Consistent Allocation Rules in Atomless Economies

Thomson, William and Lin Zhou

Working Paper No. 294
September 1991

Consistent Allocation Rules in Atomless Economies

William Thomson and Lin Zhou

Rochester Center for Economic Research
Working Paper No. 294

September 1991

Consistent Allocation Rules in Atomless Economies

William Thomson
University of Rochester
and
Lin Zhou
Cowles Foundation

and

The Rochester Center for Economic Research
Working Paper No. 294

August 1991

1. INTRODUCTION

We consider the problem of allocating a bundle of commodities among a group of agents who are collectively entitled to it. We search for systematic methods, or *solutions*, of resolving this problem. For a solution to be considered desirable, we propose that it should satisfy several requirements. First, it should select allocations that are efficient. Second, it should be equitable. A variety of distributional requirements have been considered in the literature. The requirement we will impose here is that it select allocations at which no agent would prefer the equal division of the aggregate endowment to his own consumption. In addition to these two basic efficiency and equity conditions, we require that the solution be *consistent*: the recommendation it makes for every economy is never contradicted by the recommendation it would make for any "reduced" economy obtained by allowing some subgroup of agents to leave the scene with their allotted consumptions.

The requirement of consistency is very natural. Here is a motivation for it, offered by Thomson (1988): consider all the resources available on the planet earth; one would like not only that they be distributed fairly when the planet is considered as a whole but also that whatever amount ends up in each continent, country, city, ... be fairly distributed among the members of that subpopulation, when considered in isolation.

In this paper we will study economies with a large number of agents modelled as a complete atomless measure space. Our main result is a characterization of the solutions that satisfy all three requirements of efficiency, equity, and consistency. It follows from our result that, under standard assumptions on preferences, any such solution is a subsolution of the equal-income Walrasian solution. However, our characterization holds for more general economies since we allow preferences with satiation points. Mas-Colell (1988) extended traditional equilibrium theory to such economies by introducing the notion of *Walrasian equilibria with slack.*[1] Our characterization is that for such an economy any solution satisfying our three requirements has to select allocations that are supported by equal-budget Walrasian equilibria with slack.

Although the atomless model has been used extensively in the analysis of a variety of issues (core allocations, envy-free allocations, etc.), this is the first time (to our knowledge) that the issue of consistency has been addressed in this model. This model permits stronger conclusions, and under weaker assumptions, than the finite-agent model. Indeed, let us compare the result here with a previous result of Thomson (1988). Thomson considered standard economies with finitely many agents and proved: any solution that satisfies the three requirements above and the condition of replication invariance[2] must be a subsolution of the equal-income Walrasian solution. Thomson (1990b) also characterized consistent allocation rules in finite-agent economies with single-peaked preferences under the three requirements above, together with a continuity condition. First, we reach our conclusion without the replication invariance condition, or the continuity condition. Second, a conceptual distinction is worth mentioning. Suppose that one begins with some large economy. Here we

---

[1] This notion is closely related to the notion of dividend equilibria due to Aumann and Drèze (1986).

[2] If an allocation provides a desirable resolution of the problem of fair division in some economy, then for any integer k, its k-th replica provides a desirable resolution of the problem of fair allocation in the k-th replica of the economy.

reach our conclusion by applying consistency to this economy and its reduced economies. In Thomson's earlier characterizations consistency was applied to a wider range of economies, including economies into which the original economy was first embedded, and subsequently certain reduced economies obtained from them. Hence consistency in that context was a stronger requirement than in our current model. The motivation offered above for consistency obviously supports its use in our model better, since it corresponds to the actual departure of agents initially present as opposed to the departure of agents some of whom were not in fact initially present, but "could" have been present.

Our result is based upon several recent developments in the literature of general equilibrium theory and fair allocation. The above-mentioned papers by Thomson were two prototypes of our present model. Mas-Colell's paper (1988) on general equilibrium theory with possibly satiated preferences helped us to offer a unified and general treatment of standard exchange economies and economies with single-peaked preferences, two subjects that have been discussed separately in the literature on fair allocation. Finally, we note that Zhou's "average" Lyapunov convexity theorem and his characterization of allocations that are both efficient and strictly envy-free (Zhou (1991a, b)) are key to the proof of our result: An allocation is strictly envy-free if no agent prefers the average holding of any group of agents to his own. As we will see, this concept provides a bridge that connects consistent solutions and equal-budget Walrasian equilibria with slack.

The paper is organized as follows. Section 2 presents the basic model and the statement of our main result. Section 3 contains a formal proof of the main result. Section 4 gives a variant of the main result. Finally, Section 5 concludes the paper.

2. THE MODEL AND THE MAIN RESULT

The commodity space is R^l_+, the non-negative orthant of R^l. Let $(\hat{\Omega}, \mathcal{B}, \mu)$ be a complete atomless measure space with $\mu(\hat{\Omega}) < \infty$. The basic economy is a pair $(\hat{\Omega}, \hat{e})$, where $\hat{e}$ is a vector in R^l_{++}, the interior of R^l_+. Points in $\hat{\Omega}$ are interpreted as agents and $\hat{e}$ is interpreted as the aggregate endowment to which all agents are collectively entitled.

Each agent $\omega \in \hat{\Omega}$ is equipped with a (strict) preference relation $>_\omega$ on R^l_+ that satisfies the following assumptions.

(A1) *Continuity*: For each $y \in R^l_+$, the sets $\{ x \in R^l_+ \mid x >_\omega y \}$ and $\{ x \in R^l_+ \mid y >_\omega x \}$ are open in R^l_+.

Let $B_\omega(x)$ be the upper contour set of $>_\omega$ at x: $B_\omega(x) = \{ y \in R^l_+ \mid y >_\omega x \}$.

(A2) *Local nonsatiation at nonsatiated points*: For any $x \in R^l_+$, if $B_\omega(x) \neq \emptyset$, then $B_\omega(x) \cap O(x) \neq \emptyset$ for any open ball $O(x)$ around x.

(A3) *Weak smoothness*: For each $x \in R^l_+$ with $B_\omega(x) \neq \emptyset$, there is an open ball $O(x)$ and a differentiable function $u(.)$ with $\text{grad}(u)(x) \neq 0$ such that $B_\omega(x) \cap O(x) = \{ y \in O(y) \mid u(y) > u(x) \}$.

(A4) *Measurability*: For any integrable mapping $x: \hat{\Omega} \rightarrow R^l_+$ and any $x \in R^l_+$, the set $\{ \omega \in \hat{\Omega} \mid x >_\omega x(\omega) \}$ is measurable.

Note that we do not require that preferences be transitive, complete, or convex. (A1) is standard and needs no explanation. (A2) is adapted from Mas-Colell (1988). It is obviously satisfied by locally nonsatiated preferences. But it allows for the possibility of satiation. It will be satisfied, for example, by preferences that can be represented by utility functions whose local maxima are also global maxima. (A3) further says that the upper contour set of any nonsatiated point is locally diffeomorphic to an open half space. Note that (A2) and (A3) together imply that preferences are irreflexive. (A4) is borrowed from Aumann (1964). It is a requirement imposed on the *profile* of preferences that guarantees that the economy as a whole is well-behaved.

In our model we do not specify individualized initial holdings. Instead we assume that agents are collectively entitled to the commodities available. Our objective is to identify desirable methods, or *solutions*, of performing this division. Since one of the tests that we will use to evaluate a solution involves checking the robustness of the allocations it recommends under the departure of some of the agents, the concept of a solution need to be sufficiently general for this operation to be meaningful. This motivates the choice of domain made below.

Let $(\hat{\Omega}, \hat{e})$ be given and $\mathcal{E}$ be the class of economies (Ω, e) where $\Omega \in \mathcal{B}$ and $e \leq \hat{e}$. A *feasible allocation for* $(\Omega, e) \in \mathcal{E}$ is an integrable mapping $x\colon \Omega \to R^l_{++}$ such that $\int_\Omega x = e$. Let $F(\Omega, e)$ be the set of feasible allocations of (Ω, e), and F the collection of all $F(\Omega, e)$.

A *solution on* $\mathcal{E}$ is a correspondence $\phi\colon \mathcal{E} \to F$ that associates with each $(\Omega, e) \in \mathcal{E}$ a nonempty subset $\phi(\Omega, e)$ of $F(\Omega, e)$.

An example of a solution is the *Pareto solution* P, which associates with each (Ω, e) the set of its Pareto efficient allocations:

$$P(\Omega, e) = \{ x \in F(\Omega, e) \mid \text{there does not exist } y \in F(\Omega, e) \text{ and a measurable set } S \text{ of positive measure such that } y(\omega) >_\omega x(\omega) \text{ for all } \omega \in S \text{ and } y(\omega') = x(\omega') \text{ for all } \omega' \notin S \}.$$

Another example is the solution I that associates with each economy the set of its allocations to which almost no agent prefers equal division:

$$I(\Omega, e) = \{\, x \in F(\Omega, e) \mid \text{the set } \{\, \omega \in \Omega \mid \frac{e}{\mu(\Omega)} >_{\omega} x(\omega) \,\} \text{ has zero measure} \,\}.$$

Let PI denote the intersection of P and I, i.e. :

$$PI(\Omega, e) = P(\Omega, e) \cap I(\Omega, e) .$$

From a distributional viewpoint the solution I is very natural, and since we are of course interested in achieving efficient allocations, we will regard it as a basic requirement on a solution that it should be a subsolution of PI. However, PI is in general not discriminating enough. It usually admits a very large set of subsolutions, so the question of selection arises. We will address the question of selection by focusing on the property of consistency. As explained in the introduction, a solution is consistent if the recommendations it makes for an economy are never contradicted by the recommendations it would make for any "reduced" economy obtained by imagining some of the agents leaving the scene with their allotted bundles. Here is the formal definition.

Definition. A solution ϕ: $\mathcal{E} \to F$ is *consistent* if for all $(\Omega, e) \in \mathcal{E}$, for all $x \in \phi(\Omega, e)$, and for all $\Omega' \in \mathcal{E}$ with $\Omega' \subseteq \Omega$, $x|_{\Omega'} \in \phi(\Omega', \int_{\Omega'} x)$.

It is easy to verify that although the Pareto solution P is consistent for any economy $(\hat{\Omega}, \hat{e})$, neither I nor PI is. Hence, in order to use consistency to select subsolutions of PI, the first question is: Does PI have any consistent subsolutions? To answer this question, we now consider the notion of a Walrasian equilibrium with slack (Mas-Colell (1988)).

For an economy (Ω, e), a pair (x, p), where $x \in F(\Omega, e)$ and $p \in R^l \setminus \{0\}$, is an equal-budget Walrasian equilibrium with slack, or simply an *equal-*

budget equilibrium of (Ω, e), if there is a number $\alpha \geq p \cdot \frac{e}{\mu(\Omega)}$ such that for almost all $\omega \in \Omega$,

(E1) $p \cdot x(\omega) \leq \alpha$;

(E2) $x >_{\omega} x(\omega)$ implies $p \cdot x > \alpha$.

The notion of an equal-budget equilibrium is a generalization of the notion of an equal-income Walrasian equilibrium that allows us dealing with the possibility of satiation. At an equal-budget equilibrium, agents face a common budget set and all choose a maximal consumption according to their preferences. Since preferences may be satiated, some agents may choose points in the interior of the budget set. Therefore, there is possibly a slack (indeed a surplus) between the common budget and the value of the average endowment. But if preferences are all locally nonsatiated, then the slack disappears and an equal-budget equilibrium reduces to an equal-income Walrasian equilibrium.

Mas-Colell (1988) proved the existence of equal-budget equilibria under otherwise standard assumptions. Although his proof was given in the case of an economy with a finite number of agents, there is no doubt that a similar existence result can be obtained for an atomless economy by adapting his proof in such a context. Hence we will not pursue here the issue of existence of equal-budget equilibria.

We now define the *equal-budget equilibrium solution* E_{eb}: $\mathcal{E} \rightarrow F$ as the solution that associates with each $(\Omega, e) \in \mathcal{E}$ the set of its equal-budget equilibrium allocations:

$$E_{eb}(\Omega, e) = \{ x \in F(\Omega, e) \mid \text{there is } p \in R^l \text{ such that } (x, p) \text{ is an equal-budget equilibrium of } (\Omega, e) \}.$$

It is straightforward to verify that the equal-budget equilibrium solution E_{eb} is a consistent subsolution of *PI*. This shows that we can indeed use consistency to select from *PI*.

The next natural question is: how restrictive is consistency, or in other words, what are the consistent subsolutions of *PI*? Our main theorem below provides an answer to this question. It asserts that the equal-budget equilibrium solution E_{eb} is in fact the largest subsolution of *PI* that is consistent.

Theorem 1. If a subsolution of *PI* is consistent, then it is a subsolution of the equal-budget equilibrium solution.

Before we turn to the next section which contains a formal proof of Theorem 1, we specialize Theorem 1 to two widely investigated models.

The first one is the standard exchange model in which assumption (A2) is replaced by a stronger one (A2'):

(A2') *Monotonicity*: All preferences are monotonically increasing, i.e., $x >_{\omega} y$ for any $x, y \in R^l_+$ with $x \geq y$, and $x \neq y$.

As explained above, in this case an equal-budget equilibrium is an equal-income Walrasian equilibrium. Thus Theorem 1 leads to the following conclusion.

Corollary 1. For a standard exchange economy, if a subsolution of *PI* is consistent, then it is a subsolution of the equal-income Walrasian solution.

In the second model there is only one commodity and preferences can be given single-peaked numerical representations. The finite version of this model was recently considered by Sprumont (1991) and Thomson (1990b). It is obviously a special case of our general model. But what is interesting about this model is that the equal-budget equilibrium solution has a very simple form: it is known as the "uniform rule" in the literature on fix-price equilibrium (see Benassy (1982)).

Let us then assume $l = 1$, that each agent ω has a unique most preferred consumption $b(\omega)$, and that $a' >_\omega a$ for all $a, a' \in R_+$ with $a < a' \leq b(\omega)$ or $b(\omega) \leq a' < a$. Let us also assume that the function $b: \Omega \to R_+$, which associates each ω with his most preferred consumption $b(\omega)$, is measurable.

The *uniform rule*, denoted by U, is defined as follows:

(i) If $\int_\Omega b \leq e$,

$$U(\Omega, e) = \{ x \in F(\Omega, e) \mid \text{there exists } \beta \in R_+ \text{ such that for almost all } \omega \in \Omega,\ x(\omega) = \max(b(\omega), \beta) \};$$

(ii) If $\int_\Omega b > e$,

$$U(\Omega, e) = \{ x \in F(\Omega, e) \mid \text{there exists } \gamma \in R_+ \text{ such that for almost all } \omega \in \Omega,\ x(\omega) = \min(b(\omega), \gamma) \}.$$

It is easy to verify that the uniform rule is exactly the equal-budget equilibrium solution. Case (i) above corresponds to an equal-budget equilibrium allocation with negative or zero price and case (ii) corresponds to an equal-budget equilibrium allocation with positive price. Also notice that for each $(\Omega, e) \in \mathcal{E}$, there is a unique allocation x that satisfies the definition of $U(\Omega, e)$. Therefore, the uniform rule is a well-defined single-valued solution. Applying Theorem 1, we have the following result.

Corollary 2. For a one-commodity economy in which preferences are single-peaked, the uniform rule is the only subsolution of PI to be consistent.

3. A PROOF OF THEOREM 1

Our proof of Theorem 1 will involve the auxiliary distributional concept of a strictly envy-free allocation.(Zhou (1991a)), an allocation such that the set of agents who prefer the average holding of any group of agents to their own has measure zero. In the proof, we simply refer to measurable sets as "sets." We say that agent $\omega \in \Omega$ *envies a set of agents* T (with $\mu(T) > 0$) *at an allocation* x if $\bar{x}(T) >_{\omega} x(\omega)$, in which $\bar{x}(T) = \dfrac{\int_T x}{\mu(T)}$.

Definition. A feasible allocation x is *strictly envy-free for* (Ω, e) if the set of agents who envy other sets of agents has measure zero.

We wil prove two preliminary and independent propositions. First, any consistent subsolution of I must select allocations that are strictly envy-free. Second, any allocation that is both efficient and strictly envy-free must be an equal-budget equilibrium allocation. Together, these two propositions provide a proof of Theorem 1.

Proposition 1. Assume that $\phi : \mathcal{E} \to F$ is a consistent subsolution of I. Then any allocation $x \in \phi(\Omega, e)$ must be strictly envy-free.

Proof: Let us suppose, on the contrary, that x is not strictly envy-free. We show that this supposition leads to a contradiction.

Step 1. We first show that if x is not strictly envy-free, then there are two disjoint sets S' and T' with positive measures such that all agents in S' envy T'.[3] To see this, consider the set

$$B = \{ z \in R^l_+ \mid z = \bar{x}(T) \text{ for some } T \text{ with } \mu(T) > 0 \}.$$

[3] This was established in Zhou (1991). We include the proof here for self-containedness.

Since B is separable, there are countably many sets $\{T_n\}_{n \in N}$ such that $\{\bar{x}(T_n)\}_{n \in N}$ are dense in B. Because preferences are continuous, an agent who envies any group T at all will envy some T_n. Given that the set of envious agents has a positive measure, there is a set T in $\{T_n\}$ such that S, the set of agents who envy T, has a positive measure. We now use the Lyapunov convexity theorem to find two disjoint sets $S' \subseteq S$ and $T' \subseteq T$ with positive measures such that all agents in S' envy T'. Consider the mapping x on $S \cap T$. By the Lyapunov theorem, there is a set $T_1 \subseteq S \cap T$ with $\mu(T_1) = \frac{1}{2}\mu(S \cap T)$ and $\bar{x}(T_1) = \bar{x}(S \cap T)$. Similarly, there is a set $T_2 \subseteq T \setminus S$ with $\mu(T_2) = \frac{1}{2}\mu(T \setminus S)$ and $\bar{x}(T_2) = \bar{x}(T \setminus S)$. Thus, $S' = (S \cap T) \setminus T_1$ and $T' = T_1 \cup T_2$ are the two desired sets.

Step 2. We show that there exist a set $S'' \subseteq S'$ with positive measure, and a number $\varepsilon > 0$ such that $y >_\omega x(\omega)$ for every bundle y satisfying $\| y - \bar{x}(T') \| < \varepsilon$ and every agent $\omega \in S''$. To see this, define $S'_n \subseteq S'$ by

$$S'_n = \{ \omega \in S' \mid y >_\omega x(\omega) \text{ for all } y \text{ satisfying } \| y - \bar{x}(T') \| < \frac{1}{n} \}.$$

By the continuity assumption (A1), we can write S'_n as

$$S'_n = \bigcap_{y_i} \{ \omega \in S' \mid y_i >_\omega x(\omega) \},$$

in which $\{y_i\}$ is a countable dense subset of the set $\{ y \mid \| y - \bar{x}(T') \| < \frac{1}{n} \}$. By the measurability assumption (A4), each S'_n is measurable. Moreover, by (A1), $S' = \bigcup_{n \in N} S'_n$. Since $\mu(S') > 0$, there must be some $S'_m \subseteq S'$ with $\mu(S'_m) > 0$. Take $S'' = S'_m$ and $\varepsilon = \frac{1}{m}$.

Step 3. We now choose M large enough so that $\mu(S'') \geq \frac{1}{M}\mu(T')$ and $\| \bar{x}(S'') - \bar{x}(T') \| < (M + 1)\varepsilon$. Again, by the Lyapunov theorem, there is a set

$S_M \subseteq S''$ with $\mu(S_M) = \frac{1}{M}\mu(T')$ and $\bar{x}(S_M) = \bar{x}(S'')$.

Step 4. Now let $\Omega' = S_M \cup T''$ and consider the economy $(\Omega', \int_{\Omega'} x)$. Since

$$\bar{x}(\Omega') = \frac{\mu(S_M)}{\mu(T') + \mu(S_M)}\,\bar{x}(S_M) + \frac{\mu(T')}{\mu(T') + \mu(S_M)}\,\bar{x}(T')$$
$$= \frac{1}{M+1}\,\bar{x}(S'') + \frac{M}{M+1}\,\bar{x}(T')\,,$$

we have

$$\| \bar{x}(\Omega') - \bar{x}(T') \| = \frac{1}{M+1}\, \| \bar{x}(S'') - \bar{x}(T') \| < \varepsilon .$$

By Step 3, $\bar{x}(\Omega') >_\omega x(\omega)$ for all agents in S_M. So $x|_{\Omega'} \notin \phi(\Omega', \int_{\Omega'} x)$ because ϕ is assumed to be a subsolution of I. But this contradicts the assumption that ϕ is consistent. **Q.E.D.**

We now state two lemmas that are needed in the proof of the next proposition. The first one is a version of the second welfare theorem for an atomless economy.

Lemma 1. If an economy (Ω, e) satisfies (A1) and (A3), then for any efficient allocation x, there is a vector $p \in R^l \backslash \{0\}$ (with possibly zero, or negative coordinates) such that for almost all $\omega \in \Omega$,

$$\inf_{x >_\omega x(\omega)} p \cdot x \geq p \cdot x(\omega) .$$

(The expression on the left hand side is understood to be positive infinity when $B_\omega(x(\omega)) = \varnothing$.) One can prove this lemma by following the standard proof of the Aumann core equivalence result (see Hildenbrand (1982)). We thus leave it to the readers. The second lemma is an "average" Lyapunov convexity result.

Lemma 2. Let $f: \Omega \to R^l$ be an integrable mapping on a complete atomless measure space $(\Omega, \mathcal{B}, \mu)$. Construct the set A of average integrals of f over all sets of positive measure:

$$A = \{ x \in R^l \mid x = \frac{\int_T f}{\mu(S)} \text{ for some } S \in \mathcal{B} \text{ with } \mu(S) > 0 \}.$$

Then (i) A is convex; and (ii) $f(\omega) \subseteq \text{cl}(A)$ for almost all $\omega \in \Omega$, in which $\text{cl}(A)$ is the closure of A.

For a proof of Lemma 2, see Zhou (1991a) (or Zhou (1991b) for a more general result). We now turn to Proposition 2.

Proposition 2. If an allocation $x \in F(\Omega, e)$ is both efficient and strictly envy-free, then it must be an equal-budget equilibrium allocation of (Ω, e).[4]

Proof. Since x is efficient, according to Lemma 1, we can find a set Ω_1 with $\mu(\Omega_1) = \mu(\Omega)$ such that there is a price vector $p \in R^l \setminus \{0\}$ for which

$$inf_{\, x >_\omega x(\omega)} \; p \cdot x \geq p \cdot x(\omega) \quad \text{for all } \omega \in \Omega_1 .$$

Also since x is strictly envy-free, we have another set Ω_2 with $\mu(\Omega_2) = \mu(\Omega)$ such that no agent in Ω_2 envies any other group. Now let $\Omega' = \Omega_1 \cap \Omega_2$. This set still has full measure. Given that $\mu(\Omega') = \mu(\Omega)$, in order to show that (x, p) is an equal-budget equilibrium for (Ω, e), it suffices to find a number α such that for almost all $\omega \in \Omega'$,

(E1) $p \cdot x(\omega) \leq \alpha$;

(E2) $x >_\omega x(\omega)$ implies $p \cdot x > \alpha$.

---

4 This is a generalization of Proposition 3.4 in Zhou (1991a), which dealt with standard exchange economies only.

Hence, in the following proof we will work with Ω'. For each number β, let S_β denote the set of agents whose consumption bundles at x are worth more than β when evaluated by p :

$$S_\beta = \{\ \omega \in \Omega' \mid p{\cdot}x(\omega) > \beta\ \}.$$

There are two possible cases.

Case 1. $\mu(S_\beta) = 0$ for all $\beta > p\cdot\frac{e}{\mu(\Omega')}$. In this case, since $\int_{\Omega'} x = e$, we have $p{\cdot}x(\omega) = p\cdot\frac{e}{\mu(\Omega')}$ for almost all $\omega \in \Omega'$. We now choose $\alpha = p\cdot\frac{e}{\mu(\Omega')}$. (E1) is thus satisfied. It is routine to verify that (E2) is satisfied. (First, $x >_\omega x(\omega)$ implies $p{\cdot}x \geq \alpha$ by Lemma 1. Second, $x >_\omega x(\omega)$ actually implies $p{\cdot}x > \alpha$ because preferences are continuous and e is in the interior of R^l_+.)

Case 2. The more interesting case is when $\mu(S_\beta) > 0$ for some $\beta > p\cdot\frac{e}{\mu(\Omega)}$. The following observation is crucial:

(*) If $\mu(S_\beta) > 0$ for some $\beta > p\cdot\frac{e}{\mu(\Omega)}$, then $B_\omega(x(\omega)) = \varnothing$ for almost all agents $\omega \in T_\beta$, where $T_\beta = \{\ \omega \in \Omega' \mid p{\cdot}x(\omega) < \beta\ \}$.

We now prove (*). First notice that $\mu(T_\beta) > 0$ since $\beta > p\cdot\frac{e}{\mu(\Omega)}$ and $\int_{\Omega'} x = e$. Hence, we can construct the set of average integrals of x on T_β:

$$A_x = \{\ x \in R^l \mid x = \bar{x}(T) \text{ for some } T \subseteq T_\beta\ \}.$$

Applying (ii) of Lemma 2 to A_x leads to $x(\omega) \in \text{cl}(A_x)$ for almost all $\omega \in T_\beta$. We show that $B_\omega(x(\omega)) = \varnothing$ for all those agents. Suppose, on the contrary, $B_\omega(x(\omega)) \neq \varnothing$ for one such ω. By assumptions (A2) and (A3), there is an open ball $O(x(\omega))$ and a differentiable function $u(.)$ on $O(x(\omega))$ such that

$B_\omega(x(\omega)) \cap O(x(\omega)) = \{ x \in O(x(\omega)) \mid u(x) > u(x(\omega)) \}$. Since $\inf_{x >_\omega x(\omega)} p \cdot x \geq p \cdot x(\omega)$ and $p \cdot x(\omega) < \beta < p \cdot \bar{x}(S_\beta)$, there is $y \in O(x(\omega)) \cap (x(\omega), \bar{x}(S_\beta))$ such that $y \in B_\omega(x(\omega))$, where $(x(\omega), \bar{x}(S_\beta))$ is the open segment connecting $x(\omega)$ and $\bar{x}(S_\beta)$. Now we consider the set of average integrals of x on Ω':

$$B_x = \{ x \in R^l \mid x = \bar{x}(T) \text{ for some } T \subseteq \Omega' \} .$$

By (i) of Lemma 2, B_x is convex. So is $\mathrm{cl}(B_x)$. Since we have $\bar{x}(S_\beta) \in B_x$, and $x(\omega) \in \mathrm{cl}(A_x) \subseteq \mathrm{cl}(B_x)$ ($A_x \subseteq B_x$ by definition), $(x(\omega), \bar{x}(S_\beta)) \subseteq \mathrm{cl}(B_x)$. This implies $y \in \mathrm{cl}(B_x)$. Hence, $y \in B_\omega(x(\omega)) \cap \mathrm{cl}(B_x)$. But on the other hand $B_\omega(x(\omega)) \cap \mathrm{cl}(B_x)$ should be empty. The condition that x is strictly envy-free implies $B_\omega(x(\omega)) \cap B_x = \emptyset$. It further leads to $B_\omega(x(\omega)) \cap \mathrm{cl}(B_x) = \emptyset$ because, by (A1), $B_\omega(x(\omega))$ is open. This contradiction proves (*).

We now can find the number α needed in (E1) and (E2). Consider the function $f : R \to R_+$ defined by $f(\beta) = \mu(S_\beta)$. It is easy to verify that f is nonincreasing and continuous from the right. Thus there exists a smallest number (possibly positive infinity) at which f vanishes. We denote it by α. It follows from (*) that (x, p) is an equal-budget equilibrium with α. **Q.E.D.**

4. A VARIANT OF THE MAIN RESULT

The technique developed here can also be used to prove some variants of Theorem 1. For example, let us consider a property called *everywhere replicability*. To define it rigorously, we need a more general notion of a solution, the domain $\mathcal{E}'$ of which will include all replicas of the basic economy and their subeconomies.

Definition. A solution Ψ is *everywhere replicable* if for any $(\Omega, e) \in \mathcal{E}'$, any group of agents T, and any $x \in \Psi(\Omega; e)$, the allocation $z \in F(\Omega \cup \Omega', e + \int_T x)$ defined by $z|_\Omega = x$ and $z|_{\Omega'} = x|_T$ (where Ω' is a single copy of T) satisfies $z \in \Psi(\Omega \cup \Omega', e + \int_T x)$.

We now show that everywhere replicability can replace consistency in Theorem 1 to reach a similar characterization.

Theorem 2. If a subsolution of *PI* satisfies everywhere replicability, then it is a subsolution of the equal-budget equilibrium solution.

Proof. The proof is similar to that of Theorem 1. Let Ψ be a solution that satisfies the assumptions of the theorem, $(\Omega, e) \in \mathcal{E}'$, and $x \in \Psi(\Omega, e)$. We want to show that x is strictly envy-free for (Ω, e). Then Step 2 in the proof of Theorem 1 also completes the proof here.

If x is not strictly envy-free, then, as shown in the proof of Theorem 1, there are two disjoint sets S and T with positive measures and a positive number ε such that all agents in S prefer any y satisfying $\| y - \bar{x}(T) \| < \varepsilon$. Choose a positive integer k so that $\| \frac{e}{\mu(\Omega)} - \bar{x}(T) \| < (k \frac{\mu(T)}{\mu(\Omega)} + 1)\varepsilon$.

We now consider the economy $(\Omega \cup \Omega', e + k \int_T x)$, in which Ω' is a k-th copy of T, and an allocation z with $z|_\Omega = x$ and $z|_{\Omega'}$ a k-th copy of $x|_T$. It is easy to verify that if Ψ is generally replicable then $z \in \Psi(\Omega \cup \Omega', e + k \int_T x)$. Since

$$\bar{z}(\Omega \cup \Omega') = \frac{\mu(\Omega)}{\mu(\Omega) + k\mu(T)} \frac{e}{\mu(\Omega)} + \frac{k\mu(T)}{\mu(\Omega) + k\mu(T)} \bar{x}(T),$$

we have

$$\| \bar{z}(\Omega \cup \Omega') - \bar{x}(T) \| = \frac{\mu(\Omega)}{\mu(\Omega) + k\mu(T)} \| \frac{e}{\mu(\Omega)} - \bar{x}(T) \| < \varepsilon .$$

Hence each agent in S prefers the average bundle $\bar{z}(\Omega \cup \Omega')$ to this own. This contradicts the assumption that Ψ is a subsolution of PI. **Q.E.D.**

5. CONCLUSION

In the last few years the principle of consistency has played an important role in the axiomatic analyses of a variety of models including bargaining problems (Lensberg (1987, 1988)), games in coalitional form (Sobolev (1975), Peleg (1985, 1986)), fair allocation in finite-agent exchange economies with infinitely divisible goods (Thomson (1988, 1991b)) and economies with indivisible goods (Tadenuma and Thomson (1991)), taxation problems (Young (1987)), etc (for a survey of this literature, see Thomson (1990a)). It has wide intuitive appeal and yet sufficient power often to be the central axiom in characterizations. In this paper we have shown that it can provide the basis for a characterization of the equal-budget equilibrium solution in economies with a complete atomless measure space of agents, when used in conjunction with the requirement of efficiency and the standard distributional requirement of Pareto domination of equal division.

By using the consistency principle, we have achieved a characterization of equal-budget equilibria for an economy with possibly satiated preferences. In such an economy, the set of equal-budget equilibria is usually much smaller than the core from equal division of the endowment. One can convince oneself by considering the simplest case of a one-commodity economy in which the equal-budget equilibrium allocation is unique — the uniform rule — while the core allocations from equal division of endowment are abundant. This is another manifest of the power of the consistency principle.

We close with some additional comments on the literature. Winter and Wooders (1990) considered a class of abstract games with a continuum of players and used the consistency property to characterize a concept of core along the lines of Peleg's argument (1985, 1986). Dubey and Neyman (1984) considered an atomless exchange economy with transferable utilities, but consistency played no role in their characterization of Walrasian equilibria.

REFERENCES

Aumann, R. (1964): "Markets with a continuum of traders," *Econometrica* 32, 39-50.

Aumann, R., and J. Drèze (1986): "Values of markets with satiation or fixed prices," *Econometrica* 54, 1271-1318.

Benassy, J-P. (1982): *The economics of market disequilibrium.* New York: Academic Press.

Dubey, P., and A. Neyman (1984): "Payoffs in nonatomic economies: an axiomatic approach," *Econometrica* 52, 1129-1151.

Hildenbrand, W. (1982): Core of an economy, in *Handbook of Mathematical Economics* . Amsterdam: North-Holland.

Lensberg, T. (1987): "Stability and collective rationality," *Econometrica* 55, 935-961.

________ (1988): "Stability and the Nash solution," *Journal of Economic Theory* 45, 330-341.

Mas-Colell, A. (1988): "Equilibrium theory with possibly satiated preferences," Harvard University, mimeo.

Peleg, B. (1985): "An axiomatization of the core of cooperative games without side-payments," *Journal of Mathematical Economics*14, 203-214.

______ (1986): "On the reduced game property and its converse," *International Journal of Game Theory* 15, 187-200.

Sobolev, A. I. (1975): "The characterization of optimality principles in cooperative games by functional equations," *Mathematical Methods in Social Sciences* 6, 150-165. (In Russian)

Sprumont, Y. (1991): "The division problem with single-peaked preferences: A characterization of the uniform allocation rule," *Econometrica* 59, 509-519.

Tadenuma, K., and W. Thomson (1991): "No-envy and consistency in economies with indivisible goods," forthcoming *Econometrica.*

Thomson, W. (1988): "A study of choice correspondences in economies with a variable number of agents," *Journal of Economic Theory* 46, 237-254.

__________ (1990a): "The consistency Principle," in *Game Theory and Applications* (T. Ichiishi, A. Neyman, and Y. Tauman, eds.). New York: Academic Press, pp 187-215.

__________ (1990b): "Consistent solutions to the problem of fair division when preferences are single-peaked," University of Rochester, mimeo.

Winter, E., and M. Wooders (1990): "An axiomatization of the core for finite and continuum games," Discussion Paper B149, University of Bonn.

Young, P. (1987): "On dividing an amount according to individual claims or liabilities," *Mathematics of Operations Research* 12, 398-414.

Zhou, L. (1991a): "Strictly fair allocations in large exchange economies," CFDP No. 972, Yale University.

________ (1991b): "An 'average' Lyapunov convexity theorem and some core equivalence results," CFDP No. 976, Yale University.