# A Foundation of Location Theory: Existence of Equilibrium, the Welfare Theorems and Core

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Working Paper No. 298
October 1991

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# A Foundation of Location Theory: Existence of Equilibrium, the Welfare Theorems and Core[±]

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ABSTRACT

A model of an economy with land and a finite number of traders is examined. Land is modeled as a sigma algebra of subsets of a Euclidean space. Since this commodity space has no natural convex or linear structure, standard existence results cannot be applied. The contribution of this paper is the introduction of a "convexity" assumption on preferences for such a situation. Preferences are are assumed to be "convex" in the sense that there are linear functionals bounding the preference for location and shape. When combined with continuity and monotonicity assumptions, an equilibrium is shown to exist, the core is shown to be nonempty, and supporting prices can be found for Pareto optima. A first welfare theorem is also proved.

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±The authors thank Thijs ten Raa for the first half of the title and for helpful comments, John H. Boyd III for help with a counterexample, and Steven Landsburg for comments on the exposition. Support provided by National Science Foundation Grants #SES–8420247 and #SES–8605629 and by NATO Research Grant 85/0775 is gratefully acknowledged. The authors retain full responsibility for any remaining errors and especially for the views expressed herein.

## I. Introduction

The general equilibrium approach to modeling the location of economic activities and the uses of resources distributed over a spatial dimension has largely been ignored by economic theorists over much of the history of economics. There might be several reasons for this. The problems could be uninteresting, too hard, or thought to be subsumed by more general abstract models. Indeed, examples used to justify various abstract models often involve a spatial dimension. Isard (1956, pp. 24–27) proposes that the reason for this lack of interest is that the Anglo–Saxon influence on the discipline biases economists toward an interest in the temporal rather than the spatial dimensions of economics.

Independent of the reasons for a lack of interest in the area, the result is that many fundamental questions in the field remain open. The following is a selection of such questions. How should land be modeled in a general equilibrium framework? When does equilibrium exist in a spatial model? When do the welfare theorems hold? What happens as the number of agents in the economy becomes large? Are the answers to these questions as well as comparative statics model–dependent? Analogous questions in other subfields of economics were answered long ago. Answers to these questions clearly affect the validity of applications of these models, be it comparative statics or empirical work. For example, it makes little sense to look at the comparative statics of a model if an equilibrium does not generally exist. The ultimate goal of our research is to make the basis of work in urban economics as solid as in other subfields. This does not necessarily mean that the models used in practice will be justified or rejected, but rather that the assumptions underlying any particular variety of model will be exposed. This will aid in the empirical or comparative static applications of models, especially if the implications of the various models differ, as well as in making policy prescriptions and predictions. The use of an inappropriate or even flawed model can invalidate results. Moreover, scientific objectivity leads us to examine the assumptions underlying models, independent of the investment that might already have been made in them.

The purpose of this paper is to examine one of these questions, the one pertaining to existence of equilibrium, for one model. This model is important because it seems to have the classical properties of economic models, unlike the standard model used in the New Urban Economics. In order to motivate our results and connect this work to the rest of the literature, we shall first survey the related work in urban economics as well as touch on related work in general equilibrium theory.

The standard model used by New Urban Economists, called the monocentric city model, has a continuum of consumers each of whom locates in one of a continuum of locations. A mobile good as well as a density of land can be consumed at any given distance from the city center. Prices are densities (unit costs of land) over locations. We refer to Beckmann (1969) and Wheaton (1979) for complete descriptions of the model. Mathematical economists will notice two aspects of this model immediately. First, the model has a double infinity (agents and goods), so that classical results might not be expected. Second, densities are defined with respect to location (distance from the city center), not with respect to agents. Thus mean demand and supply might not be defined on a per capita or per agent basis.

Certainly one of the most prominent features of urban models in general and the monocentric city model in particular is the use of a continuum of consumers with land. Unfortunately, little has been done to examine the consistency of models employing this framework or to look at the economies with a finite number of consumers that approximate continuum models. In essence, the justification for the use of models with an infinite number of consumers is that finite models correspond roughly to reality, continuum economies approximate reasonable large (but finite) economies and that the mathematics of the continuum are simpler. In particular, in order to justify the use of a continuum model, it is necessary that the equilibria and comparative statics of a continuum model approximate those of some reasonable finite model. It is commonly thought that the relationship between continuum and finite models with land is exactly the same as the relationship between continuum and finite models without land, the latter relationship having been examined by Hildenbrand (1974), for instance. Thus, it is assumed that the proofs of consistency of models with a continuum of consumers and land and the justifications for models with a continuum of consumers and land are analogous to those for models with a continuum of consumers without land. However, it is possible to show that the land densities cannot be interpreted as actual areas of land because there are not enough disjoint subsets in the plane to give positive area to each of a continuum of consumers. It is in this *precise* sense that there is a logical problem with such models. As a consequence, the analogy to the standard large economies literature breaks down and it can be demonstrated that continuum models with land (and in particular, monocentric city models) are approximations, in the standard sense used by Hildenbrand, only to large finite economies in which land endowments and consumption vanish almost surely. Proofs of these results and a more thorough discussion can be found in Berliant (1985a). An alternative interpretation is that the continuum of agents represents fractions of individual consumers rather than individuals themselves. This interpretation has severe limitations as well; see Berliant and ten Raa (1991). Not only does it involve an aggregation problem, but if transportation cost is involved, then one must know the equilibrium distribution of agents in the continuum economy before the preferences and endowments of the finite economy can be designed. Papageorgiou and Pines (1990) claim to have shown that the equilibria of the finite and continuum models are the same under this interpretation. However, their results are limited to a single type of agent, no transportation cost, and presume existence of equilibrium in *both* the finite and continuum models. Counterexamples when each of these assumptions is relaxed can be found in Berliant and ten Raa (1991). Precise and detailed criticisms of much of the literature that attempts to justify the standard model of urban economics by using finite approximations can be found in Berliant (forthcoming). As this is a scientific endeavor, the burden of proof is on those who wish to use continuum models.

A belief that the results of the monocentric city model hold for related finite models is clearly insufficient. Consider a model with an equal number of (identical) agents and indivisible (homogeneous) land parcels. Land consumption is clearly uniform in this model, unlike the monocentric city model. How small do parcels have to be before the results are close to those of a continuum model? If land consumption tends to zero, why isn't the consumption of other goods permitted to tend to zero as well? There are also problems with existence and welfare in such finite models (with transportation cost); see Koopmans and Beckmann (1957).

Until recently, little attention has been paid to the existence of equilibrium or the welfare theorems in the context of the model proposed in the New Urban Economics; the assumption seemed to be that the appropriate theorems hold and that their proofs are a technicality (see, for example, Arnott (1986)). Fujita and Smith (1987) have conditions on both utility and demand that yield existence of equilibrium in the context of a type economy and location–independent utility functions. Consider a monocentric city model with location–independent, Cobb–Douglas utility functions and a linear transportation cost. It seems to be analytically impossible to derive an explicit contract curve. However, it seems likely that some points on this contract curve do not have price support. The difference with the Fujita–Smith model is that this might not be an economy with a finite number of types or utility levels. Allowing location–dependent utility, there are two varieties of examples that yield no equilibrium even with identical consumers and the standard assumptions of general equilibrium theory. These examples can be found in Berliant and ten Raa (1991, example 3) and Berliant, Papageorgiou, and Wang (1990, example 1). Furthermore, the welfare theorems can fail in this model when utilities are location–dependent. Examples of this phenomenon can be found in Berliant, Papageorgiou, and Wang (1990).

Perhaps the model in this literature that best captures at least the spirit of the development below is that of Alonso (1964). This is a monocentric city model in which there is a *finite* number of consumers. An equilibrium is not shown to exist in this model, although some loosely defined algorithms are used to illustrate how one might go about finding an equilibrium. Moreover, equilibrium prices are not additive (at least in chapter 5), so there are arbitrage opportunities in equilibrium. As a consequence, the equilibrium concept is not a very good one, and equilibrium allocations, if they exist, are not necessarily Pareto optimal. An attempt is made to correct this problem in an appendix of the book, but other problems arise there. Overall, neither the shape nor the location of land parcels is treated well in this model. In any case, Berliant and Fujita (forthcoming) have an existence theorem for this model when prices are additive, all traders are identical, and preferences are independent of both location and shape. In fact, there is a continuum of equilibria in this context.

We turn next to a consideration of general equilibrium theory and its relation to spatial economics. Land has several properties associated with it that are relatively unique as compared to other commodities. Most models attempt to assume these properties away in order to be able to treat land in the same manner as other commodities. This, of course, facilitates the proofs of various propositions common to these models, but it also destroys conclusions that may be drawn concerning the special role that land plays in the real world.

The features that distinguish locational economic models from other economic models can be summed up in two words: indivisibilities and nonconvexities. They arise naturally in these models because the choice of location is always a discrete choice in some sense, and because agents will generally prefer concentrated to diversified bundles of locational commodities. These features are discussed in detail in section III below, where they are used to motivate our assumptions. Problems in demonstrating that an equilibrium exists in a model with these properties are well–known. For example, Shapley and Scarf (1974) present an example with indivisible housing goods in which the core is empty. Examples with nonconvex preferences and no equilibrium are classical.

As Debreu (1959) made clear, the attributes that can be specified in the definition of a commodity include location. Thus, in a sense, land or location can be put into a classical general equilibrium model. Problems arise only when examining the assumptions that are needed to demonstrate the existence of an equilibrium. In particular, Schweizer, Varaiya, and Hartwick (1976) point out that the assumption of convexity of preferences makes little sense, as it implies that consumers would desire to own land that is spread out rather than concentrated. There are also nonconvexities in production and consumption sets, as detailed by Koopmans and Beckmann (1957). Finally, if land is infinitely divisible, then there is an infinity of commodities. Thus, the classical general equilibrium framework is inadequate for handling land. Modern variants of the classical framework and their suitability for use in a model of land are discussed in Berliant (1985a). The important work on models with infinitely many commodities, such as Bewley (1972), Jones (1984), Mas–Colell (1986), Zame (1987), and Aliprantis, Brown, and Burkinshaw (1987), is not very useful for our purposes due to the general orientation of the assumptions made to obtain existence. All of these articles use a linear commodity space (or a convex subset thereof) in conjunction with convexity of preferences and assumptions that bound marginal rates of substitution, such as uniform properness. The nonconvex and indivisible nature of locational commodities renders such models inappropriate for our context. However, these models remain quite useful for modelling other kinds of commodities (such as commodities with temporal attributes).

For the reasons listed above as well as in Berliant (1985a), attempts to push location models into standard general equilibrium frameworks have failed. Assumptions that one might expect to be satisfied in aspatial models, such as convexity of preferences, are not expected to be satisfied in spatial models. Indeed, attempts to integrate spatial and temporal economics, such as Faden (1977), require identical assumptions on spatial and temporal commodities, and their use tends to be unnatural in one dimension or another.

The purpose of this paper is to analyze the question of existence of an equilibrium when a finite number of consumers[1] have a commodity space that is a natural representation of land, a collection of subsets of the plane. We shall explicitly address the complications that are due to the size of the commodity space and the indivisibility of the elements of the commodity space. The usual assumptions concerning the convexity of preferences will not be used, since they would generally imply that agents prefer to own land that is spread out as opposed to coherent, and this does not make sense from an intuitive viewpoint.

The model has a finite number of consumers, each of whom can own a positive area of land rather than a density. Thus, the natural consumption set to consider is a sigma algebra. Hints of such a modelling technique had appeared in the urban economics literature (see Alonso (1964) or Beguin and Thisse (1979)). Such ideas have also been used in the mathematics literature in the context of fair allocations; see Dubins and Spanier (1961) or Hill (1983). (However, the fairness concept used in this literature is not the same as the concept used in economics.) Further detail on the

[1]Of course, a finite number of consumers is used because the problems outlined above arise with a continuum of consumers. If a continuum could be employed, convexification of the economy might be possible using Lyapunov's thoerem. As remarked above, even large, finite economies must have land consumption tending to zero, so although Shapley–Folkman arguments might yield approximate convexification of such economies and existence of approximate equilibrium, these economies will have land consumption close to zero.

history of this model as well as motivation for it can be found in Berliant (1985a).

An important alternative interpretation of the model is that the commodities consist of a large number of indivisible goods. With some restrictions on preferences and a large number of goods (i.e. the model used below), the difficulties of the Shapley–Scarf housing example, in which consumers receive utility from owning several commodities, disappear. Such an interpretation could have ramifications for recent developments in the theory of uncertainty, as in Kreps (1988), where agents might have preferences over and buy sets of the options available to them. For a potential application to labor economics, we refer to Heckman and Scheinkman (1987).

In previous work with this model, one assumption that has been made is that the utility for land can be expressed by the integral or aggregation of a given marginal utility density. Necessary and sufficient conditions for preferences to have such a utility representation are given in Berliant (1982). Implicit in such a representation is that parcels of land are not complements, since the utility from the union of two disjoint parcels is equal to the sum of the utilities of the two parcels. Thus, the closeness or coherence of parcels cannot matter. Although such an assumption is quite strong, it is useful for developing techniques to deal with land. It was used to give a characterization of demand in Berliant (1984) and to demonstrate the existence of an equilibrium in Berliant (1985b) for an exchange economy and in Berliant and Jeng (1990) for an economy with production.

Once the basic techniques of proof were found, it was possible to discard the assumption of linear utilities. Berliant (1986) gives a utility or set–function representation theorem without the linearity assumption, while Berliant and ten Raa (1988) show that demand is non–empty provided that preferences or utilities are continuous with respect to a certain topology. This topology can be given by a pseudometric which accounts for both the Hausdorff metric on complements of the interiors of sets, along with the integral of some marginal utility density over sets.

Examples given in that paper show how the coherence of sets can make a large difference in utility when preferences are continuous with respect to this topology. Berliant, Dunz, and Thomson (forthcoming) examine fairness concepts in the context of this model. Dunz (1991) examines the core with non–linear utilities and proves that it is nonempty for preferences which are a special case of those satisfying the key convexity assumption given below. Berliant and Dunz (1990) study nonlinear pricing (called the "land assembly problem" in the urban economics literature) in this model and more generally.

The intent of this paper is to extend the results using non–linear utilities to the point of establishing existence of an equilibrium. There are several complications, relative to classical existence theorems, that must be addressed when using as the consumption set a topological space that is not necessarily a subset of a linear space. First, we note that the basic technique used in Berliant (1985b) cannot be used here. When utilities are linear, we can embed the space of measurable sets in a larger, linear space such as $L^\infty$ (by using their indicator functions), extending utilities in an obvious manner. Employing an existence theorem for this larger space, an extreme point of the set of equilibrium allocations can be shown to be a vector of indicator functions. When utilities are not linear, it is possible that there is no extension of utilities to a larger linear space that retains the properties needed for existence of an equilibrium. Of more importance, linearity is virtually necessary in order to show that an equilibrium exists (where quasi–concavity is used) and that an extreme point of the set of equilibrium allocations is a vector of indicator functions (where quasi–convexity is used).

The task ahead appears daunting. We have a commodity space with no notion of convexity or linearity, natural or otherwise. Nonconvex preferences are not illegitimate or even exceptional, but the rule. Commodities are indivisible, and their physical form is fluid and can affect utility.

Here we examine the land market under the assumption of perfect competition. It will be important and interesting to study the land market in the context of imperfect competition. However, as we need to solve simultaneous equations in either case (i.e. find fixed points), we concentrate on the basic mathematical techniques needed for this commodity and structure in the base case of perfect competition, leaving the study of models of imperfect competition to future research.

The outline of the remainder of the paper is as follows. Section II contains the notation, the model, and formal statements of the basic assumptions. Section III contains a further development of the themes of indivisibilities and nonconvexities for the purpose of motivating the key convexity assumption. This section also presents several examples of utility functions that satisfy our convexity assumption (as well as the other assumptions employed). Section IV contains the main results and proofs. Section V contains conclusions and suggestions for extensions and future work. An appendix contains two lemmas crucial to the proof of the main theorem below.

## II. The Model

Let K be a positive integer, let m be Lebesgue measure on $\mathbb{R}^K$, let L be a compact subset of $\mathbb{R}^K$, and let $\mathscr{B}$ be the $\sigma$–algebra of measurable subsets of L. If the framework is interpreted in location – theoretic terms, L is land, a subset of $\mathbb{R}^K$, and $\mathscr{B}$ is the consumption set of each agent. Land can be heterogeneous and anything immobile can be imbedded in it, so it is a differentiated commodity that can be divided and recombined in an infinity of varieties. Elements of $\mathscr{B}$ that are the same almost surely are *not* considered to be equivalent. Combination with a null set might create a non–equivalent parcel by virtue of, for example, the new set having a larger connected area. Furthermore, there is only one instance of each potential parcel of land, so that there is a discrete choice as to whether to purchase it or not; there is an indivisibility associated with this commodity.

For $x \in L$, let $\|x\|$ denote the standard Euclidean norm on $\mathbb{R}^K$. Capital letters will generally denote elements of $\mathscr{B}$, while script letters will generally denote subsets of $\mathscr{B}$. For $A,B \in \mathscr{B}$, define the set difference by $A\backslash B = \{x \in A \mid x \notin B\}$ and the complement of A by $A^c = L\backslash A$. If $B \in \mathscr{B}$, $\overset{\circ}{B}$ is the interior of B in the relative topology on L induced from $\mathbb{R}^K$. $\partial B$ is the boundary of B in the usual topology on $\mathbb{R}^K$; $\partial B$ is the set of points in L each of whose neighborhoods in $\mathbb{R}^K$ contains members of both B and $\mathbb{R}^K\backslash B$.

If $A \in \mathscr{B}$, let $1_A$ be the indicator function of the set A.

Define the $\epsilon$–ball around $A \in \mathscr{B}$ by $B_\epsilon(A) = \{y \in L \mid \exists\, x \in A \text{ with } \| x - y \| \leq \epsilon\}$. For $x \in L$, let $\text{rad}(x,A) = \sup\{\epsilon \geq 0 \mid B_\epsilon(x) \subseteq A\}$ and let $\delta(x,A) = \inf_{y \in A} \| x - y \|$. A <u>partition</u> of L is a collection of a finite number of sets $\{A_1,\dots,A_n\}$ with $A_j \in \mathscr{B}\ \forall\, j$, $m(L\backslash[\bigcup_{i=1}^{n} A_i]) = 0$ and $\forall i{\neq}j$, $m(A_i \cap A_j) = 0$.

Next we define the Hausdorff metric H on nonempty, closed sets in $\mathscr{B}$ (see

Hildenbrand (1974, p. 16)). Let C,D $\in \mathscr{B}$ where C and D are closed and nonempty. Then

$$H(C,D) \equiv \inf\{\epsilon \geq 0 \mid C \subseteq B_\epsilon(D),\ D \subseteq B_\epsilon(C)\}.$$

There are N consumers or traders in this exchange economy, where N is integer and finite. Consumer i has an endowment $E_i \in \mathscr{B}$, where $\{E_1,\ldots,E_N\}$ partitions L. Without loss of generality, $m(E_i) > 0\ \forall\ i$, for otherwise we can eliminate any consumer with $m(E_i) = 0$ from the model.[2] Each consumer has the consumption set $\mathscr{B}$ Consequently, we assume that consumer i has a complete preorder $\succeq_i$ over $\mathscr{B}$ that is the consumer's preference ordering.

At this point, it is convenient to discuss the assumptions on preferences or utilities employed below. The first assumption is the continuity of preferences. As discussed in the introduction of Berliant and ten Raa (1988), there are many topologies on $\mathscr{B}$ with respect to which one might assume preferences to be continuous. For example, one might restrict to closed subsets of L, and impose the Hausdorff metric topology. The introduction to the aforementioned paper provides reasons not to use this topology. As detailed in that paper, most topologies are inappropriate for our context for several reasons. In some topologies, such as the weak$^*$ topology, $\mathscr{B}$ is not closed. In other topologies, such as the Hausdorff topology, budget sets are not closed under limits. In most topologies, either $\mathscr{B}$ is not compact or the topology does not capture our intuitive notion of continuity with respect to the coherence of a parcel; it seems important that the topology allow sets that differ by only a few points to be far apart in the topology so that the utilities of the two sets can be vastly different. For example, it is possible

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[2]This is accomplished using the following argument. If consumer i has $m(E_i) = 0$, give his endowment to some other consumer and remove i from the economy. Apply the desired theorem (either 1 or 3) to the modified economy, and give i the parcel $\emptyset$. The resulting equilibrium for the modified economy, that has prices in $L^1$, will be an equilibrium for the unmodified economy. This argument relies on prices in $L^1$ and the attendant continuity assumptions on preferences, including the fact that parcels of zero measure are equivalent to the empty set under the topology.

to turn a coherent set with high utility into a badly shaped set with low utility by removing a countable number of points. Topologies that discriminate among such parcels tend to have many open sets, and thus tend to yield non–compact budget and consumption sets. In fact, such topologies are often not locally compact. For these reasons, we employ a generalization of the topology proposed in Berliant and ten Raa (1988), which makes $\mathscr{B}$ compact (see Lemma 2 in the Appendix below), preserves the budget under limits, and yet discriminates among parcels with different sizes, shapes and locations.

The topology on $\mathscr{B}$ proposed in Berliant and ten Raa (1988) can be given by the following pseudometric. Fix $i \leq N$ and let $h_i$ be an integrable function on L. For $A,B \in \mathscr{B}$, $A \neq L$, $B \neq L$, define

$$d_i(A,B) \equiv H((\mathring{A})^c,(\mathring{B})^c) + \left| \int_A h_i(x)\ dm(x) - \int_B h_i(x)\ dm(x) \right|,$$

$d_i(L,L) = 0$, $d_i(L,A) = d_i(A,L) = \sup \{d_i(A,B) \mid B \in \mathscr{B}, \mathring{B} \neq L\}$. In Berliant and ten Raa (1988), it is assumed that consumer i has a preference order that is continuous in this topology in the sense that the upper and lower contour sets of the preference ordering $\succsim_i$ are closed in the topology. Under this assumption, Theorem 1 of that paper shows that demand is nonempty while Theorem 2 demonstrates the existence of a continuous utility representation. In fact, we use a relative of this topology here. Let $\mathring{\mathscr{B}}$ be the collection of relatively open sets in $\mathscr{B}$, and give it the topology induced by the metric $H(A^c,B^c)$ for $A,B \in \mathring{\mathscr{B}}$. We note that $\mathring{\mathscr{B}}$ is compact since the Hausdorff metric on closed subsets of a compact set generates a compact topology. Let $h_i$: $L \times \mathring{\mathscr{B}} \to \mathbb{R}$ be continuous for $i = 1,2,\ldots,I$. Define a new pseudometric d exactly as above except allow each $h_i$ to depend on the interior of the set and allow for an arbitrary (but finite) number of functions $h_i$:

$$d(A,B) \equiv H((\mathring{A})^c,(\mathring{B})^c) + \sum_{i=1}^{I} \left| \int_A h_i(x,\mathring{A})\ dm(x) - \int_B h_i(x,\mathring{B})\ dm(x) \right|$$

for A,B $\in \mathscr{B}$, A $\neq$ L, B $\neq$ L. As before, let d(L,L) = 0, d(L,A) = d(A,L) = sup {d(A,B) | $\mathring{B} \neq$ L}. This allows us to use the same topological space for all consumers (each employing perhaps a different $h_i$) as well as allowing a slightly more general continuity assumption. As in the previous work, shape and location of the interior of a set are topologized by the Hausdorff metric on complements of interiors. Now we also allow the linear portion of the "marginal utility" of each point to depend on the shape, location, and area of the interior of the set as well. Clearly, if no $h_i$ depends on its second argument, we have specialized to the case examined in the previous work on preferences and demand (modulo the continuity assumption imposed on $h_i$). It is straightforward to extend the theorems in Berliant and ten Raa (1988) to admit preferences continuous with respect to the topology induced by d (as opposed to $d_i$). In fact, Lemma 2 of the appendix below implies that demand is nonempty when preferences are continuous with respect to this more general topology. Like the topology induced by $d_i$, the topology induced by d on $\mathscr{B}$ (henceforth called the outer Hausdorff topology) does not separate elements. For example, B and C might have common interior and equal $h_i$ values for all i, but different locations. Also, L is isolated in this topology.

With this topology in hand, we can talk about continuity of preferences. A preference ordering $\succeq$ is said to be <u>continuous</u> if for each A $\in \mathscr{B}$, the sets {B $\in \mathscr{B}$ | B $\succeq$ A} and {B $\in \mathscr{B}$ | A $\succeq$ B} are closed in the *outer* Hausdorff topology.

Notice that this assumption generalizes the assumption of linear utilities employed in Berliant (1984, 1985b). Suppose that the preferences of trader i can be expressed as $u_i(B) = \int_B f(x)\, dm(x)$ for some continuous density f. One can always choose $h_i \equiv f$ for the topology, which makes this utility continuous. (If one wishes to employ a discontinuous but integrable f, the topology of Berliant and ten Raa (1988) can be used below in place of the outer Hausdorff topology.) Thus, linear utilities can be made continuous with appropriate choice of $h_i$. Although it is slightly bothersome that the topology chosen for the consumption set might depend on the preferences of traders, this is the best we can do for now.

In fact, continuity of preferences is stronger than necessary. A preference ordering $\succeq$ is said to be upper semi-continuous if for each $A \in \mathscr{B}$, the set $\{B \in \mathscr{B} \mid B \succeq A\}$ is closed in the outer Hausdorff topology.

Local non-satiation of preferences is also used below to prove a first welfare theorem. We say that the preference order $\succeq$ is locally non-satiated if for all $\epsilon > 0$ for all $B \in \mathscr{B}$ with $m(B) < m(L)$, there exists $A \in \mathscr{B}$ such that $m(A\backslash B) + m(B\backslash A) < \epsilon$ and $A \succ B$. This is analogous to the standard local non-satiation assumption, except that we use the $L^1$ metric in place of the outer Hausdorff topology because value is not continuous with respect to the outer Hausdorff topology; see Berliant and ten Raa (1988).

Surprisingly, local non-satiation is not needed to prove existence of an equilibrium or a second welfare theorem. Instead, a different but related assumption is used. In place of postulating that locally there is always a better parcel, we assume that locally there is always a worse parcel. We say that the preference order $\succeq$ is locally non-minimized if for all $A,B \in \mathscr{B}$ with $m(B) > 0$ and $B \succ A$, there exists $C \in \mathscr{B}$ with $C \subseteq B$ a.s., $m(B\backslash C) > 0$, and $C \succ A$.

We continue the development of the model by defining the space of prices next. The price space corresponding to the commodity space $\mathscr{B}$ is somewhat problematic, as $\mathscr{B}$ is not linear. Hence, it does not have a natural dual. It can, however, be embedded in any one of a number of linear spaces by using the indicator functions of sets, as in Berliant (1985b). There is a natural argument for placing prices in $L^1$ or a subset of it. It is desirable to have no arbitrage in equilibrium, for otherwise traders would always wish to change their demands. In the context of the model, no arbitrage means that traders cannot put parcels together or take them apart and make a profit.

Hence prices should be at least additive set functions. If traders are not to make a profit in equilibrium by putting together or taking apart an infinity of parcels, then prices should be countably additive. Continuity of preferences will imply that parcels of zero measure are equivalent to the empty set. Hence, if a parcel of measure zero is to have zero price, then the Radon–Nikodym theorem (see Rudin (1974, p. 129)) yields a price space that is the set of all integrable functions on L (called $L^1$). In other words, we are assuming that the price space is $L^1$, and that the embedding of $\mathscr{B}$ is as indicator functions of elements of $\mathscr{B}$ in $L^\infty$. Other possible dual pairings are clearly possible, and such dual pairings will yield different continuity assumptions on preferences. We have chosen this particular dual pairing because it makes the mathematics of the problem relatively easy to handle.

For the remainder of the paper, when we write $p \cdot S$ for $p \in L^1$ and $S \in \mathscr{B}$, we mean $p \cdot 1_S$.

The maximization problem of consumer i is:

(1)
$$\text{Find } B \in \mathscr{B} \text{ such that } p \cdot B \leq p \cdot E_i$$
$$\text{and}$$
$$\forall\, C \in \mathscr{B} \text{ with } C \succ_i B,\ p \cdot C > p \cdot E_i$$
$$\text{for given prices } p \in L^1.$$

This problem is studied in detail in Berliant (1984) and Berliant and ten Raa (1988).

An <u>allocation</u> is a vector $(B_1,\dots,B_N)$ with $B_i \in \mathscr{B}\ \forall\ i$. An allocation $(B_1,\dots,B_N)$ is called <u>feasible</u> if it is a partition of L, i.e. $\bigcup_{i=1}^{N} B_i = L$ a.s. and $\forall$ i,j with $i \neq j$, $B_i \cap B_j = \emptyset$ a.s.

An <u>equilibrium</u> is $(p;B_1,\dots,B_N) \in L^1 \times \mathscr{B}^N$ such that $(B_1,\dots,B_N)$ is a feasible allocation and such that for each i, $1 \leq i \leq N$, $B_i$ solves (1) with respect to prices p, where $p \neq 0$.

An <u>equilibrium allocation</u> is $(B_1,\dots,B_N) \in \mathscr{B}^N$ such that $\exists\ p \in L^1$ such that

$(p;B_1,\ldots,B_N)$ is an equilibrium.

A feasible allocation $(B_1,\ldots,B_N)$ is called <u>an equilibrium with respect to a price system</u> p if $p \in L^1$ and if $\forall$ i and for each $C \in \mathscr{B}$ with $C \succ_i B_i$, $p \cdot C > p \cdot B_i$.

A feasible allocation $(B_1,\ldots,B_N)$ is called <u>Pareto optimal</u> if there is no other feasible allocation $(C_1,\ldots,C_N)$ such that $C_i \succsim_i B_i$ $\forall$ i with strict preference holding for some trader j.

A <u>coalition</u> is any subset $\mathcal{C}$ of $\{1,2,\ldots,N\}$. A coalition $\mathcal{C}$ is said to <u>block</u> a partition $(B_1,\ldots,B_N)$ if for each $i \in \mathcal{C}$ there is a $C_i \in \mathscr{B}$ such that $\bigcup_{i\in\mathcal{C}} C_i = \bigcup_{i\in\mathcal{C}} E_i$ a.s., $\forall i,j \in \mathcal{C}$ with $i \neq j$ $C_i \cap C_j = \emptyset$ a.s., and $C_i \succsim_i B_i$ $\forall$ $i \in \mathcal{C}$ with strict preference holding for some $j \in \mathcal{C}$. The <u>core</u> is the set of all partitions that are not blocked by any coalition.

Finally, it is important to note that "ordinary" commodities can be included in this model. Simply reserve subsets of L disjoint from the rest, and let the utility of each trader depend on ownership of these sets only through the measure of the intersection of their parcels with each set. This measure of intersection can be regarded as a quantity, where each of these extra sets is homogeneous. In this way, the model presented here generalizes the standard general equilibrium model with finitely many commodities. Our example in the next section employs this structure.

III. Convexity

The final assumption on preferences used below is that of convexity. As mentioned in the introduction, the lack of convexity in consumption sets and preferences seems to us to be a distinguishing feature of the fields of location theory and urban economics. In general, a consumer has the discrete choice of owning a parcel or not owning it, thus generating a nonconvexity in the consumption set, which is $\mathscr{B}$ or indicator functions of elements of $\mathscr{B}$ in this model. Notice that a convex combination of indicator functions is generally not an indicator function. With regard to preferences, it is not at all obvious what a linear or convex structure on $\mathscr{B}$ should look like. To see the problem, consider two disjoint sets S and T. In order to define a linear structure, $\alpha S + (1-\alpha)T$, where $0 \leq \alpha \leq 1$, must be identified with some set in $\mathscr{B}$. It is not clear what this set should be. One might think that it should be a set containing $\alpha$ of S and $1-\alpha$ of T, but there are many such sets. Even if we picked one of these sets and associated it with $\alpha S + (1-\alpha)T$, there is another problem. Convexity of preferences is not necessarily a natural assumption when the commodity is land. For example, think of S and T as representing 1 acre of land on the East and West coasts, respectively. An agent might be indifferent between S and T, but prefers either of these parcels to having 1/2 acre on both coasts. This could be true no matter how $(1/2)S + (1/2)T$ were defined. This idea becomes more and more important as one takes further convex combinations with other parcels. In essence, consumers will prefer bundles that are extreme in the sense that land is close together as opposed to average bundles in which land holdings are diversified. This notion runs opposite the standard convexity assumption. We refer to Berliant (1985a) for further discussion of these issues.

In spite of the problems presented by the imposition of convexity assumptions, some further convexity-like assumption must be imposed, as the following example illustrates.

The idea behind the example is very simple. The utility of the agents will depend on the measure of land in each part of a 2 element partition of land. This means that there are essentially two homogeneous commodities or types of land. So the example will appear to be equivalent to a standard 2 x 2 exchange economy. Formally, there is one potential difference. An additive price function need not give two parcels of equal size and type the same price. However, it will be shown that this possibility cannot occur in our example, and the intuition that the example is equivalent to a standard 2 x 2 exchange economy with nonconvex preferences is correct. It is well known that, in such economies, the second welfare theorem need not hold.

Let (A,B) be a measurable partition of the space of land with $m(A) = m(B) = 1$. Define the utility functions of agents 1 and 2 by

$u_1(C) = m(A \cap C) + \frac{1}{2}[m(B \cap C) + \frac{1}{2}]^2$ and

$u_2(D) = [m(A \cap D) + \frac{1}{2}]\cdot[m(B \cap D) + \frac{1}{2}]$.

Pick $C_1$ such that $m(C_1 \cap A) = m(C_1 \cap B) = \frac{1}{2}$ and let $C_2 = C_1^{c}$. Give agent 1 initial endowment $C_1$ and give agent 2 initial endowment $C_2$. It is easy to verify that the allocation $(C_1,C_2)$ is in the core. To see this, look at the associated 2 x 2 exchange economy where $(C_1,C_2)$ is represented by the vector $((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$, i.e. $((m(A \cap C_1),m(B \cap C_1)),(m(A \cap C_2),m(B \cap C_2)))$. Notice first that neither one–person coalition will block this allocation. To see that it is Pareto efficient, compute $u_2(D)$ given that (C,D) is an allocation. The expression obtained is $u_2(D) = \frac{13}{16} + \frac{7}{8}\cdot m(B \cap D) - \frac{5}{4}\cdot[m(B \cap D)]^2 + \frac{1}{2}\cdot[m(B \cap D)]^3$. The first order condition for maximization of $u_2$ yields

$$\frac{7}{8} - \frac{5}{2}m(B \cap D) + \frac{3}{2}[m(B \cap D)]^2.$$

It is easily verified that the zeros of this equation are at $m(B \cap D) = \frac{1}{2}$ and $m(B \cap D) = \frac{7}{6}$, and that the slope of $u_2(D)$ is positive for $m(B \cap D) < \frac{1}{2}$ and negative for $m(B \cap D) > \frac{1}{2}$, so $m(B \cap D) = \frac{1}{2}$ is the global maximum. Hence the allocation is Pareto efficient. In addition, agent 1 has nonconvex preferences in this economy, and as a consequence there do not exist prices that support this allocation. The following shows this formally.

Suppose that $p \in L^1$ supports $(C_1,C_2)$. Let $S \subseteq C_1 \cap A$, $T \subseteq C_2 \cap B$, $S' \subseteq C_2 \cap A$ and $T' \subseteq C_1 \cap B$.

First we show that if $m(S) = m(S') > 0$ then $p \cdot S = p \cdot S'$. [A similar statement holds for T and T′]. Suppose $p \cdot S > p \cdot S'$. Notice first that if $\bar{S} \subseteq S$, $m(\bar{S}) > 0$, then $p \cdot \bar{S} > 0$. For if this were not the case, then agent 2 will acquire $\bar{S}$ and increase his utility, and p will not support. Next, divide S into two subsets of equal measure, $\bar{S}_1$ and $\bar{S}_2$. Either $p \cdot \bar{S}_1 \leq \frac{1}{2} \cdot p \cdot S$ or $p \cdot \bar{S}_2 \leq \frac{1}{2} \cdot p \cdot S$. Choose a subset satisfying the inequality, and again subdivide it into two sets of equal measure. Continue this procedure for a finite number of steps until the resulting set, $\tilde{S}$, is such that $p \cdot \tilde{S} < p \cdot S - p \cdot S'$. Define $S^* = S \backslash \tilde{S}$. Since $m(\tilde{S}) > 0$, $0 < p \cdot \tilde{S} = p \cdot S - p \cdot S^* < p \cdot S - p \cdot S'$. So $p \cdot [S' \cup (C_1 \backslash S^*)] = p \cdot C_1 - p \cdot S^* + p \cdot S' < p \cdot C_1$, and $S' \cup (C_1 \backslash S^*)$ is affordable to agent 1 under the supporting prices. Also,

$$m([S' \cup (C_1 \backslash S^*)] \cap B) = \frac{1}{2}$$

$$\begin{aligned} m([S' \cup (C_1 \backslash S^*)] \cap A) &= \frac{1}{2} - m(S^*) + m(S') \\ &= \frac{1}{2} - m(S^*) + m(S) \\ &= \frac{1}{2} + m(\tilde{S}) > \frac{1}{2}. \end{aligned}$$

Hence $u_1(S' \cup (C_1 \backslash S^*)) > u_1(C_1)$.

The last inequality follows because $u_1$ depends only on the area of each type of land. So $S' \cup (C_1 \backslash S^*)$ is an affordable parcel that yields a higher utility than $C_1$, which contradicts the hypothesis that p supports $(C_1,C_2)$. If $p \cdot S' > p \cdot S$ a similar argument using $u_2$ results in the same contradiction. This result implies that if two agents have positive amounts of the same type of land, then an additive supporting price must price equal–size parcels of that type the same. Note that if one trader owned all of the land of a particular type then this argument does not apply, and it would be possible for p to price parcels of the same size and type differently.

The argument used to show that equal–size parcels of the same type must be priced the same by supporting prices is essentially a no–arbitrage argument.

Now consider the following trades involving S, S′, T, and T′ that agent 1 can make. Loosely speaking, suppose S and T are such that agent 1 prefers to give up S in exchange for T. Since p supports $(C_1,C_2)$, $p\cdot T > p\cdot S$ for all such S and T. Formally,

$u_1(T \cup (C_1\backslash S)) = \frac{1}{2} - m(S) + \frac{1}{2}(1 + m(T))^2 > u_1(C_1) = 1 \Rightarrow p\cdot T > p\cdot S.$

Similarly, for S′ and T′ such that agent 1 prefers to give up T′ for S′, $p\cdot S > p\cdot T$. Formally,

$u_1(S' \cup (C_1\backslash T')) = \frac{1}{2} + m(S') + \frac{1}{2}(1 - m(T'))^2 > u_1(C_1) = 1 \Rightarrow p\cdot S' > p\cdot T'.$

Note that for $m(S) = m(S') = .1 = m(T) = m(T')$ the first inequality of each of the above statements is satisfied. Therefore, for such S, S′, T, and T′, $p\cdot T > p\cdot S$ and $p\cdot S' > p\cdot T'$. This is impossible, as the earlier argument implies that equal–size parcels of the same type are priced the same, i.e. $p\cdot S = p\cdot S'$ and $p\cdot T = p\cdot T'$. Hence p does not support.

It is instructive to see why the standard proof of the second welfare theorem fails here. First, note that by identifying sets with their indicator functions (embedded, say, in $L^\infty$), preferences are weakly convex in the sense that the indicator of a set is not contained in the convex hull of its (strict) upper contour set. In fact, the upper contour set of any agent at any parcel can be separated from the parcel by a linear functional that is zero on the parcel in question and one everywhere else. The example shows that this condition is not sufficient for the second welfare theorem to hold. Next we shall examine the breakdown of the second welfare theorem for this example, and see that the functionals supporting upper contour sets must be related in order to have a second welfare theorem.

In general, one must extend the commodity space to a larger, linear space containing indicator functions, such as $L^\infty$. The utility on indicators must be extended in a continuous (in some sense) and quasi–concave manner because standard proofs of the second welfare theorem (see Debreu (1954)) use the <u>sum</u> across consumers of the sets of bundles at least as good as the Pareto optimal allocation to a consumer (where the interior is taken for one such set). To see how the standard proof breaks down for our example, let $A = [0,1]$, $B = [2,3]$. Let $a_1$ be the indicator of $[0,.5] \cup [2,2.5]$ and let $a_2$ be the indicator of $[.5,1] \cup [2.5,3]$. $(a_1,a_2)$ are indicators of a Pareto optimal allocation as shown above. Let $a_1'$ be the indicator of $[0,.4] \cup [2,2.6]$, let $a_1''$ be the indicator of $[0,.6] \cup [2,2.4]$, and let $a_2'$ be the indicator of $[.4,.5] \cup [.6,1] \cup [2.4,2.5] \cup [2.6,3]$. It is easy to see that trader 2 is indifferent between $a_2$ and $a_2'$ and that trader 1 prefers both $a_1'$ and $a_1''$ to $a_1$. Provided that the topology on indicators is chosen such that the utility of trader 1 is continuous, $a_1'$ and $a_1''$ are both in the (relative) interior of the set of bundles at least as good as $a_1$. Notice that

$$a_1 + a_2 = 1_L = \frac{1}{2}(a_1' + a_2) + \frac{1}{2}(a_1'' + a_2').$$

Thus, we cannot separate the sum of indicators of the Pareto optimal allocation $(1_L)$ from the convex hull of the sum of sets of bundles at least as good as the Pareto optimal bundle (with one interior taken).

Next we introduce and discuss the convexity assumption that is used in our existence proof. One way to motivate this assumption is to consider an implication of convexity of preferences in a standard finite dimensional setting. If preferences are convex in such a setting then at every point, x, there exists a hyperplane, $p(\cdot)$, such that y preferred to x implies $p(y) > p(x)$. In other words, if preferences are convex then the upper contour set of a point can be separated from the point by a hyperplane, i.e. a linear functional. This statement is well–defined even in the nonlinear infinite dimensional setting of our model and leads to the following definition.

To formulate this definition, we give the space of continuous functions on L mapping into $\mathbb{R}_+$, call it $C_+^0$, the norm topology. Let $G \subseteq C_+^0$ be a convex collection of Lipschitz functions with constant $c^*$ (i.e. $|g(x) - g(y)| \leq c^* \cdot \|x - y\|$) such that sup

$\{g(x) \mid g \in G, x \in L\} = \bar{c} < \infty$ and $\inf \{g(x) \mid g \in G, x \in L\} = \underline{c} > 0$. Note that by Ascoli's Theorem the assumptions on G imply that it is compact.[3] A preference ordering $\succsim$ satisfies <u>separation by hyperplanes</u> if for each $B \in \mathscr{B}$ there exists $g_B \in G$ such that for all $A \in \mathscr{B}$ with $A \succ B$, $\int_A g_B(x)\, dm(x) \geq \int_B g_B(x)\, dm(x)$. Note that strict preference is only required to yield a weak inequality between the integrals. However, it is easy to see that if $\succsim$ is also locally non–minimized then the inequality is strict.

Notice that this assumption uses only a one–way implication; that is, any parcel preferred to B has value at least as high under $g_B$ as B. The assumption does *not* require that any parcel with value under $g_B$ at least as high as B is more preferred. In this sense, the assumption is quite similar to the condition that a point is not in the convex hull of its preferred set, a condition that is common in the literature on economies with infinitely many commodities. However, as illustrated above, the condition that a point is not in the convex hull of its preferred set is insufficient in our model to prove the second welfare theorem (even with continuity, etc.). We shall demonstrate below that separation by hyperplanes (along with other assumptions unrelated to convexity) is sufficient to prove a second welfare theorem.

One implication of this assumption is that marginal rates of substitution are bounded in some sense. Such boundedness assumptions are common in the infinite dimensional commodity space literature (some of which was cited in the Introduction). The bounds on g imply that if $A \succ B$ then $m(B \backslash A)/\, m(A \backslash B) \leq \bar{c} \,/\, \underline{c}$. So preferences have bounded marginal rates of substitution in the sense that if a set of positive measure is subtracted from a parcel, then the minimum measure that must be added to

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[3]The assumption that G consists of Lipschitz functions is actually stronger than necessary, but it makes the condition easier to check. All that is required to prove the theorems is that G is a compact subset of $L^1$ with the norm topology. For a characterization of such subsets, see Dunford and Schwartz (1988, p. 301).

obtain something preferred is bounded away from zero. In other words, if a set of positive measure is subtracted from a parcel then it is not possible to obtain something preferred by adding arbitrarily small measure.

Next we discuss some examples of preferences that satisfy separation by hyperplanes. One example is a preference relation that can be represented by a utility function of the form: $U(B) = f(m_1(B),...,m_K(B))$, where $f:\mathbb{R}^K_+ \to \mathbb{R}$ is $C^1$ and quasi–concave and each $m_i(B) \equiv \int_B h_i(x)\, dm(x)$ is a measure. So preferences over land parcels are equivalent to convex preferences over K characteristics of land parcels. Separation by hyperplanes places additional restrictions on f and the $h_i$. To find these we find the $g_B$ for such preferences. Let $\overline{m}(B) \equiv (m_1(B),...m_K(B))$ and let $Df|_B$ be the vector $\frac{\partial f}{\partial x_i}(\overline{m}(B))$, $i=1,...,K$. Since f is $C^1$ and quasi–concave, $U(A) > U(B)$ implies $Df|_B \cdot \overline{m}(A) > Df|_B \cdot \overline{m}(B)$. So define $g_B(x) \equiv \sum_{i=1}^{K} \frac{\partial f}{\partial x_i}(\overline{m}(B)) \cdot h_i(x)$. Then $U(A) > U(B)$ implies $\int_A g_B(x)dm(x) = Df|_B \cdot \overline{m}(A) > Df|_B \cdot \overline{m}(B) = \int_B g_B(x)dm(x)$, as required by separation by hyperplanes. The $g_B$ generated this way must be bounded and Lipschitz. For example, this holds if each $h_i$ is Lipschitz with $0 < \underline{c} < h_i(x) < \overline{c} < \infty$ for all x and $0 < \underline{c} < \frac{\partial f}{\partial x_i}(\overline{m}(B)) < \overline{c} < \infty$ for all $i = 1,...,K$ and $B \in \mathscr{B}$ Actually it is not necessary that all of the derivatives of f be positive. For example, consider the utility function $U(B) = \int_B h(x)/(1+m(B))\, dm(x)$, which can be written as $f(\int_B h(x)dm(x),m(B))$, where $f(x,y) = x/(1+y)$ is the required quasi–concave function. The supporting hyperplanes can be defined by $g_B(x) \equiv [h(x)-U(B)]/[1+m(B)]$. For a more specific example take $L = [0,1]$ and $h(x) = 1 + cx$. For $0 \leq c \leq 2$ this yields preferences that satisfy separation by hyperplanes and are monotone. Another example of preferences of this general form are those represented by the negative of the variance of a set (in $\mathbb{R}$). These preferences indicate a desire for coherence of a parcel. Such utilities are clearly a quasi–concave function of two measures since variance can be written as $EB^2 - (EB)^2$, where E is the expectation operator. For suitable densities, e.g. uniform on [0,1], such preferences satisfy separation by hyperplanes.

As in much of the literature on infinite dimensional existence problems, it is rather difficult to find closed form examples of utility functions satisfying the assumptions of the theorems that are not additive across commodity indexes or quasi–concave functions of a finite number of continuous linear functionals. We have found such an example; it satisfies a rather technical but minor modification of the upper semi–continuity assumption and requires minor modifications of the proofs, so we omit it here. However, it does demonstrate that the results presented here expand the domain on which existence of equilibrium and the welfare theorems can be proved for this model, relative to previous literature.

Next we discuss how separation by hyperplanes excludes the counterexample to the existence of additive supporting prices given above. It is worthwhile to sketch a more precise argument that any preferences that depend monotonically only on a vector of positive, nonatomic measures and are nonconvex in these measures do not satisfy separation by hyperplanes. We skip the details as similar arguments appear in the proof of our main theorem. Let utility depend only on the positive, nonatomic measures $m_1(\cdot),\dots,m_k(\cdot)$. The presence of a nonconvexity in the dependence on these measures implies that there exist $A, A', B \in \mathscr{B}$ with $A, A'$ preferred to B and $\alpha \in (0,1)$ such that $m_j(B) = \alpha m_j(A) + (1-\alpha) m_j(A')$ for $j = 1,\dots,k$. Suppose that separation by hyperplanes holds. Cover G with $\epsilon$ balls and let $g_1,\dots,g_n$ be the centers of the balls in a finite subcover, where n depends on $\epsilon$. Then since preference depends only on the measures, Lyapunov's theorem implies that there exists $C \in \mathscr{B}$, $m_j(C) = m_j(B)$ $(j = 1,\dots,k)$ so $C \sim B$, $\int_C g_i(x)\, dm(x) = \alpha \int_A g_i(x)\, dm(x) + (1-\alpha) \int_{A'} g_i(x)\, dm(x)$ for $i = 1,\dots,n$, where C depends on $\epsilon$. Then for small $\epsilon$, $g_C$ from the separation by hyperplanes assumption is close to some $g_i$, and hence $\int_C g_C(x)\,dm(x)$ is close to $\alpha\int_A g_C(x)\,dm(x) + (1-\alpha)\int_{A'} g_C(x)\,dm(x)$. However, $A,A' \succ B \sim C$ and monotonicity imply $\int_A g_C(x)\,dm(x) > \int_C g_C(x)\,dm(x)$ and $\int_{A'} g_C(x)\,dm(x) > \int_C g_C(x)\,dm(x)$, so $\alpha\int_A g_C(x)\,dm(x) + (1-\alpha)\int_{A'} g_C(x)\,dm(x) > \int_C g_C(x)\,dm(x)$, a contradiction. So such preferences cannot satisfy separation by hyperplanes.

Finally, we want to point out that standard general equilibrium models with finitely many commodities satisfy all of our assumptions if marginal rates of substitution are bounded away from 0 and $\infty$. In this case the separating hyperplanes can be chosen to be constant on each subset corresponding to a commodity with values in a compact subset of $(0,\infty)$. By choosing the subsets representing commodities to be a nonzero distance apart, these supporting hyperplanes will be Lipschitz and form a compact set.

## IV. Results

The main result of the paper can now be stated and proved.

> Theorem 1: Let $\succsim_i$ be upper semi–continuous, locally non–minimized, and satisfy separation by hyperplanes for each i. Given endowments $(E_1,\ldots,E_N)$, there exists an equilibrium with positive prices.

Proof: We begin by giving the general outline of the proof. First, we define a sequence, $\{\mathscr{C}^n\}_{n=1}^{\infty}$, of measurable partitions of L and a sequence, $\{S^n\}_{n=1}^{\infty}$, of finite subsets of G. For each n, $\mathscr{C}^n$ and $S^n$ are then used to define a map, the fixed points of which can be used to find a feasible allocation and prices. Finally, limit points of these allocations and prices are shown to be equilibria.

The sequence of measurable partitions of L is constructed so as to have the following properties. Each $\mathscr{C}^n$ has n elements denoted by $(C_1^n,\ldots,C_n^n)$ with $m(C_k^n) > 0$ for all k. Also, for each $\epsilon > 0$, there exists $n^*$ such that $n > n^*$ implies for all $k = 1,\ldots,n$ there exists $x \in L$ with $C_k^n \subseteq B_\epsilon(x)$. Finally, there is an $\alpha > 1$ such that for every n and for all j and k, $m(C_k^n)/m(C_j^n) \leq \alpha$.

To construct $\{S^n\}_{n=1}^{\infty}$, fix n and cover G by taking the open ball (in G) of radius 1/n around every element of G. Our assumptions on G and Ascoli's Theorem (see Munkres (1975, p. 290)) imply that G is compact. Therefore, for each n, this cover has a finite subcover. Let $S^n$ be the collection of centers of the open balls in this subcover.

Now fix n and consider the measurable partition $\mathscr{C}^n$ of L and the finite subset $S^n$ of G. Since n is fixed, for notational simplicity we will suppress the superscripts and denote these sets by $(C_1,\ldots,C_n)$ and $S = \{g^1,\ldots,g^s\}$, respectively. Define I $\equiv$ $\{(y_1^i,\ldots,y_s^i)_{i=1}^N \in (\mathbb{R}_+^s)^N \mid$ there is a feasible allocation $(B_1,\ldots,B_N)$ with $\int_{B_i} g^j(x)\, dm(x)$

$= y_j^i$ for each i and j}. Dubins and Spanier (1961, Theorem 1) implies that I is a compact, convex subset of $\mathbb{R}^{S \cdot N}$. Let co(S) denote the convex hull of S in $C_+^0$. Next we define a map $\rho \times \sigma : I \times [co(S)]^N \to I \times [co(S)]^N$. To accomplish this, we first consider the following economies.

Given $g_1, \ldots, g_N \in G$, let $\mathcal{E}(g_1, \ldots, g_N)$ denote the exchange economy in which agent i has the utility function $u_i(B) \equiv \sum_{k=1}^{n} [m(B \cap C_k)]^{1-(1/n)} \cdot \int_{C_k} g_i(x)\, dm(x)/m(C_k)$ and the endowment $e_i \equiv [m(E_i \cap C_1), \ldots, m(E_i \cap C_n)]$. This is a standard N agent, n commodity exchange economy in which agent i's consumption of commodity k is given by $z_k^i \equiv m(B_i \cap C_k)$. Using the quantity $z_k^i$ in place of $m(B_i \cap C_k)$, the consumption set of agent i is clearly $\mathbb{R}_+^n$. Thus, we have constructed a (strictly) convex exchange economy with n homogeneous commodities. Any standard existence theorem can be applied to obtain a nonempty set of equilibrium prices. It is straightforward but tedious to calculate demand in this economy. This demand is single–valued and $C^1$ for strictly positive prices. In addition, it is straightforward to show that this economy satisfies the gross substitutes assumption (see Arrow and Hahn (1971, p. 221)). Therefore, by Arrow and Hahn (1971, Corollary 9.7), $\mathcal{E}(g_1, \ldots, g_N)$ has a unique equilibrium.

Next define $\rho$: $[co(S)]^N \to I$ by $\rho(g_1, \ldots, g_N) \equiv \{y \in I \mid$ there exists a feasible allocation, $(B_1, \ldots, B_N)$, such that for all i and k, $m(B_i \cap C_k) = z_k^i$ and $\int_{B_i} g^j(x)\, dm(x) = y_j^i$, where $(z^1, \ldots, z^N)$ is the unique equilibrium allocation of $\mathcal{E}(g_1, \ldots, g_N)\}$. Using standard arguments and Dubins and Spanier (1961, Theorem 1), it is easy to verify that $\rho$ is nonempty, convex–valued, and has a closed graph.

Define $\sigma$: $I \to [co(S)]^N$ by $\sigma(y) \equiv \{(g_1, \ldots, g_N) \in [co(S)]^N \mid$ for each feasible allocation $(B_1, \ldots, B_N)$ such that $\int_{B_i} g^k(x)\, dm(x) = y_k^i$ for each i and k, for each i $g_i$

satisfies $A \in \mathscr{B}$, $A \succ_i B_i$ implies $\int_A g_i(x)\, dm(x) \geq \int_{B_i} g_i(x)\, dm(x) - 2m(L)/n\}$. Let $(B_1,\ldots,B_N)$ be a feasible allocation with $\int_{B_i} g^k(x)\, dm(x) = y^i_k$ for all i and k. The separation by hyperplanes assumption on preferences implies that there exists $g_{B_i}$ such that $A \in \mathscr{B}$, $A \succ_i B_i$ implies $\int_A g_{B_i}(x)\, dm(x) \geq \int_{B_i} g_{B_i}(x)\, dm(x)$. Since there exists $g^k \in S$ with $\|g^k - g_{B_i}\| < 1/n$, $\int_A g^k(x)\, dm(x) + m(L)/n \geq \int_A g_{B_i}(x)\, dm(x) \geq \int_{B_i} g_{B_i}(x)\, dm(x) \geq \int_{B_i} g^k(x)\, dm(x) - m(L)/n$. Hence the vector of such $g^k$ is in $\sigma$, and $\sigma$ is nonempty. The definitions of S and $\sigma$ can be used to show that $\sigma$ has the same properties as $\rho$. Hence $\rho \times \sigma$ is nonempty and convex–valued with a closed graph, and the Glicksberg (1952) fixed point theorem implies that this correspondence has a fixed point.

A fixed point $[\bar{y},(\bar{g}_1,\ldots,\bar{g}_N)]$ of $\rho \times \sigma$ has the property that there exists an equilibrium $[(z^1,\ldots,z^N),q]$ of $\mathscr{E}(\bar{g}_1,\ldots,\bar{g}_N)$ (where $q,z^i \in \mathbb{R}^n_+$; normalize $q_1 = 1$) and a feasible allocation $(B_1,\ldots,B_N)$ with $z^i_k \equiv m(B_i \cap C_k)$ such that, for each i, $A \succ_i B_i$ implies $\int_A \bar{g}_i(x)\, dm(x) \geq \int_{B_i} \bar{g}_i(x)\, dm(x) - 2m(L)/n$. Define $p \in L^\infty$ by $p \equiv \sum_{k=1}^{n} q_k \cdot 1_{C_k}$.

Applying this argument for each n (and thus for each pair $(\mathscr{C}^n,S^n)$) yields sequences $\{(\bar{g}^n_1,\ldots,\bar{g}^n_N)\}^\infty_{n=1}$, $\{(B^n_1,\ldots,B^n_N)\}^\infty_{n=1}$, and $\{p^n\}^\infty_{n=1}$. Since G is compact, we can pass to a subsequence (without changing notation) such that for each i $\lim_{n\to\infty} \bar{g}^n_i = \bar{g}_i \in G$. Next we show that a limit point of $\{p^n\}^\infty_{n=1}$ and $\{(B^n_1,\ldots,B^n_N)\}^\infty_{n=1}$ is an equilibrium.

First we consider the sequence $\{p^n\}^\infty_{n=1}$. Fix $w,w' \in L$ and suppose (without loss of generality) that $w \in C^n_k$ and $w' \in C^n_1$ for all n. The fact that $q^n_k$ is equal to each consumer's marginal rate of substitution between goods k and 1 in the finite economy $\mathscr{E}(\bar{g}_1^n,\ldots,\bar{g}_N^n)$ implies

$$
\begin{aligned}
|p^n(w) - p^t(w)| \\
&= |q_k^n - q_k^t| \\
&= |[\partial\bar{u}^n/\partial z_k^n]/[\partial\bar{u}^n/\partial z_1^n] - [\partial\bar{u}^t/\partial z_k^t]/[\partial\bar{u}^t/\partial z_1^t]| \\
&= |[z_1^n/z_k^n]^{1/n}\cdot\{\int_{C_k^n}\bar{g}^n(x)\ dm(x)/m(C_k^n)\}/\{\int_{C_1^n}\bar{g}^n(x)\ dm(x)/m(C_1^n)\} - \\
&\quad [z_1^t/z_k^t]^{1/t}\cdot\{\int_{C_k^t}\bar{g}^t(x)\ dm(x)/m(C_k^t)\}/\{\int_{C_1^t}\bar{g}^t(x)\ dm(x)/m(C_1^t)\}|,
\end{aligned}
$$

where $z^r$ is an equilibrium consumption bundle of the consumer with utility $\bar{u}^r$; we have thus eliminated some subscripts to simplify notation. Since $z^r$ is an equilibrium consumption bundle, for each n and t we can find agents such that $z_1^n/z_k^n \leq [1/N]\cdot[m(C_1^n)/m(C_k^n)] \leq \alpha/N$ and $z_1^t/z_k^t \geq [1/N]\cdot[m(C_1^t)/m(C_k^t)] \geq 1/\alpha N$, where the last inequality in each string follows from the properties of the partition $\mathscr{C}^r$. Using the fact that G is a set of Lipschitz functions, using $\bar{g}^r \in G$, and using the properties of $\mathscr{C}^n$, for all $\epsilon > 0$ there exists $n^*$ such that for all $n,t > n^*$

$$
\begin{aligned}
|p^n(w) - p^t(w)| \leq\ & |\{(\alpha/N)^{1/n}[\bar{g}^n(w) + 2c^*\epsilon]/[\bar{g}^n(w') - 2c^*\epsilon]\} \\
& - \{(1/\alpha N)^{1/t}[\bar{g}^t(w) - 2c^*\epsilon]/[\bar{g}^t(w') + 2c^*\epsilon]\}|.
\end{aligned}
$$

Now note that as r tends to infinity, $(\alpha/N)^{1/r} \to 1$, $(1/\alpha N)^{1/r} \to 1$, $\bar{g}^r(w) \to \bar{g}(w)$, and $\bar{g}^r(w') \to \bar{g}(w')$. Therefore for every $\epsilon > 0$ there exists an $\bar{n}$ such that $n,t > \bar{n}$ implies $|p^n(w) - p^t(w)| \leq |\{(1+\epsilon)[\bar{g}(w) + \epsilon + c^*\epsilon]/[\bar{g}(w') - \epsilon - c^*\epsilon]\} - \{(1-\epsilon)[\bar{g}(w) - \epsilon - c^*\epsilon]/[\bar{g}(w') + \epsilon + c^*\epsilon]\}|$, which tends to zero with $\epsilon$. Since w and w′ are arbitrary, we have $p^n$ converging uniformly to some $p \in L^\infty$. Similar arguments show that $p^n(w) \geq (N\alpha)^{-1/n}\cdot\underline{c}\ /\ \bar{c}$, so p is bounded away from zero.

Next apply Lemma 2 of the appendix to pass to a subsequence (without changing notation) $\{(B_1^n,\ldots,B_N^n)\}_{n=1}^\infty$ such that, for each i, $\lim_{n\to\infty} B_i^n = B_i$ and $\lim_{n\to\infty} \int_{B_i^n} p(x)$

$dm(x) = \int_{B_i} p(x)\, dm(x)$, where $(B_1,\ldots,B_N)$ is a feasible allocation. Now we show that $(p;B_1,\ldots,B_N)$ is an equilibrium.

Since $p^n$ converges to p, consumer i's sequence of wealth in the finite economies, $\{q^n \cdot e_i^n\}_{n=1}^{\infty}$, converges to wealth in the actual economy, $\int_{E_i} p(x)\, dm(x)$. Also,

$$\left| \int_{B_i^n} p^n(x)\, dm(x) - \int_{B_i} p(x)\, dm(x) \right|$$

$$\leq \left| \int_{B_i^n} p^n(x)\, dm(x) - \int_{B_i^n} p(x)\, dm(x) \right| + \left| \int_{B_i^n} p(x)\, dm(x) - \int_{B_i} p(x)\, dm(x) \right|$$

$$\leq \int_{B_i^n} |p^n(x) - p(x)|\, dm(x) + \left| \int_{B_i^n} p(x)\, dm(x) - \int_{B_i} p(x)\, dm(x) \right|.$$

The first term in the last expression tends to zero as n tends to infinity since $p^n$ converges uniformly to p. The definition of $B_i$ implies that the second term in the last expression tends to zero as n tends to infinity. Hence $\int_{B_i} p(x)\, dm(x) = \lim_{n\to\infty} \int_{B_i^n} p^n(x)\, dm(x) = \lim_{n\to\infty} q^n \cdot z^{i,n}$, where $z^{i,n}$ is agent i's consumption bundle at the equilibrium of $\mathscr{E}(\bar{g}_1^n,\ldots,\bar{g}_N^n)$. Since for all n, $q^n \cdot z^{i,n} = q^n \cdot e_i^n$, we have $\int_{B_i} p(x)\, dm(x) = \int_{E_i} p(x)\, dm(x)$. Finally, we must show that any $A \succ_i B_i$ is not affordable. Suppose the contrary, $\int_A p(x)\, dm(x) \leq \int_{B_i} p(x)\, dm(x)$. Then, using local non-minimization of preferences twice, we can find $A'', A' \in \mathscr{B}$, $A'' \subsetneq A' \subsetneq A$ a.s., with $m(A'') < m(A') < m(A)$ and $A' \succ_i B_i$, $A'' \succ_i B_i$. Hence $\int_{A'} p(x)\, dm(x) < \int_{B_i} p(x)\, dm(x)$. So for large n, $\int_{A'} p^n(x)\, dm(x) < \int_{B_i^n} p^n(x)\, dm(x)$ and $A'$ defines an affordable bundle in the economy

$\mathcal{E}(\bar{g}_1^n,\ldots,\bar{g}_N^n)$. Also, using upper semi–continuity of preferences, $\int_{A''} \bar{g}_i^n(x)\ dm(x) \geq \int_{B_i^n} \bar{g}_i^n(x)\ dm(x) - 2m(L)/n$. Hence, for large n, $\int_{A'} \bar{g}_i^n(x)\ dm(x) > \int_{B_i^n} \bar{g}_i^n(x)\ dm(x)$. So for large n, $A'$ is preferred to $B_i^n$ and affordable in $\mathcal{E}(\bar{g}_1^n,\ldots,\bar{g}_N^n)$, which contradicts that $(B_1^n,\ldots,B_N^n)$ is an equilibrium allocation for this economy. Therefore A is not affordable and $(p;B_1,\ldots,B_N)$ is an equilibrium.

Q.E.D.

Note that Lemma 2 of the appendix shows that the set of partitions of L with N elements is compact. If it is assumed that each preference order $\succeq_i$ or utility $u_i$ is upper semi–continuous with respect to our topology, then the existence of Pareto optima of any variety (utilitarian or Rawlsian, for example) is guaranteed. Here we examine price support for such allocations.

> Theorem 2: Under the conditions of Theorem 1, every Pareto optimal allocation is an equilibrium relative to some positive price system.

The assumptions used for the second welfare theorem are necessary in the sense that there are counterexamples without them, one of which is given in section III above.

Proof: Let $(B_1,\ldots,B_N)$ be a Pareto optimal allocation, and set endowments $(E_1,\ldots,E_N) \equiv (B_1,\ldots,B_N)$. Apply Theorem 1 to this vector of endowments to obtain an equilibrium $(p^*;C_1,\ldots,C_N)$. We claim that $(p^*;B_1,\ldots,B_N)$ is also an equilibrium. For if not, for each i $B_i$ is affordable to i at prices $p^*$, so there exists j such that $C_j \succ_j B_j$. By individual rationality, $C_i \succeq B_i$ for all i. Hence $(C_1,\ldots,C_N)$ Pareto dominates $(B_1,\ldots,B_N)$, a contradiction. Therefore $(p^*;B_1,\ldots,B_N)$ is an equilibrium, and we have found supporting prices $p^*$ for $(B_1,...,B_N)$.

Q.E.D.

Theorem 3: If for all i, i = 1,2,...,N, $\succeq_i$ is locally non–satiated, then any equilibrium allocation is in the core, and hence is Pareto optimal.

Proof: Standard.

Theorem 4: If for all i, i = 1,2,...,N, $\succeq_i$ satisfies the assumptions of Theorems 1 and 3, then the core is nonempty.

Proof: Follows immediately from Theorems 1 and 3.

## V. Conclusions

The main goal of this paper was to demonstrate the existence of competitive equilibrium in models with land under reasonably general conditions. The main difficulty in dealing with land is that the natural commodity space, subsets of the plane, has no convex structure. This makes determining general, reasonable assumptions on preferences problematic. Our key assumption, separation by hyperplanes, seems to be a natural analog of standard convexity assumptions and therefore appears "necessary" for competitive equilibrium to exist in general. Of course, it would be useful to know more about the kinds of preferences that satisfy this assumption.

Much work remains to be done to extend the results presented here. First, it will be useful to extend the model to allow transportation cost as, for example, a continuous function of parcels. Second, the model could be extended to allow production, as in Berliant and Jeng (1990). Third, it would be convenient to derive some rules for generating comparative statics in this model. These comparative statics properties could be useful for testing the finite model against the continuum model. Fourth, the mathematics could easily be extended to a more general underlying space. L could simply be a compact subset of a metric space and m could be a positive, nonatomic measure defined on the Borel subsets of L. The proofs of Theorems 1 – 4 would hardly change, since the arguments and mathematical theorems employed can handle a more abstract structure. However, from the point of view of applications, this generalization does not add much.

Finally, it would be interesting to set up a finite model approximately analogous to a monocentric city model. Let L be a homogeneous disk of land in the plane and measure transportation cost as distance to the point at the center of the disk. Let all consumers have identical utilities and endow them with income. Then we will see if the equilibrium and comparative statics are the same (even in their signs or magnitudes) as those derived for the monocentric city model as in Wheaton (1979). Berliant and Fujita (forthcoming) make a first step in this direction by developing Alonso's classical model with a finite number of consumers using the tools of contemporary microeconomics.

## REFERENCES

Aliprantis, C. D., D. J. Brown, and O. Burkinshaw (1987): "Edgeworth Equilibria," *Econometrica*, 55, 1109–1137.

Alonso, W. (1964): *Location and Land Use.* Cambridge: Harvard University Press.

Arnott, R. (1986): "Economic Theory and Housing," manuscript, Queens University.

Arrow, K., and F. Hahn (1971): *General Competitive Analysis.* Amsterdam: North–Holland.

Beckmann, M. (1969): "On the Distribution of Urban Rent and Residential Density," *Journal of Economic Theory*, 1, 60–67.

Beguin, H., and J. F. Thisse (1979): "An Axiomatic Approach to Geographical Space," *Geographical Analysis*, 11, 325–341.

Berliant, M. (1982): "A General Equilibrium Model of an Economy with Land," Ph.D. Thesis, University of California–Berkeley.

________________ (1984): "A Characterization of the Demand for Land," *Journal of Economic Theory*, 33, 289–300.

________________ (1985a): "Equilibrium Models with Land: A Criticism and an Alternative," *Regional Science and Urban Economics*, 15, 325–340.

________________ (1985b): "An Equilibrium Existence Result for an Economy with Land," *Journal of Mathematical Economics*, 14, 53–56.

________________ (1986): "A Utility Representation for a Preference Relation on a $\sigma$–Algebra," *Econometrica*, 54, 359–362.

________________ (forthcoming): "Comments on 'On the Foundations of Land Use Theory: Discrete versus Continuous Populations' by Y. Asami, M. Fujita, and T. Smith," forthcoming in *Regional Science and Urban Economics.*

Berliant, M., and K. Dunz (1990):"Nonlinear Supporting Prices: The Superadditive Case," *Journal of Mathematical Economics*, 19, 357–367.

Berliant, M., K. Dunz, and W. Thomson (forthcoming): "On the Fair Division of a

Heterogeneous Commodity," forthcoming in *Journal of Mathematical Economics.*

Berliant, M., and M. Fujita (forthcoming): "Alonso's Discrete Population Model of Land Use: Efficient Allocations and Competitive Equilibria," forthcoming in *International Economic Review.*

Berliant, M., and H.–W. Jeng (1990): "Supply and Equilibrium in an Economy with Land and Production," *International Economic Review*, 31, 799–828.

Berliant, M., Y.Y. Papageorgiou, and Ping Wang (1990): "On Welfare Theory and Urban Economics," *Regional Science and Urban Economics*, 20, 245–261.

Berliant, M., and T. ten Raa (1988): "A Foundation of Location Theory: Consumer Preferences and Demand," *Journal of Economic Theory*, 44, 336–353.

Berliant, M., and T. ten Raa (1991): "On the Continuum Approach of Spatial and Some Local Public Goods or Product Differentiation Models: Some Problems," *Journal of Economic Theory*, 55, 95–120.

Bewley, T. (1972): "Existence of Equilibria in Economies with Infinitely Many Commodities," *Journal of Economic Theory*, 4, 514–540.

Debreu, G. (1954): "Valuation Equilibrium and Pareto Optimum," *Proceedings of the National Academy of Sciences of the U.S.A.*, 38, 588–592.

________________ (1959): *Theory of Value.* New Haven: Yale University Press.

Dubins, L., and E. Spanier (1961): "How to Cut a Cake Fairly," *American Mathematical Monthly*, 68, 1–17.

Dunford, N. and J.T. Schwartz (1988): *Linear Operators, Part I: General Theory.* New York: John Wiley and Sons.

Dunz, K. (1991): "On the Core of a Land Trading Game," *Regional Science and Urban Economics*, 21, 73–88.

Faden, A. (1977): *Economics of Space and Time.* Ames: Iowa State University Press.

Fujita, M., and T. Smith (1987): "Existence of Continuous Residential Land–Use Equilibria," *Regional Science and Urban Economics*, 17, 549–594.

Glicksberg, I. (1952): "A Further Generalization of the Kakutani Fixed Point Theorem, with Application to Nash Equilibrium Points," *Proceedings of the American Mathematical Society*, 3, 170–174.

Heckman, J., and J. Scheinkman (1987): "The Importance of Bundling in a Gorman–Lancaster Model of Earnings," *Review of Economic Studies*, 54, 243–255.

Hildenbrand, W. (1974): *Core and Equilibria of a Large Economy.* Princeton: Princeton University.

Hill, T. (1983): "Determining a Fair Border," *American Mathematical Monthly*, 90, 438–442.

Isard, W. (1956): *Location and Space–Economy.* Cambridge: M.I.T. Press.

Jones, L. (1984): "A Competitive Model of Commodity Differentiation," *Econometrica*, 52, 507–530.

Koopmans, T., and M. Beckmann (1957): "Assignment Problems and the Location of Economic Activities," *Econometrica*, 25, 53–76.

Kreps, D. (1988): "Static Choice in the Presence of Unforeseen Contingencies," manuscript.

Mas–Colell, A. (1986): "The Price Equilibrium Existence Problem in Topological Vector Lattices," *Econometrica*, 54, 1039–1053.

Munkres, J. (1975): *Topology: A First Course.* Englewood Cliffs: Prentice–Hall.

Papageorgiou, P. and D. Pines (1990): "The Logical Foundations of Urban Economics are Consistent," *Journal of Economic Theory* 50, 37–53.

Rudin, W. (1974): *Real and Complex Analysis.* New York: McGraw–Hill.

Schweizer, U., P. Varaiya, and J. Hartwick (1976): "General Equilibrium and Location Theory," *Journal of Urban Economics*, 3, 285–303.

Shapley, L., and H. Scarf (1974): "On Cores and Indivisibility," *Journal of Mathematical Economics*, 1, 23–37.

Wheaton, W. (1979): "Monocentric Models of Urban Land Use: Contributions and

Criticisms," in *Current Issues in Urban Economics*, ed. by P. Mieszkowski and M. Straszheim. Baltimore: Johns Hopkins University Press.

Zame, W. R. (1987): "Competitive Equilibria in Production Economies with an Infinite–Dimensional Commodity Space," *Econometrica*, 55, 1075–1108.

## APPENDIX

Lemma 1: Let $B^j \in \mathscr{B}\ \forall\ j,\ j = 1,2,\ldots,\infty$ with $\lim_{j\to\infty} B^j = B$, then $\forall\ x \in \mathring{B},\ \exists\ J$ such that $\forall\ j \geq J,\ x \in \mathring{B}^j$.

Proof: By definition of outer Hausdorff convergence, $\lim_{j\to\infty} H((\mathring{B})^c, (\mathring{B}^j)^c) = 0$, so $\forall\ \epsilon > 0$, $\exists\ J$ such that $\forall\ j \geq J$, $(\mathring{B}^j)^c \subseteq B_\epsilon((\mathring{B})^c)$. Now $B_\epsilon((\mathring{B})^c) \to (\mathring{B})^c$ pointwise as $\epsilon \to 0$, so $\forall\ x \in \mathring{B}$, $\exists\ \epsilon > 0$ such that $x \notin B_\epsilon((\mathring{B})^c)$ (since $\mathring{B}$ is open). Let J correspond to this $\epsilon$. Then $\forall\ j \geq J$, $x \notin (\mathring{B}^j)^c$, so $x \in \mathring{B}^j$.

Q.E.D.

Lemma 2: Let $\{p_s\}_{s=1}^S$ be a finite set of functions integrable over L and let $\{(B_i^n)_{i=1}^N\}_{n=1}^\infty \subseteq \mathscr{B}$ be a sequence of partitions of L. Then there exists a subsequence $\{(B_i^{n_r})_{i=1}^N\}_{r=1}^\infty$ and a partition $(B_i)_{i=1}^N \in \mathscr{B}$ of L such that $\forall i \lim_{r\to\infty} B_i^{n_r} = B_i$ and $\lim_{r\to\infty} \int_{B_i^{n_r}} p_s(x)\, dm(x) = \int_{B_i} p_s(x)\, dm(x)$.

Proof: If for some i $\mathring{B}_i^n = L$ infinitely often, the lemma is trivial. Hereafter we only consider the case $\mathring{B}_i^n \neq L$ for all i and n. Then for each i, $\{(\mathring{B}_i^n)^c\}_{n=1}^\infty$ is a sequence of nonempty, compact subsets of L. By Hildenbrand (1974, p.17), there is a subsequence converging in the Hausdorff topology. Since N is finite, there is a subsequence that converges for all i simultaneously. We pass to this subsequence, but call it $\{(\mathring{B}_i^n)^c\}_{n=1}^{\infty}$ for notational simplicity.

The next task is to find a limit. Let $D_i$ be the Hausdorff limit of $\{(\mathring{B}_i^n)^c\}_{n=1}^{\infty}$; this limit exists and is unique. Notice that if $m(D_i \cap D_{i'}) > 0$ for some $i' \neq i$, then by Lemma 1 and Lebesgue's dominated convergence theorem, $m(\mathring{B}_i^n \cap \mathring{B}_{i'}^n) > 0$ for n sufficiently large, a contradiction. Hence $m(D_i \cap D_{i'}) = 0 \ \forall \ i \neq i'$. For notational simplicity, let $C_i = D_i^c$ and let $A = L \setminus (\bigcup_{i=1}^{N} C_i)$. Let $\mathscr{A}$ be the $\sigma$–algebra of measurable subsets of A.

We will now employ Theorem 1 of Dubins and Spanier (1961), which says that if a vector measure $\mu = \mu_1,\ldots,\mu_T$ is nonatomic, the range $\mathscr{R}$ of TxN matrices with elements $\mu_t(A_i)$, where $A_1,\ldots,A_N \in \mathscr{A}$ partitions A, is a convex and compact set of matrices. This theorem is an extension of Lyapunov's theorem. In fact, we only need the compactness part of the result. Notice that for each n, $(A \cap B_i^n)_{i=1}^N$ forms a partition of A. Let $\mu_1 = \int h_1(\cdot,C_1)\, dm$, $\mu_2 = \int h_1(\cdot,C_2),\ldots,\mu_{I\cdot N} = \int h_I(\cdot,C_N)\, dm$, $\mu_{(I\cdot N)+1} = \int p_1\, dm,\ldots,\mu_{(I\cdot N)+S} = \int p_S\, dm$. Using the theorem cited just above, we can draw a subsequence $\{(B_i^{n_r})_{i=1}^N\}_{r=1}^{\infty}$ and a partition $(K_i)_{i=1}^N \in \mathscr{A}$ of A such that

$$\lim_{r\to\infty} \int_{B_i^{n_r} \cap A} h_v(x,C_i)\, dm(x) = \int_{K_i} h_v(x,C_i)\, dm(x),\ v = 1,2,\ldots,I,\ i = 1,2,\ldots,N, \text{ and}$$

$$\lim_{r\to\infty} \int_{B_i^{n_r} \cap A} p_s(x)\, dm(x) = \int_{K_i} p_s(x)\, dm(x),\ s = 1,2,\ldots,S,\ i = 1,2,\ldots,N.$$

Now $K_i \cap C_i = \emptyset$ a.s. $\forall i$. Let $\mathscr{O}$ be a countable dense subset of A. Define $Y_i \equiv C_i \cup (K_i \setminus \mathscr{O})$. Let $B_i = C_i \cup (K_i \setminus \{\mathscr{O} \cup [\mathring{Y}_i \setminus C_i]\})$, as in Berliant and ten Raa (1988, p.347). This is done to make sure that no interior is added to $B_i$ by $K_i$. Then $\mathring{B}_i = C_i$, $B_i \setminus \mathring{B}_i = K_i$ a.s. Now

$$\left| \int_{B_i^{n_r}} h_v(x,C_i)\, dm(x) - \int_{B_i} h_v(x,C_i)\, dm(x) \right|$$

$$= \left| \int_{B_i^{n_r} \cap A} h_v(x,C_i)\, dm(x) + \int_{B_i^{n_r} \cap C_i} h_v(x,C_i)\, dm(x) \right.$$

$$+ \int_{B_i^{n_r} \cap (L \setminus [A \cup C_i])} h_v(x,C_i)\, dm(x)$$

$$- \int_{B_i \cap A} h_v(x,C_i)\, dm(x) - \int_{B_i \cap C_i} h_v(x,C_i)\, dm(x) \Big|$$

$$\leq \left| \int_{B_i^{n_r} \cap A} h_v(x,C_i)\, dm(x) - \int_{K_i} h_v(x,C_i)\, dm(x) \right|$$

$$+ \left| \int_{B_i^{n_r} \cap C_i} h_v(x,C_i)\, dm(x) - \int_{C_i} h_v(x,C_i)\, dm(x) \right|$$

$$+ \left| \int_{B_i^{n_r} \cap (L \setminus [A \cup C_i])} h_v(x,C_i)\, dm(x) \right|$$

The first term tends to zero by choice of $K_i$, while the second term tends to zero by Lemma 1 and Lebesgue's dominated convergence theorem. Using the fact that for each r, $(B_i^{n_r})_{i=1}^{N}$ partitions L, Lemma 1 and Lebesgue's dominated convergence theorem imply that the third term tends to zero as r tends to infinity. An identical argument will work for $p_s$, s = 1,2,...,S, in place of $h_v$. Applying the uniform continuity theorem to each $h_v$, it follows that

$$\lim_{r \to \infty} \int_{B_i^{n_r}} h_v(x,\mathring{B}_i^{n_r})\, dm(x) = \int_{B_i} h_v(x,\mathring{B}_i)\, dm(x).$$

Thus, the appropriate integrals converge. By choice of $B_i$, $\{B_i\}_{i=1}^{N}$ partitions L, and $\lim_{r \to \infty} H((\mathring{B}_i^{n_r})^c,(\mathring{B}_i)^c) = 0$, so $\lim_{r \to \infty} d(B_i,B_i^{n_r}) = 0$.

Q.E.D.