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Dutta, Prajit and Mukul Majumdar

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Prajit Dutta and Mukul Majumar

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Prajit Dutta
Columbia University
and
University of Rochester

Mukul Majumdar
Cornell University

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1. Introduction

Let ϕ be a family of real valued functions on a metric space S and Ψ a family of probability measures on its Borel σ–field. The "integration to the limit" problem (Billingsley (1968, p 31) is the following: suppose that a sequence f_n of elements in ϕ (respectively a sequence μ_n of elements in Ψ) "converges" to an element f in ϕ (respectively μ in Ψ); under what conditions will $\int f_n d\mu_n$ converge to $\int f d\mu$? It is known that a positive answer to this question can be obtained if, informally speaking, either f_n converges to f "strongly" (uniformly over S) or if $\mu_n(A)$ converges to $\mu(A)$ for every Borel set A. A more general result is available in Billingsley (1968) [see Theorem 3 below]. In this paper we study this problem as well as the problem of establishing a variation of Fatou's lemma ($\limsup_{n\to\infty}\int f_n d\mu_n \le \int f d\mu$ in the above framework) when convergence is "weak". Our motivation comes from applications to "parametric variation" problems of dynamic programming and stochastic optimization models of mathematical economics. In such applications, S can usually be taken to be a separable normed linear space endowed with a partial order and the relevant functions are monotone and uniformly bounded. In Section 2 we collect the basic definitions. In Section 3 we establish a variation of Fatou's lemma for such monotone functions (Theorem 1). In Section 4 we prove a theorem on integration to the limit (Theorem 4). Examples indicating the critical role of the monotonicity assumptions are also given. In Section 5 we briefly outline possible applications of our results to the dynamic programming framework.

2. Notation and Basic Definitions

Let S be a convex subset of a normed linear space with norm denoted $\|\cdot\|$. Let $\succ$ be a transitive, antisymmetric, irreflexive relation on S. The following assumptions on the partially ordered space $(S,\succ)$ are maintained throughout:

(A1) $s' \succ s$ implies that there are neighborhoods U and V (of s' and s respectively) such that for all $x \in U$, $y \in V$ one has $x \succ y$.

(A2) For all $s \in S$, (i) $\bar{P}(s) = \{s' \in S: \ s' \succ s\}$ is nonempty, (ii) $\underline{P}(s) = \{s' \in S: \ s \succ s'\}$ is nonempty.

(A3) For all $s, s' \in S$, $s' \succ s$ implies $s' \succ \lambda s' + (1 - \lambda)s \succ s$, if $\lambda \in (0,1)$.

Example 1: $S = \mathbb{R}^m$ with any of the equivalent norms on this space. Let $\succ$ be the partial order defined as $s' \succ s \Leftrightarrow s_i' > s_i$, $i = 1,\dots,m$. Clearly we could also take S to be any nonempty open convex set in $\mathbb{R}^m$.

Example 2: $S = C([0,1])$, the space of continuous real–valued functions on $[0,1]$, under the sup–norm. Define $f \succ g \Leftrightarrow f(x) > g(x)$, for all $x \in [0,1]$. Again any nonempty open convex subset of $C([0,1])$ also suffices.

It is straightforward to check that each example satisfies (A1) – (A3).

All functions on S that we consider are real–valued. A function f is ***non–decreasing*** if $s' \succ s$ implies $f(s') \geq f(s)$. f is ***upper semi–continuous*** at s if $s_n \to s$ implies that $\overline{\lim}_{n \to \infty} f(s_n) \leq f(s)$. f is upper semi–continuous on S if it is upper semi–continuous at each $s \in S$. f is ***lower semi–continuous*** at s (on S) if $-f$ is upper semi–continuous at s (on S). Finally, f is said to be ***continuous*** on S if it is both upper and lower semi–continuous on S.

Consider a sequence of functions $(f_n)_{n \geq 0}$ and a candidate "limit" function f. There are two senses in which we examine functional convergence. We say that f_n converges ***weakly*** to f if $f_n(s) \to f(s)$ at all continuity points of f.[1] Alternatively, we say f_n converges ***pointwise*** to f if $f_n(s) \to f(s)$ at all $s \in S$.

We say that a family of functions (f_n, f) is ***uniformly bounded above*** by $K < \infty$ if $f_n(s) \leq K$, $f(s) \leq K$, for all $s \in S$, $n \geq 0$. (f_n, f) is ***uniformly bounded below*** if $(-f_n, -f)$ is uniformly bounded above by some $K < \infty$. (f_n, f) is ***uniformly bounded*** if it is uniformly bounded above and below.

Let (μ_n,μ) be a family of probability measures (on the Borel σ–field of S). Then μ_n is said to converge ***weakly*** to μ (denoted $\mu_n \Rightarrow \mu$) if $\overline{\lim}_{n\to\infty} \mu_n(C) \leq \mu(C)$ for all closed sets C. We say that a probability measure μ' ***stochastically dominates*** a probability measure μ if $\int f d\mu' \geq \int f d\mu$ for all bounded non–decreasing functions f (for a discussion see Heyman–Sobel (1982)).

3. A Variation of Fatou's Lemma

In this section we prove the following variation of Fatou's lemma:

THEOREM 1: *Let $(f_n)_{n\geq 0}$ be a family of non–decreasing, upper semi–continuous functions that are uniformly bounded above. Suppose that f_n converges weakly to a non–decreasing upper semi–continuous function f. Let $(\mu_n)_{n\geq 0}$ be a sequence of probability measures that converge weakly to a probability measure μ. Then,*

$$\overline{\lim_{n\to\infty}} \int f_n d\mu_n \leq \int f d\mu \tag{1}$$

PROOF: Theorem 1 is proved by way of lemmas 1–5.

Lemma 1. The set of continuity points of f is dense in S.

Pf. Suppose per absurdum that there is $s \in S$ and an ϵ–neighborhood of s, $N_\epsilon(s)$, such that every point in $N_\epsilon(s)$ is a point of discontinuity of f. By (A2(i)) there is $s' \in S$ such that $s' \succ s$. Further, by (A3), $s(\lambda) \succ s$ where $s(\lambda) \equiv \lambda s + (1-\lambda)s'$, $\lambda \in [0,1)$. Without loss of generality suppose that $s(\lambda) \in N_\epsilon(s)$, for all $\lambda \in [0,1]$. We now prove that on the "ray" $R \equiv \{s(\lambda): \lambda \in [0,1]\}$ there can be at most a countable number of discontinuity points. This would of course contradict the maintained hypothesis that each of the uncountable elements of R is a point of discontinuity of f.

<u>Claim (i)</u>. Let $s(\lambda) \in R$, $\lambda < 1$, be a discontinuity point of f. Then for any

sequence $\lambda_n \downarrow \lambda$, it must be the case that $\lim_{n\to\infty} f(s(\lambda_n)) < f(s(\lambda))$.

Pf. Since $s(\lambda)$ is a discontinuity point of f there is a sequence $z_n \to s(\lambda)$ such that $\lim_{n\to\infty} f(z_n) < f(s(\lambda))$. Note that (z_n) need not be a sequence along the ray R, i.e. $z_n \notin R$ in general. However by (A1), $z_n \succ s$ (without loss of generality) for all n. By (A1) again $z_n \succ s(\lambda)$ for all λ sufficiently "close" to 1. In fact, define

$$\lambda_n' \equiv \inf \{\lambda: \; z_n \succ s(\lambda)\} \tag{2}$$

and let $\lambda_n \equiv \lambda_n' + \frac{1}{n}$ (which is in [0,1) for "large" n). Clearly the definition (2) can be modified straightforwardly to make λ_n' a monotonically decreasing function of n. In fact, by (A3), $z_n \succ s(\lambda)$ for all $\lambda > \lambda_n'$ and of course λ_n is a monotonically decreasing function of n. Finally by (A1) and (A3) it directly follows that $\lambda_n \downarrow \lambda$, as $n \to \infty$. Hence, along this sequence $f(s(\lambda_n)) \le f(z_n)$ and $s(\lambda_n) \to s(\lambda)$. The claim follows for this specification of the sequence λ_n. More generally consider any sequence $\lambda_m \downarrow \lambda$. For every m there is n(m) such that $\lambda_m \ge \lambda_{n(m)}$ and $\lambda_{n(m)}$ is an element of the sequence λ_n. In particular, $s(\lambda_{n(m)}) \succ s(\lambda_m)$ and hence $f(s(\lambda_{n(m)}) \ge f(s(\lambda_m)$. The claim follows now for this general sequence. ∎

<u>Claim (ii)</u>. R has at most a countable set of discontinuity points.

Pf. By claim (i), discontinuities are generated through sequences which lie entirely on the completely ordered single dimensional set R. However f is monotone on this set. The claim immediately follows. In turn Lemma 1 is completely proved. ∎

Without loss of generality, set the uniform upper bound K to zero. Note from lemma 1 that this upper bound also applies to f.

Lemma 2 [Bourbaki (1966, p. 155)]. There is a sequence of bounded ***continuous*** functions $(g_k)_{k\geq 0}$ on S, converging pointwise to f, such that for all s,

$$f(s) \leq g_{k+1}(s) \leq g_k(s) \leq 0, \qquad \text{for all } k \geq 0 \tag{3}$$

Pf. Consider $h_k(s) \equiv \sup \{f(s') - k \|s - s'\|\}$. Then define $g_k(s) = \max (h_k, -k)$. It is straightforward to verify that (6) holds and that each g_k is a bounded function.■

Lemma 3 [Billingsley (1968, p. 117)]. For a fixed k,

$$\lim_{n\to\infty} \int g_k \, d\mu_n = \int g_k \, d\mu \tag{4}$$

Lemma 4. Suppose $\lim_{n\to\infty} s_n = s$. Along any sequence $q(n)$, $n \geq 0$, $q(n) \to \infty$,

$$\overline{\lim_{n'\to\infty}} f_{q(n')} (s_{n'}) \leq f(s) \tag{5}$$

where $s_{n'}$ is a subsequence.

Pf. From the monotonicity and upper semi-continuity of f it follows that whenever $z_n \succ s$ and $\lim_{n\to\infty} z_n = s$, then $\lim_{n\to\infty} f(z_n) = f(s)$. From Lemma 1 and assumptions (A2) – (A3) it then follows that for all $\epsilon > 0$ there is a continuity point of f, s', such that $s' \succ s$ and

$$f(s) \geq f(s') - \epsilon \tag{6}$$

By (A1), $s' \succ s_n$ for all n (without loss of generality). Hence,

$$f_{q(n')} (s') \geq f_{q(n')} (s_{n'}) \tag{7}$$

Since s' is a continuity point of f, $\lim_{n\to\infty} f_{q(n')} (s') = f(s')$. Combining this with (6) – (7) yields

$$\overline{\lim_{n\to\infty}} f_{q(n')} (s_{n'}) \leq f(s) + \epsilon \tag{8}$$

Since (8) holds for all $\epsilon > 0$, the lemma is proved. ∎

Lemma 5. For a fixed k,

$$\overline{\lim_{n\to\infty}} \int f_n \, d\mu_n \;\le\; \lim_{n\to\infty} \int g_k \, d\mu_n \tag{9}$$

Pf. Fix some $\epsilon > 0$ and define

$$E_m \;=\; \{s: \; f_n(s) \le g_k(s) + \epsilon, \;\; \text{for all } n \ge m\} \tag{10}$$

We now prove: E_m is open, $E_{m+1} \supset E_m$ and $\bigcup_m E_m = S$. We first show that the complement of E_m, $\tilde{E}_m$ is in fact closed. To this end, let s_n be a sequence in $\tilde{E}_m$ converging to s. Two cases are to be distinguished.

<u>Case (i)</u>. There is a common index $\bar{n} \ge m$ and a subsequence $s_{n'}$, such that $f_{\bar{n}}\,(s_{n'}) \ge g_k\,(s_{n'}) + \epsilon$, for all s_n'. By the upper semi-continuity of $f_{\bar{n}}$ and the continuity of g_k, it then follows that $f_{\bar{n}}\,(s) \ge g_k(s) + \epsilon$. which implies that $s \in \tilde{E}_m$.

<u>Case (ii)</u>. There are distinct indices q(n), such that $f_{q(n)}(s_n) \ge g_k(s_n) + \epsilon$.

Define for each n, q(n) to be the first index greater than m such that

$$f_{q(n)}(s_n) \;\ge\; g_k(s_n) + \epsilon \tag{11}$$

Set $n_1 \equiv 1$ and write

$$n_2 \;\equiv\; \min\,\{n > 1: \;\exists\; q(n) > q(1) \text{ s.t. } f_{q(n)}(s_n) \ge g_k(s_n) + \epsilon\} \tag{12}$$

Since we are in Case 2 there exists such a n_2. Similarly for $i = 3,4,\ldots$ define

$$n_i \;=\; \min\,\{n > n_{i-1}: \exists\; q(n) > q(n_{i-1}) \text{ s.t. } f_{q(n)}(s_n) \ge g_k(s_n) + \epsilon\} \tag{13}$$

Since we are in Case 2, the subsequence in (13) is well-defined and $q(n_i)$ is a strictly monmotone increasing sequence with $q(n_i) \to \infty$, as $n_i \to \infty$. By Lemma 5,

$\overline{\lim} \int f_{q(n_i)} (s_{n_i}) \le f(s)$. Hence, $f(s) \ge g_k(s) + \epsilon$. This is clearly a contradiction of Lemma 2. Hence case (ii) cannot hold, i.e., $s \in \tilde{E}_m$ and E_m is open.

It is clear that $E_{m+1} \supset E_m$. Finally, suppose that $s \in \tilde{E}_m$ for all m. Then,

$$f_{n'}(s) \ge g_k(s) + \epsilon \tag{14}$$

along a subsequence $n' \to \infty$. Lemma 4 again yields a contradiction.

We now prove (9). Note that for each m and $n \ge m$,

$$\begin{aligned} \int f_n d\mu_n &= \int_{E_m} f_n d\mu_n + \int_{\tilde{E}_m} f_n d\mu_n \\ &\le \int_{E_m} f_n d\mu_n \qquad \text{(since } f_n \le 0\text{)} \\ &\le \int_{E_m} g_k d\mu_n + \epsilon \\ &= \int g_k d\mu_n - \int_{\tilde{E}_m} g_k d\mu_n + \epsilon \end{aligned} \tag{15}$$

Since $\tilde{E}_m \downarrow \phi$, one can find some m such that $\mu(\tilde{E}_m) < \epsilon$. Since $\tilde{E}_m$ is closed, by weak convergence of μ_n there is some N such that $\mu_n(\tilde{E}_m) < \epsilon$, for all $n \ge N$. From Lemma 2 and (15) it then follows that

$$\int f_n d\mu_n \le \int g_k d\mu_n + k\epsilon + \epsilon$$

Hence,

$$\overline{\lim_{n\to\infty}} \int f_n d\mu_n \le \lim_{n\to\infty} \int g_k d\mu_n + (k+1)\epsilon \tag{16}$$

Since $\epsilon > 0$ is arbitrary, from (16), Lemma 5 follows. ∎

The proof of the theorem is now a consequence of Lemmas 3, 5 and the monotone convergence theorem. ∎

In addition to (A1) – (A3) suppose we make the following further assumption:

(A4) $(S,\|\cdot\|)$ is separable.

Let f_n be a sequence of non–decreasing, upper semi–continuous functions that are uniformly bounded above. The proof of Helly's theorem [Billingsley (1985, p. 392)] carries over to S. Hence there is a non–decreasing, upper semi–continuous function f which is bounded above such that along a subsequence $(f_{n'})$ [of (f_n)], $f_{n'}$ converges weakly to f. Hence, we have:

THEOREM 2. *Let (f_n) be a sequence of non–decreasing, upper semi–continuous functions that are uniformly bounded above. Let (μ_n) be a sequence of probability measures converging weakly to a probability measure μ. Then, there is a non–decreasing, upper semi–continuous function* f *that is bounded above such that*

i) *a subsequence $f_{n'}$ converges to* f *weakly*

ii) $$\overline{\lim_{n'\to\infty}} \int f_{n'} d\mu_{n'} \le \int f \, d\mu$$

Remarks 1. For a result related to Theorem 1, see Royden (1968, Proposition 17, p. 231). (1) is established there without making any continuity or monotonicity assumptions on (f_n) and f, but under a considerably stronger convergence restriction on the probability measures (μ_n): $\mu_n(E) \to \mu(E)$ for all Borel sets E.

2. If $\mu_n = \mu$ for all n, Fatou's lemma yields (1). This is so since from Lemma 4 we know that $\overline{\lim}_{n\to\infty} f_n(s) \le f(s)$, for all $s \in S$. Fatou's lemma says:

$$\overline{\lim_{n\to\infty}} \int f_n d\mu \le \int \overline{\lim_{n\to\infty}} f_n d\mu \tag{17}$$

From (17) and the above observation, (1) follows. On the other hand if $f_n = f$

for all n, then (1) follows from the weak convergence of the probability measures, provided that f is an upper semi–continuous function which is bounded above. Hence the additional requirement in Theorem 1 is precisely the monotonicity conditions on f_n, and this allows both integrands and measures to vary.

3. Theorems 1 and 2 generalize a result proved in Dutta (1990) where the domain is $S = \mathbb{R}$.

4. Theorems 1 and 2 are both true without (A2(ii)). We require it only in Theorem 4 but state it with the other assumptions for compactness of exposition.

To clarify the role of the monotonicity assumptions, consider the following example.

Example 3. $S = \mathbb{R}$ and $\succ$ is the usual strict inequality ordering. Define

$$f_n(s) = (ns)\, I_{\{0 \le s \le \frac{1}{n}\}} + (2 - ns)\, I_{\{\frac{1}{n} \le s \le \frac{2}{n}\}}, \qquad \text{and } f \equiv 0$$

$$\mu_n\{\tfrac{1}{n}\} = 1 \quad \text{and} \quad \mu\{0\} = 1$$

Clearly $\int f_n d\mu_n = 1$ and $\int f d\mu = 0$. Note that f_n is continuous but not non–decreasing. ■

On the other hand consider the following example which shows that the inequality in (1) cannot be tightened to an equality.

Example 4. $S = \mathbb{R}$ and $\succ$ the strict inequality order. Define

$$f_n = f = I_{\{s \ge 0\}}$$

$$\mu_n\,(-\tfrac{1}{n}) = 1$$

$$\mu(0) = 1$$

Clearly, $\int f_n d\mu_n = 0$ and $\int f d\mu = 1$. Of course f is monotone but only upper semi–continuous. ■

5. Integration to the Limit

In this section we revert to assumptions (A1) – (A3). When a function f is bounded and ***continuous***, rather than merely upper semi–continuous, $\mu_n \Rightarrow \mu$ implies that $\lim_{n\to\infty} \int f d\mu_n = \int f d\mu$ (e.g., see the Portmanteau theorem [Billingsley (1968), p. 17]). The question we now turn to is: if the integrands are taken to be continuous functions, can (1) be improved to yield an exact equality?

We recall the basic result of Billingsley (1968, p. 34) which specializes to our framework. Let (f_n), f be measurable functions on S. Let E be the (measurable) set of s on which $f_n(s_n) \to f(s)$ fails to hold for some sequence $s_n \to s$. We then have:

Theorem 3 [Billingsley (1968), Theorem 5.5] *Let $\mu_n \Rightarrow \mu$ and suppose $\mu(E) = 0$. Then*

$$\lim_{n\to\infty} \int f_n d\mu_n = \int f d\mu \tag{19}$$

An application of this result leads to the following theorem.

Theorem 4. *Suppose that (f_n) is a uniformly bounded sequence of non–decreasing continuous functions on S such that f_n converges weakly to a function f, where f is continuous. Let μ_n converge weakly to μ. Then,*

$$\lim_{n\to\infty} \int f_n d\mu_n = \int f d\mu \tag{20}$$

Proof: We want to show that $\lim_{n\to\infty} f_n(s_n) = f(s)$, for any sequence $s_n \to s$. In view of Lemma 4, it suffices to show that $\underline{\lim}\ f_n(s_n) \geq f(s)$. A direct adaptation of the proof of Lemma 4 (but using now a comparison point s′ such that $s \succ s'$) establishes this inequality. But then we have shown that the set E is empty. Hence, $\mu(E) = 0$

and Theorem 3 applies and so (20) follows. ∎

<u>Remarks</u> 1. The proof of Theorem 1 can be easily adapted to prove Theorem 4 without any appeal to Theorem 3.

2. For related results see Royden (1968, Proposition 18, p. 232), Billingsley (1968, p. 17, problems 7–8) and Parthasarathy (1967, p. 51).

3. Examples 3 and 4 show that neither monotonicity nor continuity of the functions (f_n) can be relaxed.

5. Parametric Continuity of Dynamic Programming Problems

In this section we report and briefly discuss two applications of the results to establish parametric continuity in dynamic programming problems. In Section 5.1 the structure and assumptions of dynamic programming are detailed. Then we examine the continuity of the value function and the continuity of optimal actions, in a sense made precise below, in some exogeneous parameter θ. The first result is in discounted dynamic programming. This result is an application of the integration to the limit theorem (Theorem 4). Next we report on a result on continuity between discounted and undiscounted problems (Dutta (1990)). This is an application of the variation of Fatou's lemma (Theorem 1).

5.1 Dynamic Programming Problems

A parametric dynamic programming problem is specified by a sextuple $<S,A,q,r,\delta,H>$. S is the set of states of a dynamical system and is taken to be a nonempty separable Banach space. A represents the set of actions available to a decision maker at any time and is assumed to be a compact, metric space. H should be thought of as a set of exogeneous parameter values, and is taken to be a metric space. The triple (s,a,θ) is a generic element of $S{\times}A{\times}H$. q is the law of motion of the system — it associates (Borel measurably) with each triple (s,a,θ), a probability measure $q(\cdot|s,a,\theta)$ on the Borel σ–field of S. If the exogeneous parameter has value θ, and this value remains fixed throughout, then whenever the system is in state s and action a is chosen, the system moves to state s′ according to the distribution $q(\cdot|s,a,\theta)$. It will be assumed that the transition probabilities are separately continuous on SxA and HxA, i.e., $(s_n,a_n) \to (\hat{s},\hat{a})$ (respectively $(\theta_n,a_n) \to (\hat{\theta},\hat{a})$) implies $q(\cdot|s_n,a_n,\theta) \Rightarrow q(\cdot|\hat{s},\hat{a},\theta)$ for all θ (respectively $q(\cdot|s,a_n,\theta_n) \Rightarrow q(\cdot|s,\hat{a},\hat{\theta})$ for all s). Further, it is also assumed that $s' \succ s$ implies that $q(\cdot|s',a,\theta)$ stochastically dominates $q(\cdot|s,a,\theta)$ for all (a,θ). The one period return function is r, which associates with every (s,a,θ) a return $r(s,a,\theta)$. In other words, if the exogeneous parameter has value θ, then whenever the system is in state s and action a is chosen, the immediate payoff is $r(s,a,\theta)$. We shall assume that r is bounded and separately continuous on S x A and H x A and further that $r(\cdot,a,\theta)$ is a non–decreasing function of S. Finally, $\delta(\theta) \in [0,1]$ is the discount factor, also determined by the exogenous parameter θ and this relationship is taken to be continuous.

The structure above is standard and is described in greater detail in Maitra (1968), Heyman–Sobel (1982) and Stokey–Lucas (1989).[2] The only difference in our formulation is that we make the underlying decision–problem dependent on an exogeneous parameter θ and hence generate a family of decision problems, one associated with each value of θ. The continuity requirements on q, r and δ are modified accordingly. Finally, we impose some monotonicity restrictions on the problem which are motivated by examples in economics and operations research.

A policy <g> is a sequence $g_0, g_1, \ldots$ where g_t selects the action at the t–th period as a function of the previous history $h = (s_0,a_0, \ldots, a_{t-1},s_t)$ of the system by associating with each h (Borel measurably) an action $g_t(h)$. A policy g^∞ is ***stationary*** if $g_t = g$, for all t, for some Borel measurable g. Each policy induces a distribution on the state and action at each period, conditional on the initial state s (and the exogeneous parameter θ). Let $r_\theta(<g>)_t(s)$ denote the expected return in period t under policy <g>, if the initial state is s and the exogeneous parameter has value θ. In the **discounted problem**, i.e. when $\delta(\theta) < 1$, the total discounted expected return is

$$I_\theta^{<g>}(s) = \sum_{t=0}^{\infty} \delta(\theta)^t \, r_\theta \, (<g>)_t(s) \tag{21}$$

If $\delta(\theta) = 1$, the decision criterion will be the **long-run average** which is defined as

$$I_\theta^{<g>}(s) = \underline{\lim}_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} r_\theta \, (<g>)_t(s) \tag{22}$$

A policy <g*> is ***optimal***, under θ, if $I_\theta^{<g^*>}(s) \geq I_\theta^{<g>}(s)$ for all policies <g> and all initial states $s \in S$. The ***value function*** for initial state s and parameter θ is

$$V(s,\theta) = \sup_{<g>} I_\theta^{<g>}(s) \tag{23}$$

5.2 Parametric Continuity in Discounted Problems

Suppose that we look at a class of discounted problems. To this end consider the dynamic programming problem of Section 5.1, $< S,A,q,r,\delta,H >$. Recall (Maitra (1968)) that for each θ, the value function, $V(\cdot,\theta)$, is the unique bounded continuous solution to the Bellman equation:

$$V(s,\theta) = \max_{a} \{r(s,a,\theta) + \delta \int V(\cdot,\theta) dq(\cdot\,|\,s,a,\theta)\} \tag{24}$$

Let $\alpha(s,\theta)$ be the set of maximizers in the right side of (24) for all s. For a given θ, optimal policies in the dynamic programming problem are selections from the correspondence $\alpha(\cdot,\theta)$. If $g(\cdot,\theta)$ is a (measurable) selection from $\alpha(\cdot,\theta)$, then standard techniques show that g^∞ is a stationary optimal policy. The parametric continuity result for discounted problems is:

THEOREM 5. Consider (A1)–(A3). *Suppose that* $\theta_n \to \theta$ and $\delta(\theta) < 1$. *Then,*

i) *Value continuity:* $V(s,\theta_n) \to V(s,\theta)$, *for all* $s \in S$

ii) *Policy upper semi-continuity:* $a_n \in \alpha(s,\theta_n)$, $a_n \to a$ *implies that* $a \in \alpha(s,\theta)$, *for all* $s \in S$.

Proof : Let $C^*(S\times H)$ denote the space of bounded functions on $S \times H$ which are separately continuous and non-decreasing on S.[3] Let $W \in C^*(S\times H)$ and define

$$TW(s,\theta) = \max_a \{r(s,a,\theta) + \delta \int W(\cdot,\theta)dq(\cdot|s,a,\theta)\} \tag{25}$$

Suppose that $\theta_n \to \theta$ and $a_n \to a$ for some sequence (θ_n,a_n). Then $q(\cdot|s,a_n,\theta_n) \Rightarrow q(\cdot|s,a,\theta)$. Further, $W_n \equiv W(\cdot,\theta_n)$ is a continuous non-decreasing function such that $W_n(s) \to W(s) \equiv W(s,\theta)$, for all $s \in S$. By Theorem 4, $\int W_n\, dq_n \to \int W\, dq$. Since $r(s,\cdot)$ is continuous on $A \times H$, an application of the maximum theorem (Berge (1963)), then establishes that TW is continuous on H for fixed s. The arguments that show that TW is continuous on S and bounded are standard (see, e.g., Maitra (1968)) as also the argument that establishes that TW is non-decreasing on S. Hence, $TW \in C^*(S\times H)$. But $C^*(S\times H)$ as a closed subset of a complete metric space is itself a complete metric space. Standard contraction mapping arguments then establish that there is a unique element of $C^*(S\times H)$, say V, such that $TV = V$. That proves i). The Berge maximum theorem applied to TV yields (ii). ∎

Remark Dutta–Majumdar–Sundaram (1990) give a number of other conditions under which value and policy continuity hold in discounted dynamic programming problems.

5.3 Continuity Between Discounted and Undiscounted Problems

Consider now a problem in which only the discount factor changes. So $r(s,a,\theta) \equiv r(s,a)$ and $q(\cdot|s,a,\theta) \equiv q(\cdot|s,a)$ but $\delta(\theta) = \theta$, where $\theta \in [0,1]$. In particular we are interested in the following question: suppose $\theta_n \uparrow 1$, i.e., we go from a discounted to an undiscounted problem. Does value and policy continuity hold? In general the answer is no (and Ross (1983) contains a number of counter–examples) but in monotone problems one can give a positive answer. We need however a boundedness condition and in order to state it, let us define:

<u>A Normalized Value</u>: Fix $z \in S$ and $\delta < 1$. Then, define ψ_z by:

$$\psi_z(s,\delta) \equiv [V(s,\delta) - V(z,\delta)]$$

(A5) <u>Value Boundedness</u>: There is $z \in S$ and a bound M such that

$$|\psi_z(s,\delta)| < M \qquad \text{for all } s \text{ and } \delta < 1$$

Theorem 6: Suppose (A1)–(A5); *There is* $\nu \in \mathbb{R}$ *such that*

i) *Value continuity:* $\nu = \lim_{\delta \uparrow 1} (1-\delta)\, V(s,\delta)$ *for all* s *and* ν *is the long–run average value of the undiscounted problem.*

ii) *Policy upper semi–continuity:* $a_n \in \alpha(s,\delta_n)$, $a_n \to a$, *implies that* $a \in \alpha(s,1)$.

A closely related result is proved in Dutta (1990) and so we only sketch some steps to indicate the applicability of our result.

Proof (sketch): The optimality equation (24) can be re–written as

$$\psi_z(s,\delta) + (1-\delta)\, V(z,\delta) = \max_a \{r(s,a) + \delta \int \psi_z(\cdot,\delta) dq(\cdot|s,a)\} \tag{26}$$

Under our assumptions, $\psi_z(\cdot,\delta)$ is a non–decreasing, continuous function on S, for each $\delta \in [0,1)$. By Theorem 2, there is a weak limit of $\psi_z(\cdot,\delta_n)$, say $\hat{\psi}$ which is non–decreasing, upper semi–continuous and $\overline{\lim}_{n\to\infty} \int \psi_z(\cdot,\delta_n) dq(\cdot|s,a_n) \le \int \hat{\psi}(\cdot) dq(\cdot|s,a)$, where $a_n \in \alpha(s,\delta_n)$ and a is a (subsequential) limit of a_n. Write $\nu = \lim_{n\to\infty} (1-\delta_n)$ $V(z,\delta_n)$ possibly on a subsequence. Suppose that s is a continuity point of $\hat{\psi}$. Then we have

$$\hat{\psi}(s) + \nu \le \max_a \{r(s,a) + \int \hat{\psi}(\cdot) dq(\cdot|s,a)\} \tag{27}$$

Indeed (27) actually holds for all $s \in S$ since the continuity points of $\hat{\psi}$ are dense in S and $\hat{\psi}$ is upper semi–continuous. To see this let s_n be a continuity point of $\hat{\psi}$, $s_n \succ s$ and $s_n \to s$. By the maximum theorem, $\max_a \{r(s_n,a) + \int \hat{\psi}(\cdot)dq(\cdot|s_n,a)\} \to \max_a \{r(s,a) + \int \hat{\psi}(\cdot)dq(\cdot|s,a)\}$.

Since $\hat{\psi}$ is upper semi–continuous, by the Dubins–Savage selection theorem (Maitra (1968)) there is actually a (measurable) function h such that

$$\hat{\psi}(s) + \nu \;\leq\; r(s,h(s)) + \int \hat{\psi}(\cdot)dq(\cdot|s,h(s))$$

$$= \max_a \{r(s,a) + \int \hat{\psi}(\cdot)dq(\cdot|s,a)\} \tag{28}$$

A finite iteration of (28) yields

$$\hat{\psi}(s) + T\nu \;\leq\; \sum_{t=0}^{T-1} r(h^{\infty})_t(s) + E_T^h\,\hat{\psi} \tag{29}$$

where $E_T^h\,\hat{\psi}$ is the expectation of $\hat{\psi}$ under the T–1 period distribution on S, induced by h^{∞}. Dividing both sides of (29) by T and letting $T \to \infty$ implies that the long–run average returns from h^{∞} are at least ν, i.e., that the long–run average value is at least ν.

Similar arguments establish that ν is also at least as large as the long–run average value. Finally, the state z that was used to normalize the value function is arbitrary (as can be checked from (A5)) and hence we actually have $\nu = \lim_{n\to\infty} (1-\delta)$ $V(s,\delta)$. Finally, policy continuity is proved as in Theorem 5. ■

<u>Remarks</u> 1. The critical requirement that both Theorems 5 and 6 exploited is the fact that the discounted value functions are monotonic on S. This is a natural assumption in many economic models where starting from a "higher" initial state cannot be any worse than starting from a "lower" initial state. Examples of economic models that satisfy value monotonicity may be found in Burdett–

Mortensen (1980), Brock–Mirman (1972), Dechert–Nishimura (1983), Reed (1974), Stokey–Lucas (1989), among others.

2. Counter–examples for models without monotonicity can be constructed. For instance, modify example 4 in the following way. Let $S = \mathbb{R}$, $A = \{\bar{a}\}$ and $H = [0,1]$. Define $r(s,\bar{a},\theta) = \frac{s}{\theta} I_{\{0\leq s\leq\theta\}} + (2 - \frac{s}{\theta}) I_{\{\theta\leq s\leq 2\theta\}}$ for $\theta > 0$ and $r(s,\bar{a},0) \equiv 0$. Further, $q(\{\theta\}|s,\bar{a},\theta) = 1$, for all s. Then, $V(s,\theta) = r(s,\bar{a},\theta) + \frac{\delta}{1-\delta}$, whenever $\theta > 0$, and $V(s,\theta) \equiv 0$ for $\theta = 0$.

Footnotes

[1]For this definition to have any bite clearly the set of continuity points of f needs to be "large." We show that for non–decreasing functions this set can be shown to be dense in S. The definition is motivated by the definition of weak convergence of distribution functions.

[2]The latter authors' formulation is slightly different from the one presented here (see Stokey–Lucas chapter 9 for details). However the assumptions are virtually identical to those made above with the exception that Stokey–Lucas require S to be a subset of $\mathbb{R}^m$.

[3]Of course these functions are then jointly continuous on $S \times H$.

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