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Abstract: This paper studies sensitivity and continuity of finite horizon optimal plans in a general multi-sectoral model of intertemporal optimization which admits non-convexities and technological change. Unlike the aggregative model, investment choices are sensitive to the length of the decision horizon. A new concept of value insensitivity is proposed and shown to hold even in the multi-sectoral model. Finally, continuity of investment choices and values, in going from the finite to the infinite horizon problem, is established.

*This paper contains some results from an earlier manuscript, Dutta (1987). I have benefitted from, recent, helpful conversations with Tapan Mitra, Roy Radner and Itzhak Zilcha as well as the comments of Mukul Majumdar and Manfred Nermuth on the earlier research.

1. Introduction

This paper provides some new answers to two old questions in the theory of optimal intertemporal allocation. Consider a set of optimal investment plans, all of which start from the same initial capital stock but which differ in terms of the decision horizon. The ***insensitivity question*** is: are finite horizon optimal investment plans, and the associated level of maximized utility, close to each other for different specifications of horizon length? On the other hand, the ***continuity question*** is: is each finite horizon optimal plan close to some infinite horizon optimum?[1]

The first question is important if one believes that the "true" horizon is finite but, either the exact horizon length is not known with certainty or that the decision-maker has incomplete information about technology and preferences beyond some point in time. Insensitivity of optimal choices and values has great practical significance since it allows a decision-maker to plan for the distant future by making shorter-run decisions which can be updated in the light of new information. On the other hand, if we believe that the logical horizon is infinite, and many authors have argued that this must be so especially at the national level, we have to ask how good an approximation is a long but finite horizon. Again, we might be restricted to finite horizon decision-making either because of the informational reasons discussed above or simply because of greater computational ease. (These arguments can be found in more detailed form in, for instance, Chakravarty (1969, Chapters 2 and 3) and Mckenzie (1976)). Note that the two questions are conceptually distinct and an affirmative answer to one does not necessarily imply a similar conclusion for the other.[2]

Both of these questions have been comprehensively studied in the one-sector or aggregative capital accumulation model (Chakravarty (1962), Manneschi (1966), Brock (1971), Majumdar-Nermuth (1982), Mitra (1983) and Mitra-Ray (1984)). Brock's classic treatment, and the subsequent generalizations of his results, have established insensitivity and continuity of finite horizon decision-making in the one-sector model provided preferences are convex. Unfortunately, the analyses and crucial intermediate results of the aggregative model do not extend to the multi-sectoral context- indeed the results are not true under identical hypotheses (see the further discussion in Sections 3 and 4).

The multi-sectoral model has been studied more sparsely. Gale (1967) established insensitivity and continuity results in a convex, stationary (i.e. with time-independent technology and preferences) framework and his results were extended to a non-stationary model (under some strong hypotheses) by Mckenzie (1976). The only treatment that fully accomodates technological and preference changes and allows for non-convexities is contained in an important paper by Nermuth (1978). Nermuth confined himself to the continuity question alone. Using an innovative topological approach to the problem he tried to show that under some strong conditions on infinite horizon preferences but otherwise mild restrictions, continuity does indeed obtain. However, Nermuth's main theorem is false (Example 4.3 below). Hence, both the sensitivity and continuity questions remain open in general multi-sectoral models, which are precisely the models we study in this paper. We now briefly describe our results.

We start with the ***investment sensitivity*** question: if the decision horizon were T' rather than T, how much of an effect would that have on the optimal investment levels, especially in early years of either plan. We argue by way of an example, (Example 3.1), that investment insensitivity is a specialized feature of the aggregative model which will not hold in a general multi-sectoral framework. Given substitution possibilities between sectors, the composition and time profile of investment may vary significantly between horizons T and T' - to best exploit technological progress or increasing returns. We then propose that the appropriate measure for such models is ***value sensitivity***- how do maximized utilities (normalized for horizon length in some fashion) change with the length of the decision horizon? We take as our criterion average utility or utility per time period. We show that if the technologies are productive and over time technologies and preferences tend to some limit, then the maximum average utility for any two long plan horizons T and T' are indeed close to each other (Theorem 3.2). We further show that the two assumptions are minimally sufficient in that the conclusion is false in the presence of exhaustible resources (Example 3.4) or arbitrary time-dependence in technology and preferences (Example 3.3).

The ***continuity*** question is known not to have a positive answer in some well-known examples (for instance the cake-eating examples of Gale (1967)). Note that a number of alternative criteria have been employed in the infinite horizon problem- and we investigate a number of the alternatives. We show that in the original Ramsey (1928) formulation of infinite horizon preferences, such continuity does indeed obtain (Theorem 4.4). A similar conclusion holds if infinite horizon preferences are continuous (as would be the case under discounting) (Theorem 4.1). Unlike the aggregative model however, the existence of a unique catching-up optimal plan does not imply that any finite horizon optimal plan is close to it (Example 4.5).

The main insensitivity and continuity results are easily generalizable to other intertemporal problems (in addition to the multi-sectoral growth model studied here). For instance, all of these results will be seen to carry over to a non-stationary dynamic programming problem, with the appropriate modifications in hypotheses.

Section 2 sets up the model. Sections 3 and 4 analyze the sensitivity and continuity issues while Section 5 concludes.

2. The Model

2.1 Notation and Definitions

Let $\mathbb{R}^n$ be n-dimensional real space with $\| \cdot \|$ denoting the max norm on this space ($\mathbb{R}^n_+$

will denote the non-negative orthant). For any x, y in $\mathbb{R}^n$, $x \geq y$ means $x_i \geq y_i$, i = 1,..n; $x > y$ means $x \geq y$, $x \neq y$. Let N be the set of non-negative integars. Finally, $e \in \mathbb{R}^n$ will denote the unit vector; e = (1,1....1).

For any set Y, let P(Y) denote the collection of all subsets of Y. A correspondence Γ from a topological space X to (subsets of) a topological space Y is said to be upper semicontinuous (usc) at x_0, if for each open set G containing $\Gamma(x_0)$, there is a neighborhood $U(x_0)$ such that $x \in U(x_0) \Rightarrow \Gamma(x) \subset G$. Γ is upper semicontinuous on X if it is usc at each point of X and if, in addition, $\Gamma(x)$ is a compact set for each x in X. Γ is lower semicontinuous (lsc) at x_0 if for each open set G intersecting $\Gamma(x_0)$, there is a neighborhood $U(x_0)$ such that $x \in U(x_0) \Rightarrow \Gamma(x) \cap G \neq \phi$. The correspondence is lsc on X if it is lsc at all $x_0 \in X$. When Γ is both usc and lsc it will be said to be a continuous correspondence. Suppose that Y is a metric space. A sequence of correspondences Γ_n will be said to converge ***uniformly*** to a correspondence Γ if for all $\varepsilon > 0$, there is $n' < \infty$ such that $\rho(\Gamma(x),\Gamma_n(x)) < \varepsilon$, for all x, whenever $n \geq n'$ (where ρ is the familiar Hausdorff metric on subsets of Y).

2.2 Feasible Plans

Production relations in the multi-sectoral model are specified by a sequence of production correspondences, $(F_t)_{t \in N}$, $F_t : \mathbb{R}^n_+ \to P(\mathbb{R}^n_+)$. $F_t(x)$ is the set of possible outputs in period t from an input x in period t-1. Following standard notation, we will use x_t, c_t, y_t to refer to the investment, consumption and output in period t. Further, x^i_t (resp. c^i_t, y^i_t) will denote the investment (resp. consumption and output) of commodity i in period t.

A finite horizon allocation problem is characterized by a triplet of parameters $\xi \equiv (x,a,T)$ where $x \in \mathbb{R}^n_+$ is the inital capital stock, $a \in \mathbb{R}^n_+$ is the target stock and $T \in N$ is the decision horizon. Much of the analysis that follows will involve alternative specifications of the horizon T for fixed x and a=0. A ξ-feasible plan or program is $(x_t, c_t)_{t=0}^T$ such that

$$x_0 + c_0 \in F_0(x) \tag{2.1}$$

$$x_t + c_t \in F_t(x_{t-1}),\ t = 1,...T \tag{2.2}$$

$$x_T \geq a \tag{2.3}$$

$$x_t \geq 0,\ c_t \geq 0 \qquad t = 0,...T \tag{2.4}$$

An infinite horizon feasible plan is $\tilde{x},\tilde{c} \equiv (x_t, c_t)_{t=0}^{\infty}$ such that (2.1), (2.2) and (2.4) are satisfied for all t in N. We will use the notation Q to refer to the set of infinite horizon feasible plans (respectively Q_T for T-horizon feasible plans). A pure accumulation program $(\bar{x}_t, \bar{c}_t)_{t=0}^{\infty}$ is defined by $\bar{c}_t = 0,\ t \geq 0$ and $\bar{x}_t \in F_t(\bar{x}_{t-1}),\ t \geq 0,\ \bar{x}_{-1} = x$.

2.3 Preferences and Optimization Problems

The preference structure is defined by a sequence of time-dependent utility functions $(u_t)_{t\in N}$, where $u_t\colon\ \mathbb{R}_+^n \to \mathbb{R}$. Utility is defined on consumption.

The finite horizon optimization problem is to choose a ξ-feasible plan $(x_t^*, c_t^*)_{t=0}^{T}$ such that

$$\sum_{t=0}^{T} u_t(c_t^*) \geq \sum_{t=0}^{T} u_t(c_t) \tag{2.5}$$

for all ξ-feasible $(x_t, c_t)_{t=0}^{T}$. Clearly, this optimization problem is equivalent to maximizing the average or per period utility, i.e. max $\frac{1}{T+1} \sum_{t=0}^{T} u_t(c_t)$ over ξ-feasible plans. For $\xi = (x,0,T)$, denote the maximized utilities or value function $V_T(x)$ (and $v_T(x)$ for the average value).

There are several different ways in which infinite horizon preferences can be specified. We outline here the alternatives which we discuss.

Infinite-Sum Utility Functions

The obvious extension of finite horizon preferences is to define horizon utility as

$$U(\tilde{x}, \tilde{c}) = \sum_{t=0}^{\infty} u_t(c_t), \tag{2.6}$$

The problem with (2.6) is of course that the infinite sum may not be well-defined, or finite, for all feasible programs. If it is, then optimality is defined in the usual manner and we shall denote the associated value function, V(x). The best-known example of well-defined preferences under this criterion is that of discounted utilities; $u_t = \delta^t u$.

The Ramsey Utility Function

Denote $b_t = \sup u_t(c_t)$, $c_t \in F_t(\bar{x}_{t-1})$, where $(\bar{x}_t, \bar{c}_t)_{t\in N}$ is some pure accumulation program. Ramsey (1928) considered the following:[3]

$$U(\tilde{x}, \tilde{c}) = \sum_{t=0}^{\infty} [u_t(c_t) - b_t] \tag{2.7}$$

(2.7) is non-positive, under standard monotonicity conditions. If U is finite for some subset of infinite horizon plans, optimality can be defined in the obvious way on this subset; let W(x) denote the value function (and $W_T(x)$ its T-horizon analog).

Catching-Up Preferences

$\tilde{x}^*, \tilde{c}^*$ catches-up to another feasible plan $\tilde{x}, \tilde{c}$ if:

$$\overline{\lim_{T\to\infty}} \sum_{t=0}^{T} [u_t(c_t) - u_t(c_t^*)] \leq 0 \tag{2.8}$$

An optimal program is one that catches up to all other programs. Clearly, optimality under either of the first two criteria, implies optimality under the catching-up.

2.4 Assumptions

The maintained assumptions will be: For all t

(A1) i) (Closedness) F_t is a non-empty continuous correspondence

ii) (Boundedness) $\exists \beta > 0$ s.t. $\| x \| > \beta \Rightarrow \| y \| \le \| x \|, \forall y \in F_t(x)$.

iii) (Monotonicity) $y \in F_t(x) \Rightarrow y' \in F_t(x')$ if $x' \ge x$, $y' \le y$

(A2) i) (Continuity) u_t is a continuous function.

ii) (Monotonicity) $c' \ge c \Rightarrow u_t(c') \ge u_t(c)$.

(A1) - (A2) are the standard compactness-continuity and free disposal assumptions. Note that we do not make any convexity assumptions on production and preferences. Time-dependence admits the possibility of technological progress, although we do not allow unbounded progress (note A1ii). In the sequel we make some further assumptions.

3. Sensitivity of Finite Horizon Plans and Values

In the aggregative, convex capital accumulation model, the following monotonicity result has been proved for optimal investment choice: suppose $\underset{\sim}{x}(T)$ and $\underset{\sim}{x}(T')$ are optimal plans for horizons T and T' and terminal stock zero. If $T > T'$ then $x_t(T) \ge x_t(T')$, for all t (Brock (1971), Theorem 2)[4]. An immediate corollary is that there is a limit program $\underset{\sim}{\bar{x}}$, whose period t investment, $\bar{x}_t$, is defined as $\lim_{T \to \infty} x_t(T)$. In particular, this implies insensitivity of the initial investment choices; for all $\varepsilon > 0$ and $m \in N$, there is $\hat{T}$ such that $\|x_t(T) - x_t(T')\| < \varepsilon$, $t = 0,1..m$, whenever $\min(T,T') > \hat{T}$. Unfortunately, this critical monotonicity result, and the implied insensitivity, of the aggregative model simply does not carry over to the multi-sectoral context. We present a brief illustrative example.

Example 3.1[5] F_t and u_t satisfy (A1)-(A2) and additional convexity properties. But period 0 investment is very sensitive to horizon length. In particular, $\|x_0(T+1) - x_0(T)\| = 1$, for all $T \ge 1$.

Details: Let $n = 2$. Recall that investment (resp. consumption) in period t of the two commodities is denoted x_t^1, x_t^2 (resp. c_t^1, c_t^2). Suppose that:

$$F_t(x^1_{t-1}, x^2_{t-1}) = \{(y^1, y^2) \in \mathbb{R}^2_+ : y^i \le x^i_{t-1}, i = 1,2\}, t \ge 1$$

$$F_0(x^1, x^2) = \{(y^1, y^2) \in \mathbb{R}^2_+ : y^1 + y^2 \le x^1 + x^2\} \tag{3.1}$$

Let $(m_t)_{t \in N}$ be any strictly increasing, bounded sequence. The preferences are defined as:

$$u_t(c^1_t, c^2_t) = m_t c^i_t, \text{ i= 1 if t odd and i=2, if t even} \tag{3.2}$$

$$u_0(c^1_o, c^2_0) \equiv 0$$

Finally, let the initial stock x = (1/2, 1/2) and the terminal stock a = (0, 0). It is easy to see from (3.1) and (3.2) that (A1) - (A2) are satisfied and further that technology and preferences are convex.

Claim: For T odd, the optimal period 0 investment is given by $x^1_0 = 1$, $x^2_0 = 0$ whereas for T even, $x^1_0 = 0$, $x^2_0 = 1$.

Given (3.1)-(3.2) and any (x^1_0, x^2_0), in an optimal policy the only consumption that takes place is at the terminal and penultimate dates ; for instance when T is odd, $c^1_T = x^1_0$, $c^2_{T-1} = x^2_0$ and all other consumption is zero. But a unit of consumption yields greater utility in period T than at T-1. Hence, given the substitution possibilities in period 0, the claim follows.■

Remark It is clear that one could modify the example to one in which utility is time-invariant. Further, there are more complex examples in which technologies admit progress (i.e. $F_{t+1}(x) \supset F_t(x)$ for all t and x).

Example 3.1 is neither cause for despair nor surprising as far as multi-sectoral insensitivity is concerned. In the presence of substitution possibilities investment insensitivity is asking for too much in any case. Besides, from the point of view of a decision-maker who is unsure about the length of his decision horizon, the relevant question should surely be: is the level of ***maximized utility*** insensitive to the specification of plan horizon? Surprisingly, all of the extant analyses are concerned with investment insensitivity alone. Of course, for the question to be meaningful, we have to normalize the sum of utilities appropriately for different values of T. The most obvious normalization is to take averages. Recall that $v_T(x)$ is the maximum average utility from initial state x. The insensitivity question we now analyze is: under what conditions are $v_T(x)$ and $v_{T'}(x)$ close, for long but distinct plan horizons T and T'?

We present a positive result on value insensitivity. For this result we need two additional assumptions and a further piece of notation. Let $B = \beta e \in \mathbb{R}^n_+$, where β is the upper bound on the production correspondence, given by A1 ii).

(A3) **(Uniform Productivity)** $0 < x < B \Rightarrow \exists\, y(x) \in F_t(x)$, for all t, s.t. $x < y(x)$

(A3) essentially says that that growth is possible for all low stocks - i.e. for all x s.t. $0 < x < B$. Since utility and technology is time-dependent, we also need the following:

(A4) **(Limiting Technology- Preferences)** i) On the compact set $\{c \in \mathbb{R}^n_+ : 0 \le c \le B\}$, the functions u_t converge uniformly to a function u^*, as $t \to \infty$.
ii) On the compact set $\{x \in \mathbb{R}^n_+ : 0 \le x \le B\}$, the correspondences F_t converge uniformly to a correspondence F^*, as $t \to \infty$.

For any infinite horizon feasible plan $(\tilde{x}, \tilde{c})$, define the long-run average utility as:

$$u(\tilde{x}, \tilde{c}) = \underline{\lim}_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} u_t(c_t) \tag{3.3}$$

The long-run average value for initial state x, is defined as $v(x) = \sup u(\tilde{x}, \tilde{c})$, $(\tilde{x}, \tilde{c})$ feasible from x. We are now ready to prove the following value insensitivity result:

<u>Theorem 3.2</u> <u>Under (A1)-(A4), there is</u> $v^* \in \mathbb{R}$ <u>such that</u>

i) $$v^* = \lim_{T \to \infty} v_T(x), \quad 0 < x \le B \tag{3.4}$$

ii) v^* <u>is the long-run average value, for all</u> $0 < x \le B$. <u>In particular, average values are</u>

<u>insensitive to the length of the plan horizon (and initial non-zero stock) provided the horizon is sufficiently long</u>; $\forall \epsilon > 0$ and $0 < x \leq B$, $0 < x' \leq B$, there is $\hat{T} < \infty$ such that $|v_T(x) - v_{T'}(x')| < \epsilon$, whenever $\min(T, T') \geq \hat{T}$.

Proof: See Appendix A. ■

A brief discussion of the two assumptions (A3) and (A4) is in order. It is clear that if the average values have to satisfy some limiting behavior, then the environments of decision-making (the utility function and production correspondence) have to tend to some limiting behavior as well. Indeed, Example 3.3 shows that if such is not the case, the average values could behave arbitrarily over time. Uniform convergence is possibly too strong a requirement for the result- we think that pointwise convergence might suffice but we have been unsuccessful in proving that. Of course, the stationary model trivially satisfies (A4). The growth assumption, (A3), guarantees that the asymptotic behavior of average values are independent of the initial state and in its absence such independence may not hold (Example 3.4 below). It should be noted that the growth assumption is standard.[6]

From the proof it can be seen that the result holds more generally for (non-stationary) dynamic programming problems. There are well-known examples in dynamic programming (for instance, see Ross (1983)) in which the time average values do not converge to the long-run average value as the horizon length goes to infinity. Theorem 3.2 establishes then that in continuous, compact problems satisfying productivity such convergence does obtain. It should be noted that in a stationary dynamic programming model, Dutta (1990, Theorem 3) gives conditions under which (infinite horizon) **discounted** average values converge to the long-run average value when the discount factor goes to one. That result is used in the establishing Theorem 3.2. We turn now to the examples:

<u>Example 3.3</u> <u>Technology and preferences satisfy (A1) - (A3), but average values are sensitive to the horizon.</u>

Details: n=1 and the production function is

$$f_t = f(x) = \begin{cases} 2x & x \le 1/2 \\ 1 & x > 1/2 \end{cases} \tag{3.5}$$

Let $\tilde{u}_t$ be any sequence of functions that are individually continuous (C^∞ even), strictly increasing and strictly concave and which satisfy the following property

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{t=0}^{T-1} \tilde{u}_t\left(\frac{1}{2}\right) > \underline{\lim_{T\to\infty}} \frac{1}{T} \sum_{t=0}^{T-1} \tilde{u}_t\left(\frac{1}{2}\right) \tag{3.6}$$

Now define

$$u_t(c) = \begin{cases} \tilde{u}_t(c) & c \le 1/2 \\ \tilde{u}_t(1/2) & c > 1/2 \end{cases} \tag{3.7}$$

So the specification satisfies (A1) - (A3). It does not however satisfy (A4). An easy way to see that is to notice that if $u_t(1/2)$ does in fact converge, then so must the averages and hence one cannot satisfy (3.6). It is clear that with initial state x = 1 (and hence f(x) = 1), the optimal T-period plan is $x_t = c_t = 1/2$, t = 0, ... T-1. But then (3.6) implies that average values are sensitive to the horizon. Indeed, one could modify the example in an obvious fashion to yield up to a countable set of limit points for the sequence of averages, $v^T(1)$. One could also modify the example straightforwardly to show that with a stationary utility function but a non-stationary production correspondence, which additionally violates (A4ii), a similar sensitivity holds.[7] ■

Example 3.4 Technology and preferences satisfy (A1) - (A2) and (A4), but average values are sensitive to the initial stock.

Details: $$f_t(x) = \begin{cases} \max(x, -1/2 + 2x), & x \le 3/4 \\ 1 & x > 3/4 \end{cases}$$

$$u(c) = c$$

It is easy to check that $v^T(x) \to 0$, for $x \le 1/2$, $v^T(x) \to 1/4$ for $x > 1/2$. Clearly, the technology and preferences satisfy (A1)-(A2) and (A4).

4. Continuity of Finite Horizon Plans and Values

We turn now to the continuity question - is every finite horizon optimal plan (for some admissible set of terminal stocks) "close" to some infinite horizon optimal plan, for long but finite horizons? We know that some conditions, in addition to the maintained assumptions (A1) - (A2), will need to be placed in order to obtain affirmative answers to these questions. This can be seen from the Gale "cake-eating" example (Gale (1967), Example 2) in which the finite horizon optimal policies converge to the pure accumulation program, which is clearly not infinite horizon optimal.

We present two positive results: Theorems 4.1 and 4.4. Theorem 4.1 generalizes and corrects the only available continuity result in multi-sectoral, non-convex models - that of Nermuth (1978). Theorem 4.4 pertains to the original Ramsey formulation and is related to the Gale (1967) result for the convex, stationary model. The analysis in this section is carried out in a spirit similar to that in Nermuth (1978). In particular, we treat the set of optimization problems as a parametric family with the horizon as the relevant parameter. Continuity is established by showing that the correspondence of optimal choices (as a mapping from T to the set of feasible plans), is upper semi-continuous at $T = \infty$.

Theorem 4.1 Suppose that (A1) - (A2) hold and further that on the set of feasible infinite horizon plans from initial state x, denoted Q, $\sum_{t=0}^{\infty} u_t(c_t)$ is finite and upper semi-continuous with respect to the product topology. Then,

i) As $T \to \infty$, $V_T(x) \to V(x) \equiv \max \sum_{t=0}^{\infty} u_t(c_t)$, s.t. $\underset{\sim}{x}, \underset{\sim}{c} \in Q$

ii) for all $\epsilon > 0$ and $T < \infty$, there is T_ϵ such that whenever the horizon length is $T' \geq T_\epsilon$, for any T'-optimal program to target stock zero, $\underset{\sim}{x}(T')$, $\underset{\sim}{c}(T')$, there is an infinite horizon optimal plan $(\tilde{x}^*, \tilde{c}^*)$ satisfying

$$\| x_t(T') - x_t^* \| < \epsilon, \quad t = 0, \ldots T \tag{4.1}$$

$$\| c_t(T') - c_t^* \| < \epsilon, \quad t = 0, \ldots T \tag{4.2}$$

Corollary 4.2 Under the hypotheses of Theorem 4.1, if the infinite horizon optimal plan is unique, then all optimal T'-period programmes with zero terminal stock, agree up to ϵ in the first T periods, for T' sufficiently large.

Proof: See Appendix B. □

Remark 1 As the reader can verify from the proof of Theorem 4.1, (A1) - (A2) are unnecessarily strong requirements. Define an attainable stock z as $z \in F_t(x)$ for all t and x. Weaken (A1) to (A1)': F_t is a non-empty usc correspondence, for all t. Then, under only (A1)' and the restrictions on infinite horizon preferences stated above, Theorem 4.1 holds if the terminal stock for the finite horizon problems is required to be attainable.

Remark 2 Nermuth (1978) sought to prove the theorem under (A1)', (A2i); and a considerably stronger condition on infinite horizon preferences ($\sum_{t=0}^{T} u_t(c_t)$ converges uniformly to a continuous function $\sum_{t=0}^{\infty} u_t(c_t)$). However, he did allow a larger set of terminal stocks. A target stock b is called Nermuth-attainable if there is a feasible infinite horizon plan $\tilde{x}, \tilde{c}$ with $x_t \geq b$, for some $t \in N$.[8] The theorem is false, however, under these hypotheses. The set of admissible target stocks is too large, as is shown by the following example.

Example 4.3 $F_t(x) = \{y \in \mathbb{R}_+ : y \leq x\}, \ t \in N$

$$u_t(c) = \begin{cases} 0, & t \geq 1 \\ c, & t = 0 \end{cases}$$

Let b = 1 and initial stock x = 1. Clearly, the only feasible finite horizon plan is x_t =

1, $c_t = 0$, $t = 0,..T$ while the infinite horizon optimal plan is $x_t = 0$, $t \geq 0$, $c_0 = 1$, $c_t = 0$, $t > 0$.■

<u>Remark 3</u> The restriction to attainable stocks is possibly unnecessarily strict. The set of terminal stocks for which Theorem 4.1 holds is likely to be larger. Given any sequence $(\tilde{x}(T), \tilde{c}(T))$ of optimal plans, let $\tilde{x}'$, $\tilde{c}'$ be a subsequential limit, as $T \to \infty$. Let $\underline{x}' = \underline{\lim}_{t \to \infty} x'_t$ and let $\underline{x}$ = inf $\underline{x}'$ over the set of "limit plans", $\tilde{x}'$, $\tilde{c}'$. We conjecture that the theorem holds for terminal stocks $0 \leq b < \underline{x}$. This result is already known in the aggregative model and was proved by Brock (1971).

We now prove a continuity result when the Ramsey utility function, (2.7) is well-defined.

<u>Theorem 4.4</u> <u>Suppose that (A1) - (A2) hold and suppose further that</u> $\sum_{t=0}^{\infty} [u_t(c_t) - b_t] > -\infty$ <u>for some feasible program</u> $(\tilde{x}, \tilde{c}) \in Q$. <u>Then,</u>

i) as $T \to \infty$, $W_T \equiv \text{Max} \sum_{t=0}^{T} [u_t(c_t) - b_t]$ <u>(over finite horizon feasible plans), converges to</u> $W \equiv \text{Max} \sum_{t=0}^{\infty} [u_t(c_t) - b_t]$ <u>(over infinite horizon feasible plans).</u>

ii) <u>Theorem 4.1 ii) holds</u>.

□

Proof: See Appendix B

<u>Remark 1</u> Of course a corollary identical to Corollary 4.2 is valid in this case as well.

<u>Remark 2</u> The condition that there are feasible plans such that $\sum_{t=0}^{\infty} [u_t(c_t) - b_t] > -\infty$, is precisely the original Ramsey (1928) construction, to circumvent the problem of ill-defined infinite horizon preferences.

<u>Remark 3</u> The condition is also related to the criteron for "good" programmes employed by Gale (1967) in the convex, stationary model. Gale employed a weaker notion in which b_t =

b was the utility from golden-rule consumption. In his model he proved the existence of good programmes and the consequent insensitivity of finite horizon choices. In the non-convex, time-dependent model there does not seem any way out but to assume this condition.

<u>Remark 4</u> As Corollary 4.2 shows, in the presence of uniqueness of optimal choices, continuity implies insensitivity. It is easy to construct examples in which without such uniqueness, optimal choices are continuous at infinity but sensitive to horizon length. Of course, the cake-eating example is an instance of the converse: insensitivity and discontinuity.

The hypotheses of Theorems 4.1 and 4.4 include restrictions on infinite horizon preferences which guarantee the existence of infinite horizon optima. An alternative question is: suppose that an optimal plan is known to exist, for instance under the catching-up criterion. Are the conclusions of Theorems 4.1 and 4.4 valid under this weaker hypothesis? The answer is known to be yes in the aggregative model (Brock (1971)). We now show that the answer is, in general, no in the multi-sectoral model.

<u>Example 4.5</u> <u>There is a unique catching-up optimal plan</u> $(\underset{\sim}{x}^*, \underset{\sim}{c}^*)$ <u>and unique finite horizon optimal plans</u> $(\underset{\sim}{x}(T), \underset{\sim}{c}(T))$. <u>However, no finite horizon optimal plan is close to the infinite horizonal optimum. In particular,</u> $\| c_0(T) - c_0^* \| = 1$, <u>for all T</u>

<u>Details</u> n= 2. Let x = (1/2, 1/2)

$$F_0(x) = \{y^1 + y^2 \le x^1 + x^2\}$$

$$F_t(x) = \{(y^1, y^2) : y^1 \le f(x^1), y^2 \in \Gamma_t(x^2)\} \qquad (4.3)$$

where

$$f(x^1) = \begin{cases} 2x^1 & x^1 \le 1 \\ 2 & x^1 > 2 \end{cases} \qquad (4.4)$$

$$\Gamma_t(x^2) = \begin{cases} 0 & x^2 < \gamma_{t-1} \\ [0, \gamma_t] & x^2 \ge \gamma_{t-1} \end{cases} \qquad (4.5)$$

and γ_{t-1} is an increasing sequence such that $\gamma_t > t$, $\gamma_{-1} = 1$.[9] Finally,

$$u_t(c^1, c^2) = c^1 + c^2 \tag{4.6}$$

Essentially the two commodities are perfectly substitutable in production in period zero and thereafter follow totally independent processes. Moreover, for commodity 2's production to get off the ground, the aggregate of the commodities has to be used in the second production process in period zero. So the choices are: a) only produce commodity 1 from period 1 onwards (and then the best policy is $x_t = c_t = 1$, for t=0,..T-1) or b) switch to commodity 2 and the discrete alternatives are $\underset{\sim}{x} = (1, \gamma_0, \gamma_1, ...\gamma_{T-1}, 0, 0...)$ with an associated $\underset{\sim}{c} = (0, 0, ...0, \gamma_T, 0)$. Since $\gamma_t > t$, the finite horizon optimum is b), for T = T'. But clearly the unique catching-up optimum is $c_t^1, c_t^2 = 1, 0$, for all t.■

5. Conclusions

This paper established insensivity and continuity results in general multi-sectoral intertemporal allocation problems. We proposed that the appropriate measure of insensitivity ought to be value insensitivity. We further showed that in growth models, under standard assumptions, such insensitivity does indeed obtain. The principal difference with aggregative analyses is that investment insensitivity can fail easily and unsurprisingly. For continuity, it suffices to have some structure on infinite horizon preferences and we studied these structures for some often used preferences. All of the results were proved by primal topological arguments. This allowed us to dispense with convex structures in both technologies and preferences. Much of the analysis extends straightforwardly to other intertemporal problems in economics.

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Appendix A

In this appendix, we prove Theorem 3.2. The proof is long and en route we prove two other results which are of independent interest. Some additional notation needs to be introduced at this point. Consider any optimization problem with time-independent utility function and production correspondence $(\hat{u},\hat{F})$. For discount factor $\delta \in [0,1)$, define

$$\hat{v}_\delta(x) = \max \sum_{t=0}^{\infty} (1-\delta)\, \delta^t\, \hat{u}(c_t) \text{ s.t.} \tag{A.1}$$

$$x_{-1} = x,\ x_t + c_t \in \hat{F}(x_{t-1}),\ t = 0,1\ldots \tag{A.2}$$

So $\hat{v}_\delta(x)$ is the discounted average value under the stationary problem defined by $\hat{u}$, $\hat{F}$. For the same specification of utility and production, let $\hat{v}(x)$ (resp. $\hat{v}_T(x)$) denote the long-run average (resp. T-period average) value. So the superscript on the values will refer to the utility-production specification whereas the subscript will describe the optimization parameter. In particular, for (u*,F*) - the limiting utility and production for the sequences $(u_t)_{t\in N}$ and $(F_t)_{t\in N}$ - the notation will be (v^*_δ, v^*_T, v^*). Finally, abusing convention somewhat, we will use the notation (0,B] (respectively [0,B]) to denote the set $\{x \in \mathbb{R}^n: 0 < x \leq B\}$ (respectively $\{x \in \mathbb{R}^n: 0 \leq x \leq B\}$). Theorem 3.2 will be proved by way of the following lemmas:

Lemma A.1 $\quad \exists\, v^* \in \mathbb{R}$ s.t. i) $v^* = v^*(x),\ \forall\, x \in (0,B]$.

ii) $v^* = \lim_{\delta \uparrow 1} v^*_\delta(x),\ \forall\, x \in (0.B]$

i.e. the long-run average value is state-independent and the limit of discounted average values, if the utility - production specification is u^*, F^*.

Lemma A.2 $\quad \lim_{T\to\infty} v^*_T(x) = \lim_{\delta\uparrow 1} v^*_\delta(x) = v^*,\ \forall\, x \in (0,B]$

i.e. the time average and discounted average values have an identical limit and this limit is the long-run average value.

Recall that $v_T(x)$ is our notation for the time-average value, under (u_t,F_t).

Lemma A.3 $\lim_{T\to\infty} v_T^*(x) = \lim_{T\to\infty} v_T(x) = v^*, \forall x \in (0,B]$

i.e. the time average values in the original problem have the same limits as those under the limiting problem (u^*,F^*) and this limit is the long-run average value.

Remark Lemmas A.1 and A.2 are results of independent interest which will be seen to hold for any stationary problem whose data satisfy assumptions (A1)-(A3). For the aggregative convex model, Lemma A.1 follows from Dutta (1990, Proposition 1). To the best of my knowledge these results were however unknown for the general multi-sectoral model. Note incidentally, that an implicit assertion in Lemmas A.1 - A.3 is that all of the relevant limits are well-defined.

Proof of Lemma A.1: From hereon, we take the state space to be [0,B]. Lemma A.1 will be proved by appealing to the following implication of Dutta (1990, Theorem 3):

Fact 1: Consider $(\hat{u},\hat{F})$ which satisfy (A1)-(A2). Suppose that there is a real-valued function M(x) such that

$$[\hat{v}_\delta(x) - \hat{v}_\delta(B)] \geq M(x)(1-\delta), \forall x \in (0,B], \delta \in [0.1). \tag{A.3}$$

Then, $\exists \hat{v} \in \mathbb{R}$ such that $\lim_{\delta \uparrow 1} \hat{v}_\delta(x) = \hat{v}(x) = \hat{v}, \forall x \in (0,B]$

The condition (A.3) we will refer to as the **uniform value boundedness** condition. We will show that this condition is satisfied, and hence the conclusion of Fact 1 follows, for arbitrarily close approximations of (u^*,F^*). That in turn will prove Lemma A.1. Let

$$u^n(c) = \begin{cases} u^*(c), & \text{if } c \geq (1/n)\, e, \text{ where } n \in N \\ 0 & \text{else} \end{cases}$$

$$F^n \equiv F^*$$

Writing B_n for B - (1/n)e and normalizing $u^n(0) = 0$, it follows from the monotonicity assumptions that

$$v_\delta^n(B) \leq u^n(B)(1-\delta) + \delta v_\delta^n(B_n) \tag{A.4}$$

The productivity assumption (A.3) clearly implies that for every $x \in (0,B]$, there is a pure accumulation programme which attains B_n in a finite number of steps, say $T(x)$ (i.e. $B_n \in F^n(\bar{x}_{T(x)-1})$ for some pure accumulation programme $(\underset{\sim}{\bar{x}},\underset{\sim}{\bar{c}})$ with $\bar{x}_{-1} = x$). It then folows that

$$v^n_\delta(x) \geq \delta^{T(x)} v^n_\delta(B_n) \tag{A.5}$$

(A.4) and (A.5) imply, after some tedious algebra,

$$v^n_\delta(x) - v^n_\delta(B) \geq \delta v^n_\delta(B_n) [\delta^{T(x)}-1] - u^n(B)(1-\delta) \tag{A.6}$$

Noting that $v^n_\delta(B_n) \leq u^n(B)$, (A.6) yields

$$v^n_\delta(x) - v^n_\delta(B) \geq - u^n(B) \left[\delta \frac{1-\delta^{T(x)-1}}{1-\delta} + 1\right] (1-\delta)$$

Since the tern in the brackets goes to $T(x)$, value boundedness is satisfied by (u^n,F^n) and hence $\lim_{\delta\uparrow 1} v^n_\delta(x) = v^n(x) = v^n$. By construction, $\lim_{\delta\uparrow 1} v^*_\delta(x) \geq \lim_{\delta\uparrow 1} v^n_\delta(x)$. Further, $\lim_{n\to\infty} v^n = v^*$. Hence, $\lim_{\delta\uparrow 1} v^*_\delta(x) \geq v^*$. On the other hand, it is immediate that for all $\varepsilon > 0$, there is $n(\varepsilon)$ such that $v^n_\delta(x) \geq v^*_\delta(x) - \varepsilon$, whenever $n \geq n(\varepsilon)$ and for all $x \in (0,B]$, $\delta \in [0,1)$. From this the reverse inequality follows and therefore Lemma A.1. ■

<u>Proof of Lemma A.2:</u> Throughout this proof, the initial state x is going to remain fixed. Unfortunately, we need a little more notation. Let $p_t(\delta) \equiv (1-\delta)\delta^t$ and $q_t(T) \equiv 1/T$ for $t=0,1..T-1$ and $q_t(T) = 0$, else. Let $(\underset{\sim}{x},\underset{\sim}{c})$ be any feasible programme and write u_t for $u^*(c_t)$. Then,

$$\left| \sum_{t=0}^{\infty} [p_t(\delta) - q_t(T)]u_t \right| \leq u^*(B) \sum_{t=0}^{\infty} |p_t(\delta) - q_t(T)| \tag{A.7}$$

We now prove the following:

<u>Sub-Lemma 4</u>: For any sequence $T_n\uparrow\infty$, there is a sequence of discount factors $\delta_n\uparrow 1$ s.t.

$$\lim_{n\to\infty} \sum_{t=0}^{\infty} |p_t(\delta_n) - q_t(T_n)| = 0 \tag{A.8}$$

It should be clear that (A.7) and (A.8) in combination imply Lemma A.2. We suppress momentarily the subscript n. For every T, we can find $\delta \in [0,1)$ s.t. i) $1-\delta > 1/T > (1-\delta)\delta^T$ and ii) there is an integar T' defined by $(1-\delta)\delta^{T'} = 1/T$. Hence,

$$\sum_{t=0}^{\infty} |p_t(\delta) - q_t(T)| = \sum_{t=0}^{T'-1} (p_t - q_t) + \sum_{t=T'}^{T-1} (q_t - p_t) + \sum_{t=T}^{\infty} (p_t - q_t)$$

$$= 2\,[1 + \delta^T - \frac{1}{T(1-\delta)} + \frac{\ln T(1-\delta)}{T \ln\delta}] \tag{A.9}$$

(A.9) follows by simple algebraic manipulation on the previous step by way of using the definition of T'. (A.8) will be proved therefore if we can find a sequence of discount factors, $\delta(T)$ such that $\delta(T)^T - \frac{1}{T(1-\delta(T))} + \frac{\ln T(1-\delta(T))}{T \ln\delta(T)}$ goes to -1 and $\delta(T)\uparrow 1$ as $T\uparrow\infty$. Write $\delta(T) \equiv \exp(\hat{\varphi}/T)$, where $\hat{\varphi} \in \mathbb{R}_-$ and its exact value is given by,

$$e^{\hat{\varphi}} + 1/\hat{\varphi} + \ln(-\hat{\varphi})/\hat{\varphi} = -1 \tag{A.10}$$

This is possible to do since the function in (A.10) is a continuous function on $\mathbb{R}_-$ with value 0 when $\varphi = -\infty$ and value $-\infty$ when $\varphi = 0$. Since $\ln \delta(T)^T = T(1-\delta(T))\,[\ln \delta(T) / (1-\delta(T))]$, $-T(1-\delta(T))$ has the same limit as $\ln \delta(T)^T$. Hence from (A.9) it follows that

$$\lim_{T\to\infty} \sum_{t=0}^{\infty} |p_t(\delta(T)) - q_t(T)| = 2[1 + e^{\hat{\varphi}} + 1/\hat{\varphi} + \ln(-\hat{\varphi})/\hat{\varphi}] \tag{A.11}$$

The sub-lemma is seen to follow immediately from (A.11). In particular, that also completes the proof of Lemma A.2. ∎

Proof of Lemma A.3: The proof is essentially a continuity argument. Let $\bar{F}^n$ and $\underline{F}^n$ be two sequences of production correspondences satisfying (A1) and (A3) and converging uniformly to F^*. Further, $\underline{F}^n(x) \subset F^*(x) \subset \bar{F}^n(x)$, for all $x \in (0,B]$ (where the subset inclusion is strict in each case) so that for all $t \geq T(n)$, $\underline{F}^n(x) \subset F_t(x) \subset \bar{F}^n(x)$, for all x. Similarly, let $\bar{u}^n$ and $\underline{u}^n$ be upper and lower sequences of utility functions, satisfying (A2) converging uniformly to u^*, such that for all $t \geq T(n)$, $\underline{u}^n(c) < u_t(c) < \bar{u}^n(c)$, for all c. Let the associated time average and long-run average values be defined as $\underline{v}_T^n$ and $\underline{v}^n$ (resp. $\bar{v}_T^n$ and $\bar{v}^n$).

It was clear from the proofs of Lemmas A.1 and A.2 that they apply to any stationary production-utility specification which satisfy (A1)-(A3) and hence to the upper and lower approximating sequences. In particular, therefore $\lim_{T\to\infty} \underline{v}_T^n(x) = \underline{v}^n$, for all $x \in (0,B]$ (resp. $\lim_{T\to\infty} \bar{v}_T^n(x) = \bar{v}^n$, for all $x \in (0,B]$). Further, it is clear that $\lim_{n\to\infty} \underline{v}^n = v^*$ (resp. $\lim_{n\to\infty} \bar{v}^n = v^*$). So pick n such that

$$v^* - \varepsilon/4 < \underline{v}^n < \bar{v}^n < v^* + \varepsilon/4, \tag{A.12}$$

From hereon n (in addition to x) remains fixed. Let $\hat{T}$ be given by the requirement that

$$\underline{v}^n - \varepsilon/4 < \underline{v}_T^n(x) < \bar{v}_T^n(x) < \bar{v}^n + \varepsilon/4, \ T \geq \hat{T} \tag{A.13}$$

Clearly, there is T' such that for all $T \geq T'$,

$$\underline{v}_T^n(x) - \varepsilon/2 < v_T(x) < \bar{v}_T^n(x) + \varepsilon/2 \tag{A.14}$$

Combining (A.12)-(A.14), for $T \geq \max(\hat{T}, T')$, we have

$$v^* - \varepsilon < v_T(x) < v^* + \varepsilon \tag{A.15}$$

Since (A.15) holds for all $\varepsilon > 0$, the proof of Lemma A.3 is complete.■

Appendix B

In this appendix we prove Theorems 4.1 and 4.4. Throughout, the initial state x remains fixed and we suppress this variable in all functions and correspondences that follow; e.g. the T-period value function is written V_T rather than $V_T(x)$ etc.

Proof of Theorem 4.1: Recall that Q_T (resp. Q) denotes the set of finite horizon (resp. infinite horizon) feasible plans. For any finite horizon feasible plan $(\underset{\sim}{x}(T),\underset{\sim}{c}(T))$ define the infinite horizon extension, $\underset{\sim}{x}_e,\underset{\sim}{c}_e$ as the programme which corresponds to it over the first T periods and then maintains zero investment and consumption levels thereafter. Let $\Gamma_T \equiv \{(\underset{\sim}{x},\underset{\sim}{c}) \in Q\text{: } x_t = c_t = 0, t > T\}, \Gamma_\infty \equiv Q$.

Lemma B.1 $\hat{\underset{\sim}{x}}(T),\hat{\underset{\sim}{c}}(T)$ solves Max $\sum_{t=0}^{T} u_t(c_t)$, subject to $\underset{\sim}{x}(T),\underset{\sim}{c}(T) \in Q_T$, if and only if $\hat{\underset{\sim}{x}}_e,\hat{\underset{\sim}{c}}_e$ solves Max $\sum_{t=0}^{\infty} u_t(c_t)$, subject to $\underset{\sim}{x},\underset{\sim}{c} \in \Gamma_T$.

Proof: Note that $\sum_{t=0}^{\infty} u_t(0)$ is finite (since zero consumption forever is a feasible option in the infinite horizon and hence preferences are well-defined for it). From this the lemma trivially follows. □

So the optimization problem for any horizon, finite or infinite can be compactly expressed as

$$\hat{V}_T = \text{Max} \sum_{t=0}^{\infty} u_t(c_t) \quad \text{s.t. } \underset{\sim}{x},\underset{\sim}{c} \in \Gamma_T \tag{B.1}$$

So $\hat{V}_T = V_T + \sum_{t=T+1}^{\infty} u_t(0)$, whenever $T < \infty$ and $\hat{V}_\infty = V$. In turn, the optimal programmes are given by $\Psi_T \equiv \{(\underset{\sim}{x},\underset{\sim}{c}) \in \Gamma_T\text{: } \sum_{t=0}^{\infty} u_t(c_t) \geq \hat{V}_T\}$.

We shall treat the function $\hat{V}_T$ and the correspondence Ψ_T as maps from the extended natural numbers, $N \cup \{\infty\}$ into $\mathbb{R}$ and P(Q) respectively. We shall then investigate the continuity of $\hat{V}_T$ and the upper semi-continuity of Ψ_T with respect to the horizon length T. Note that $N \cup \{\infty\}$ is a Hausdorff space if we take as a base for a topology all sets of the form

$$\{t\} \quad t = 0,1,2... \tag{B.2}$$

$$N_T = \{t \in N: \; t \geq T \text{ or } t = \infty\} \quad T = 0,1,2...$$

(see Kelly (1955, p.47,Theorem 11)). The topology on Q will be the pointwise convergence or product topology. Recall that a base for the neighborhoods of $\underset{\sim}{x},\underset{\sim}{c} \in Q$ is given by

$$U_{T,\epsilon} = \{(\underset{\sim}{x}'\underset{\sim}{c}') \in Q: \|c_t'-c_t\| < \epsilon, \|x_t'-x_t\| < \epsilon, 0 \leq t \leq T\} \tag{B.3}$$

where $\epsilon > 0$, $T = 0,1,...$ For a subset $M \subset Q$ we can define a neighborhood $U_{T,\epsilon}(M)$ as the union of all $U_{T,\epsilon}(\underset{\sim}{x},\underset{\sim}{c})$, $\underset{\sim}{x},\underset{\sim}{c} \in M$. From (B.2)-(B.3) it should be clear that the theorem is proved if we succeed in showing that Ψ is in fact usc at $T = \infty$.

Note that Q is compact in the product topology. This follows from an induction argument for details of which the reader should consult Nermuth (1978, Lemma 2). Clearly, given A1i), Γ_T is a closed subset of Q and hence compact. From that and the upper semi-continuity of the infinite horizon preferences, it follows that Ψ_T is non-empty for all T. It is also immediate that Γ_T is an increasing correspondence in T. That yields as an immediate corollary the upper semi continuity of the correspondence at $T = \infty$. As a second implication we have the fact that $\hat{V}_T$ is a monotonically increasing sequence. Finally, a standard argument yields

<u>Lemma B.2</u> As $T \to \infty$, $\hat{V}_T \uparrow V$ <u>where</u> V <u>is the infinite horizon value.</u>

We now show

<u>Lemma B.3</u> ψ <u>is usc on</u> $N \cup \{\infty\}$

Pf: Define

$$\Omega_T = \{(\underset{\sim}{x},\underset{\sim}{c}) \in Q: \sum_{t=0}^{\infty} u_t(c_t) \geq \hat{V}_T\} \tag{B.4}$$

We show that Ω is a usc correspondence. It is clear from (B.2) that given the topology on the set of extended naturals, upper semi-continuity really only needs to be checked at T = ∞. By the assumption of upper semi-continuity on the function $\sum_{T=0}^{\infty} u_t(c_t)$ and Lemma B.2, it follows that Ω is a closed graph correspondence; since Q is compact, it is in fact usc (Berge (1963)). Finally, $\Psi_T = \Gamma_T \cap \Omega_T$ as an intersection of two usc correspondences, is also usc. (Berge (1963), p. 112, Theorem 7). □

<u>Proof of Theorem 4.4</u> The ideas of the proof will be similar to those used in proving Theorem 4.1. In particular, we will again show that the correspondence of maximizers, Ψ_T is usc at T = ∞. The constructions of the feasible correspondence and objective function will, however, be dual to those employed above. Define the feasible correspondence $\Gamma_T \equiv Q$ and the objective function $U_T(\underset{\sim}{x},\underset{\sim}{c},) = \sum_{t=0}^{T} [u_t(c_t) - b_t]$. For T $\in$ N, let

$$W_T = \text{Max } U_T(\underset{\sim}{x},\underset{\sim}{c},) \text{ , } (\underset{\sim}{x},\underset{\sim}{c}) \in Q \tag{B.5}$$

$$\Psi_T = \{\underset{\sim}{x},\underset{\sim}{c} \in Q : U_T(\underset{\sim}{x},\underset{\sim}{c},) = W_T\}$$

Note that Ψ_T is non-empty given (A2i). Clearly, W_T is a monotonically declining sequence. Let

$$W = \text{Sup} \sum_{t=0}^{\infty} [u_t(c_t) - b_t], \quad (\underset{\sim}{x}, \underset{\sim}{c}) \in Q \tag{B.6}$$

By Brock-Gale (1969, Lemma 2), the supremum is in fact achieved, i.e. there exists a feasible plan $(\underset{\sim}{\overset{*}{x}}, \underset{\sim}{\overset{*}{c}})$ such that $\sum_{T=0}^{\infty} [u_t(\overset{*}{c}_t) - b_t] = W$. So Ψ_∞ is non-empty as well.

We show that Ψ is usc by establishing that it is a closed graph correspondence (and since the range Q is compact, that sufficies). So let $(\underset{\sim}{x}_n, \underset{\sim}{c}_n) \in \Psi(T_n)$, $(\underset{\sim}{x}_n,\underset{\sim}{c}_n)$ converges pointwise to $(\hat{\underset{\sim}{x}}, \hat{\underset{\sim}{c}})$. (Note that $T_n = \infty$ for infinitely many indices is admissible). Suppose in fact

$$W' \equiv \sum_{t=0}^{\infty} [u_t(\hat{c}_t) - b_t] < W \tag{B.7}$$

Pick $T < \infty$ such that $\sum_{t=0}^{T} [u_t(\hat{c}_t) - b_t] < W' + \epsilon/2$, for $\epsilon > 0$ chosen so that $W' + \epsilon < W$.

Then, for n large,

$$\sum_{t=0}^{T} [u_t(c_{tn}) - b_t] < \sum_{t=0}^{T} [u_t(\hat{c}_t) - b_t] + \epsilon/2 < W' + \epsilon$$

But $\sum_{t=0}^{T_n} [u_t(c_{tn}) - b_t] \leq \sum_{t=0}^{T} [u_t(c_{tn}) - b_t] < W' + \epsilon < W$. This contradicts the fact that $\underset{\sim}{x}_n, \underset{\sim}{c}_n \in \psi(T_n)$. □

Footnotes

1 If insensitivity and continuity is simultaneously satisfied, i.e. finite horizon optima are close to each other and to all infinite horizon optima, then authors as Mckenzie (1976) term this property an "initial turnpike."

2 For instance, finite horizon optima may all be clustered together, but "far" from any infinite horizon optimum. On the other hand, each finite horizon optimum may be close to some infinite horizon optimum, but be "far" from each other (examples will be discussed in the sequel). Of course, if optimal choices are unique, then continuity implies insensitivity. One important motivation for this research is to admit increasing returns and other non-convexities, in the presence of which optimal choices are typically not unique.

3 To be exact, Ramsey took $b_t \equiv \sup u_t(c_t)$, without any restriction to pure accumulation programmes. Clearly, our definition is more likely to result in summable utility.

4 This result has been appropriately and succesively generalized to non-convex technologies (maintaining still the convexity of preferences), by Majumdar-Nermuth (1982) and Mitra-Ray (1984).

5 A similar but somewhat less transparent example may be found in Nermuth (1978).

6 For the aggregative model it merely says that the production function lies above the 45° line for an initial set of investment levels. For multi-sectoral convex models, the usual productivity assumption is that (A3) holds for some $x > 0$. Exploting convexity that can be seen to imply precisely (A3).

7 This example satisfies the assumptions of the aggregative model studied by Mitra-Ray (1984) (and under appropriate modifications to accomodate differentiability and strict concavity-monotonicity, the assumptions of Brock (1971) and Majumdar-Nermuth (1982) models). Consequently, investment insensitivity obtains in this example. So this example, in combination with Theorem 3.2 proves that value insensitivity neither implies nor is implied by investment insensitivity.

8 Nermuth also assumed $y \in F_t(y)$ for all y and t, which makes the definition reasonable.

9 The production correspondences do not satisfy uniform boundedness and lower semicontinuity. However, boundedness assumptions are not required for the corresponding positive result in the aggregative model. The example can be easily adapted to yield lower semicontinuity.