Maximizing or Minimizing Expected Returns Prior to Hitting Zero

Dutta, Prajit

Working Paper No. 267
February 1991

Maximizing or Minimizing Expected Returns prior to hitting Zero

Prajit Dutta

Rochester Center for Economic Research
Working Paper No. 267

Maximizing or Minimizing Expected Returns Prior to Hitting Zero*

Prajit K. Duttta**
Columbia University

Rochester Center for Economic Research
Working Paper No. 267

July 1990
Revised: February 1991

Abstract: This paper analyzes the optimal control of a diffusion with absorption at the origin. Unlike previous studies, controls themselves may have immediate payoffs. Existence and characterization results are proved and some illustrative examples are presented.

* I have benefitted from extremely helpful conversations with Ioannis Karatzas. Roy Radner also provided valuable suggestions. All errors of course are of my doing.
** This paper was completed while visiting the University of Rochester.

1. Introduction

Consider a process $[Y(t) : t \geq 0]$ on $[0, \infty)$ given by a stochastic differential equation

$$dY(t) = m(t) + v^{1/2}(t)\ dB(t),\ Y(0) = y \tag{1.1}$$

where $[B(t) : t \geq 0]$ is a standard Brownian motion and $[m(t), v(t)]$ are nonanticipative controls chosen from a set A. Suppose further that the process is absorbed at the origin and let T denote the random time of absorption. The choice $[m(t),v(t)]$ yields a flow payoff at instant t which is denoted $U(m(t),v(t))$. At the time of absorption the decision maker receives a terminal payment φ (and from hereon we normalize the severance payoff to zero). Future payoffs are discounted at a rate $\delta \in (0, \infty)$ and the control problem is to pick an admissible strategy to maximize expected discounted lifetime payoffs:

$$\text{Maximize } E\ \delta\int_0^T e^{-\delta t}\ U(m(t),v(t))\ dt \tag{1.2}$$

Let $\overline{U} = \sup U(m,v)$, $(m,v) \in A$. Clearly for the problem to be interesting it must be the case that $\overline{U} > 0$. From hereon we maintain this specification and call this a **survival** problem. A variant of this problem is one in which $\overline{U} \leq 0$ and the state space is $(-\infty,0]$ (with absorption at zero again). This variant we will call the **race** problem. In the former case optimization involves a tradeoff between instantaneous payoffs and a movement away from zero while in the latter the tradeoff is between flow payoffs and a rapid approach to zero.

A number of papers have analyzed the pure survival or pure race version of this problem: the control problem in which $U(m,v) \equiv c$ for all controls in A, where c is some non–zero constant. (See Heath–Orey–Pestien–Sudderth (1987), Orey–Pestien–Sudderth (1987), Dupuis–Kushner (1989), Majumdar–Radner (1989)).[1] In many applications of this problem, in economics, finance and game theory, the pure survival formulation is unduly restrictive. (We provide some specific examples below).[2] This paper analyzes the general control problem (1.2). As will become clear in the sequel, allowing controls to have direct returns increases the analytical complexity in a non–trivial manner.

In the pure survival or race problem there is no tradeoff between immediate payoffs and continuation values. This allows an explicit computation of the optimal policy (which turns out to be the usage of a constant control regardless of history). When controls have direct payoffs constant controls are typically not optimal. This has two well-known implications: a) it is impossible except in special cases to directly "guess" an optimal policy (and use a verification theorem to establish its optimality). Hence we have to establish the existence of a value function with appropriate differentiability properties and the existence of optimal policies by analytical methods. b) since optimal policies depend on history (and stationary optimal policies depend on the state) one has to characterize this dependence. In this paper we will investigate the existence and characterization issues. We will also present two examples of explicitly computable problems.

Section 2 presents some basic notation and definitions. Sections 3–5 discuss the survival problem. The main results on existence and characterization are stated and discussed in Section 3. The proofs of these results are contained in Section 4. We present examples in Section 5. Section 6 contains the analysis of the race problem. Section 7 discusses possible applications and extensions.

2. Basic Definitions and Assumptions

Let $[B(t):t \geq 0]$ be a standard Brownian motion on some probability space $(\Omega, \mathfrak{F}, P)$. Let $\mathfrak{F}_t$ be the smallest family of sub σ–fields generated by the Brownian motion. Let $[f(t): t \geq 0]$ be a $\mathfrak{F}_t$*–adapted process*[3] which further satisfies

i) $P[\omega : \int_0^t f^2(\omega, s)\, ds < \infty] = 1$, for each $t \geq 0$.

A **strategy** is a stochastic process $\pi \equiv [m(\omega,t),v(\omega,t): \omega \in \Omega,\ t \geq 0]$ with values in A, which is progressively measurable. It will be convenient to also develop a subset of strategies that directly condition on the set of histories of the controlled process. Let $C[0,\infty)$ be the space of continuous functions Y_t, $t \in [0,\infty)$; let $\mathcal{H}_t$ be the smallest σ–algebra of subsets of $C[0,\infty)$ which contains all sets of the form $\{Y[0,\infty): Y_s \leq a,\ s\leq t,\ a\in R\}$. Let H_t denote the set of histories, i.e. $H_t = \{ Y(s): s\leq t\}$. A **natural strategy**, $[\pi(t) = m(t),v(t): t \geq 0]$ is a specification $\pi(t):H_t \to A$, for all $t\geq 0$, such that π takes values in A, is $\mathcal{H}_t$–adapted and there exists atleast one solution to the stochastic differential equation (1.1) which is $\mathfrak{F}_t$ measurable. A natural strategy is said to be a stationary Markov strategy if $\pi(t) = \beta(y(t))$ for some function β.

As explained above the problem is terminated when the process hits the origin, with a terminal payment of zero.[4] Prior to this date, the agent's instantaneous payoff or utility is $U(m(\pi(t)),v(\pi(t)))$. The agent's optimization problem is:

$$\text{(P)} \qquad \underset{\pi}{\text{Max}}\ E\delta \int_0^{T(\pi)} e^{-\delta t}\ U(m(\pi(t)),\ v(\pi(t))\ dt$$

where $T(\pi)$ is the first hitting time of zero,(with $Y(0) = y > 0$ in the survival problem and $Y(0) = y < 0$ in the race problem.), under the strategy π. (P) is clearly a stationary dynamic programming problem, and we shall denote its value function by $V(y)$. Any solution of (P) will be called an optimal strategy or policy. If an optimal policy picks controls which depends only on the level of the current state, i.e. if a Markov policy is optimal in the class of all strategies, it will be called a stationary Markov optimal policy.

The following **assumptions** on U and A are maintained (except in Section 5):

(A1) The utility function $U(m,v) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is continuous and strictly concave.

(A2) Variances are bounded away from zero: there is $\alpha > 0$, s.t. $(m, v) \in A \Rightarrow v \geq \alpha$.

(A3) The set of feasible controls is convex and compact.

3. The Survival Problem: Existence and Characterization

Actions are chosen by the agent in order to optimally trade–off the maximization of instantaneous utility against future payoffs. Future payoffs depend on the expected continuation values whose determination involves the marginal valuations and the degree of local risk aversion exhibited by the value function (i.e. the first and second derivative of the function). Hence optimal policies are likely to be state dependent. Further, this dependence will not generally be as simple as a constant or linear relationship. Indeed even the problem of determining the "efficient frontier" in the space of controls (i.e. determining which controls will never be used) is a non–trivial one when the agent's action has effects on three dimensions; mean, variance and immediate utility. In short, with a three–dimensional choice space the simple trade–off between instantaneous utility and future prospects has no simple trade–off in control space to determine the efficient frontier of controls. Of course determining such a frontier is only part of the solution anyway since the more difficult problem is to determine the order in which the controls on this frontier are utilized as y increases. We present now existence and characterization results. Note that all proofs are in Section 4.

The following theorem is the basic existence result:

Theorem 3.1 **The value function is C^2, $V(0)=0$ and $\lim_{y\to\infty} V(y)=\overline{U}$ and satisfies**

$$\operatorname*{Max}_{(m,v)\in A} \{ 1/2\ v\ V''(y) + m\ V'(y) - \delta\ V(y) + \delta\ U(m,v)\} = 0 \quad (3.1)$$

Furthermore, V is the unique solution of (3.1) in the class of functions which are twice continuously differentiable on $[0,\infty)$ and equal to the endpoints 0 and $\overline{U}$.

Finally, there is a continuous function β^*, such that the stationary Markov strategy formed by this function is an optimal policy.

Remark 1: Theorem 3.1 is similar to a result proved in Krylov (1981, Theorem 1.4.5). There are two main differences between the results. Firstly, Krylov's result is valid when the domain of the state space is a compact interval (and indeed it does not seem that the proof would extend to unbounded domains). Secondly the result proved there requires the immediate payoff function U to satisfy a Lipschitz condition (which Theorem 3.1 does not require). On the other hand, in Krylov's formulation, state dependence in the payoffs is admissible and he does not impose the condition of strict concavity on the payoff function.

Remark 2: The sufficiency or verification result, that any solution to (3.1) is in fact atleast as great as the value function is of course true even if the immediate payoffs are not concave.

We turn now to the characterization results:

Theorem 3.2 **i) The value function is strictly increasing in y.**
ii) Suppose that π^* is an optimal natural strategy. Then,

$$V'(y) = V'(0)\ Ee^{-\delta T^*(y)} \quad (3.2)$$

where,

$$T^*(y) = \min \{t\colon Y(t) = 0 \mid Y(0) = y,\ \pi^*\}.$$

iii) The value function is strictly concave. Further V'' is increasing in y.

Remark: The characterization (3.2) is extremely useful, as we shall see in the sequel. Its proof critically uses the fact that the level of the state variable does not directly affect the instantaneous utility of the agent. An interesting point to note is that the monotonicity and strict concavity properties of the value function are true even when the immediate payoff function is not concave.

Armed with the optimality equation characterization, it is easy now to develop several alternative characterizations of the optimal policy. Allowing a choice over the diffusion coefficient creates many analytical difficulties. This may be seen concretely from (3.1). We know from Theorem 3.2 that the value function is strictly concave and strictly increasing and has an increasing second derivative. However in order to derive properties of β^*, the selection of maximizers of (3.2), we additionally need to know how V''/V' behaves in y. In general, this information is impossible to deduce. We now present results which develop alternative and increasingly more detailed characterizations of β^*, under some further restrictions.

Proposition 3.3 **Suppose y' > y. Let $\beta^*(y) \equiv$ m,v and $\beta^*(y') \equiv$ m',v'. Then, it must be the case that either or both of the following monotonicity relations are satisfied: i) v'>v ,ii) m'<m and m'/v' < m/v.**

Remark: The proof of Proposition 3.3 does not use the concavity of the utility function. Indeed the result is true for any selection from the optimality equation and holds without any assumptions whatsoever on U. It is our conjecture that under monotonicity and strict concavity restrictions on U, both i) and ii) will hold for the optimal policy. We have however been able to prove this only under some further conditions that are discussed in the immediate sequel. We do have counterexamples to show that, without concavity assumptions, only i) or ii) but not both need hold (see Section 5).

We shall say that β^* is an interior optimal policy if $\beta^*(y) \in$ int A, for all y$\in$ $\mathbb{R}_+$. Suppose that U is differentiable and define H_W: A $\rightarrow \mathbb{R}$ as H(m, v) = U(m,v) − vU_2(m,v) − mU_1(m,v).

Proposition 3.4. **Suppose β^* is interior. Then, y' > y implies that**

i) H(m′,v′) > H(m,v) **(3.3)**

ii) $U_1(m',v') > U_1(m,v)$

iii) $U_2(m',v') < U_2(m,v)$

Further, the optimal policy β^* is a one to one function.

So (3.3) establishes that there is an order of usage of the optimal controls. However, in general that order is nothing as straightforward as the mean–variance ratio or some other such simple index (unlike the pure survival problem).

We shall say that the utility function is separable if there exist functions $\xi(m)$ and $\phi(v)$ such that

$$U(m, v) = \xi(m) - \phi(v).$$

Proposition 3.5 **Suppose that β^* is interior and U is separable. Further suppose that utility is decreasing in m and increasing in v. Then, $y' > y$ implies that $m' < m$, $v' > v$ and $U(m',v') > U(m,v)$.**

Incidentally, boundary conditions that are widely used in economics and finance (sometimes referred to as Inada conditions)[5] would guarantee interiority of the optimal policy. If the feasible set A has more structure, we can even dispose of the interiority assumption...We say that A is a rectangle if $A = [\underline{m}, \overline{m}] \times [\underline{v}, \overline{v}]$, $0 < \underline{v} \leq \overline{v}$.

Proposition 3.6 **Suppose that A is a rectangle and U is separable. Then $y' > y$ implies $m' < m$, $v' > v$, $U(m',v') > U(m,v)$.**

4. Proofs of Section 3

It is simpler for a logical development of the proofs of Theorems 3.1 and 3.2 to actually prove them in approximately the reverse order. More precisely we will first prove that the value function is increasing and if C^1 it satisfies (3.2). This will be used to prove Theorem 3.1 after which we will conclude by proving Theorem 3.2iii).

Proof of Theorem 3.2 i) Consider two initial states y and y' with $y' > y$. A candidate policy from y' is: use the constant control that generates $\overline{U}$ (say $\overline{a}$) till the first time the state hits y. Thereafter, use π where π is ϵ–optimal from y. Then,

$$V(y') \geq \overline{U}\,(1 - Ee^{-\delta \overline{T}}) + Ee^{-\delta \overline{T}}\,(V(y) - \epsilon) \tag{4.1}$$

where $\overline{T} = \min\{t\colon Y(t) = y \mid Y(0) = y',\ \pi \equiv \overline{a}\}$

Since ϵ is arbitrary and $\overline{U} \geq V(y)$, it follows from (4.1) that the value function is monotonic (it will in fact be seen to be strictly monotonic from the strict concavity property that we will shortly prove). ∎

Proof of (3.2) when the value function is C^1 and there is a natural optimal strategy:

Let π be any natural strategy. We will first describe a Θ–translate of π, where $\Theta \in \mathbb{R}$. To begin with, suppose in fact that $\Theta > 0$. For any partial history $\{Y(s) : 0 \leq s \leq t\}$, a Θ–translate of the history is $\{Y(s) - \Theta : 0 \leq s \leq t\}$. A Θ–translate of π is now defined as: for every partial history take its Θ–translate. If the Θ–translate has not yet hit zero, i.e. $Y(s) - \Theta > 0$, for all $0 \leq s \leq t$. then the action for $\{Y(s) : 0 \leq s \leq t\}$ is identical to the action under π for partial history $\{Y(s) - \Theta : 0 \leq s \leq t\}$. If $Y(s) - \Theta = 0$, for some $0 \leq s \leq t$, then the action may be somewhat arbitrarily chosen. For concreteness, suppose it is $\hat{\alpha} \in A$ where $U(\hat{m},\hat{v})>0$. If $\neg$ is H_t measurable then so is a θ–translate of π.

Fix $y > 0$. Consider an alternative starting state $y + \Theta$, $\Theta > 0$. Let π^* be an optimal policy. Let the Θ–translate of π^* be the policy from $y + \Theta$. Then letting $T^*(y)$ denote the first time of hitting θ with initial state $y + \theta$ and the θ–translate of π^* as policy, we have,

$$V(y + \Theta) - V(y) \geq Ee^{-\delta T^*(y)}\ V(\Theta) \tag{4.2}$$

It immediately follows that

$$V'(y) \geq V'(0)\ Ee^{-\delta T^*(y)} \tag{4.3}$$

Consider instead the starting state $y - \Theta$. Again consider the $(-\Theta)$–translate of π^*, as a policy for starting state $y - \Theta$. By arguments as above we have

$$V(y) - V(y - \Theta) \leq Ee^{-\delta T^*(y,\Theta)}\ V(\Theta), \tag{4.4}$$

where

$$T^*(y, \Theta) = \min\ \{t : Y(t) = \Theta \mid Y(0) = y,\ \pi^*\}$$

From (4.4) it follows that

$$V'(y) \leq V'(0)\ Ee^{-\delta T^*(y)} \tag{4.5}$$

(4.5) is true by the dominated convergence theorem, utilizing the fact that $T^*(y,\Theta) \uparrow T^*(y)$ a.e. From (4.4) and (4.5), (3.2) follows. In the immediate sequel we prove that there is a stationary Markov optimal policy and the value function is C^2. (3.2) then immediately yields the strict concavity of the value function. □

Proof of Theorem 3.1: In proving Theorem 3.1 the following notation will be useful:

$$F(a, b, c) \equiv \max_{m, v \in A} \left\{ \frac{1}{2} va + mb - \delta c + \delta U(m,v) \right\} \tag{4.6}$$

$$F_1(b,c) = \max_{mv \in A} \left\{ \frac{2m}{v} b - \frac{2\delta}{v} c + \frac{2\delta}{v} U(m, v) \right\} \tag{4.7}$$

For a stationary Markov policy $\beta: \mathbb{R}_+ \to A$ and a $C^2[0,\infty)$ function w, define

$$L_\beta w(y) = \tfrac{1}{2}\beta_2(y) w''(y) + \beta_1(y) w'(y) - \delta w(y) \tag{4.8}$$

where $\beta(y) \equiv \beta_1(y), \beta_2(y) = m, v$

$$F[w](y) = \max_{m, v \in A} \left\{ \tfrac{1}{2} v w''(y) + m w'(y) - \delta w(y) + \delta U(m,v) \right\}$$

$$= F(w''(y), w'(y), w(y)) \tag{4.9}$$

The proof of Theorem 3.1 will be in two steps.

Step 1. Consider the k–step truncated optimization problem, in which the agent controls the diffusion process starting at some $y \in (0, k)$, where $k > 0$. The process is absorbed the first time it hits either 0 or k. Let V_k denote the value function for this problem. Continue to assume that returns after absorption are normalized to zero, irrespective of whether the absorption was at $y = 0$ or $y = k$.

Lemma 4.1 i) V_k is C^2 and is the unique solution to the constrained Bellman equation

$$F[V_k](y) = \max_{m, v \in A} \left\{ \tfrac{1}{2} v V_k''(y) + m V_k'(y) - \delta V_k(y) + \delta U(m,v) \right\} = 0, \; y \in [0, k) \tag{4.10}$$

$$V_k(0) = V_k(k) = 0$$

ii) **There is $M < \infty$, independent of k, such that**

$$\|V_k\| + \|V_k'\| + \|V_k''\| < M \tag{4.11}$$

where $\|\cdot\|$ denotes the sup–norm.

Step 2. We shall then let $k \uparrow \infty$, and argue that limits are well–defined and indeed define the value function of the $k = \infty$ problem.

Lemma 4.2 **There is a C^2 function $\tilde{V}$, such that $(V_k, V_k') \to (\tilde{V}, \tilde{V}')$. Moreover, $F[\tilde{V}](y) = 0$, for all $y \in [0, \infty)$. Further $\tilde{V} = V$.**

Proof of Lemma 4.1: The proof proceeds by way of several auxiliary lemmas. The underlying idea is the Bellman–Howard improvement routine. Parts of the proof are adapted from Krylov (1980, Theorem 1.4.5).

Lemma 4.3 **Suppose $\beta : [0, k] \to A$ is a continuous function. Then**

i) **there is a unique C^2 function w_β such that**

$$L_\beta w_\beta(y) + \delta U(\beta(y)) = 0, \quad y \in [0, k] \tag{4.12}$$

$$w_\beta(0) = w_\beta(k) = 0$$

ii) **there is $M_k < \infty$, such that**

$$\|w_\beta\| + \|w_\beta'\| + \|w_\beta''\| < M_k \tag{4.13}$$

Proof: This result is proved as Lemma 1.4.6 in Krylov (1980). □

The next result is standard and shows that the function w_β given by Lemma 4.3 is in fact the lifetime returns from using β as a stationary Markov policy.

Lemma 4.4 **Consider the stationary Markovian policy in which $\pi(y_{[0,\ t]}) = \beta(y_t)$ where β is continuous. Let**

$$I_\beta(y) = E_\beta\, \delta \int_0^T e^{-\delta s}\, U(\beta(y_s))\, ds$$

where $T = \min\{t : Y(t) = 0 \text{ or } k \mid Y(0) = y, \beta\}$

Then, I_β is the unique C^2 function that solves $L_\beta I_\beta + \delta U(\beta) \equiv 0$, $y \in [0, k]$, and further satisfies $I_\beta(0) = I_\beta(k) = 0$.

Proof: By Lemma 4.3, there is a unique C^2 solution of (4.12). Since β is continuous, from Theorem 5.2 of Skorohod (1982), the stochastic differential equation (1.1) has a solution $[Y(t): t \geq 0]$,unique in probability law. By an application of Ito's lemma to $e^{-\delta t} w_\beta(y_t)$ we get

$$w_\beta(y) = E_\beta \left\{ \delta \int_0^{T \wedge t} U(\beta(y_s))\, e^{-\delta s}\, ds + e^{-\delta(T \wedge t)}\, w_\beta(y_{T \wedge t}) \right\}$$

Letting $t \to \infty$, $T \wedge t \to T$, and an application of the dominated convergence theorem yields

$$w_\beta(y) = E_\beta\, \delta \int_0^T e^{-\delta s}\, U(\beta(y_s))\, ds$$ ■

Let β_0 be an arbitrary continuous function, and denote its associated lifetime returns w_0. Consider,

$$F[w_0]\,(y) = \max_{m,\ v \in A} \left\{ \frac{1}{2} v w''_0(y) + m w'_0(y) - \delta w_0(y) + \delta U(m,v) \right\} \quad (4.13)$$

By the strict concavity of the utility function and the convexity of A, the argmax in (4.13) is single–valued. By the Maximum Theorem (Berge (1963), p. 116), the function of maximizers is in fact a continuous function. Denote this function β_1 and the associated returns w_1. In this manner, we can construct a sequence of C^2 functions w_n and stationary policy functions β_n, $n \geq 0$ such that

$$L_n w_n + \delta U_n = 0 \leq L_{n+1} w_n + \delta U_{n+1}$$

(where $L_n w_n \equiv L_{\beta_n}$, $U_n \equiv U(\beta_n)$ etc.)

Lemma 4.5 **For all n, $y \in [0, k]$, $w_{n+1}(y) \geq w_n(y)$**

Proof: Let the continuous function $h_n \geq 0$ be defined by

$$L_{n+1}(w_{n+1} - w_n) + \delta h_n \equiv 0$$

Then, an application of Ito's lemma to the function $w_{n+1} - w_n$, yields

$$w_{n+1}(y) - w_n(y) = E\,\delta \int_0^{T_n} e^{-\delta s}\, h(y_s)\, ds \geq 0 \qquad \blacksquare$$

To continue the proof of Lemma 4.1, let us now define $\tilde{w} = \lim_{n\to\infty} w_n$. This limit is well–defined by Lemma 4.3, for all $y \in [0,k]$. Further w_n' is uniformly bounded. Since

$$w_n(y) - w_n(z) = \int_z^y w_n'(x)\, dx \tag{4.14}$$

it follows that w_n is in fact an equicontinuous family. Hence, $\tilde{w}$ is a continuous function and $w_n \to \tilde{w}$ uniformly on any compact subset of $\mathbb{R}_+$. Similarly,

$$w_n'(y) - w_n'(z) = \int_z^y w_n''(x)\, dx$$

and hence w_n' is also an equicontinuous (and uniformly bounded) family. By the Arzela–Ascoli theorem, there is a subsequence (retain notation) such that w_n' is convergent on it, to ψ say. Taking limits, along this subsequence,

$$\tilde{w}(y) - \tilde{w}(z) = \int_z^y \phi(x)\, dx \tag{4.15}$$

We further used the dominated convergence theorem in arriving at (4.15). Moreover, (4.15) establishes that $\tilde{w}'(Y) = \phi(y)$. Hence, along the full sequence, $w_n' \to \tilde{w}'$, and of course $\tilde{w}$ is C^1. We have hence proved

Lemma 4.6 **$w_n \to \tilde{w}$, where $\tilde{w}$ is a C^1 function. Further, $w_n' \to \tilde{w}'$.**

It remains to show that in fact $\tilde{w}$ is C^2 and further that $F[\tilde{w}] = 0$. This last step of the proof of Lemma 4.1 is identical to the proof of the analogous step in Krylov (1981) Theorem 1.4.5 and hence is not reproduced here. Clearly, $\tilde{w}(0) = \tilde{w}(k) = 0$.

A standard application of Ito's lemma now shows that $\tilde{w} \geq V_k$. Further, let β be argmax $F[\tilde{w}]$ (and hence it is a continuous function). Then, a second application of Ito's lemma to the diffusion process generated by the stationary policy β, establishes that $I_\beta \geq \tilde{w}$. The first half of Lemma 4.1 is completely proved. We now turn to the proof of Lemma 4.1 ii).

Lemma 4.7 **Let β_k denote the stationary Markovian optimal policy for the kth truncated problem. Define**

$$T_k^0(y) = \min \{t > 0\colon Y(t) = 0 \mid Y(0) = y, \beta_k\}$$

$$T_k^k(y) = \min \{t \geq 0\colon Y(t) = k \mid Y(0) = y, \beta_k\}$$

$$T_k(y) = \min (T_k^0(y), T_k^k(y))$$

$$V_k'(y) = E_{\beta_k} [e^{-\delta T_k^0(y)} \cdot P(T_k^0 = T_k)V_k'(0) + e^{-\delta T_k^k(y)} \cdot P(T_k^k = T_k)V_k'(k)] \tag{4.16}$$

Proof: The proof of Lemma 4.7 is very similar to the proof of Theorem 3.2 ii), and hence we omit it. ∎

It is immediate from (4.16) that $V_k'(y) \leq \max(V_k'(0), V_k'(k))$. To establish an upper bound on $\|V_k'\|$ it clearly suffices to establish such a bound on $V_k'(0)$ and $V_k'(k)$. We now show that a bound exists which is in fact independent of k.

Consider a problem in which the stochastic process starts at $y > 0$, is terminated only if it ever hits zero and each control yields constant flow payoffs of $\overline{U}$. The pure survival optimization problem is

$$(P') \quad \underset{\pi}{Max} \quad \overline{U} \left[E_{\overline{\alpha}}\, \delta \int_0^T e^{-\delta s} \right]$$

It is straightforward to check that an optimal solution to (P') involves the choice of a constant control. Denote the value function for this problem W and the optimal control $(\overline{m},\overline{v})$. Further it is clear that $W(y) \geq V_k(y)$ for all y and k and $W(0) = 0$. Finally, elementary calculations show that

$$W(y) = 1 - e^{-\overline{\lambda} y}$$

where $\overline{\lambda} = \dfrac{\overline{m} + \sqrt{\overline{m}^2 + 2\delta\overline{v}}}{2\overline{v}}$.

So we have

$$V_k(y) - V_k(0) \leq W(y) - W(0),$$

Form this it follows that

$$V_k'(0) \leq W'(0) = \overline{\lambda}$$

Of course, by definition $V_k'(0) \geq 0$. It is easy to see that a symmetric set of arguments could be repeated for $V_k'(k)$.

Of course $\|V_k\| \leq \overline{U}$. Finally note that

$$V_k''(y) = -\frac{2m_k}{v_k} V_k'(y) + \frac{2\delta}{v_k} V_k(y) - \frac{2\delta}{v_k} U(m_k, v_k) \qquad (4.17)$$

Clearly, the uniform upper bounds on $\|V_k'\|$, $\|V_k\|$ and $\|U\|$, imply a uniform upper bound on $\|V_k''\|$, an upper bound independent of k. Hence, Lemma 4.1 is now fully proved. □

We now turn to the second step in our proof of Theorem 3.1. The objective here is to use the properties of the k–th truncation value function to establish analogous properties for the value function V. First of all, clearly V_k is a monotone sequence of functions, and hence there is $\tilde{V}$ such that $V_k \uparrow \tilde{V}$, as $K \to \infty$. By an argument identical to that in Lemma 4.6, we get

Lemma 4.8 $\tilde{V}$ **is a** C^1 **function. Moreover,** $V_k' \to \tilde{V}'$**, as** $k \to \infty$**.**

Note that $F[V_k](y)=0$ is equivalent to $F_1(V_k',V_k) + V_k''=0$. Hence,

$$V_k'(y) - V_k'(0) + \int_0^y F_1(V_k(x),\, V_k'(x))\, dx = 0 \tag{4.18}$$

But $V_k'(y)$ converges to $\tilde{V}'(y)$ for all y and by the maximum theorem $F_1(V_k',V_k)$ converges to $F_1(\tilde{V}',\tilde{V})$. From (4.18) it then follows that

$$\tilde{V}'(y) - \tilde{V}'(0) + \int_0^y F_1(\tilde{V}(x),\, \tilde{V}'(x))\, dx = 0 \tag{4.19}$$

From the fundamental theorem of calculas, the maximum theorem and (4.19) it follows that $\tilde{V}$ has a continuous second derivative $\tilde{V}'' = F_1(\tilde{V},\tilde{V}')$. Further $F[\tilde{V}] = 0$. Standard applications of Ito's lemma then establishes that $\tilde{V} = V$. Further given the strict concavity of the utility function, there is a unique element to argmax F[V] and this selection, by the maximum theorem, is a continuous function. A further use of Ito's lemma establishes that this selection in fact achieves the returns V, i.e. is optimal. The proof of Lemma 4.2 and Theorem 3.1 is complete. □

From the strict concavity and monotonicity of V, $V'' + F_1(V',V) = 0$ implies that V'' increases with y. The proof of Theorem 3.2 is complete. ■

Proof of Proposition 3.3: Suppose $y' > y$. From the fact that $(m,v) = \text{argmax } F[V](y)$ and $(m',v') = \text{argmax } F[V](y')$ it follows that

$$(v - v')(V''(y) - V''(y')) + (m - m')(V'(y) - V'(y')) \geq 0 \qquad (4.20)$$

Further $F[V] = 0$ is equivalent to $V'' + F_1(V',V) = 0$ and there is a common argmax for F and F_1. It then follows that

$$(m/v - m'/v')(V'(y) - V'(y')) - \delta(1/v - 1/v')(V(y) - V(y')) \geq 0 \qquad (4.21)$$

From (4.20) and (4.21) the proposition follows. ■

Proof of Proposition 3.4: First–order conditions yield

$$\frac{1}{2} V''(y) = - \delta U_2(m,v) \qquad (4.21)$$

$$V'(y) = -\delta U_1(m,v) \qquad (4.22)$$

Substituting (4.21) – (4.22) back into $F[V] = 0$, yields

$$H(m,v) = V(y)$$

From the monotonicity and strict concavity of V and the further property that V'' is increasing in y, the proposition follows. □

Proof of Proposition 3.5: Follows from (4.21),(4.22)and the fact the utility function is assumed to decrease in m and increase in v. ■

Proof of Proposition 3.6: Since $A = [m, \overline{m}] \times [v,\overline{v}]$ and U is separable, F[V](y) can be re–written as

$$\max_{(m\ ,\ v)\epsilon A} \left\{\frac{1}{2} vV''(y) + mV'(y) + \delta U(m,v)\right\} = \max_{m} \{mV'(y) + \delta\xi(m)\} +$$

$$\max \left\{\frac{1}{2}vV''(y) - \delta\phi(v)\right\} = \delta V(y)$$

Let $W_1(y) \equiv \max_{m} \{V'(y) + \delta\xi(m)\}$. From the strict concavity of V it immediately follows that $m' < m$. For similar reasons $v' > v$ and hence $U(m,v) < U(m',v')$. □

5. Some Examples

In this section we present two computable examples of survival problems. The intention is to show that one may not be able to do much better than the characterization results presented in Section 3 and that quite a variety of policies are consistent with optimality. As is well known computable examples are hard to come by when controls have direct payoffs. Hence the examples will have an artificial flavor in that strong resrictions will be placed on the payoff function U and feasible set A to facilitate computation. In particular the convexity assumptions which were used to prove Theorem 3.1 may not be satisfied by these examples. Note however that the verification result, that a solution Of (3.1) and an associated stationary Markov policy are in fact optimal, is true even without convexity restrictions.

Example 5.1: $\mathbf{U(m,v) = m^{\gamma}\, v^{\theta},\ 2\theta + \gamma \in (0,1).\quad A = \{(m,v){:}\ v = m^2,\ m \geq 0\}}$.

The solution we will check for is $V(y) = ky^{\alpha}$ and $\beta(y) = (y/c,\ (y/c)^2)$, where $k > 0$ and $c > 0$ and $\alpha \in (0,1)$. There are two conditions to verify: i) $L_{\beta}V + \delta U_{\beta} = 0$ and ii) $L_{\beta}V + \delta U_{\beta} = F[V]$. It is straightforward to check that i) implies that $\alpha = 2\theta + \gamma$ and further that $k = \delta(\alpha-2)c^{1-\alpha}\,[\alpha - 2\delta c]^{-1}$. ii) implies that $k = 2\delta c^{2-\alpha}[\alpha(1-\alpha) + \delta c^2 - \alpha c]$. It is a somewhat tedious exercise in algebra to then show that these last two equations have positive solutions for all large δ.

So this example exhibits increasing mean (and consequently increasing variance) but a decreasing mean–variance ratio as y increases.

Example 5.2: $\mathbf{U(m,v) = |m|^{\gamma}\, v^{\theta},\ 2\theta + \gamma \in (0,1).\quad A = \{(m,v){:}\ v = m^2,\ m \leq 0\}}$

By a procedure identical to that of the previous example it is possible to show that $V(y) = ky^{\alpha}$ and $\beta(y) = (-cy,\ (cy)^2)$ for $k > 0$, $c > 0$ and $\alpha \in (0,1)$. So in this example, the mean decreases, the variance increases and the mean–variance increases in y.

Remark: For both of the above examples a solution of the Bellman equation is in fact a stationary Markov optimal strategy. It follows from a theorem in Karatzas and Shreve (1987,pp. 352–355) that there are diffusions consistent with each of the exhibited strategies. A standard argument then establishes V to be an upward bound for the returns to arbitrary strategies and a bound realizable by β. The argument needs to be in two steps: going from a truncated problem on state space [0,k] to the full problem.

6. The Race

We turn finally to the minimization problem (the race to zero). Recall that the state space is $(-\infty,0]$ and $\bar{U} \leq 0$. The proofs of the results in this section mirror the proofs in Section 4 and hence will be omitted. The existence and value function characterization are:

Theorem 6.1: i) The value function V is twice continuously differentiable, satisfies the limit properties $V(0) = 0$ and $\lim_{y \to -\infty} V(y) = \bar{U}$ and solves the Bellman optimality equation (3.1). Further it is the unique solution of (3.1) in the space of C^2 functions with the given limit properties.
ii) The value function is a strictly increasing, strictly concave function and V'' increases in y.

On the other hand the optimal policy "works less" the closer the state gets to zero, i.e. we have

Proposition 6.2: i) There is a continuous function β which constitutes the unique stationary optimal policy.
ii) Let $y' > y$ and denote $\beta(y) = m,v$, $\beta(y') = m',v'$. Then, either a) $v' \geq v$ or b) $m'/v' \leq m/v$ and $m' \leq m$ or c) both a) and b) hold.
iii) Under the hypotheses of Propositions 3.5 or 3.6, $v' \geq v$, $m'/v' \leq m/v$ and $m' \leq m$.

7. Applications and Extensions

There are several applications in economics, finance and game theory of the general survival problem studied in this paper. Consider the following:

i) **Incentive Schemes:** Suppose the actions of an economic agent (e.g. the manager of a firm) are not observable by some other agent (e.g. the owner/s) whose payoffs are determined (with some noise) by these actions. A widely observable class of incentive schemes are those in which the manager is paid a salary (w) and retained on the job provided aggregate returns (Y(t)) are above a critical threshhold ($Y(t) \geq 0$). A variant requires a minimum positive rate of return k (i.e. the standard is $Y(t) - kt \geq 0$). These schemes were introduced in Radner (1986) and further studied in Dutta–Radner (1990). The analysis here suggests the optimal response of such a manager when faced with such "up or out" schemes. (For further details see Radner (1986) and Dutta–Radner (1990)). Note that different actions may be interpreted as different choices of projects or effort levels and hence imply different direct payoffs to the manager.

ii) **Minimal Consumption:** A voluminous literature in economics addresses the issues of poverty and malnutrition in which a critical minimal consumption level plays a central role. The idea here would be to choose actions so as to keep aggregate wealth (Y(t)) above aggregate consumption requirements (kt). Again it is critical to allow control dependent utility. (For a further discussion see Gersovitz (1983)).

iii) **Portfolio Management:** The choice of an optimal portfolio subject to bankruptcy constraints is another possible application. Aggregate wealth (Y(t)) has to be non–negative to participate in the market (in the absence of perfect capital markets). A portfolio determines the distribution of current returns. In this application the log of the wealth process follows the diffussion (1.1) (see Heath et. al. for details).

iv) **Strategic Election Models:** A number of models in political science consider the following situation: a candidate is elected and maintained in office provided his performance is satisfactory. While on the job he is paid a constant wage. The issue is to find an appropriate definition of satisfactory performance given that candidates respond differently to alternative reelection procedures. The survival problem outlined here provides one reelection procedure in which the aggregate of outcomes in previous periods is the determining factor. (For an example of such a model see Rogoff (1990).

A very useful generalization of our results would be to the case where U is a function of Y as well as (m, v). This problem seems a dimension up in difficulty although the techniques and results developed here will hopefully help in the analysis.

Footnotes

[1]Additionally, almost all of these papers have analyzed exclusively the undiscounted case: the objectives in Heath et.al are, respectively, maximize ET and minimize ET. Dupuis–Kushner also take as an objective maximize ET while Majumdar–Radner use maximize $P(T = \infty)$. One usage of a discounted criterion may be found in Orey et.al. who analyze the folowing two cases :i) Maximize $E\beta^T$ where $\beta \in (0,1)$ and ii) Minimize $E\beta^T$ with $\beta > 1$. Pestien–Sudderth (1985, 1988) and Sudderth–Weerasinghe (1989) have also looked at closely related objectives, but again in the pure survival or race case.

[2]Another example of a general survival analysis is Sheng (1980). However she restricts herself to a problem in which there are only two drift–diffusion pairs to choose from. Her computation–based approach does not seem to generalize beyond the binary control problem.

[3]A stochastic process $[f(t) : t \geq 0]$ on $(\Omega, \mathfrak{F})$ is $\mathfrak{F}_t$ adapted if i) $f(\omega, t)$ is jointly measurable in ω and t, and ii) $f(\cdot, t)$ is $\mathfrak{F}_t$–measurable, for each $t \geq 0$.

[4]Zero is of course just a normalization and any constant severance reward would do as well.

[5]Define $\overline{m}(v) = \max \{m: (m,v) \in A\}$ (respectively, $\underline{m}(v) = \min \{m: (m,v) \in A\}$; similarly $\overline{v}(m)$ and $\underline{v}(m)$. The Inada conditions are: $\lim_{m \to \overline{m}(v)} U_1(m,v) = -\infty$ and $\lim_{m \to \underline{m}(v)} U_1(m,v) = 0$, for all v. Further, $\lim_{v \to \underline{v}(m)} U_2(m,v) = \infty$, $\lim_{v \to v(m)} U_2(m,v) = 0$ for all m.

References

1. Berge, C., 1963. Topological Spaces, McMillan, New York.

2. Dupuis, P. and H. Kushner, 1989. Minimizing Escape Probabilities: A Large Deviations Approach, **SIAM Journal of Control and Optimization**, 27: 432–445.

3. Dutta, P.K. and R. Radner, 1990. Optimal Principal Agent Contracts for a Class of Incentive Schemes: A Characterization and the Rate of Approach to Efficiency, mimeo, University of Rochester.

4. Gersovitz, M. 1983, "Savings and Nutrition at Low Incomes", **Journal of Political Economy**, Vol. 91, pp. 841–855.

5. Heath, D., S. Orey, V. Pestien, W. Sudderth 1987, "Maximizing or Minimizing the Expected time to Reach Zero", **SIAM Journal of Control and Optimization**, vol. 25, no. 1, pp. 195–205.

6. Karatzas, I. and S.Shreve, 1987, Brownian Motion and Stochastic Calculus, Springer–Verlag, NY.

7. Krylov, N., 1981. Controlled Diffusion Processes, Springer–Verlag, NY.

8. Kushner, H.J., 1978. Optimality Conditions for the Average Cost Per Unit Time Problem with a Diffusion Model, **SIAM Journal of Control and Optimization**, 16: 330–346.

9. Majumdar, M. and R. Radner, 1988. "Survival under Production Uncertainty" in Equilibrium and Dynamics (eds. W. Brock and M. Majumdar), Macmillian (forthcoming).

10. Majumdar, M. and R. Radner, 1990. Linear Models of Economic Survival Under Production Uncertainty, Working Paper # 427, Cornell University, Ithaca, NY.

11. Orey, S., V. Pestien and W. Sudderth, 1987. Reaching Zero Rapidly, **SIAM Journal of Control and Optimization**, 25, 1253–1265.

12. Pestien, V. and W. Sudderth, 1984. Continuous Time Red and Black: How to Control a Diffusion to a Goal, **Mathematics of Operations Research**, 10, 599–611.

13. Pestien, V. and W. Sudderth, 1988. Continuous Time Casino Problems, **Mathematics of Operations Research**, 13, 364–376.

14. Radner, R., 1986. Repeated Moral Hazard with Low Discount Rates, in: Heller, W., R. Starr and D. Starrett, eds., Essays in Honor of Kenneth Arrow Cambridge University Press, Cambridge, U.K..

15. Rogoff, K., 1990, Equilibrium Budget Political Cycles, **American Economic Review**, 80, 21 – 37.

16. Sheng, D., 1980. Two–Mode Control of Absorbing Brownian Motion, Bell Laboratories Technical Memorandum, Murray Hill, NJ.

17. Skorokhod, A.V., 1982. Studies in the Theory of Random Processes, Dover, NY.

18. Sudderth, W. and A. Weerasinghe, 1989. Controlling a Process to a Goal in Finite Time, **Mathematics of Operations Research**, 14, 400–409.