.

Proof. For each i, let m_i be such that $u_i(1,m_i) = u_i(0,(M-m_i)/(|Q|-1))$; under our assumptions, m_i exists and is unique. Now, let i_0 be such that $m_{i_0} \leqq m_i$ for all $i \in Q$. The allocation $z \in Z(e)$ defined by $z_{i_0} = (1,m_{i_0})$ and $z_j = (0,(M-m_{i_0})/(|Q|-1))$ is envy-free.

Q.E.D.

If money holdings were restricted to be non-negative, or more generally bounded below, then existence would require that there be a sufficient amount of money available to compensate the losers.

We now turn to our main objective which is to identify well-behaved subsolutions of the no-envy solution. Indeed, economies frequently admit many envy-free allocations, in fact, a continuum of such allocations, and in such cases the question naturally arises whether some of these allocations are more desirable than others from the viewpoint of fairness. Achieving single-valued selections would of course be best but that might be difficult. The reason is that on the domains commonly analyzed in economics, most solutions φ have the property that if an allocation z is "φ-optimal for some economy e", (that is, belongs to $\varphi(e)$), other (often, all) allocations that are Pareto-indifferent to it (that is, indifferent to it for all agents), belong to $\varphi(e)$ as well. In our search for a solution that would make a precise recommendation, it is therefore natural to expect no more than single-valuedness "up to Pareto-indifference". However, solutions that are single-valued up to Pareto-indifference are themselves actually quite rare. It is only because of the special structure of the problem under study that we are entitled to be somewhat more optimistic.

Here, we will consider solutions satisfying the following very mild condition introduced in TT. This condition is satisfied by all existing solutions, and all the new solutions that we will encounter below also satisfy it. It says that if z is φ–optimal for some e and a permutation of some of the components of z leaves all agents indifferent, then that permuted allocation is also φ–optimal for e.

Neutrality: For all $e = (Q,M,u_Q)$, for all $z \in \varphi(e)$, for all permutations π: $Q \to Q$, if $u_i(z_i) = u_i(\pi_i(z))$ for all i (we will say that ***$\pi(z)$ is obtained from z by an indifferent permutation*** and we will write $z \simeq \pi(z)$), then $\pi(z) \in \varphi(e)$.

The following lemmas are fundamental to the understanding of the problem under study.

Lemma 1. If $z \in N(e)$, then $z \in P(e)$.

The proof that no–envy implies efficiency follows Svensson (1983) who proved this property for the general model when the number of indivisible goods is equal to the number of agents. For completeness, we provide it in the appendix.

The following lemma sheds some light on the structure of the set of envy–free allocations. It states that the assignment of the indivisible good is essentially the same at all envy–free allocations. This was already observed for the two–person case by TT. The lemma slightly generalizes this observation.

Lemma 2. If $z, z' \in N(e)$, then $\delta = \delta'$ or $z \simeq z'$.

Proof. Let $z, z' \in N(e)$ and suppose that $\delta \neq \delta'$. To fix the ideas, suppose that $\delta_1 = 1$ and $\delta_2' = 1$. If $m_2' > m_1$, then $m_1' < m_2$ by feasibility, and therefore, $u_1(z_1') < u_1(z_2) \leqq u_1(z_1) < u_1(z_2')$, which says that agent 1 envies agent 2 at z′. If $m_2' < m_1$, then $m_1' > m_2$ by feasibility, and therefore $u_2(z_2') < u_2(z_1) \leqq u_2(z_2) < u_2(z_1')$, which says that agent 2 envies agent 1 at z′. We conclude that $m_2' = m_1$ and $m_1' = m_2$. This implies on the one hand that $z_i' = z_i$ for all $i \neq 1,2$ and on the other hand, since $z, z' \in N(e)$, that $u_1(z_1) = u_1(z_1')$ and $u_2(z_2) = u_2(z_2')$. Altogether, we have $z \simeq z'$.

Q.E.D.

A consequence of Lemma 2 is that the set of envy–free allocations can essentially (i.e. up to indifferent permutations) be parametrized by how much money the winner receives, w. There is indeed an interval $[\underline{w},\overline{w}]$ of amounts of money received by the winner at allocations in N(e).[6] The allocation attributing $(1,\underline{w})$ to him and $(0,(M-\underline{w})/(|Q|-1))$ to each of the losers is the worst for him and the best for the losers in that set, whereas the opposite holds for the allocation attributing $(1,\overline{w})$ to him and $(0,(M-\overline{w})/(|Q|-1))$ to each of the losers. The former is obtained when the winner is indifferent between what he receives and what each of the losers receives. At the latter, (at least) one of the losers is indifferent between what he receives and what the winner receives.

Another equity notion that has played an important role in the literature is that of an egalitarian–equivalent allocation. As we will see, this notion will emerge naturally here.

Definition (Pazner and Schmeidler, 1978). Given $e = (Q,M,u_Q)$, the allocation $z \in Z(e)$ is ***egalitarian–equivalent for e*** if there is $z_0 \in \{0,1\}\times\mathbb{R}$ such that for all i, $u_i(z_i) = u_i(z_0)$. Let ***E*(e)*** be the set of egalitarian–equivalent allocations of e and ***E*P(e)*** its intersection with the set of efficient allocations.

***Lemma* 3.** Except in degenerate situations, there are two egalitarian–equivalent and efficient allocations.[7]

[6] In some degenerate situations, $\underline{w}$ and $\overline{w}$ are equal. This fact, which is exploited in the proof of Proposition 1, is explained there in detail.

[7] The most serious degeneracy occurs when the two reference bundles are judged

The proof of Lemma 3 is similar to the proof of the existence in classical exchange economies of egalitarian–equivalent and efficient allocations with reference bundle in a prespecified direction. Here, there are exactly two possible reference bundles, one containing the object, one not. Lemma 3 is also a consequence of Svensson's (1983) existence and characterization proof for the general model. For completeness, we give a proof in the appendix for our special case.

In what follows the egalitarian–equivalent and efficient allocation(s) whose associated reference bundle contains the null object will play a crucial role:

Definition. Given $e = (Q,M,u_Q)$, let $\varphi^*(e) \equiv \{z \in P(e) | \exists\ m_0 \text{ s.t. } \forall\ i,\ u_i(z_i) = u_i(0,m_0)\}$.

The solution φ^* is a member of the family, commonly considered in welfare economies, of solutions defined in the following three steps: (i) choose a numeraire good together with some basket of the other goods, (ii) assign to each bundle in an agent's consumption space a "utility" equal to the quantity of numeraire necessary, when combined with the complementary basket, to make the individual indifferent to that bundle, (iii) choose allocations that are efficient and at which utilities are equal. Although there are many possible choices of a numeraire good and of a complementary basket, the choice of the "zero" complementary basket is often made. That choice corresponds in the current model to the null object, and it gives us φ^*.

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indifferent by all agents. This can occur only if for some m_0, and for all i, $u_i(1,m_0) = u_i(0,(M-m_0)/(|Q|-1))$. Then, there are $|Q|$ allocations in $E^*P(e)$ of the form $z_W = (1,m_0)$, $z_\ell = (0,(M-m_0)/(|Q|-1))$ for all ℓ, the winner being any one of the $|Q|$ agents, with either $(1,m_0)$ or $(0,(M-m_0)/(|Q|-1))$ serving as reference bundle. These allocations are all Pareto–indifferent to each other. Other, less severe, degeneracies occur when $z \in E^*P(e)$ and there exists z' with $z' \simeq z$. But again, any such z' will be Pareto–indifferent to z.

The solution φ^* has the advantage of making very precise recommendations. Formally, it satisfies the following property:

Single–valuedness up to indifferent permutations: For all $e = (Q,M,u_Q)$, for all $z, z' \in \varphi(e)$, $z \simeq z'$.

Other properties of φ^* are given in the next lemma.

***Lemma* 4.** If $z \in \varphi^*(e)$, then $z \in N(e)$. Moreover, z is the worst allocation in $N(e)$ for the winner.[8]

Proof At z, all losers receive a bundle equal to the reference bundle $z_0 = (0,m_0)$ and therefore no envy exists between them. Since the winner is indifferent between his bundle and z_0, which is the common bundle of all losers, he does not envy any of them. Finally, assume that a loser were to envy the winner. Then, by exchanging bundles between the two of them, and again using the fact that the winner is indifferent between his bundle and z_0, we could achieve a Pareto improvement, but this is impossible since $\varphi^* \subseteq P$.

The fact that the winner is indifferent between his bundle and the losers' common bundle means that z is the worst allocation in $N(e)$ for the winner.

Q.E.D.

Lemma 4 is of particular interest since it shows that egalitarian–equivalence and no–envy are compatible. In the case of arbitrary numbers of objects and agents, this is not true, as shown in Thomson (1990b). Note that it is precisely by identifying a point in $\varphi^*(e)$ that we established the existence of envy–free and efficient allocations in Theorem 1.

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[8]Again, in some degenerate cases, there could be several distinct allocations at which the amount of money associated with the object is the smallest in the envy–free set. Then, the envy–free set is a singleton, up to indifferent permutations, as will be made clear in the proof of Proposition 1.

3. ***Variable population.*** Here we turn to an examination of the variable population case. For the purposes of this section, we slightly enlarge the domain of admissible economies by allowing economies in which there is only money to distribute. In such an economy, the problem of fair allocation is trivial: the no–envy solution recommends the unique allocation at which all agents receive the same amount of money. The resources available in a given economy are now denoted (Δ,M) with $\Delta = 1$ if the object is present, and $\Delta = 0$ otherwise. A typical economy is denoted $(Q,(\Delta,M),u_Q)$. Feasibility of an allocation $z = (\delta_i,m_i)_{i\in Q}$ for $(Q,(\Delta,M),u_Q)$ is written as $\sum_Q(\delta_i,m_i) = (\Delta,M)$ or $\sum_Q z_i = (\Delta,M)$.

We will look for subsolutions of the no–envy solution with the property that the desirability of an allocation is left unaffected by the departure of some of the agents with their alloted bundles. Specifically, given an economy e and given an allocation z which is φ–optimal for it, consider any subgroup of agents and ask whether the restriction of the allocation to that subgroup constitutes a φ–optimal way of allocating among them the resources they have collectively received (the object plus some amount of money if the subgroup includes the winner, only some amount of money otherwise). If the answer is always yes, the solution is said to be *consistent.*[9]

Consistency: For all $e = (Q,(\Delta,M),u_Q)$, for all $Q' \subset Q$, for all $z \in \varphi(e)$, $z_{Q'} \in \varphi(Q',\sum_{Q'} z_i,u_{Q'})$.

***Lemma* 5.** The Pareto solution, the no–envy solution, and the solution φ^* satisfy *consistency.*

[9]For a survey of the various uses to which the "Consistency Principle" has been put in recent literature, see Thomson (1989a).

Proof. The solutions P and N enjoy this property on the general domain. Checking that φ^* also does is easy and we omit the proof.

Q.E.D.

Lemma 5 is very encouraging. Indeed, on the general domain, TT showed that no proper selection from the no–envy solution satisfies *consistency* and *neutrality*. Not only is φ^* a selection from the no–envy solution but, as we noted earlier, φ^* is single–valued up to indifferent permutations. We show below that φ^* is in fact the only selection from the no–envy solution satisfying all three properties. This characterization of φ^* follows directly from the following result.

***Proposition* 1.** If $\varphi \subseteq$ N satisfies *consistency* and *neutrality*, then $\varphi \supseteq \varphi^*$.

Proof. Let $e = (Q,(\Delta,M),u_Q)$ be given and $z \in \varphi^*(e)$. Let $i_0 \notin Q$ be given and u_{i_0} be such that $u_{i_0}(z_i) = u_{i_0}(z_j)$ for all i, j $\in$ Q. Let ℓ (resp. w) be the amount of money received by each loser (resp. the winner) at z. Let $e' = (Q\cup\{i_0\}, (\Delta,M+\ell),u_{Q\cup\{i_0\}})$. It is clear that $z' \in [\{0,1\}\times\mathbb{R}]^{|Q\cup\{i_0\}|}$ defined by $z'_Q = z$ and $z'_{i_0} = (0,\ell)$ belongs to $N(e')$.

In fact, if $z'' \in N(e')$, then $z'' \simeq z'$. To see this, let ℓ'' (resp. w") be the amount of money received by each loser (resp. the winner) at z". First suppose that $\ell'' > \ell$, which implies $w'' < w$. For all $i \in Q\cup\{i_0\}$, $u_i(1,w'') < u_i(1,w) \leqq u_i(0,\ell) < u_i(0,\ell'')$ and therefore no i could be the winner at z" without envying the losers. Suppose next that $\ell'' < \ell$, which implies $w'' > w$. If i_0 is not the winner at z", he will envy that winner. If he is, then the winner at z', who is one of the losers at z", will envy him at z".

Since $\varphi \subseteq N$ and φ satisfies *neutrality*, it now follows that $z' \in \varphi(e')$. Then, by *consistency*, $z'_Q = z \in \varphi(Q,\sum_Q z_i,u_Q) = \varphi(e)$.

Q.E.D.

The following theorem is a direct corollary of Proposition 1.

***Theorem* 2.** The solution φ^* is the only subsolution of the no–envy solution satisfying *consistency, neutrality,* and *single–valuedness up to indifferent permutations.*[10]

To see that the requirement of single–valuedness is important, note that φ^* is not the only proper subcorrespondence of N satisfying *consistency* and *neutrality.* Indeed, consider the following family: let $k \in \mathbb{N}$ be given and for each $e = (Q,(\Delta,M),u_Q)$ let $\varphi^k(e) = \varphi^*(e)$ if $|Q| \geq k$ and $\varphi^k(e) = N(e)$ if $|Q| < k$. Any solution so defined satisfies *consistency* and *neutrality.*

A weakening of *consistency* is obtained by restricting its application to subgroups of cardinality 2. Let us call this condition *bilateral consistency.* On the general domain there is an important difference between *consistency* and *bilateral consistency.* Indeed, TT showed that on that domain, there is no proper subsolution of the no–envy solution satisfying *consistency* and *neutrality* whereas there is an infinite class of proper subsolutions satisfying *bilateral consistency* and *neutrality.* These solutions coincide with the no–envy solution for 2–person economies, but otherwise, they are arbitrary up to indifferent permutations; therefore, the requirement of *single–valuedness up to indifferent permutations* cannot be met. Here, it can. And in fact, when it is imposed,

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[10]Because φ^* satisfies *neutrality*, if $\varphi \supseteq \varphi^*$ and $\varphi \neq \varphi^*$, then φ cannot satisfy *single–valuedness up to indifferent permutations.*

in conjunction with *bilateral consistency* and *neutrality*, it is satisfied by only one solution.

***Theorem* 3.** The solution φ^* is the only subsolution of the no–envy solution satisfying *bilateral consistency, neutrality,* and *single–valuedness up to indifferent permutations.*

Proof. If, in the proof of Proposition 1, the point of departure is a two–person economy e, then the conclusion that $\varphi(e) \supseteq \varphi^*(e)$ can be reached by applying *bilateral consistency* instead of *consistency.* If φ is required to satisfy *single–valuedness up to indifferent permutations* as well, then $\varphi = \varphi^*$ on the class of two–person economies.

To show that this conclusion holds for all cardinalities, suppose by way of contradiction that for some $e = (Q,(\Delta,M),u_Q)$ with $|Q| > 2$, there exists $z \in \varphi(e)$ that does not belong to $\varphi^*(e)$. Let i be the winner at z, and j be one of the losers. Then $u_j(z_i) > u_j(z_j)$. Let $Q' = \{i,j\}$ and $e' = (Q',z_i+z_j,u_{Q'})$. Then, by *bilateral consistency,* $z_{Q'} = (z_i,z_j) \in \varphi(e')$. Since $|Q'| = 2$, $\varphi(e') = \varphi^*(e')$. But, $z_{Q'} \notin \varphi^*(e')$.

Q.E.D.

We now consider a condition that is dual to *consistency.* It says that if a feasible allocation is such that its restriction to any subgroup of cardinality 2 constitutes a φ–optimal way of allocating the resources this subgroup has received, then it is itself φ–optimal for the whole economy.

Converse consistency. For all $e = (Q,(\Delta,M),u_Q)$, for all $z \in Z(e)$, if for all $Q' \subset Q$ with $|Q'| = 2$, $z_{Q'} \in \varphi(Q',\sum_{Q'} z_i,u_{Q'})$, then $z \in \varphi(e)$.

***Proposition* 2.** The Pareto solution, the no-envy solution, the egalitarian-equivalence solution and its intersection with the Pareto solution, and the solution φ^* all satisfy *converse consistency*.

Proof. The solution N satisfies this property on the general domain.[11] The proof that φ^* does too is straightforward. Turning now to E^*, let $e = (Q,(\Delta,M),u_Q)$ and $z \in Z(e)$ be given. For simplicity, assume $Q = \{1,2,\ldots,q\}$. For each $i \in Q$, let m_{0i} and m_{1i} be such that $u_i(z_i) = u_i(0,m_{0i}) = u_i(1,m_{1i})$. By hypothesis, $m_{01} = m_{02}$ or $m_{11} = m_{12}$. Assume that (i) $m_{01} = m_{02} = \ldots = m_{0k}$ or (ii) $m_{11} = m_{12} = \ldots = m_{1k}$ for $k \in \mathbb{N}$ such that $2 \leq k \leq q-1$. By hypothesis, (iii) $m_{0(k+1)} = m_{01}$ or (iv) $m_{1(k+1)} = m_{11}$. If (i) holds and (iii) does not hold, then for no $i \in \{1,\ldots,k\}$, $m_{0(k+1)} = m_{0i}$, and by hypothesis, we must have $m_{1(k+1)} = m_{1i}$ for every $i \in \{1,\ldots,k\}$. Thus, $m_{11} = m_{12} = \ldots = m_{1k} = m_{1(k+1)}$. Similarly, if (ii) holds and (iv) does not hold, then $m_{01} = m_{02} = \ldots = m_{0k} = m_{0(k+1)}$. We have shown that $m_{01} = \ldots = m_{0k} = m_{0(k+1)}$ or $m_{11} = \ldots = m_{1k} = m_{1(k+1)}$. By induction we conclude that $m_{01} = \ldots = m_{0q}$ or $m_{11} = \ldots = m_{1q}$, and therefore $z \in E^*(e)$.

To prove that P satisfies *converse consistency* as well, let $e = (Q,(\Delta,M),u_Q)$ and $z \in Z(e)$ be such that for all $Q' \subset Q$ with $|Q'| = 2$, $z_{Q'} \in P(Q',\sum_{Q'} z_i,u_{Q'})$. To show that $z \in P(e)$, suppose by way of contradiction that there exists $z' \in Z(e)$ such that for all $i \in Q$, $u_i(z'_i) \geqq u_i(z_i)$, strict inequality holding for at least one i. If $\delta = \delta'$, then z' differs from z only in that all agents receive at least as much money at z' as at z and at least one agent strictly more, and this contradicts the assumption that $z, z' \in Z(e)$. Therefore, $\delta \neq \delta'$: this means that the winner at z is not the same as the winner at z'. Let Q' be the two-person group consisting of the winners at z

[11]Note that it is because subeconomies not containing the winner are allowed that we can conclude that the losers all receive the same amount.

and z'. By hypothesis, $z_{Q'} \in P(Q', \sum_{Q'} z_i, u_{Q'})$. Therefore, since (i) $u_i(z_i') \geqq u_i(z_i)$ for all $i \in Q'$, it follows that (ii) $\sum_{Q'} m_i' \geqq \sum_{Q'} m_i$. Then, (iii) $\sum_{Q\setminus Q'} m_j' \leqq \sum_{Q\setminus Q'} m_j$. Now, if inequality (i) is strict for at least one $i \in Q'$, then inequality (ii) is strict, which in turn implies that inequality (iii) is strict. This is impossible since all agents $j \notin Q'$ receive the null object at z and at z' and therefore Pareto–domination of z by z' implies $\sum_{Q'} m_j' \geqq \sum_{Q'} m_j$. If $u_i(z_i') > u_i(z_i)$ for some $i \notin Q'$, then (iii) is violated.

Finally, to prove that E*P satisfies *converse consistency*, we simply note that the intersection of two correspondences satisfying *converse consistency* also does, and we use our earlier conclusions concerning E^* and P.

Q.E.D.

Proposition 2 establishes another important difference between the general domain and the domain considered here. On the general domain, neither the Pareto solution nor its intersection with the egalitarian–equivalent solution satisfy *converse consistency* (again, see TT).

4. ***Monotonicity properties of φ^*.*** Having established the crucial role of the solution φ^*, we now engage in a further analysis of it. We focus on two monotonicity properties, which pertain to changes in the amount of money available on the one hand, and in the number of agents on the other. The solution φ^* satisfies both. Moreover, it is "essentially" the only one to satisfy the latter.

4.1. ***Money monotonicity.*** First, we consider the requirement that all agents benefit from increases in the amount of the divisible good available. The implication of this requirement for classical economies were examined by Roemer (1986), Chun and Thomson (1988) and Moulin and Thomson (1988).

Money–monotonicity: For all $e = (Q,M,u_Q)$ and $e' = (Q',M',u'_Q)$ with $(Q,u_Q) = (Q',u'_Q)$ and $M' > M$, for all $z \in \varphi(e)$ and $z' \in \varphi(e')$, $u_i(z'_i) \geqq u_i(z_i)$ for all $i \in Q$.

The solution φ^* as well as the solution φ^{**}, which chooses for each economy its set of egalitarian–equivalent and efficient allocations associated with a reference bundle of the form $(1,m)$, satisfy this property.[12] (Note that φ^{**} is single–valued up to indifferent permutations, just like φ^*). The existence of money monotonic selections from the no–envy solution on the general domain is investigated by Alkan, Demange, and Gale (1988). Here, we simply record that φ^* satisfies the property.

***Proposition* 3.** The solution φ^* satisfies *money–monotonicity.*

4.2. ***Population monotonicity.*** Finally, we consider the following property: an increase in the number of agents, resources being kept fixed, should affect all agents initially present in the same direction.

Population–monotonicity: For all $e = (Q,M,u_Q)$ and $e' = (Q',M',u'_{Q'})$ with $Q \subset Q'$, and $(M,u_Q) = (M',u'_Q)$, for all $z \in \varphi(e)$ and $z' \in \varphi(e')$, $u_i(z'_i) \leqq u_i(z_i)$ for all $i \in Q$ or $u_i(z'_i) \geqq u_i(z_i)$ for all $i \in Q$.

Thomson (1983) considered the following requirement: if the number of agents increases but resources remain fixed, all agents initially present lose.

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[12]They are not the only ones.

This property cannot be satisfied here. The following example[13] shows that the arrival of additional agents may have to benefit some of the agents originally present.

Example. Let $Q = \{1,2,3\}$ and u_Q be such that

$u_1(1,m) = u_1(0,m+1)$ for all m

$u_2(1,m) = u_3(1,m) = u_2(0,m+6) = u_3(0,m+6)$ for all m.

Let $e = (\{1\},0,u_1)$ and $e' = (Q,0,u_Q)$. Then, $N(e) = \{(1,0)\}$ and $N(e') = \{((0,2),(1,-4),(0,2)),((0,2),(0,2),(1,-4))\}$. Agent 1 benefits from the arrival of agents 2 and 3.

However, it is possible to guarantee that the agents initially present either all lose or all gain, as required by *population monotonicity.* This weaker version, which was examined by Chun (1986) in the context of quasi–linear social choice problems, is natural in particular when there are public goods or other forms of externalities. Alkan (1990) analyzes yet other monotonicity properties.

***Proposition* 4.** The solution φ^* satisfies *population monotonicity.*

Proof. The solution φ^* chooses allocations at which the canonical utilities of all agents present are equal. So as new agents come in, the utilities of the agents initially present are all equal to a larger number or all equal to a smaller number.

Q.E.D.

The solution φ^W, "dual" to φ^*, which associates with every economy the envy–free allocation the most favorable to the winner, is worth investigating.

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[13]We owe this example to A. Alkan (1988). Moulin (1990) has constructed similar examples.

However, it does not satisfy *population monotonicity*, as established by the next example.[14]

Example. Let $Q = \{1,2\}$, $Q' = \{1,2,3\}$ and

$u_1(0,m) = u_1(1,m)$ for all m,

$u_2(0,m) = u_2(1,m+3)$ for all m,

$u_3 = u_1$.

Let $e = (Q,3,u_Q)$ and $e' = (Q',3,u_{Q'})$. We have $\varphi^W(e) = ((1,3),(0,0))$ and $\varphi^W(e') = \{((1,1),(0,1),(0,1)),((0,1),(0,1),(1,1))\}$. Agent 1 loses and agent 2 gains as new agents come in.

The fact that φ^* satisfies *population monotonicity* is a very nice feature of this solution. Moreover, φ^* is essentially the only population monotonic selection from the no-envy solution that satisfies the following property, which says that the choice of the origin should not matter.

Translation invariance: For all $e = (Q,M,u_Q)$, for all $t \in \mathbb{R}$, for all $z \in \varphi(e)$, $(z_i-t(0,1))_{i\in Q} \in \varphi(Q,M-|Q|t,u_Q')$ where for all $i \in Q$ and for all $(\delta_i,m_i) = \{0,1\}\times\mathbb{R}$, $u_i'(\delta_i,m_i) = u_i(\delta_i,m_i+t)$.

This is a very weak property which is satisfied by all the solutions that we have encountered so far.

***Theorem* 4.** The solution φ^* is the the only subsolution of the no-envy solution satisfying *population monotonicity, translation invariance*, and *neutrality.*

Proof. Let φ be a subsolution of N satisfying the above properties and suppose that $\varphi \neq \varphi^*$. Then there is $e = (Q,M,u_Q)$, and $z \in \varphi(e)$ such that $u_w(1,m_w) > u_w(0,m_\ell)$, where m_w and m_ℓ denote the amounts of money received by the

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[14]The analysis of this solution was suggested to us by B. Dutta. This solution, which a priori would seem to deserve as much attention as φ^*, suffers in comparison since it satisfies neither *consistency* nor *population monotonicity.* Note however that it satisfies *neutrality, money monotonicity,* and *converse consistency.*

winner and each of the losers respectively. Let $\overline{m}$ be such that $u_w(1,m_w) = u_w(0,\overline{m})$. Then, $m_\ell < \overline{m}$. By *translation invariance*, we can suppose that $m_\ell \geqq 0$.

Now, let $m^* \in]m_\ell,\overline{m}[$ be given and let i and j be two new agents such that $u_i(1,M-m^*(|Q|+1)) = u_i(0,m^*)$, and $u_j = u_i$.

Let $e' = (Q\cup\{i,j\},M,u_{Q\cup\{i,j\}})$, and $z' \in N(e')$. We claim that $u_w(z_w) < u_w(z'_w)$ and $u_\ell(z_\ell) > u_\ell(z'_\ell)$ for all ℓ. Indeed, if $z' \in N(e')$, then $u_i(z'_i) = u_i(z'_j)$ and $u_i(z'_i) \geqq u_i(z'_k)$ for all $k \in Q$, $k \neq j$.

This is true only if

(i) $z'_i = z'_j = z'_\ell = (0,m')$ for $m' \geqq m^*$ and $z'_w = (1,M-m'(|Q|+1))$

or

(ii) either i or j receives $(1,M-m^*(|Q|-1))$, and all other agents, including the old winner, receive $(0,m^*)$.

In either case, the old winner is worse off at z' than he was at z whereas the opposite holds for each of the old losers.

Q.E.D.

The population monotonicity of the solution φ^* should be contrasted with what happens on the general domain. Let $e = (Q,A,M,u_Q)$ denote a "general" economy, where A is a set of objects and each u_i is defined on $A \times \mathbb{R}$. As before, whoever does not receive an object in A receives one of the null objects.

***Proposition* 5.**[15] On the general domain, there is no subsolution of the no-envy solution satisfying *population monotonicity.*

Proof. Let $e = (Q,A,M,u_Q)$ be such that $Q = \{1,2\}$, $A = \{\alpha,\beta\}$, $M = 0$, and for all $m \in \mathbb{R}$

[15]Proposition 5 was conjectured by H. Moulin.

$u_1(\alpha,m) = u_1(\beta,m) = u_1(0,m+1)$ and

$u_2(\alpha,m) = u_2(\beta,m) = u_2(0,m+10)$.

It is easy to check that $NP(e) = \{((\alpha,0),(\beta,0)),((\beta,0),(\alpha,0))\}$.

Let $e' = (Q',A,M,u_{Q'})$ be such that $Q' = \{1,2,3,4\}$ and for all $m \in \mathbb{R}$, $u_3(\alpha,m) = u_3(\beta,m) = u_3(0,m+4)$, and $u_3 = u_4$.

We claim that if $z' \in NP(e')$, then $u_1(z_1') > u_1(z_1)$ and $u_2(z_2') < u_2(z_2)$. Indeed, at z',

(i) Agent 2 receives one of the two objects (this is because $z' \in P(e')$).

(ii) Either agent 3 or agent 4 receives the remaining object (this is because $z' \in P(e')$). Since they have identical preferences, the one who does not receive an object should receive 4 more units of money than the other (this is necessary and sufficient for envy not to exist between them).

(iii) The two agents receiving the objects receive the same amount of money (this is necessary and sufficient for envy not to exist between them). The same holds for the two agents who receive no object.

Thus, $z_1' = (0,2)$ and $z_2' = (\alpha,-2)$ or $(\beta,-2)$, and therefore, $u_1(z_1') > u_1(z_1)$ and $u_2(z_2') < u_2(z_2)$.

Q.E.D.

5. ***Concluding comments.*** First, we note that our results also hold on the restricted class of "quasi-linear economies", i.e. economies in which each agent i has preferences representable by a function u_i satisfying: for all m, m', $t \in \mathbb{R}$, $u_i(0,m) = u_i(1,m')$ implies $u_i(0,m+t) = u_i(1,m'+t)$.

We have identified a particular selection from the no-envy solution that satisfies a variety of appealing properties. First of all, it has a familiar interpretation as an "egalitarian-type" solution: it can be described as achieving "equal welfares". It is the smallest subsolution of the no-envy solution satisfying consistency and a minor neutrality condition. It also satisfies the converse of consistency. Finally, it is monotonic with respect to changes in the amount of money to be divided and with respect to changes in the number of participants. It is the only translation invariant subsolution of the no–envy solution to have that last property.

Since this solution picks the envy–free allocation the most unfavorable to the winner of the indivisible good, the fact that no other solution seems to satisfy so many appealing properties may be interpreted as a new form of the winner's curse!

Appendix

***Proof of Lemma* 1.** Let $e = (Q,M,u_Q)$ be given and $z \in N(e)$. Let w be the amount of money received by the winner at z. For the losers not to envy one another, they should all receive the same amount of money $\ell \equiv (M-w)/(|Q|-1)$. Now, suppose by way of contradiction that $z \notin P(e)$. Then there is $z' \in Z(e)$ such that $u_i(z_i') \geqq u_i(z_i)$ for all $i \in Q$, with strict inequality for at least one i. Let $\overline{m}$ be such that the winner at z be indifferent between his bundle and $(0,\overline{m})$.[16] For each loser i at z, let $\overline{\overline{m}}_i$ be such that agent i be indifferent between z_i and $(1,\overline{\overline{m}}_i)$. Since $z \in N(e)$, $\overline{m} \geqq \ell$ and for each such i, $\overline{\overline{m}}_i \geqq w$. Since z' Pareto–dominates z, we have $m_i' \geqq w$ if i is the winner at z' and $m_i' \geqq \ell$ if i is a loser at z', with strict inequality for at least one i. But then $\sum_Q m_i' > w + (|Q|-1)\ell = M$, which contradicts the feasibility of z'.

Q.E.D.

***Proof of Lemma* 3.** Let $\delta \in \{0,1\}$ be given. For each i, let $v_i^\delta \in \{0,1\}\times\mathbb{R} \to \mathbb{R}$ be the numerical representation of agent i's preferences obtained as follows: given $z_i \in \{0,1\}\times\mathbb{R}$, $u_i(\delta,v_i^\delta(z_i)) = u_i(z_i)$. Let $t^\delta = \{\max t \mid \exists z \in Z(e)$ with $v_i^\delta(z_i) = t$ for all $i\}$. Finally, let $\varphi^\delta(e) \equiv v^{-1}(t^\delta)$. Then $E^*P(e) = \bigcup_{\delta\in\{0,1\}} \varphi^\delta(e)$.

Q.E.D.

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[16]The existence of $\overline{m}$ follows from our maintained assumption that $\lim_{m\to\infty} u_i(\delta,m) = \infty$. However, the result is true even if this assumption is not made. If there is no ***finite*** $\overline{m} \in \mathbb{R}$ such that the winner is indifferent between his bundle and $(0,\overline{m})$, then set $\overline{m} = \infty$ (similarly, we might set $\overline{\overline{m}}_i = \infty$).

Note that the allocation associated with $\delta = 0$ is envy–free. This lemma provides us with another proof of the existence of envy–free allocations.

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