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Abstract

This paper models trading as a dynamic game between strategic agents with private information, paying particular attention to the role of trading arrangements. We analyze the impact on intertemporal price formation of two institutional features: first, price continuity-depth requirements that restrict transaction-to-transaction price changes, and second, the form of order submission. Private information is a common resource that insiders 'over-exploit' by trading too quickly. Price continuity rules enable insiders to slowly exploit their information over time. Paradoxically, more stringent price continuity requirements may actually improve market efficiency indirectly by increasing insider profits and inducing more traders to become informed at cost.

We prove that insiders cannot collude to trade more slowly and thereby increase their profits, even if we allow history-dependent strategies that threaten retaliation for deviating from 'cartel' behavior. Trading arrangements, however, have a substantial effect on the character of equilibrium. If traders can submit price-contingent limit orders, collusive behavior can be maintained. Finally, we examine the strategic response of insiders to competition from outsiders who use simple technical trading rules to 'free-ride' off their private information.

1 Introduction

In many problems of economic interest, some agents possess more information about market variables than others. Two interesting questions arise as a consequence: First, how do less informed agents infer the nature of private information from public signals? Second, how do informed agents compete with one another to optimally exploit their informational advantage? Both these issues have an added dimension of complexity in a securities market, where interactions between agents take place over time. The first question is the principal focus of the literature on rational expectations.[1] The second question has been largely ignored in a dynamic framework. In this paper, we model the dynamic competition among a group of strategic agents with private information, paying particular attention to the importance of institutional arrangements.

We analyze the following model. A group of traders (or insiders) possess the same private information concerning the full-information value of a security. Insiders and other (uninformed) traders trade in the market until a random date at which time the full-information price is publicly revealed. In each period, an optimizing market maker who acts as a trader of last resort determines the asset's price, subject to certain institutional constraints that we discuss shortly.[2] We examine two alternative forms for an insider's actions: (i) *Market orders*, which specify the quantity to be traded, and (ii) *Limit orders*, which specify a demand schedule. The interaction between traders is modelled as a dynamic game.[3]

[1]See, for example, Grossman (1976), Grossman and Stiglitz (1980), and Hellwig (1980) for competitive static models and Dubey, Geanakoplos, and Shubik (1989) and Kyle (1989) for imperfectly competitive static models. Models of financial markets where agents attempt to infer the private information of insiders include, among others, Glosten and Milgrom (1985), Kyle (1985) and Admati and Pfleiderer (1988).

[2]Market makers stand ready to trade on demand, absorbing order imbalances created by the asynchronous arrival of traders, and providing *immediacy* to the market.

[3]Our model differs from previous studies of insider trading in several respects. First, unlike Kyle (1985), Grinblatt and Ross (1985), and Laffont and Maskin (1990), we do not focus exclusively on the single insider case. Second, unlike other papers with multiple insiders, we consider intertemporal trading strategies including history-dependent strategies. Finally, we focus attention on the importance of explicitly modeling trading arrangements.

Three critical features of the model should be emphasized. First, private information is a resource commonly owned by the insiders. Trading by any one insider generates a negative externality for other insiders through the effect on the current and future prices of the security. Our intuition suggests that multiple insiders will generally over-exploit their common information, leading to more rapid price convergence and higher trading volume than a cartel that acts as a monopolistic insider.[4]

The second feature of our model is that it is dynamic. With repeated interactions over trading rounds, it is not immediate that 'excessive' trading is inevitable. Indeed, the literature on repeated games demonstrates that dynamic interactions can sustain collusive behavior by the threat to move to a suboptimal Nash equilibrium for all future periods, provided the players are sufficiently patient.[5] However, our model is, strictly speaking, not a repeated game since the price of the security changes over time with the volume of trading. There is no general theory available that characterizes the equilibrium payoff sets in (non-repeated) dynamic games.[6] The intuition from repeated games suggests that collusive trading is sustainable if the discount rate is sufficiently low.

Third, we explicitly incorporate some of the institutional features of securities markets. In many markets, the prices set by market makers must satisfy price continuity-depth requirements. These restrictions place limits on transaction-to-transaction price movements for a given volume of trade. For example, New York Stock Exchange (NYSE) Rule 104 requires that the specialist (on the NYSE a single market maker or specialist is assigned to each stock) maintain a 'fair and orderly market.' To accomplish this, the NYSE provides price

[4]For example, Meulbroek (1991) examines data on illegal insider trading obtained from the SEC and concludes that insider trading leads to large price movements. Further, in Meulbroek's sample, an episode of insider trading typically involves multiple insiders. Since her data includes only insiders who were prosecuted, the actual number of insiders per episode is presumably larger.

[5]See, for example, Friedman (1971). Fudenberg and Maskin (1986) analyze the sustainability of collusive (and other) outcomes through more efficient threats.

[6]Two recent papers by Abreu and Dutta (1990) and Lockwood (1990) each suggest that broad classes of dynamic games have qualitative features similar to repeated games.

continuity-depth guidelines to its specialists that vary across stocks and are determined by the price range and normal trading volume of the stock. For a NYSE stock trading between \$20 and \$29 $\frac{7}{8}$ with average daily share volume in the previous month (excluding trades of 25,000 or more) of 10,000–24,999, the maximum price change for 3,000 share volume is \$ $\frac{7}{8}$.[7]

Specialists who perform poorly according to these measures risk having their stocks reassigned to others or not being assigned more profitable stocks in the future. Usually there are special provisions for suspending continuity requirements following important information events such as earnings announcements. For example, NYSE specialists can exceed their price limits after obtaining permission from a floor official.[8] Incorporating price continuity rules into our model permits us to analyze their effect on market efficiency and enables us to derive closed-form solutions for the dynamic strategies of insiders.

The second set of institutional features we accommodate concern the types of orders permitted. We begin with the case where traders submit market orders, i.e., orders to execute at the prevailing market price. We analyze Markov perfect equilibria, i.e., equilibria in which traders adopt strategies that depend only on the current price.[9] We show the existence of Markov perfect equilibria in which the speed at which prices converge to the full information value is increasing in the number of insiders. Curiously, for a given number of insiders, price continuity requirements have no affect on the rate of convergence of prices. As a result, if private information can be acquired at cost, more stringent price continuity requirements actually *increase* market efficiency by increasing the rewards to becoming informed. However, this leads to the duplication of investment in the production of short-term information that

[7]The NYSE reports that in 1988, 92.1% of all transactions of 1,000 shares or less traded with a price change of 0 or $\frac{1}{8}$ from the immediately preceding trade.

[8]An independent question concerns why exchanges institute price continuity rules. Black (1971) suggests a trade-off between efficiency and price stability: "There is a right amount of price continuity for every stock under any given set of market conditions, and either more or less than that is undesirable. Large changes in price caused by the arrival of new information affecting the value of the stock are desirable. Large changes in price that are caused by a temporary imbalance between supply and demand are undesirable."

[9]It is well known that Markov perfect equilibria are subgame perfect equilibria in the space of all admissible strategies. See, for example, Fudenberg and Tirole (1991).

will soon become public, imposing social costs.

When we broaden our analysis to include history-dependent (non-Markov) strategies we find that collusive trading can never be sustained in equilibrium. This result holds for all positive discount rates and all trading environments. Underlying this surprising result is the fact that the game environment changes over time in the sense that price moves closer to the full-information value over time.[10]

We then consider the effect of competition from uninformed traders who pursue simple strategies that effectively free-ride on insiders' information. We demonstrate that outsiders can profit from the autocorrelation in prices induced by insiders' dynamic trading strategies by adopting technical analysis. The effect of such competition is to increase price efficiency and lower the profits of the informed traders. In this case, price continuity requirements affect the rate of convergence of prices.

The above results are for a trading protocol in which traders submit quantity orders that execute at the prevailing market price. We focus attention on the importance of order form by allowing traders to submit price-contingent limit orders. We show that, unlike the equilibria when only market orders are allowed, multiple insiders can avert over-exploitation of their common informational resource if they can place price contingent orders. In fact, we exhibit equilibria with multiple insiders under which the rate of price convergence is *slower* than the rate with a single insider. Intuitively, with limit orders, any abnormal price movement is caused only in part by the trader who deviates from the cartel.[11]

The rest of the paper proceeds as follows: In Section 2 we set up the model, and describe the strategic choices of the players. Section 3 analyzes the equilibrium strategies of

[10]Another way to view this result is that it reflects the exhaustible resource nature of commonly-owned private information. Hence, the trading game is analogous to a repeated game in which the payoffs per period shrink after every play of the one-shot game.

[11]The argument requires some subtlety. Merely placing limit orders that execute if price movements are 'abnormal' is not sufficient, since there is the additional requirement that the specialist also finds it in his best interest to set the collusive price if there is no deviation by any player.

a monopolistic insider, and in Section 4, we analyze the multiple insider case. The analysis is extended in Section 5 to the case where traders can acquire information at cost. In Section 6, we discuss how these strategies are affected by competition from outsiders who use simple trading strategies to free- ride on the autocorrelation in prices induced by insider trading. Finally, Section 7 summarizes the paper and offers suggestions for further research. All proofs are in the appendix.

2 The Model

2.1 The Structure of Trading

Consider the market for a single risky asset that can be traded at dates $t = 1, 2, \ldots,$ into the future. The security is a claim to the cash flows from a project which begins to yield revenue or dividends at a random date $\tilde{\tau}$ in the future. The dividend paid at time $t (t > \tau)$ is denoted by $\tilde{d}_t$. We assume $\tilde{d}_t$ is independently drawn from a distribution with unknown mean $d > 0$. The event date τ is an integer-valued random variable; on this date the mean of the dividend process d is publicly announced and the first (random) dividend payment is made.[12] We assume that the probability that the event occurs in the next period, given no announcements to date, is $\phi \in (0, 1)$.[13]

With these assumptions, the expected value of the security is simply the expected present value of the stochastic stream of dividends. Let $r > 0$ denote the risk-free rate of interest, and denote by v the expected value of future dividends given private information about d. It is easy to show that $v = \frac{\phi}{\phi + r}\left(\frac{d}{r}\right)$, so the expected value of the asset is constant in the pre-event period.[14]

[12]The announcement is made just before trading in that period begins.

[13]In other words, the time to revelation is exponentially distributed, and is thus independent of the elapsed time.

[14]This is a direct consequence of our assumption that the probability of an event (announcement) is independent of the current time period. The model can be extended to allow for time-dependence, without affecting our qualitative conclusions in any way. Note that the expected value is computed at the beginning of each period given the event outcome.

We turn to a formal description of the objectives and information of market participants. The stochastic structure of the economy is common knowledge. The focus of the paper is on the behavior of a group of traders, referred to as *insiders* or *informed traders*, who obtain private information regarding the mean of the dividend process, d, at time 0.[15] Insiders are assumed to be risk-neutral. In particular, we are interested in the question of how insiders profitably exploit their information in the pre-event epoch and the effect of their strategies on price dynamics. There are $N \geq 1$ insiders, indexed by $i = 1, \ldots, N$. Let q_{it} represent the order quantity of informed trader i with the convention that $q > 0$ denotes a trader purchase and $q < 0$ a trader sale.

Insiders may face competition from other agents, referred to as *outsiders or uninformed traders* who have no private information. These traders pursue simple dynamic strategies (technical analysis) that the insiders must take into account in their trading. We detail these strategies later. Let x_t denote the aggregate uninformed trade at time t, where $x > 0$ ($x < 0$) denotes net purchase (sale) as before. From the viewpoint of the insiders and the specialist, uninformed trading is a random variable, denoted by $\tilde{x}_t$. In particular $\tilde{x}_t$ will be a (stochastic) function of observable public data such as the size of trade and price changes. The strategies we shall consider for uninformed traders will attempt to capture popular rules of thumb including variants of technical analysis. For simplicity, we ignore noise induced by exogenous liquidity trading.[16] This could be motivated by arguing that since it is common knowledge an information event is to occur in the future, extremely risk averse liquidity traders choose to postpone their trading until the information asymmetry has been resolved.

The trading protocol follows Kyle (1985), and Admati and Pfleiderer (1988, 1989). De-

[15] Dow and Gorton (1990) present a model where two insiders learn each other's information by observing past trades, suggesting that the previous trading history conveys information. Here, insider's have homogeneous beliefs, either because they know the full-information value of the asset or receive a common information signal.

[16] Noise traders are a standard feature of models of trading such as Kyle (1985), serving to camouflage the actions of insiders and offset the losses of market makers. Such trading merely generates an additional stochastic element without altering our basic conclusions.

note by p_t the security's price at time t. This price is determined by a market maker or specialist, who takes the opposite side of all transactions, on the basis of the order flow at that time, and we write $p_t = p_t(Q_t)$, where $Q_t = \sum_i q_{it} + x_t$ is the total order flow originating from informed and uninformed traders. For the moment, we will assume, as is common in the microstructure literature, that only market orders can be submitted.[17] The market maker announces the price *schedule* $p_t(\cdot)$ at the start of each period, choosing the schedule optimally. Since the specialist is at an information disadvantage relative to traders, he suffers expected losses from trade in the pre-event period. In the periods after the information event, all traders know the value of the security, and the market maker deals only with liquidity-motivated traders. The market maker's pre-event losses must be offset by profits made off liquidity traders during these 'normal' times when there is no impending information event on the horizon. We do not analyze the post-event or symmetric information market in detail in this paper since it has been extensively considered elsewhere.[18]

2.2 Price Continuity Requirements

We assume that the maximum transaction-to-transaction price change is bounded above by a linear function of the quantity traded. Specifically, we assume that the market maker is required to satisfy the following condition: $|p_t - p_{t-1}| \leq \lambda |Q_t|$. The coefficient $\lambda > 0$ summarizes the liquidity of the market, with lower values of λ implying more shares can be traded for a given price change.[19] We define *market depth* as λ^{-1}, the *minimum* order flow associated with a unit price change, a definition analogous to that in Kyle (1985). When the dividend announcement is made at time τ, we assume the specialist can halt trading and prices adjust to the new level implied by the release of dividend information. This rule

[17]This assumption is relaxed later when we consider a market structure that permits price contingent orders.

[18]If the market maker is to make profits at times when there is no information asymmetry, there must be non-information or liquidity motivations for trading, such as portfolio hedging, life-cycle consumption needs, and so on.

[19]See also Hakansson, Beja, and Kale (1985) for alternative criteria.

is consistent with most exchange protocols. The continuity-depth requirement, if binding, reduces price variability after the event, i.e., when all information asymmetries have been resolved. However, as noted above, the situation before the event is the main focus of this paper. As an aside, we note that all of our results go through if price revisions are a linear function of order imbalances. Such a price adjustment rule may simply be dictated by the exchange as part of an automated trading system that does not rely on specialists to set prices. For some information structures, such as Kyle's (1985) model, a linear price adjustment rule arises endogenously out of the learning process of an uninformed market maker.[20]

2.3 Trading Strategies and Equilibrium

Suppose there are N informed traders. At time t, each trader submits an order q_{it} based on all available information including the previous trading history (denoted by h_t) generated by past prices and quantities and the anticipation of the price ($p_t(Q_t)$) set by the specialist given the strategies of the other traders. Simultaneously, the specialist announces a price schedule for the round, again conditioning on all available information and expectations of the strategies of the informed traders. Let H_t denote the set of histories (i.e., $H_t = \{h_t\}$), the set of all possible realizations of prices and quantities in previous trading rounds. A *strategy* for an informed trader is a decision rule to select the order quantity q_{it} as a function of the particular history observed, h_t. Similarly, the specialist's strategy is the price adjustment rule $p_t(Q_t)$ in period t given h_t. The decision rules are said to be *Markov* (or history independent) if they depend only on the beginning period price, p_{t-1}.

A vector of strategies forms a *Nash equilibrium* if each player's strategy maximizes discounted profits evaluated at date 0, conditional upon the strategies of all other players being correctly conjectured. A vector of strategies form a *sub-game perfect equilibrium* if, after

[20]Foster and Vishwanathan (1990) present a continuous time version of Kyle's (1985) model where market depth is constant, as we assume here.

each history, the continuation strategies, form a Nash equilibrium in the remaining game. A sub-game perfect equilibrium is a *Markov perfect equilibrium* if the decision rules are Markov.

In this paper we consider subgame perfect equilibria in the trading game under a variety of scenarios. We begin with the familiar case of a monopolistic insider, generalize our results to the case of competition among several insiders, and finally consider the impact of introducing strategic outsiders who attempt to free-ride on the trading of insiders.

3 The Monopolist Insider

In this section we analyze the optimal trading strategy of a single insider in the periods preceding an information event. The market also contains a profit-maximizing specialist who conditions correctly on all public information. We will transform the problem into a dynamic game between the specialist and insider and analyze the Markov perfect equilibria of this game. We examine first the optimal trading sequence of the insider and the implied price dynamics.

The logical sequence of the argument will be as follows: we first investigate the insider's trading pattern when the price adjustment rule, from the insider's viewpoint, has the form $p_t(Q_t) = p_{t-1} + \lambda Q_t$ in the pre-event period.[21] The initial price p_0 is the unconditional expectation of the value of the security at time 0. We then show that in equilibrium the optimal actions of the market maker gives rise to such a price adjustment function. The optimization problem faced by a monopolist insider is:

$$\max_{\{q_t\}} E\left[\sum_{t=0}^{\tilde{\tau}} \frac{(v - \tilde{p}_t)q_t}{(1+r)^t}\right] \tag{1}$$

where $\tilde{p}_t = p_{t-1} + \lambda q_t$ with probability $1-\phi$ and v with probability ϕ. The expectation in (1) is taken over $\tilde{\tau}$. Profits beyond τ are zero since the insider no longer possesses any private information. Observe that the price continuity requirement on price adjustment implies the

[21] Following the announcement, the price is set equal to the present value of future expected dividends.

insider's trading problem is intertemporal in nature. If the security is undervalued initially, insider purchases lead to price increases, lowering potential profits in the future. We will show that the optimal dynamic strategy is to trade a fixed proportion of the deviation between the market price and the value of the asset.

Let $V(p_{t-1})$ denote the value function corresponding to the optimization problem (1) for any time $t < \tau$. The associated optimality equation is (using the Bellman principle):

$$V(p_{t-1}) = \max_{q_t} E\left[(v - p_{t-1} - \lambda q_t)q_t + \beta V(p_{t-1} + \lambda q_t)\right] . \tag{2}$$

where $\beta \equiv \frac{1-\phi}{1+r}$ is the effective discount factor. Note that in (2) we have used the fact that if the announcement is made, $V(p) = 0$, since prices adjust immediately without trade. It will be analytically convenient to consider as a basic state variable the deviation of price from true value rather than the price itself. Accordingly, define $z_{t-1} \equiv v - p_{t-1}$ as the state variable and denote by W the associated value function. Of course, $W(z_{t-1}) = V(v - z_{t-1})$.

Proposition 1 *For $t < \tau$, the optimal trading strategy of the monopolist insider is a linear function of the price deviation:*

$$q_t^*(z_t) = \hat{\theta} z_t \tag{3}$$

where $\hat{\theta}$ is inversely proportional to λ. Further, the value function is:

$$W(z_t) = \left(\frac{1 - \lambda\hat{\theta}}{2\lambda}\right) z_t^2 \tag{4}$$

i.e., discounted profits are a quadratic function of the initial price deviation. Further, expected prices converge geometrically to the true value of the asset:

$$E[z_t] = (\alpha(1-\phi))^t z_0$$

where $\alpha = \alpha(\beta) \in (0,1)$ is a constant described in the appendix. In particular, α is independent of the continuity-depth parameter λ.

Remarks: The parameter α is a metric for market efficiency since it is the rate at which prices converge to the full information value in the pre-event period. Prices jump to the present value of future dividends following the earnings announcement.[22] Lower values of α are associated with greater market efficiency, i.e., faster rates of convergence. Proposition 1 shows that the continuity-depth parameter λ does not effect the rate of convergence of prices, α.

Proposition 1 forms an important benchmark case where the insider is a monopolist. This situation can be regarded as an extreme case of risk aversion on the part of uninformed agents. Uninformed traders are aware that there are insiders with private information concerning the dividend announcement and postpone their trades until a time when they are no longer at an informational disadvantage. Since the insider will buy if the security is under- valued and sell if it is over-valued and never in quantities that move prices past the fundamental value in a single round, the market maker's optimal strategy is to set $p_t = p_{t-1} + \lambda q_t$, i.e., to set the maximum permissible price when faced with a buy order and the minimum when faced with a sell. So, the monopolist's strategy given in Proposition 1 together with the linear price adjustment rule of the specialist form a Markov perfect equilibrium.[23] The proposition shows that the trader's strategy, given the actions of the market maker, are such that the rate of price convergence is independent of market depth. Of course, the insider's profits despite such an adjustment are increasing in the depth; discounted profits are a linear function of λ^{-1}. So, the effect of more stringent price continuity rules (i.e., lower λ) is not to slow the rate at which private information is impounded in prices, but to increase the informational rent of the insider.

[22]Meulbroek (1991) examines the stock price run-up before takeovers and finds price paths suggestive of the slow exploitation of information by insiders. In particular, she finds that 43 percent of the run-up in the 20 days before the announcement occurs on days when insiders traded.

[23]In out of equilibrium situations, the market maker may wish to adjust prices non-linearly. However, such eventualities are zero probability events and by virtue of the simultaneous move structure, the schedule is chosen before the order flow is observed.

4 Multiple Insiders

4.1 Markov Strategies

In the next two sections we analyze the price dynamics resulting from competition among insiders. We examine whether competition among insiders necessarily lead to faster convergence of prices to the full information value of the asset. To insiders, their shared private information resembles a 'common property' resource. This creates two types of externalities. The first type resembles a 'consumption' externality since each trader's current trade partly determines the current price for all traders. The second, and perhaps more important, externality can be described as a 'production' externality. This externality arises because a marginal decrease in the amount traded benefits all traders by increasing the expected deviation between next period's beginning price and the value of the security. Intuition suggests that this externality may lead to a 'tragedy of the commons' problem, where the common resource (information) is overexploited (by increases in order size) when there are many traders. In this section, we examine the validity of this argument.

Purely for notational ease we discuss the case where $N = 2$. The generalization of our results to $N > 2$ is straightforward. Again for notational ease we ignore uninformed traders in the present section, and focus exclusively on the competition between informed traders.

Our starting point is the class of *Markov strategies*, i.e., strategies where all informed traders' actions depend on the current state (i.e., price) alone. We discuss non-Markov or history-dependent strategies later. Recall that trader i's best response problem, given the trading strategy of trader j, denoted by $q_j(p)$, is:

$$\max_{\{q_{it}\}} E\left[\sum_{t=0}^{\tilde{\tau}} \frac{(v - p_t)q_{it}}{(1+r)^t}\right] \tag{5}$$

where $p_t = p_{t-1} + \lambda(q_{it} + q_{jt}(p_{t-1}))$. As in the monopolist insider case, it is convenient to think of trader i as the residual trader who picks the deviation of price from fundamental in the next period. Let $z_{t-1} = v - p_{t-1}$ be the current the price deviation. Then, writing W_i

for the value function for trader i, some algebra yields the following Bellman equation[24] for the trader:

$$W_i(z_{t-1}) = \max_{z_t} \left[\frac{z_t(z_{t-1} - z_t - \lambda q_j(z_{t-1}))}{\lambda} + \beta W_i(z_t) \right] \tag{6}$$

where $\beta = (1 - \phi)/(1 + r)$ and z_t is the next period's price deviation. Suppose now that trader j's strategy is a linear function of the price deviation:

$$q_j(z_t) = \theta_j z_t \tag{7}$$

where $\theta_j > 0$. Such a strategy is motivated by the linearity of the monopolist insider's optimal trading strategy. We explore two elements of the analysis: the optimal trading strategy or best response of insider i as a residual trader and, secondly, the existence and character of equilibria in linear strategies.

The following result establishes that there exists a $\theta^* \in (0, \frac{1}{2\lambda})$ such that if $\theta_j = \theta^*$, then the best response of trader i is a linear trading rule with $\theta_i = \theta^*$. In this equilibrium, prices converge geometrically to v at a rate faster than the monopolist insider case.

Proposition 2 *There is a proportional trading rule $\theta^* z$ for the insiders, and the linear adjustment rule $p_t = p_{t-1} + \lambda Q_t$ for the specialist which constitutes a Markov-perfect equilibrium. The expected rate of price convergence is geometric:*

$$E[z_t] = (\alpha_N(1 - \phi))^t z_0 \tag{8}$$

where $\alpha_N \in (0, 1)$ is a constant. Further, $\alpha_N < \alpha$, i.e., the rate of convergence of prices is faster with N traders than with a monopolist insider.

[24]The value function W_i will depend on trader j's trading strategy q_j, and should therefore be written as $W_i(\cdot; q_j)$. However, in the immediate sequel we will construct a particular form for q_j and the value function will refer to this strategy. Further (5)–(6) implicitly assume that the price continuity constraint is binding, i.e., $p_t = p_{t-1} + \lambda Q_t$. As in Section 3 we assume this to be the case at first and then show that it is an optimal response on the part of the specialist.

This is the unique symmetric equilibrium when traders adopt linear Markov strategies.[25] Expected discounted profits are given by:

$$W_i(z_t) = \frac{(1-\lambda\theta^*)(1-2\lambda\theta^*)}{\lambda} z_t^2 \; . \tag{9}$$

We discuss the relationship between the continuity-depth parameter λ and the rate of price convergence later. We now consider alternative equilibria.

4.2 History-Dependent Strategies

In the equilibrium strategies we constructed in the previous section, insiders systematically ignored any information other than the current price in determining the size of their transactions. In particular, they ignored observable historical variables such as order flow and the timing of the order flow. Does the threat of moving to more rapid price convergence (i.e., the Markov equilibria) support a price path that leads to slower convergence than under a monopolistic insider? Formally, the threat strategies of each player are as follows: each player takes an action $\theta' z$, if both traders have used this strategy in the past. In particular, θ' may be the collusive or first-best strategy. Upon any deviation, both players switch to a non-cooperative strategy. We refer to such strategies as *linear threat strategies.* We know from the work of Abreu (1988) that all equilibria of a game are recovered by using as a threat point the worst subgame perfect equilibrium outcome. In this case, the corresponding worst outcome is public revelation of the security's value, followed by immediate adjustment of prices.[26]

It seems clear that the effectiveness of any deterrent will depend on the price at which cartel behavior is sought to be enforced since the gains from 'defecting' from the cartel strategy vary with price. Further, the sustainability of the cartel might depend on the time

[25]In principle, it is possible that there are non-linear Markov perfect equilibria. We do not examine such equilibria here, but we do examine equilibria where strategies are not Markov.

[26]The strategy outlined has been called a "trigger strategy" in the repeated game literature. In the context of non-repeated games such as ours, one other attempt at analyzing more general trigger strategies, like the ones we study, may be found in Benhabib and Radner (1989).

preferences of its members. So, our intuition from the repeated game literature suggests that if insiders are sufficiently patient and if the deviation of current price from value is sufficiently great, the threat of moving to the suboptimal equilibrium of the previous section may be enough to ensure that the traders as a group act as if they were a cartel. Prices would initially converge at the same rate as they would under a monopolist insider. Eventually, as the price deviation narrowed, the benefits to breaking from cheating on the proposed cartel strategies would lead to faster price convergence as traders switch to the strategies under the non-cooperative Markov equilibrium.

The following proposition shows that this reasoning, derived from the repeated game context, is not correct for the market structure under consideration. In particular, there does not exist any history-dependent strategy that supports collusive trading, even with the worst threat: full disclosure.

Proposition 3 *The rate of price convergence to the full information value of the security is faster with multiple insiders than with a monopolistic insider. In particular, collusive behavior cannot be sustained in equilibrium for any non-zero discount rate and any current price deviation.*

Proposition 3 is remarkable in that it holds irrespective of the 'threat' strategy or the discount rates of the players. So, irrespective of the initial price deviation, the players' discount rates, and the threat point, prices converge faster with multiple insiders than in the monopolistic insider regime. In the result above, the fact that traders could not submit orders that were complex enough to punish deviations without a lag was crucial to the result. We show next that the way in which we model trading arrangements can affect the equilibrium outcome in this dynamic game.

4.3 The Importance of Order Form

So far, we have assumed a very simple trading protocol where traders can submit only market orders. In reality, most real-world securities markets permit price contingent orders. For example, in a market with a trading floor, traders may submit complicated orders to their brokers so that order quantities can be revised during each auction round as prices are formed. If orders are submitted electronically, they may be made contingent on prices or volume, again providing more flexibility than we have currently assumed. In this section, we provide an example of a particular trading arrangement under which cartel trading is sustainable under the threat strategy discussed above. Our objective is not to model a specific institutional structure, but rather to show that subtle differences in trading practices can have a substantial effect on theoretical outcomes.

We begin with a discussion of the market protocol and then move to a discussion of trading strategies. Suppose traders can submit price-contingent (limit) orders of the form $q_i(p)$. We say that a price $\hat{p}$ is an *indicated price* if it satisfies the price-continuity requirement:

$$|\sum_i q_i(\hat{p})| \geq \frac{|\hat{p} - p_{t-1}|}{\lambda} \tag{10}$$

Let $Q(\hat{p}) = \sum_i q_i(\hat{p})$. If the quantity executed by the specialist is $\hat{q}$ at price $\hat{p}$, where $|\hat{q}| < |Q(\hat{p})|$, we assume the quantities are rationed by the size of the limit order. So, assuming Q and q_i have the same sign, trader i receives $q_i^r(\hat{p})$, where:

$$q_i^r(\hat{p}) = \frac{q_i(\hat{p})}{Q(\hat{p})}\hat{q}. \tag{11}$$

In actual practice, since trading is continuous and orders do not arrive simultaneously, time priority could be used as a rationing device. Here we abstract from these complications. We also assume that $\hat{q}$ must satisfy the price-continuity requirement, i.e., that:

$$\lambda\hat{q} \geq |\hat{p} - p_{t-1}|. \tag{12}$$

We also assume that individual orders cannot violate price continuity requirements. Given this restriction on order form, the specialist is free to choose any indicated price (subject to (10)) and quantity (subject to (12)) to maximize his own profits. Excess demand is rationed through (11).

Profit maximization by the specialist implies that $\hat{q} = \frac{\hat{p}-p_{t-1}}{\lambda}$ at all times since there is no point in buying (selling) any more over-valued (under-valued) stock than is necessary.[27] We now consider candidate limit order strategies for the insiders. Let p_t^* be the collusive price in period t. Consider a strategy of the form: $q_i(p)$ is $\frac{p-p_{t-1}}{2\lambda}$ if $p = p^*$ and $q_i(p) = \frac{p-p_{t-1}}{2\lambda} - \xi(p)$ if $p \neq p^*$, where $\xi(p) > 0$ is defined in the appendix. Now, if these strategies form an equilibrium in price-contingent orders, then the only solution satisfying (10) is p^*. The limit order functions are indexed by p_{t-1}. The candidate strategies we consider have the following simple structure. Each trader plays $q_i(p; p_{t-1})$ as long as no one deviates from the first-best price. A deviation leads to a price adjustment to v. For the market maker, at any p_{t-1}, the optimal strategy is to pick the highest (lowest) price when faced with a net buy (sell) imbalance. The following proposition shows that these candidate strategies can implement the equilibrium that obtains under a monopolist insider, provided the probability of a public announcement (i.e., ϕ) or the discount rate (i.e., r) is sufficiently low.

Proposition 4 *There is a $\beta^* \in (0,1)$ such that, for all $\beta > \beta^*$, the candidate strategies $\{q_i(p)\}$ form a trigger-strategy equilibrium. The rate of price convergence in this equilibrium is identical to the rate of convergence with a single insider. Further, the joint profits of insiders are maximized under this strategy.*

Clearly, other outcomes, perhaps involving slower rates of convergence than under a single insider can also be implemented. Further, the result does not depend critically on our

[27]This is true because insiders do not place orders that take prices beyond the full information value in equilibrium.

assumed rationing scheme. Other alternatives, including modeling each round as a call (verbal) auction, could yield the same result.

5 Comparative Statics

We now investigate the variation in equilibrium behavior as the market primitives, the number of informed traders N and the market liquidity λ, change. For concreteness we only analyze the Markov perfect equilibria of Section 4. We examine the effect of varying (N, λ) on price efficiency and the size of informational rents.

Recall that the (symmetric) equilibrium strategies are given by: each informed trader trades $\theta(N,\lambda)[v - p_{t-1}]$ at price p_{t-1} and discounted expected profits are $k(N,\lambda)(v-p_0)^2$ if the initial price is p_0. Moreover, in the rounds before disclosure, prices converge geometrically at the rate $1 - N\lambda\theta(N,\lambda) \equiv \alpha(N,\lambda)$.[28] Proposition 5 examines the effect of changes in N and λ on market efficiency and profits.

Proposition 5 *In a symmetric Markov equilibrium:*

(a) The rate of price convergence α is independent of market liquidity λ.

(b) However, α is a declining function of the number of insiders N.

(c) The value function $W(z_0; N, \lambda)$ is decreasing in N and λ for every initial price deviation $z_0 = v - p_0$.

(d) The total losses of market makers is decreasing in N.

The proposition shows that for a fixed number of informed traders, price efficiency, as measured by α, is independent of the continuity-depth parameter λ. As expected, prices are more efficient with a greater number of informed traders. An immediate corollary of proposition 5 is that the trading proportion $\theta(N,\lambda)$ is inversely proportional to λ, i.e., given a

[28]We have formally proved the existence of such an equilibrium only for the case $N = 2$. The arguments for the general case merely mimic the duopoly case.

more continuous price regime, in equilibrium *all* traders exactly offset the potential increase in smoothing by larger volume of transactions. Of course, this increases informational rents. On the other hand an increase in the number of informed traders reduces informational rents.

In our analysis so far, we have assumed that some traders are exogenously endowed with private information. Suppose now that traders can endogenously determine whether or not to become informed at cost. We assume that there is a pre-game round in which the number of insiders is determined endogenously. The payoffs to this game are determined by insider profits in the subsequent trading rounds, which depends on the total number of traders who choose to become informed. The Nash equilibrium for the pre-game determines the number of insiders.

Formally, suppose at time 0, a trader can become informed at cost $c > 0$. If the current price is p and the number of other informed traders is $N - 1$, then the expected returns to becoming informed is:

$$R_N(p) = \int_0^\infty W(v - p; N, \lambda) df(v) - c \quad (13)$$

$$= k(N, \lambda) \int_0^\infty (v - p)^2 df(v) - c \quad (14)$$

where f is the prior density of $\tilde{v}$ given public information.

Writing $\psi(p) \equiv \int_0^\infty (v-p)^2 df(v)$, the number of traders who choose to acquire information is given implicitly by $k(N^*(p, \lambda), \lambda) = \frac{c}{\psi(p)}$. Clearly the larger the expected deviation of current price from the true value, the greater the number of informed traders. The number of insiders will also depend on the continuity-depth parameter λ.

Proposition 6 *If traders can acquire information at cost, stricter price continuity-depth requirements (i.e., lower values of λ) are associated with faster price convergence (i.e., with lower values of α).*

The proposition appears counter-intuitive at first. The essential idea is that lower λ generates higher insider profits and hence induces a higher N^*, implying more rapid price convergence. From a policy viewpoint this is not necessarily the best way to increase market efficiency, however, since it induces a larger number of socially wasteful searches.

6 Strategic Outsiders

So far, we have assumed that the only active traders in the pre-event period are insiders.[29] The equilibrium strategies of these traders generates, as we have seen, positive autocorrelation in returns. Rational outsiders, although uninformed, who observe such autocorrelation may attempt to make inferences about the private information of insiders. Unless the outsiders know the number of insiders, this inference is non-trivial. Of course, insiders will take into account the learning strategies of outsiders and modify their trading accordingly. Fully rational outsiders in turn will condition on this in making inferences, leading to a complicated learning problem. In this section we examine the effect of outsiders who attempt to free-ride on the private information of insiders. We confine our analysis to the case where outsiders use simple trading strategies, precisely to avoid the difficulties created by having to model intertemporal learning.

Suppose then that instead of withdrawing from the market altogether in the face of an impending information event, uninformed traders do stay to trade but do so in some boundedly rational manner. By this we simply mean that such outside traders do realize that publicly observable variables as aggregate order flow and price changes contain the information known to insiders. However they are boundedly rational in that they employ simple rules based on such observables to determine their trades. To simplify matters, we ignore trading by noise traders who trade for exogenous reasons. The effect of noise trading is to generate an additive disturbance term to prices, but does not alter our conclusions in any major way. For simplicity, we restrict our attention to the case of a single insider.[30]

[29]None of our results are affected qualitatively by the addition of pure noise traders of the type considered by Kyle (1985) or Admati and Pfleiderer (1988).

[30]The key distinction is that in Section 4 fully informed fully rational insiders play Nash against each

Suppose that the outsiders look at price changes to determine their trades. In particular suppose

$$\tilde{x}_t = \delta(1 + \tilde{\omega})(p_t - p_{t-1}) \tag{15}$$

where ω is a random variable that satisfies $\Pr[|\tilde{\omega}| \leq 1] = 1$ $E(\tilde{\omega}) = 0$ and δ is a constant. The random variable ω can be thought of as a proxy for a stochastic number of uninformed traders. The constant δ can be positive or negative; if $\delta > 0$, the uninformed traders are 'chartists' attempting to trade on monotonic trends. Conversely, $\delta < 0$ corresponds to the behavior of 'contrarians' who hope to profit from cyclical movements in prices. We also assume that $\delta < \frac{1}{2\lambda}$, a condition needed to rule out explosive price paths. Suppose the optimal action of the market maker is to charge the highest permissible price when faced with a net buy imbalance and the lowest permissible price when faced with a net sell imbalance. We show below that this is in fact the optimal action of the market maker. Re-arranging terms yields:

$$p_t = p_{t-1} + \tilde{\lambda} q_t \tag{16}$$

where $\tilde{\lambda} = \frac{\lambda}{1-\lambda\delta(1+\tilde{\omega})}$. Stochastic numbers of outsiders following technical trading rules gives rise to a framework similar to that examined in Proposition 1, but with a stochastic price adjustment rule. To apply these results, however, an equilibrium argument is needed. First, we introduce some additional notation. Let $E[\tilde{\lambda}] = \lambda_0$ and $\text{Var}[\tilde{\lambda}] = \nu$, where these parameters, of course, depend on the particular strategy being used by outsiders.

The following proposition characterizes the insider's strategy when faced with a stochastic price adjustment schedule.

Proposition 7 *For $t < \tau$, the optimal trading strategy of the monopolist is linear in z:*

$$q^*(z_t) = \theta(\nu, \lambda_0) z_t$$

other, whereas in this section the informed trader plays a Stackelberg leader against the boundedly rational uninformed traders.

Under this strategy, prices converge geometrically to the true value of the asset:

$$E[z_t] = (\alpha(\beta, \frac{\nu}{\lambda_0^2})(1-\phi))^t z_0.$$

Further, the value function is a quadratic function of z:

$$W(z_t) = k(\beta, \frac{\nu}{\lambda_0^2}) z_t^2$$

where α, k, and θ are described in the appendix.

Proposition 7 is analogous to Proposition 1, but the parameters now depend on the price adjustment parameter λ.

We finally move to the argument that establishes that adjusting prices to the maximum extent permissible by price continuity regulations, is the optimal action of an intertemporal profit maximizing market-maker. To make this argument, the distribution of $\tilde{\lambda}$ must satisfy certain requirements. In particular, we require:

$$\frac{\sqrt{(1-\beta)\left(1-\beta\frac{\nu}{\lambda_0^2}\right)} - (1-\beta)}{\beta\left(1-\frac{\nu}{\lambda_0^2}\right)} \leq \frac{\lambda_0}{\overline{\lambda}} \tag{17}$$

where $\overline{\lambda} \equiv \sup \tilde{\lambda}$.

This fact is established in the proof of proposition 8, and we shall assume in the remainder of this section that (17) is always satisfied. Equation (17) implicitly restricts the class of boundedly rational strategies uninformed traders follow.[31] Note also that when there is no uninformed trading, i.e., $\nu = 0$, $\overline{\lambda} = \lambda_0$, (17) automatically holds. Condition (17) bounds the variation of $\tilde{x}_t$ relative to total order flow. Under this condition the market maker finds it dynamically optimal to adopt the myopic profit maximizing policy; setting the maximum allowable price for net buys and the lowest allowable price for net sells. We discuss this in more detail in the appendix.

[31]For high values of β, the left hand side of (17) approaches 0, providing a crude sufficient condition.

Proposition 8 *There is a sub-game perfect equilibrium in the securities market in which:*

(a) The market-maker adjusts prices according to the price schedule $p_t = p_{t-1} + \lambda Q_t$.

(b) The insider trades according to the linear strategy, $q_t = \theta(\nu, \lambda_0)[v - p_{t-1}]$.

(c) Uninformed traders employ technical trading rules summarized by $\tilde{x}_t = \delta(1 + \tilde{\omega})(p_t - p_{t-1})$.

(d) The implied expected price dynamics are geometric convergence to true value, $E(p_t - v) = (\alpha(1 - \phi))^t(p_0 - v)$.

Since best responses of specialist and the insider are unique, we have in fact a *strong perfect* equilibrium. Note that in the pre-event period, contrarian strategies generate negative profits for outsiders, but chartist strategies that exploit the positive autocorrelation in price changes created by insiders are profitable. The result provides a possible explanation for the persistence of technical trading strategies.[32]

7 Conclusions

This paper examines the dynamic strategies of traders with private information concerning the value of a risky security in a game-theoretic framework. Insiders face competition from other insiders or from outsiders who adopt simple strategies that mimic the insider's actions. We show that the nature of this competition is critically dependent on trading arrangements. In particular, we model price continuity-depth requirements on market makers that restrict transaction-to-transaction price movements for a given volume of trade, and the types of orders permitted by the market structure. Price continuity requirements allow insiders to

[32]We have assumed δ is prespecified, but in reality it is endogenously determined the long-run. Higher levels of δ imply higher outsider profits in periods prior to announcements. However, in periods when there are, in fact, no insiders, higher values of δ imply higher losses as outsiders 'chase' false trends. Thus, the long-run equilibrium δ is determined by likelihood that insiders are present.

follow dynamic strategies that slowly capitalize on their information. The result of these strategies are price trends preceding the release of information, and outsiders can profit from the autocorrelation in prices by adopting technical analysis. The effect of such competition is to increase price efficiency and lower the profits of the informed. We show that if there is a single insider, the rate of convergence of prices to the full-information value is independent of the price continuity requirements.

We then consider the price dynamics with a finite number of strategic insiders. We analyze the resulting price dynamics when traders adopt Markov strategies and show that greater numbers of informed traders leads to faster price convergence. Private information is a common resource that insiders 'over-exploit' by trading too quickly. As in the single agent case, the rate of price convergence is independent of price continuity requirements. Paradoxically, more stringent price continuity requirements may improve market efficiency indirectly by increasing insider profits and inducing more traders to become informed at cost.

We prove that insiders can never collude to trade more slowly and thereby increase their profits. The standard results on repeated games consequently may not apply to trading games which are dynamic in nature. As in a repeated game, deviation in a dynamic game generates immediate rewards. In a dynamic game, however, deviation also alters the future environment of the game. This secondary effect may persist in the limit as players become more patient. Consequently, collusion may not be sustainable in non-repeated dynamic games. Trading arrangements are crucial to this result; if price-contingent orders are permitted, insiders can immediately punish any attempt to over-trade, slowing the rate of price convergence.

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Appendix

Proof of Proposition 1

We will prove a more general version of the proposition from which several other results will follow as special cases. Suppose that, from the insider's perspective, the price schedule is stochastic and takes the form:

$$p_t = p_{t-1} + \tilde{\lambda} q_t \tag{A.1}$$

where $\tilde{\lambda}$ is a random variable with mean λ_0 and variance ν. We will formally derive equation (A.1) in the proof of Proposition 7 when we consider the case where outsiders participate in trading. Proposition 1 is a special case of equation (A.1) where $\lambda_0 = \lambda$ and $\nu = 0$. The optimization problem faced by a monopolist insider is:

$$\max_{\{q_t\}} E\left[\sum_{t=0}^{\tilde{\tau}} \frac{(v - \tilde{p}_t)q_t}{(1+r)^t}\right] \tag{A.2}$$

where from equation (A.1), $\tilde{p}_t = p_{t-1} + \tilde{\lambda} q_t$ with probability $1-\phi$ and v with probability ϕ. Let $V(p_{t-1})$ denote the value function corresponding to the optimization problem (A.2) for any time $t < \tau$. The Bellman condition yields:

$$V(p_{t-1}) = \max_{q_t} E\left[(v - p_{t-1} - \tilde{\lambda} q_t)q_t + \beta V(p_{t-1} + \tilde{\lambda} q_t)\right] . \tag{A.3}$$

where $\beta \equiv \frac{1-\phi}{1+r}$. Rewriting (A.3) we obtain:

$$W(z) = \max_{q} E\left[(z - \tilde{\lambda} q)q + \beta W(z - \tilde{\lambda} q)\right] . \tag{A.4}$$

Here $\tilde{y} = z - \tilde{\lambda} q$ is the (random) price deviation that results from a trade of size q given a deviation of z at the start of the period. The trader's choice problem then is equivalent to the choice of the expected deviation $z - \lambda_0 q \equiv y$ conditional upon there being no information announcement. Equation (A.4) yields:

$$W(z) = \max_{y} E\left[\tilde{y}\left(\frac{z-y}{\lambda_0}\right) + \beta W(\tilde{y})\right]$$

which can be simplified to

$$W(z) = \max_{y}\left[y\left(\frac{z-y}{\lambda_0}\right) + \beta E W\left(\frac{\tilde{\lambda}}{\lambda_0} y + \frac{\lambda_0 - \tilde{\lambda}}{\lambda_0} z\right)\right] . \tag{A.5}$$

Note that in (A.5) the expectation is taken over $\tilde{\lambda}$. Let the stationary optimal policy of the above problem be denoted by $h(z)$. This optimal choice satisfies (provided W is differentiable):

$$\frac{z - 2h(z)}{\lambda_0} + \beta E\left[W'(\tilde{y}) \cdot \frac{\tilde{\lambda}}{\lambda_0}\right] = 0 . \tag{A.6}$$

The envelope theorem yields:

$$W'(z) = \frac{h(z)}{\lambda_0} + \beta E\left[W'(\tilde{y}) \cdot \frac{\lambda_0 - \tilde{\lambda}}{\lambda_0}\right] . \tag{A.7}$$

Here (A.6) and (A.7) are stochastic difference equations. Consider solutions of the form $W(z) = \frac{k}{2}z^2$ and $h(z) = \alpha z$. Substituting these in (A.6)–(A.7) yields

$$z(1-2\alpha) + \beta k E\left[\left(\frac{\tilde{\lambda}}{\lambda_0}\alpha z + \frac{\lambda_0 - \tilde{\lambda}}{\lambda_0} z\right)\tilde{\lambda}\right] = 0$$

which upon simplification leads to

$$1 - 2\alpha + \beta k\left[\lambda_0 - (1-\alpha)\frac{\nu + \lambda_0^2}{\lambda_0}\right] = 0\,. \tag{A.8}$$

Similarly from (A.7) we get

$$kz = \frac{\alpha}{\lambda_0}z + \beta k E\left[\left(\frac{\tilde{\lambda}}{\lambda_0}\alpha z + \frac{\lambda_0 + \tilde{\lambda}}{\lambda_0} z\right)\left(\frac{\lambda_0 - \tilde{\lambda}}{\lambda_0}\right)\right]$$

which simplifies to

$$k = \frac{\alpha}{\lambda_0} + \frac{\beta k}{\lambda_0^2}(1-\alpha)\nu\,. \tag{A.9}$$

(A.8) and (A.9) solve to yield optimal values of k and α. In particular, a little algebra reveals that α may be derived from the following quadratic function

$$\beta(\nu - \lambda_0^2)\alpha^2 + 2(\lambda_0^2 - \beta\nu)\alpha - (\lambda_0^2 - \beta\nu) = 0.$$

The quadratic has two roots. One root yields $\alpha > 1$ and implies an expected price path which is explosive. This clearly cannot be part of an optimal trading strategy of the insider since it implies that he consistently buys overvalued stock and sells, in turn, undervalued stock. Therefore, the only meaningful solution to the quadratic is:

$$\alpha = 1 - \frac{\sqrt{(1-\beta)\left(1 - \beta\frac{\nu}{\lambda_0^2}\right)} - (1-\beta)}{\beta\left(1 - \frac{\nu}{\lambda_0^2}\right)}\,. \tag{A.10}$$

Either (A.8) or (A.9) can then be solved for k. From standard arguments we then know that as a solution to the optimality equation, W is uniquely defined as the value function. The first order condition (A.6) is also a sufficient condition for optimality if the maximand is concave. The only term of any consequence in the right hand side of (A.5) is the coefficient of y^2, which is $\frac{E(\tilde{\lambda}^2)}{2\lambda_0}\beta k - 1$. If this coefficient is negative, the maximization problem is in fact strictly concave and we shall assume that the distribution of $\tilde{\lambda}$ is such that this is indeed the case. An additional argument is required to show that the value function is the unique solution of the optimality equation. This is so since the one period returns are unbounded. Further αz is the unique trading strategy which maximizes a risk-neutral informed trader's expected profits when faced with a linear price adjustment schedule operated by the specialist.[1]

[1] From the strict concavity of the objective in (A.5), the uniqueness of the optimal trading strategy follows.

Recall $\alpha = h(z)/z$ where $h(z) = z - \lambda_0 q^*(z)$. Here $q^*(z)$ is the optimal trading volume at price deviation z. In turn this implies that $\alpha = 1 - \lambda_0 \frac{q^*(z)}{z}$, which means that:

$$q^*(z) = \frac{(1-\alpha)}{\lambda_0} z \equiv \theta z \, . \tag{A.11}$$

where:

$$\theta = \frac{\sqrt{(1-\beta)\left(1-\beta\frac{\nu}{\lambda_0^2}\right)} - (1-\beta)}{\beta\left(1-\frac{\nu}{\lambda_0^2}\right)\lambda_0} . \tag{A.12}$$

What remains to be proved is (4). Note that (A.7) implies that:

$$W'(z) = \frac{\alpha\lambda_0}{\lambda_0^2 - \beta(1-\alpha)\nu} z.$$

So:

$$W(z) = \frac{\alpha\lambda_0}{2[\lambda_0^2 - \beta(1-\alpha)\nu]} z^2 + C \tag{A.13}$$

where C is a constant. Since $W(0) = 0$, $C = 0$. Proposition 1 follows from equations (A.10), (A.11)–(A.13) by setting $\nu = 0$ and $\lambda_0 = \lambda$. ∎

Proof of Proposition 2:

We first characterize the best response of trader i given that j adopts a linear strategy. Then we discuss the equilibrium conditions. From this point on, the price adjustment rule will be taken as the deterministic rule $p_t = p_{t-1} + \lambda q_t$. Rewriting (6) we obtain:

$$W_i(z) = \max_y \left[\frac{y((1-\lambda\theta_j)z - y)}{\lambda} + \beta W_i(y)\right] . \tag{A.14}$$

Denote the stationary policy for this problem by $h_i(z)$. Assuming for the moment that W_i is in fact differentiable, the optimal choice must satisfy the following first-order and envelope conditions:

$$\frac{(1-\lambda\theta_j)z - 2h(z)}{\lambda} + \beta[W_i'(h(z))] = 0 \tag{A.15}$$

and

$$W_i'(z) = \frac{(1-\lambda\theta_j)h(z)}{\lambda} . \tag{A.16}$$

The Euler equation follows from (A.15) and (A.16):

$$(1-\lambda\theta_j)z - 2h(z) + \beta(1-\lambda\theta_j)h^2(z) = 0 \, . \tag{A.17}$$

As in the monopoly case, we conjecture a price adjustment rule that is linear, i.e., $h(z) = \alpha_2 z$. Then (A.17) yields:

$$(1-\lambda\theta_j) - 2\alpha_2 + \beta(1-\lambda\theta_j)\alpha_2^2 = 0 \, . \tag{A.18}$$

From (A.18) it follows that:

$$\alpha_2 = \frac{1 \pm \sqrt{1-(1-\lambda\theta_j)^2\beta}}{\beta(1-\lambda\theta_j)} . \tag{A.19}$$

Equation (A.19) suggests the possibility of cyclical behavior (i.e., $\alpha_2 < 0$), stable monotone price convergence (i.e., $\alpha_2 \in (0,1)$), or even unstable monotone explosive behavior (i.e., $\alpha_2 \geq 1$). We analyze the cyclical and monotone explosive cases below and show in particular that such behavior cannot arise in equilibrium. We begin with the case: $1 - \lambda\theta_j \in (0,1)$, i.e., a situation in which, in the absence of any trade by i, $\operatorname{sgn}(z_t) = \operatorname{sgn}(z_{t-1})$.

Lemma 1 *Suppose $(1 - \lambda\theta_j) > 0$. Then, under the best response of trader i,*

(a) If $z_{t+1} > z_t$, then $z_{t+2} > z_{t+1}$. Conversely, if $z_{t+1} < z_t$, then $z_{t+2} < z_{t+1}$, i.e., the price deviation is either monotonically increasing or decreasing. Further, $\operatorname{sign}(z_t) = \operatorname{sign}(z_{t+1})$.

(b) The rate of convergence of prices is:

$$\alpha_2 = \frac{1 - \sqrt{1 - (1 - \beta\lambda\theta_j)^2}}{\beta(1 - \lambda\theta_j)} < \frac{1 - \sqrt{1 - \beta}}{\beta} = \alpha. \tag{A.20}$$

(c) The rate of adjustment α_2 is decreasing in θ_j.

Proof of Lemma 1:

Consider the Bellman equation (A.14). Denote the maximand by $U(z, y)$. We will show that U is a strictly *super-modular* function.[2] A sufficient condition is $U_{12} > 0$. Note that $U_{12} = \frac{(1-\lambda\theta_j)}{\lambda} > 0$. From strict super-modularity, a well-known result (see, e.g., Ross (1983)) implies that the policy function h is monotone. Then $z_{t+1} = h(z_t) > z_t$ implies that $z_{t+2} = h^2(z_t) > h(z_t) = z_{t+1}$. The converse is identical and this establishes (a) since $\alpha_2 > 0$.

To prove part (b), note that if $\alpha_2 > 1$, the implied expected price path is explosive. This implies strict losses for trader i, clearly an inoptimal strategy given the option to not trade at all. It is easy to check, by simple but tedious calculus that $\alpha_2 < \alpha$, implying faster convergence. From the expression for α_2, it is straightforward to check that α_2 is a decreasing function of θ_j, establishing part (c). ∎

Remark: It should be kept in mind that we have analyzed only trader i's best response. The rate of convergence, α_2, is dependent on the parameter λ. Recall that $\alpha_2 = \frac{h(z)}{z}$. Since $h(z) = (1 - \lambda\theta_j)z - \lambda q_i^*(z)$, we see that:

$$q_i^*(z) = \frac{[(1 - \lambda\theta_j) - \alpha_2]z}{\lambda} \equiv \theta_i(\theta_j)z$$

i.e., demand is a linear function of the price deviation. To complete the argument we must show that there are linear strategies which are best responses to each other. Lemma 2 provides the basis for this result.

Lemma 2 *Suppose that $1 - \lambda\theta_j > 0$. Then:*

[2] A function $U(z, y)$ is *super-modular* if, for all $z' > z, y' > y$, $U(z', y') - U(z', y) > U(z, y') - U(z, y)$.

(a) *The best response of trader i to trader j's strategy is:*

$$q_i^*(z) = \theta_i(\theta_j)z \tag{A.21}$$

where:

$$\theta_i(\theta_j) = \frac{\sqrt{1-\beta(1-\lambda\theta_j)^2}}{\beta(1-\lambda\theta_j)\lambda}(1-\sqrt{1-\beta(1-\lambda\theta_j)^2}).$$

(b) *The demand coefficient is strictly positive, i.e., $\theta_i(\theta_j) > 0$.*

(c) *The demand coefficient $\theta_i(\theta_j)$ is a continuous function of θ_j, and satisfies the inequality $(1 - \lambda\theta_i(\theta_j)) > 0$.*

(d) *The value function for trader i is:*

$$W_i(z) = \frac{1-\sqrt{1-\beta(1-\lambda\theta_j)^2}}{2\lambda}z^2$$

i.e., the profits are a quadratic function of price deviation.

Proof of Lemma 2: For (a), Equation (A.21) is immediate from the preceding discussion. Parts (b) and (c) are immediate. Note that $\alpha_2 > 0$ implies that $1 - \lambda\theta_i(\theta_j) > 0$. For part (d), the characterization of the value function follows from (A.16) and arguments identical to those in proposition 1. ∎

To finish the proof, we now consider the case where $(1 - \lambda\theta_j) < 0$. In this case, the price deviation in the next period, $z(1 - \lambda\theta_j)$, is the opposite sign of the initial deviation, and prices are cyclical or explosive. It is straightforward to demonstrate that such price paths cannot emerge in equilibrium.[3] Then, the proposition follows directly from Lemmas 1 and 2. ∎

Proof of Proposition 3:

Proposition 3 concerns the viability of collusive trading strategies under which the rate of price convergence is slower than that of the Markov equilibrium. Adapting the arguments of Abreu (1988), it suffices to examine strategies which revert to the worst equilibrium outcome in the event of a deviation. To keep the analysis simple, suppose that the security's value is revealed to the specialist. The following strategies thus form an equilibrium: each insider trades $\lambda^{-1}z_{t-1}$ and the specialist adjusts prices in a single step to v. We ask whether the threat of reverting to this 'zero rent' equilibrium is sufficient to sustain collusive trading. We emphasize that we have provided the conditions most favorable to collusive behavior by admitting immediate rent dissipation as an equilibrium outcome. We now show that even with this possibility, collusive behavior cannot be sustained in equilibrium.

Now consider the following strategy. Each player's action is $\theta^* z$ if both played this in the past. Upon any deviation, the players immediately switch to the zero rent equilibrium. Then the optimal deviation can be computed as follows:

$$p_t = p_{t-1} + \lambda\theta' z_{t-1} + \lambda q_t \tag{A.22}$$

[3]In the special case where $(1 - \lambda\theta_j) = 0$, prices converge in a single step.

which implies that:

$$q_t = \frac{-z_t + (1-\lambda\theta')z_{t-1}}{\lambda} \tag{A.23}$$

Then, it is easily shown that the optimal deviation is:

$$z_t = \frac{(1-\lambda\theta')}{2} z_{t-1} \tag{A.24}$$

Denoting the value of the optimal deviation by $M(z)$, we have:

$$M(z) = \frac{(1-\lambda\theta')^2}{4\lambda} z^2 \tag{A.25}$$

From Proposition 1, a cartel trades $\hat{\theta}z$ and the associated profits for each cartel member is:

$$W(z) = \frac{(1-\lambda\hat{\theta})}{4\lambda} z^2 \tag{A.26}$$

For a sustainable cartel, $M(z) \leq W(z)$ when $\theta' = \hat{\theta}/2$. However, this implies that $(1-\lambda\hat{\theta}/2)^2 \leq 1-\lambda\hat{\theta}$, which is a contradiction. To complete the proof, note that if we consider any trading rule $\theta' z$ that generates slower price convergence than that of the monopolist insider case (i.e., $\theta' < \hat{\theta}/2$), then the deviation profits $M(\cdot)$ increases. By definition, since W is the maximum attainable joint profit, the return to this alternative rule is less than W and hence the proposition follows. ∎

Proof of Proposition 4:

First, consider the optimal deviation of trader j if trader i plays as hypothesized. Suppose for the moment that $\xi(p) = 0$, and denote j's deviation strategy by $q_j^d(p)$. If $\hat{p}$ is a candidate price then from (10), we have:

$$q_j^d(\hat{p}) \geq \frac{\hat{p}-p_{t-1}}{\lambda} - \frac{\hat{p}-p_{t-1}}{2\lambda} = \frac{\hat{p}-p_{t-1}}{2\lambda} \tag{A.27}$$

Given (11), if $\hat{p}$ is the market clearing price, it must be that trader j places the largest buy order possible if $\hat{p} < v$ and the largest sell order possible if $v > \hat{p}$. So, we have:

$$q_j^d(p) \geq \frac{\hat{p}-p_{t-1}}{\lambda} \tag{A.28}$$

From (11) it then follows that of the quantity traded, two-thirds is j's and one-third is i's. Consequently, the quantity j actually trades is:

$$q_j^r(p) \geq \frac{2}{3}\left(\frac{\hat{p}-p_{t-1}}{\lambda}\right). \tag{A.29}$$

Given (A.29), the optimal price choice for j solves:

$$\max_{\{p\}} \left(\frac{2}{3}\right) \frac{(v-p)(p-p_{t-1})}{\lambda}. \tag{A.30}$$

Then, the optimal deviation implies that $v - p = \frac{1}{2}(v - p_{t-1})$. The maximum returns to deviation is $M(z)$ where:

$$M(z) = \frac{1}{6\lambda} z^2 \tag{A.31}$$

where $z = v - p$. The first-best profits are:

$$W(z) = \frac{1}{4\lambda}(1 - \lambda\hat{\theta})z^2 \tag{A.32}$$

From our calculations, $W(z) > M(z)$ implies that $1 - \lambda\hat{\theta} \geq \frac{2}{3}$. From our previous calculations:

$$1 - \lambda\hat{\theta} = 1 - \frac{\sqrt{1-\beta} - (1-\beta)}{\beta} \tag{A.33}$$

It is easy to show that there exists $\beta^* \in (0,1)$ such that for $\beta \geq \beta^*$, $1 - \lambda\hat{\theta} \geq \frac{2}{3}$ and hence $W(z) \geq M(z)$. Then, for any $\beta > \beta^*$, there exists a function $\xi(p)$ such that $W(z) > M(z)$ for all z. The proposition follows immediately. ∎

Proof of Proposition 5

Given that $\theta(N,\lambda)(v - p_{t-1})$ is the trading strategy of the other $N-1$ traders, the i-the trader's maximization problem as a residual trader picking next period's price deviation y, given the current price deviation z, is

$$\max_y \left[\frac{(1-(N-1)\lambda\theta)z - y}{\lambda}y + \beta W(y; N, \lambda)\right] . \tag{A.34}$$

In (A.34) and henceforth in the proof we write θ instead of $\theta(N,\lambda)$. The first-order condition and envelope theorem condition are the exact analogues of (A.15) and (A.16) in the two insider case but we report them here for completeness

$$\frac{(1-(N-1)\lambda\theta)z - 2y}{\lambda} + \beta W'(y; N, \lambda) = 0 \tag{A.35}$$

$$W'(z) = \frac{1-(N-1)\lambda\theta}{\lambda}y . \tag{A.36}$$

Substituting (A.36) in (A.35) and using the fact that θz is a best response in the above problem (i.e., that $y = (1 - N\lambda\theta)z$), we get

$$\beta(1-N\lambda\theta)^3 + \beta\lambda\theta(1-N\lambda\theta)^2 - (1-N\lambda\theta) + \lambda\theta = 0 . \tag{A.37}$$

Write $1 - N\lambda\theta \equiv \alpha$, i.e., $\lambda\theta = \frac{1-\alpha}{N}$. Then,

$$\beta(N-1)\alpha^3 + \beta\alpha^2 - (N+1)\alpha + 1 = 0 . \tag{A.38}$$

Denote the function on the left hand side of (A.38), $\zeta(N,\alpha)$. Elementary calculus yields $\zeta_2 = 3\beta(N-1)\alpha^2 + 2\beta\alpha - (N+1)$. It is easy to see that this quadratic can have at most one root in (0,1), i.e., $\zeta(N,\cdot)$ has a single extremum in $\alpha \in (0,1)$. Further, $\zeta_{22} = 6\beta(N-1)\alpha + 2\beta > 0$, i.e., the extremum is in fact a point at which ζ attains a minimum. Finally note that $\zeta(N,0) = 1$ and $\zeta(N,1) = N(\beta - 1) < 0$. So there is exactly one $\alpha^*(N) \in (0,1)$ such that $\zeta(N,\alpha^*(N)) = 0$.

We have proved (a). Notice further that $\zeta_1(N,\alpha) = \beta\alpha^3 - \alpha < 0$. Hence, $N' > N$ implies that $\alpha^*(N') < \alpha^*(N)$. So (b) is proved.

If N is fixed, the price path is independent of λ, i.e., θ is inversely proportional to λ. This immediately implies that the informational rents, W, is inversely proportional to λ as well. On the other hand if λ is fixed, increasing N to $N+1$ leads to faster convergence than in the N trader case, i.e., the aggregate informational rents are lower. That directly implies that per capita rents are lower since a smaller total rent is now split between a larger set of traders. So the proposition is proved. ∎

Proof of Proposition 6

Recall from Proposition 5 that the greater the number of insiders the more efficient are prices. So requiring more price continuity (i.e., lower λ) raises potential informational rents and hence attracts a large number of informational searches and faster price convergence. ∎

Proof of Proposition 7

From the discussion in Section 6, the insider faces a stochastic price adjustment function of the type given by equation (A.1). The proposition then follows directly from equations (A.10), (A.12), and (A.13) in the proof of Proposition 1. ∎

Proof of Proposition 8

Suppose, without loss of generality, that $z_t > 0$. An informed trader would never unilaterally place an order large enough to push potential prices above v. This is so since his maximum returns $W(z)$ are symmetric around $z = 0$. Hence he does strictly better by reducing buy orders in order that the maximum potential price rises only to $v-\epsilon$, rather than to $v+\epsilon$, for $\epsilon > 0$, and consequently making money rather than losing money on the immediate trade. This implies that the optimal response of the specialist is $z_t = z_{t-1} - \lambda Q_t$. In other words, the event: "buy orders so large that $z_{t-1} - \lambda Q_t < 0$ if $z_{t-1} > 0$ " is a zero probability event in the absence of uninformed trading. The same is true in the presence of "some" noise trading. In particular, note that $z_{t-1} - \tilde{\lambda} Q_t \geq 0$ if and only if $1 - (1-\alpha)\frac{\tilde{\lambda}}{\lambda_0} \geq 0$ i.e., $(1-\alpha)\tilde{\lambda} \leq \lambda_0$. So, it follows that:
Fact 1: Under the informed trader's optimal trading strategy the maximum permissible price stays below (resp. above) the fundamental value v if it is currently below (resp. above) v if and only if:

$$\frac{\sqrt{(1-\beta)\left(1-\beta\frac{\nu}{\lambda_0^2}\right)} - (1-\beta)}{\beta\left(1-\frac{\nu}{\lambda_0^2}\right)} \leq \frac{\lambda_0}{\overline{\lambda}} \tag{A.39}$$

where $\overline{\lambda} \equiv \sup \tilde{\lambda}$. We assume this condition is always satisfied.

So, if the specialist believes that the informed buy or sell according to the linear trading strategies (3), and that the uninformed transact according to some boundedly rational rule as given by (15), then observed trades give conditional expectation of v above (below) $p_{t-1} + \lambda Q_t$ if $Q_t > 0$ ($Q_t < 0$), implying the linear adjustment is the optimal response for the market-maker. We can now apply the results of Proposition 1 directly. Note that the rate of price convergence will in this case depend on market liquidity λ as well as on the discretionary component of uninformed trading, δ. For the result to be meaningful we need $\delta < \frac{1}{2\lambda}$. ∎