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Abstract: In this paper we study a repeated moral hazard problem for the following incentive schemes: an agent is retained on the job and paid a fixed wage provided he has maintained a specified rate of output during his tenure. We characteize the optimal contract for this class and show that the payoffs under such a contract approach the (first-best) efficient outcome at the rate $O(\delta^{1/2} \ln\delta)$ where δ is the common discount rate of principal and agent.

1. Introduction and Summary

In this paper we study a repeated moral hazard problem in which an agent is paid a constant wage every period and retained on the job provided his past performance has met the following standard: he has maintained a specified rate of profits (or output) during his tenure or if he has not done so, the cumulative losses, net of this rate, are no larger than an acceptable maximum. Within this class of incentive schemes we characterize the optimal contract, show that such a contract approximates the (first-best) efficient outcome if principal and agent are sufficiently patient and, most importantly, give an estimate of the rate of approach to efficiency as the discount rate goes to zero.

A notable feature of many real life contracts is that they specify simple compensation rules; i.e. they identify only a small set of contingencies on which the agent's rewards are conditioned. This is difficult to reconcile with the theoretical work on second-best contracts, i.e the optimal contract when no restrictions have been placed on the types of incentive schemes that are admissible (see Rogerson (1985), Spear-Srivastava (1988) and Lambert (1983)). These papers demonstrate that the second-best contract should subtly condition on various elements of an agent's past performance.[1] Perhaps this seeming paradox can be resolved by modelling the costs of contracting explicitly.

In this paper we start instead by restricting ourselves to the set of simple incentive schemes described above. These schemes have some of the stylised features of observable contracts; in particular they employ the threat of dismissal as an incentive device and use a simple statistic of past performance to determine the necessity of dismissal. Many managerial compensation packages have a similar structure; evaluations may be based on the industry-average of profits.[2] Insurance contracts in which full indemnity coverage is provided only if the number of past claims is no larger than a prespecified number is a second example. Sales or franchise contracts which are renewed only if the volume of past business is sufficiently large is a third example.

The simple version of dismissal-contingent schemes that we study here were introduced by Radner (1986) who called them bankruptcy schemes. He showed that such incentive schemes generate almost efficient outcomes if principal and agent are sufficiently patient. We add to the analysis of that paper by explicitly characterizing the optimal contract within the class of bankruptcy schemes. Such a characterization demonstrates the optimal mix of incentive and insurance considerations in such contracts.

One difference between complete-information repeated games and repeated moral hazard is that as long as players are not infinitely patient, i.e. have discount rates of zero, exact optimality cannot be sustained under moral hazard. Consider the case of a risk-neutral principal and risk-averse agent. Clearly, even in the repeated context, all Pareto optimal outcomes require that the agent's compensation be constant regardless of the output consequences of his actions. So no "punishment" is possible since compensation has to be independent of the only observable variable. Hence, any asymptotic efficiency result is necessarily an approximate one. We know from a number of papers (Fudenberg-Maskin (1986), Radner (1981, 1985, 1986), Rubinstein (1979) and Rubinstein-Yaari (1983))[3] that there are long-term contracts that are asymptotically approximately efficient, as the players' common discount rate goes to zero. If we believe, however, that the "true" model involves discounting then the natural question to ask is, how good are these approximations? For any $\varepsilon > 0$, how patient do principal and agent need to be in order to sustain ε-optimality? In other words, what is the rate of convergence to optimality in a repeated moral hazard situation? A principal purpose of this paper is to give a first answer to this question, within the restricted class of bankruptcy contracts. To the best of our knowledge this is the first estimate in the literature of the rate of convergence to efficiency under asymmetric information.

We turn now to a discussion of the model and results. The action of the agent in any period determines a distribution of returns in that period. Hence cumulative returns up to any period follow a random walk. For reasons discussed in the sequel we analyze the model in continuous time; i.e. cumulative returns follow a controlled diffusion process (with the distribution of increments at any instant determined by the agent's action at that instant). The agent's action also gives him instantaneous utility. The agent trades off myopic utility maximization against improvement of tenure prospects. When the services of any one agent are terminated the principal is free to hire another identical agent and offer a second (possibly different) bankruptcy contract. Further, we assume throughout that the principal commits to his offer and cannot dismiss an agent who is performing satisfactorily. The principal picks a bankruptcy contract to maximize his (risk-neutral) returns subject to incentive and individual rationality considerations.

Optimal Choice of an Agent. The agent's optimal choice problem is an example of a more general survival problem in stochastic control (see Dutta (1990)). We use results from the general formulation to give a characterization of the optimal choice when the set of feasible drift-variance choices is an arbitrary convex, compact set in $\mathbb{R}^2$ (Theorems

3.1-3.3). We then explicitly compute the optimal policy of the agent in the binary case, in which the agent has only two actions (Theorem 3.4). The charcterization in the binary case serves to illustrate the general results. It is shown that the optimal policy conditions the current action on the current level of net aggregative output alone. The optimal policy progressively shirks, that is the higher is aggregative output the higher is the instantaneous utility and (under some additional conditions) the smaller the incremental mean of the control employed. A risk-neutral principal would like to have the control with the maximum mean used throughout, and the coasting by the agent at "safe" output levels is precisely a measure of the inefficiency of moral hazard, from the principal's point of view.

Principal's Contract Choice The principal's return, with a (stochastic) succession of agents, is derived in Section 5. It is shown that in an optimal contract the allowable shortfall is the smallest one consistent with individual rationality. This is not a priori obvious since a lower acceptable loss (or insurance level) implies quicker bankruptcies and hence associated inefficiencies for the principal. On the other hand we get an efficiency-wage-type result, that the optimal wage is **higher** than the minimum wage consistent with individual rationality.

Pareto Optimality The first-best arrangement involves a constant control exercised by the agent, no dismissal and full insurance by the risk-neutral principal, by way of paying an outcome-independent wage (Propositions 5.1 and 5.2).

Asymptotic Efficiency We conclude the investigation of bankruptcy contracts by showing that the values under the optimal bankruptcy contract converge to the first best at a rate at least as fast as $O(\delta^{1/2} \ln \delta)$, as $\delta \to 0$, where δ is the (common) discount rate of principal and agent.

The principal-agent model is described in detail in Section 2. Section 3 discusses the optimal response of an agent to a bankruptcy compensation scheme. The principal's choice-of-contract problem is analyzed in Section 4. Section 5 contains the characterization of the Pareto optimal policies, whereas the analysis leading to a derivation of the rate of convergence to Pareto optimality is in Section 6. Bibliographic notes and a discussion of possible extensions of the current analysis may be found in Section 7.

2. The Model

2.1 Some Preliminaries

Let $[B(t): t \geq 0]$ be a standard Brownian motion on some probability space $(\Omega, \mathfrak{I}, P)$. Let $\mathfrak{I}_t$ be the smallest family of sub σ-fields generated by the Brownian motion, i.e. $\mathfrak{I}_t$ is the smallest σ-field with respect to which $B(s)$, $s \in [0,t]$ is measurable. Let $[f(t): t \geq 0]$ be a $\mathfrak{I}_t$-adapted process[4] which further satisfies

i) $P[\omega : \int_0^t f^2(\omega,s)ds < \infty] = 1$, for each $t \geq 0$.

The stochastic integral $\int_0^t f(s)dB(s)$ is well-defined for all $t \geq 0$ a.e. A stochastic process $[\hat{Y}(t) : t \geq 0]$ is said to be a diffusion if it can be written as:

$$\hat{Y}(t) = \hat{Y}(0) + \int_0^t m(s)ds + \int_0^t v^{1/2}(s)dB(s), \tag{2.1}$$

where $[m(t) : t \geq 0]$ and $[v(t) : t \geq 0]$ are $\mathfrak{I}_t$-adapted and satisfy i), and $\hat{Y}(0)$ is some constant. The functions $m(\cdot)$ and $v(\cdot)$ are respectively, the drift and variance components of the process.

2.2 A Principal-Agent Model and Bankruptcy Schemes

Let us start with a description of bankruptcy schemes in discrete time. At any period, say nh, an agent picks an action which conditions a distribution for the uncertain output in that period; call this (random) output R(nh). The agent is expected to maintain a rate of output, say k, during his tenure. In particular, the net output in that period is then R(nh) - kh. If we denote the cumulative net output till period nh as Y(nh) then this grows according to the following equation

$$Y(nh + h) = Y(nh) + R(nh) - kh \tag{2.2}$$

If the agent is able to maintain the required output rate then Y(nh) is evidently nonnegative. Suppose that the agent is unable to maintain this output rate, possibly on account of "bad luck". To allow for this eventuality a bankruptcy scheme allows the agent to run up some losses, say up to y, before his tenure is terminated. Equivalently, the agent is set up with an initial output level $y > 0$ and has his services terminated the first time at which Y(nh) is less than or equal to zero. (2.2) clearly defines a controlled random walk.

One problem with discrete time is that the agent could go bankrupt with any non-positive level of cumulative output. The agent's continuation value should be made contingent on the level of terminal output but there is no obvious way in which to assign this value. In turn any assignment clearly affects in a fundamental way the agent's optimal choices while on the job. To avoid such "overshooting at the boundary" problems we choose to model the principal agent question in a continuous time framework where such problems are absent. We turn now to the continuous time analog of (2.2).

An agent controls a diffusion (the cumulative output) process. The agent's action is the choice of a feasible instantaneous drift-variance pair [m(t),v(t)]. Let the set of feasible mean-variance choices be denoted A. A choice at t conditions on the observable history of output during [0,t). An admissible strategy for the agent is a pair of $\mathfrak{I}_t$-adapted processes [m(t):t≥0] and [v(t):t ≥ 0] in which $(m(t,\omega),v(t,\omega)) \in A$ for all (t,ω) and which lead to a solution of the following stochastic differential equation:

$$Y(t) = Y(0) + \int_0^t m(Y(s))ds + \int_0^t v^{1/2}(Y(s))dB(s) - kt, \quad t \geq 0.$$

Note that there are several interpretations possible for the formulation in which the agent directly picks instantaneous drift and variance. One interpretation is that the agent chooses from a menu of available projects or techniques, each involving different levels of supervision or skill or effort and having a mean and a variance. An alternative interpretation would be that the agent chooses (possibly multi-dimensional) effort and each level of effort corresponds to an instantaneous drift-diffusion pair. To define a termination date, for any strategy π and initial output $y > 0$, let

$$T_\pi(y) \equiv \inf \{t \geq 0: Y(t) = 0 | \; Y(0) = y, \pi\}.$$

Let the constant wage paid be denoted w and let the agent's instantaneous utility function be called U. For any given (w,k,y) the discounted utility over an agent's uncertain lifetime, for a strategy π, is[5]

$$g_\pi(y) = E \; \delta \int_0^{T_\pi(y)} e^{-\delta s} \; U(w,m(s),v(s))ds.$$

To complete the formulation of the moral hazard problem let the principal's discounted lifetime earnings under a compensation triple (w,k,y) and agent's strategy π be denoted $H(w,k,y;\pi)$.[6] (In Section 4 we will explicitly derive $H(\cdot)$.) Then the optimal contract choice problem for the principal is

$$\text{Max } H(w,k,y;\pi),$$

$$\text{s.t.} \quad g_\pi(y) \geq g_{\pi'}(y) \quad \text{for any admissible } \pi', \tag{2.3}$$

$$g_\pi(y) \geq \hat{U} \tag{2.4}$$

Condition (2.3) is the incentive constraint and (2.4) is the individual rationality constraint.

The Pareto-optimality or first-best problem is that of maximizing the principal's discounted lifetime earnings H subject only to the individual rationality constraint, i.e. in the absence of moral hazard. For this problem we shall not restrict the set of feasible contracts. The precise formulation is discussed in Section 5.

3. Incentive Constraint Analysis: The Agent's Problem

The agent's best response problem is: given w, k and y, maximize $g_\pi(y)$ over the set of admissible policies. This is clearly a stationary dynamic programming problem, and we shall denote its value function by V(y;w,k). In much of what follows, we shall concentrate on the effect of changes in the aggregate output level Y(t) (equivalently changes in allowable shortfall). Hence the dependence of the value function on w and k will frequently be suppressed and it shall be written simply as V(y). Any solution will be called an optimal strategy or policy. If an optimal policy picks controls that depend only on the level of current aggregate output, it will be called a stationary Markov optimal policy.

We make the following assumptions throughout:

(A0) $\quad \sup_{(m,v)\,\varepsilon A} U(w,m,v) \equiv \bar{U}(w) > 0 \quad \text{for all } w \geq 0$

(A1) $\quad \text{Inf } \{v\text{: } (m,v) \in A\} > 0$

Since the severance pay has been normalized to zero, (A0) is a minimal necessary assumption for a bankruptcy scheme to have any incentive effects at all. (A1) says that the agent's actions lead to uncertain outcomes. Clearly this is necessary for the principal's inference problem to be non-trivial. We also assume

(A2) The set of feasible controls A is a convex, compact set.

(A3) The utility function U(w,m,v) is continuous and strictly concave in the last two arguments.

The agent's best response exercise is an example of a general survival problem in stochastic control (it is in fact a version of the gambler's ruin problem). In the formulation here, an instantaneous choice is being made simultaneously along three dimensions: drift, variance and utility. In previous investigations authors have allowed a choice over drift and variance (holding utility constant) or have allowed a choice over drift and utility (holding variance constant).[7] Three dimensional trade-offs are extremely difficult to characterize in a transparent way. Dutta (1990) has investigated the general control problem and we use those results to describe some basic properties of the agent's optimal choice. To add to the intuition we then explicitly compute the optimal policy in the case where the agent has only two actions available at every instant.

Let (w,k) be fixed until further notice. The following characterization of the value function holds:

Theorem 3.1 i) **The value function V(y) is strictly increasing in y.**
ii) **(Bellman equation) V is C^2 and satisfies the optimality equation**

$$\max_{(m\,,\,v)\in A} \left\{\frac{1}{2}vV''(y) + (m - k)V'(y) - \delta V(y) + \delta U(w, m, v)\right\} = 0 \quad y \geq 0 \qquad (3.1)$$

iii) **The marginal valuation satisfies the following**

$$V'(y) = V'(0)Ee^{-\delta T^*(y)}$$

where $T^*(y)$ is the termination date under the agent's optimal policy. Consequently, the value function is strictly concave.

Remark: The proof of Theorem 3.1 may be found in Dutta (1990).

Letting the parameters (w,k) vary temporarily, the following comparative statics and boundary properties of the value function are easy to see:

Proposition 3.2 i) **The value function V(y; w,k) is increasing in w, provided the utility function is increasing in w, and decreasing in k. Further, it is continuous in (w,k).**
ii) $\lim_{y\to\infty} V(y;\, w,\, k) = \bar{U}(w)$, **for all k, and** $V(0;\, w,\, k) = 0$, **for all (w,k).**

Turning to a characterization of the optimal strategy, we first define a stationary Markov policy β: $\mathbb{R}_+ \to A$ to be interior if $\beta(y) \in \text{int}.A$ for all $y \in \mathbb{R}_+$. Further, the utility function is separable if there exist functions $\xi_w(m)$ and $\phi_w(v)$ such that U(w, m, v)

$= \xi_w(m) - \phi_w(v).$

Theorem 3.3 i) **There is a unique stationary Markov optimal policy $\beta^* \equiv (m^*, v^*)$: $\mathbb{R}_+ \to A$, and this policy is given by the maximizers from (3.1). Furthermore, β^* is a continuous function.**

ii) $y' > y$ implies that either or both of the following conditions hold: a) $v^*(y') \geq v^*(y)$ or b) $m^*(y') \leq m^*(y)$, $\frac{m^*(y')-k}{v^*(y')-k} \leq \frac{m^*(y)-k}{v^*(y)-k}$. In words, as the cumulative output level grows, the agent switches to high variance and/or low mean options.

iii) Suppose that U is separable and β^* is interior. Then, $y' > y$ implies that $m^*(y') < m^*(y)$ and $v^*(y') > v^*(y)$. If U is decreasing (resp. increasing) in m (resp. v), then $U(w,m^*(y'),v^*(y')) > U(w,m^*(y),v^*(y))$; i.e. at high cumulative output levels the agent employs high variance-low mean actions that give him higher instantaneous utility.

Proof: That any selection from the maximizers correspondence of the optimality equation defines a stationary Markov optimal policy follows from a standard argument via Ito's lemma (e.g. see Krylov (1981, 1.1 and 1.4)). By the Maximum theorem of Berge (1963) and the fact that the value function is C^2, this correspondence is upper hemi-continuous. From the strict concavity of the utility function, the set of maximizers is actually single valued for every y. Hence this function, β^*, is continuous.

Suppose we denote the optimal choice at y' by (m',v') (respectively the optimal choice at y by (m,v)). Then it follows from the optimality equation that

$$\frac{1}{2}(v - v')[V''(y) - V''(y')] + (m - m')[V'(y) - V'(y')] \geq 0 \tag{3.2}$$

$$\left(\frac{m-k}{v} - \frac{m'-k}{v'}\right)[V'(y) - V'(y')] - \delta\left(\frac{1}{v} - \frac{1}{v'}\right)[V(y) - V(y')] \geq 0 \tag{3.3}$$

Dutta (1990) Theorem 3.1 establishes that V'' increases in y. That combined with Theorem 3.1 and (3.2)-(3.3) yields the second part of the theorem. In the separable utility case, first-order conditions yield

$$V'(y) = -\delta\xi_w'(m^*(y)) \qquad V''(y) = \delta\phi'(v^*(y))$$

From the strict concavity of V (Theorem 3.1 iii), the third part of the theorem follows. □

The order of usage of the drifts points directly to the inefficiency, from the principal's point of view, that persists under a bankruptcy incentive scheme. At low cumulative output levels, with the threat of dismissal near, the agent does in fact forego instant gratification to boost immediate returns for the principal. However at higher and safer levels, after a run of good luck or "hard work," the agent coasts on his laurels. Of course, if the principal could renege on his commitment to the bankruptcy contract, this is precisely when he would like to do so, and dismiss an agent in order to hire a new one for whom the threat of dismissal is more effective.[8]

In order to compute the agent's optimal policy in a specific case we now examine the **binary choice** problem; the set of feasible actions contains two elements (m_1,v_1) and (m_2,v_2). This problem was first studied by Sheng (1980) and the results that follow are variants of her results. Our formulation is somewhat different. For that reason and for completeness, the relevant computations are reported in Appendix 1. Denote $U_i \equiv U(w,m_i,v_i)$, $i = 1,2$ and suppose that $U_2 \geq U_1$.

The principal result states that faced with a bankruptcy scheme the agent finds it optimal to employ a **swichpoint strategy** of the following kind: above a critical aggregate output level $\hat{y}$ the agent uses control 2 while below $\hat{y}$ the agent switches to the other control in order to improve tenure prospects. As long as the preference between the two controls is strict, i.e. $U_2 > U_1$, the agent must eventually shirk, i.e. $\hat{y} < \infty$. Typically the high utility action will also be the control with a lower mean, and hence that which the principal does not want employed. We compute the optimal switchpoint as a function of w and k (and the rate of impatience δ).

A stationary Markov policy $\beta : \mathbb{R}_+ \rightarrow A$ is called a switchpoint policy if

$$\beta(y) = I[0,\hat{y})\ (m_1,v_1) + I[\hat{y},\infty)\ (m_2,v_2), \quad \hat{y} \in \mathbb{R}_+ \cup \{\infty\}$$

where I(C) is the indicator function on a set C. Consider the quadratic function $\frac{1}{2} v_i x^2 + (m_i - k)x - \delta = 0$ and denote its negative (resp. positive) root $-\lambda_i$ (resp. Θ_i)

Theorem 3.4 i) **There is a unique stationary Markov optimal policy for the agent's problem, and this policy is a swichpoint strategy.**

ii) **Suppose that $U_2 > U_1$. Then the optimal switchpoint $\hat{y}$ is finite. It is zero iff**

$$\xi_1(\lambda_2) \geq \frac{\delta(U_1 - U_2)}{U_2} \tag{3.4}$$

where $\xi_1(\lambda_2) = \frac{1}{2}v_1(-\lambda_2)^2 + (m_1 - k)(-\lambda_2) - \delta$.

iii) Suppose $U_2 = U_1$. The optimal policy is: exclusive use of control 1 (i.e. $\hat{y} = \infty$) if $\lambda_1 > \lambda_2$ or exclusive use of control 2 (i.e. $\hat{y} = 0$) if $\lambda_2 > \lambda_1$. If $\lambda_1 = \lambda_2$, the agent is indifferent at all output levels, between the two controls.

iv) The value function satisfies all of the properties that hold for the general case (Theorem 3.1).

Consider the interesting case: $U_2 > U_1$ and $m_2 < m_1$. Simple algebra shows that $\xi_1(\lambda_2) \lesseqgtr 0 \Leftrightarrow \lambda_1 \gtreqless \lambda_2$. Further as $\delta \downarrow 0$, $\lambda_i(\delta) \rightarrow \max\left[\frac{m_i - k}{v_i}, 0\right]$. So (3.4) implies that both controls are used by a patient agent if $\frac{m_1 - k}{v_1} > \max\left[\frac{m_2 - k}{v_2}, 0\right]$. It will be shown in appendix 1 that eventually only the principal's preferred action (control 1) is used; i.e. as $\delta \downarrow 0$, $\hat{y} \rightarrow \infty$.

Note that using Theorem 3.4iv) it can be shown that all of the subsequent analysis, which is proved for the general case, will also hold for the binary case.

4. Optimal Principal-Agent Contracts

4.1 Principal's Problem

Given the agent's best response and the individual rationality constraint, the principal picks a contract triple (w,k,y) to maximize net receipts. There are two alternative ways to model the principal's returns. In the first, a "cash reserve" interpretation which we now detail, the specified rate of return k is an actual outflow. The principal pays the agent compensation w and receives a dividend k - w every period. The cumulative index Y is then a cumulative cash reserve. The principal sets the agent up with initial cash y and dismisses him when the cash reserve runs down to zero, and hires in turn a second agent. There is every period an interest payment on the initial cash y. Let us suppose the principal's discount rate is also δ (this is unnecessary for the analysis in the current section but will be required in Section 5). The principal's net reciepts, denoted $H(w,k,y;\pi)$, when the agent follows a strategy π and so do succesive agents, is

$$H(w,k,y;\ \pi) = E\delta \int_0^{T_\pi} e^{-\delta s}[k - w - \delta y]ds - Ee^{-\delta T_\pi}[\delta y - H(w,k,y;\ \pi)]$$

Note that there is of course no loss in generality in restricting successive agents to the same best response strategy. Collecting terms

$$H = k - w - \frac{\delta y}{1 - Ee^{-\delta T}} \tag{4.1}$$

where $1 - Ee^{-\delta T} = E\delta \int_0^{T} \pi\, e^{-\delta s}\, ds$, is the expected discounted time to failure by the agent. No matter which generation of agent is currently employed, the principal always gets per period returns of k - w. However, the discounted average cash outlay, $\frac{\delta y}{1 - Ee^{-\delta T}}$, depends on the agent's best response. The principal's problem is:

$$\max_{(w,k,y) \in \mathbb{R}^3_+} \quad k - w - \frac{\delta y}{1 - Ee^{-\delta T^*(y)}}$$

$$\text{s.t.} \quad V(y;\ w,k) \geq \hat{U} \geq 0$$

where $\qquad T^*(y) = \min\ \{t\colon\ Y(t) = 0;\ Y(0) = y,\ \pi = \beta^*(k,w)\}.$

A second interpretation of the principal's returns is one in which the rate of return k is not an actual outflow but is used to keep a "score" of the agent's performance. All of the incremental return, dY + k accrues to the principal out of which he pays the agent the constant compensation w. The conclusions under these two interpretations are similar (indeed as $\delta \to 0$, the two returns converge to the same limit) and so in this paper we pursue only the cash reserve interpretation.[9]

4.2 Individual Rationality

The requirement that the agent be able to make at least the reservation utility $\hat{U}$, restricts the set of feasible contracts. Recall that $\breve{U}(w)$ is defined as the highest instantaneous utility when the prevailing wage is w. Define the minimum wage $\underline{w}$ as

$$\breve{U}(\underline{w}) = \hat{U}.$$

Note that the minimum wage is independent of the rate of return k. We know from Proposition 3.2 that the the agent's value in a bankruptcy scheme is bounded above by $\breve{U}(w)$. Clearly, any compensation scheme offered by the principal must pay a wage at least as large as $\underline{w}$. Further, define y*(w,k) as the minimum security level for fixed (w,k)

$$V(y^*(w,k);\ w,k) = \hat{U}, \qquad w \geq \underline{w}$$

Given Theorem 3.1 and Proposition 3.2, y*(w,k) is well-defined, and indeed is decreasing in w and increasing in k. The set of feasible compensation schemes then is

$$B = \{(w,k,y) \in \mathbb{R}^3_+ : \; w\in[\underline{w}, k], \;\; y \geq y^*(w,k)\}$$

4.3 Loss Level Choice

For fixed (w,k), the principal picks a loss level $y \geq y^*$, to minimize the expected discounted per period setup costs $\frac{\delta y}{1 - Ee^{-\delta T}}$. The loss level is one mechanism by which the principal transfers risk to the agent. Note the a priori conflicts: a lower loss level implies smaller interest payments for the principal but also quicker failure on the part of the agent and hence a more rapid outlay of initial capital by the principal.

Proposition 4.1. **For fixed (w,k) the optimal loss level choice is $y^*(w,k)$.**

Proof. From Theorem 3.1, $V'(y) = V'(0)\, Ee^{-\delta T^*(y)}$. Since V is C^2, it follows that $V''(y) = V'(0) \frac{d}{dy} Ee^{-\delta T^*(y)}$. Dutta (1990, Theorem 3.1) shows that V" increases in y. It then follows that $Ee^{-\delta T^*(y)}$ is a decreasing, convex function, or equivalently that $1 - Ee^{-\delta T^*(y)}$ is an increasing, concave function. Hence $\frac{\delta y}{1 - Ee^{-\delta T^*(y)}}$ is minimized over $[y^*,\infty)$, at y^*.■

It is somewhat surprising to find that in the model under study, under reasonable general conditions, the principal finds it optimal to transfer all the risk that can be feasibly transferred through the loss level mechanism. The result may not hold when there are costs to new hires, e.g. when there are training costs for new agents. However, it is still the case that an optimal loss level choice will exist in general. This is so since the principal's returns tend to $-\infty$, as $y \uparrow \infty$.

4.4 Compensation Level Choice

Given the results of the previous sub-section the optimal choice of a compensation level involves the maximization of $(k - w) - \frac{\delta y^*}{1 - Ee^{-\delta T}}$. A lower compensation w increases the net dividend to the principal, k - w (for k fixed). Since the agent's value increases in w (Proposition 3.2), in order to guarantee the agent expected utility $\bar{U}$, the tolerable loss level y^* has to increase, thereby raising interest payments for the principal. This is the direct cost of lowering w. There is a further indirect cost, in that the agent's best response is affected, and he may be moved to take actions which lead to (stochastically) more frequent failure.

Proposition 4.2 **For fixed k, there is an optimal choice of compensation level w^*,**

with $\underline{w} < w^* \leq k$.

Proof. The optimization problem is: Minimize $w + \frac{\delta y^*(w)}{1 - Ee^{-\delta T_w(y)}}$, over w in $(\underline{w},k]$, where we write $T_w(y)$ to denote the termination date when an agent uses his optimal response for wage w and initial output y. It is easy to see that as w is lowered to $\underline{w}$, $y^*(w) \to \infty$. Since the expected discounted time to failure $1 - Ee^{-\delta T_w(y^*)}$ is bounded between 0 and 1, the minimand goes to ∞, as $w \downarrow \underline{w}$. The agent's value function V is a continuous function of w and hence so is y^* (and $1 - Ee^{-\delta T}$). So a minimum is achieved over $(\underline{w},k]$. □

Lowering the agent's compensation increases the principal's dividend linearly but also increases the expected debt and the latter increases "infinitely" fast as the compensation is lowered to the minimum wage. The result, that wages are strictly higher than minimum wage, looks like an efficiency wage conclusion although the explanation here is a combination of incentive and individual rationality arguments and therefore different from the standard purely incentive-based argument.

4.5 Rate of Return Choice and the Optimal Contract

The final component of the principal's choice problem is to pick an expected rate of return k to maximize $k - w^* - \frac{\delta y^*}{1 - Ee^{-\delta T}}$. The incentive and individual rationality considerations are similar to the case of w. An increase in the standard makes the agent more receptive to tenure considerations (which the principal prefers). On the other hand the (binding) individual rationality constraint implies that the allowable shortfall has to be larger. It is our conjecture that the optimal choice of lies between the highest and lowest drifts. We have however not been able to prove this.

5. First-Best Analysis

The first-best or Pareto-optimality problem is one of maximizing the prindipal's net receipts subject to the individual rationality constraint, but in the absence of moral hazard. There are two differences consequently: firstly, since the agent's actions are observable (and agents are identical) there is no need for dismissal as an incentive device. Secondly, actions are taken so as to maximize a (weighted) sum of principal and agent utilities. The principal result of this section shows that if the agent's utility is separable in compensation and action and the agent is risk averse, then the Pareto-optimal policy is to choose always the control that maximizes an appropriate weighted sum of instantaneous returns. Formally, the first best problem is,

$$\text{Maximize} \quad E\delta \int_0^\infty e^{-\delta s} \left\{[dY + kds] - w(s)ds\right\}$$

$$\text{s.t.} \quad E\delta \int_0^\infty e^{-\delta s}\, U(w(s),(m(s),v(s)))ds \geq \hat{U},$$

where the compensation scheme $[w(t) : t \geq 0]$ is some $\mathfrak{I}_t$-adapted process. In the remaining sections we shall strengthen our assumptions on the agent's instantaneous utility.

(A.4) U is separable in compensation and action; $U(w,m,v) = u(w) + q(m,v)$. Further u is increasing and strictly concave, and $\lim_{w \to \infty} U'(w) = 0$.

Recall that the principal and agent's discount rates are the same. Given the agent's risk-aversion and identical discount rates standard arguments show that the principal should completely insure the agent in the first-best strategies.

Proposition 5.1. **Under (A.4) the agent's compensation is constant in the first-best strategies, i.e., $w(\omega,t) = w$, for all (ω,t) in $\Omega \times [0, \infty)$.**
Proof. See appendix 2.

Define the weighted first-best problem as

$$\text{Max } (1 - \lambda)\left\{E\delta \int_0^\infty e^{-\delta s} [dY + kds] - w\right\} + \lambda\left\{E\delta \int_0^\infty e^{-\delta s} q(m(s),v(s))ds + u(w)\right\} \qquad (5.1)$$

where λ is in [0, 1].

It is well-known that the principal-agent values generated by the weighted first-best problem as λ varies, are exactly the Pareto optimal values. Further we have

Proposition 5.2. **For any λ in [0, 1), a solution to (5.1) is a strategy using control $(\tilde{m},\tilde{v})$ exclusively, and a compensation $\tilde{w}(\lambda)$ where,**

i) $(\tilde{m},\tilde{v}) \in \text{argmax } [(1 - \lambda)m + \lambda q(m,v)]$

ii) $\tilde{w}(\lambda)$ is the (unique) maximizer of $\lambda u(w) - (1 - \lambda)w$, $w \geq 0$.

Proof. See appendix 2.

The reason that constant use of a single control is optimal is clear. The principal, in the formulation of the first-best problem above, is assumed to have an infinite pocket. Hence principal (and agent) at every instant face an infinite horizon problem which is invariant over the cumulative profits to-date. So myopic optimization, i.e. maximization of (weighted) one-period utilities, is dynamically optimal. With possible firm bankruptcy the simple results here would no longer hold; note however that in an optimal solution the probability of the principal and agent accumulating negative infinite wealth is zero.[10]

6. Convergence to First-Best Utilities

The following result bounds the rate at which principal-agent values under bankruptcy contracts approach first-best efficiency. Clearly this is a lower bound for the rate at which second-best values approach efficiency. It is an open question as to how tight these bounds are.

Proposition 6.1 For any first-best values (G, H), there exist principal-agent contracts $(w(\delta),k(\delta),y(\delta))$ such that

$$1 - \frac{V_\delta(y(\delta);\ w(\delta),k(\delta)}{G} = O(\delta^{1/2} \ln \delta)$$

$$1 - \frac{H_\delta(y(\delta);\ w(\delta),k(\delta)}{H} = O(\delta^{1/2} \ln \delta).$$

Proof. See Appendix 2.

Remark: The arguments in the proof of Proposition 6.1 are completely independent of the particular Pareto optimal point that is being approximated. So, the result is really a statement on the rate of uniform convergence of the principal-agent value frontier to the Pareto optimal first-best frontier.

7. Discussion and Extensions

Second-best contracts have been characterized by Lambert (1983), Rogerson (1985), Holmstrom-Milgrom (1987) and Spear-Srivastava (1988). The work of Rogerson (1985) and Lambert (1983) has shown that second-best incentive schemes will, in general, have "memory"; compensations in any period will depend in a subtle manner on previous compensations and/or outcomes. The theoretical reason for this is the fact that although optimal contracts will depend in the expected manner on the information revealed by observed outcomes, this information can however be linked quite arbitrarily to the outcomes themselves. Spear-Srivastava (1988) provide some reduction in the dimension of contingent variables. They show that the second-best scheme conditions on current output and the agent's expected continuation value. Unfortunately this last statistic is not easy to relate to any aggregate of outcomes. As mentioned in the introduction these results are difficult to reconcile with the simplicity of observed contracts.

Holmstrom-Milgrom (1987) show that if principal and agent utilities are multiplicatively separable and exponential then the second-best contract has the attractive feature of being a succession of short-term contracts (and indeed is linearly related to observed outcomes). Fellingham-Newman-Suh (1985) isolated a couple of other configurations of principal and agent preferences for which the same conclusion holds. Unfortunately, the linear short-term characterization is very delicately predicated on the constant absolute risk aversion specification of preferences.

As described in the introduction, a line of research has indeed looked at some simple schemes, and shown that any single-period efficient utility level can be attained arbitrarily closely by such schemes in the limit (Radner (1981,1985,1987) Rubinstein (1979) and Rubinstein and Yaari (1983)). This is the literature that motivated us directly. In particular, we have tried in this paper to complement the findings of this line of inquiry by providing a direct analysis of the optimal principal-agent contracts (within a class of simple schemes) and by providing an estimate of the rate of approach to efficiency. Note also that Fudenberg-Maskin (1986) employ ideas used in the oligopoly context by Abreu-Pearce-Stachetti (1986) to study the entire set of sustainable payoffs in settings of imperfect information more general than the repeated moral hazard problem. They establish the asymptotic sustainability of all individually rational payoffs (and hence first-best payoffs) under some conditions.

In an interesting paper Fudenberg-Holmstrom-Milgrom (1986) argue the general point that if the agent is allowed to insure himself, then some of the insurance that the principal has to provide in standard contracts without this feature, becomes unnecessary. In particular they show that when the preferences of principal and agent are additively separable and of the constant absolute risk aversion class, then long-term contracts can be replaced with a succession of second-best short-term contracts. In this context it is worth noting that Yaari (1976) has shown that, for some specifications of bankruptcy, a patient risk-averse agent subject to income fluctuations finds it optimal to consume every period the expected income; i.e. to behave as a risk-neutral agent. This suggests the conjecture that if the agent is allowed to self-insure and is made the residual claimant in our model, then the resulting outcomes would again approximate the first-best, provided principal and agent are sufficiently patient.[11]

Two possible generalizations of the model can be attempted. First one can study compensation schemes in which the agent's compensation is linked directly to immediate performance (as well as indirectly through the possibility of being fired), salary plus bonus schemes. Secondly, as discussed above, the agent can be allowed to insure himself, allowing for the smoothing of consumption across periods even when income is erratic. In a model incorporating these features, it is thus far possible to derive some general results,[12] but not enough to allow an explicit characterization of the optimal contract choice.

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Footnotes

1 Fellingham-Newman-Suh (1985), Holmstrom-Milgrom (1987) and Fudenberg-Holmstrom-Milgrom (1986) have however established that under some specifications of preferences for principal and agent, history-independent short-term contracts are (constrained) optimal. See the discussion in Section 7.

2 Of course, managerial compensations typically also contain bonus provisions which directly reward immediate performance.

3 With no discounting and an infinite horizon, Rubinstein (1979) showed that exact optima could be sustained, whereas Radner (1981) established the sustainability of approximate optimality in sufficiently long but finite horizon contexts. Fudenberg-Maskin (1986) and Radner (1985, 1986) show that there exist contracts which approximate efficiency in discounted models, for sufficiently small discount rates. Indeed in all of these papers the contracts are additionally incentive compatible for the principal as well.

4 A stochastic process $[f(t) : t \geq 0]$ on $(\Omega, \Im)$ is said to be $\Im_t$-adapted if $f(\omega, t)$ is jointly measurable in ω and t, and ii) $f(\cdot, t)$ is $\Im_t$-measurable, for each $t \geq 0$.

5 Implicit in the formulation is the assumption that once fired the agent receives a severance pay or a reassignment to different sinecured position etc., i.e., options which yield some constant value. That value has been normalized to zero.

6 Strictly speaking, since the principal hires a new agent if and when the current agent fails to meet performance requirements, his returns are derived from a succession of compensation schemes offered and a succession of strategies followed by different agents. As we shall see in Section 4, the Optimality Principle implies stationarity in the compensation schemes and strategies and allows us to write H as a function of a single compensation triple (w, k, y) and a single strategy π.

7 Heath et al. (1987), Orey et al. (1987) and Majumdar-Radner (1990), among others, analyze related versions of the pure survival case where all controls have the same utility, whereas Benes (1973)and Davis (1977), among others, analyze problems in which all controls have the same variance. See Dutta (1990) for further references.

8 Such breach of contract brings into the picture further considerations of reputation effects for the principal. Further a rational agent foreseeing such a possibility would also adjust his behavior. At some cost of complexity the present analysis could be extended to generate the commitment of the principal as a self-enforcing outcome. Given our focus we prefer just to assume that such a breach of contract is not possible.

9 In the second formulation

$$H = E\delta \int_0^T e^{-\delta s} [dY + (k - w)\, ds] + Ee^{-\delta T} H. \qquad \text{(i)}$$

Using Ito's lemma (i) can be rewritten as

$$H = E\delta \int_0^T e^{-\delta s}[\delta Y(s) + k - w]\, ds - \delta y + Ee^{-\delta T}H$$

$$= k - w - \frac{\delta y}{1 - Ee^{-\delta T}} + \frac{E\delta \int_0^T e^{-\delta s}\, \delta Y(s)\, ds}{1 - Ee^{-\delta T}}\,. \qquad \text{(ii)}$$

The difference in the net receipts (4.1) and (ii), is the last term in (ii) which reflects the fact that in the "score" approach the principal gets, on average, the dividend k - w plus the excess profits that would go into the cash reserve.

10 However note that the infinite-pocket assumption is also implicit in the moral hazard formulation of Section 4 and hence in order to compare moral hazard and first-best values, as we shall do shortly, we need to maintain this consistency in assumption.

11 There are two important differences between our formulation and Yaari's on account of which we cannot immediately infer that the Yaari result holds here. First, Yaari allows only a consumption choice for the agent; stochastic returns are generated every period with mean μ and the agent picks a consumption rate c thereby determining an effective mean of m. In our framework, the agent picks from a given menu a parti ular project or effort level (corresponding to a mean μ) and additionally picks consumption c (and therefore a net mean m). Second, Yaari formulates bankruptcy in his (finite-horizon) model in a way that has no analog in the infinite horizon problem.

12 Dutta, unpublished notes.

Appendix 1: The Binary Case

In this appendix we prove Theorem 3.4. Let $g(\cdot,\hat{y})$ denote the lifetime returns to any switchpoint policy, with a switchpoint at $\hat{y}$. The following Bellman equation is standard and is proved for the binary choice context by Sheng (1980, Theorems 3 and 4):

Proposition A.1 (Sheng) i) $g(\cdot,\hat{y})$ is the unique function, C^2 except possibly at $\hat{y}$ s.t.

$$\frac{1}{2}v_i g''(y) + (m_i\text{-}k)g'(y) - \delta g(y) + \delta U_i = 0,$$

$i=1$ on $[0,\hat{y})$, $i=2$ on $[\hat{y},\infty)$, $g(0,\hat{y}) = 0$, $g(\infty, \hat{y}) = U_2$ if $\hat{y} < \infty$ and $g(\infty, \infty) = U_1$, if $\hat{y} = \infty$.

ii) **A switchpoint strategy is optimal if and only if its return function satisfies the Bellman equation**

$$\max_{(m_i,\ v_i)} \left\{\frac{1}{2} v_i g''(y) + (m_i\text{-}k)g'(y) - \delta g(y) + \delta U_i\right\} = 0, \quad y\epsilon[0, \infty) \qquad \text{(A.1.1)}$$

The above proposition will be now used to prove the theorem. We start by proving Theorem 3.4 iii), i.e. the optimal choice characterization when $U_1 = U_2$. Then we prove the characterization of Theorem 3.4ii) assuming that there is an optimal switchpoint strategy, i.e. assuming that Theorem 3.4i) holds. We conclude by proving the first part of the theorem.

Proof of Theorem 3.4 iii) Suppose that $\lambda_1 > \lambda_2$. Consider $\hat{y} = 0$. For notational ease from here on we write m_i instead of $(m_i\text{-}k)$ and suppress reference to the switchpoint $\hat{y}$; i.e. we write $g(y)$ instead of $g(y;\hat{y})$. Then,

$$\frac{1}{2} v_2 g''(y) + m_2 g'(y) - \delta g(y) + \delta U_2 = 0, \quad y \geq 0 \qquad \text{(A.1.2)}$$

From the elementary theory of differential equations, the solution to (A.1.2) is of the form

$$g(y) = \alpha_2 e^{-\lambda_2 y} + b_2 e^{\Theta_2 y} + U_2$$

where $-\lambda_2$ (resp. θ_2) is the negative (resp. positive) root of the quadratic $1/2\ v_2 x^2 + m_2 x - \delta$. The boundary conditions imply that $b_2 = 0$, $\alpha_2 = -U_2$. Hence,

$$g(y) = U_2(1 - e^{-\lambda_2 y})$$

To check that this return satisfies the Bellman equation (A.1.1) we need to show that

$$\left[\frac{1}{2} v_1(-\lambda_2)^2 + m_1(-\lambda_2) - \delta\right] U_2 e^{-\lambda_2 y} \geq 0$$

The hypothesis $\lambda_2 > \lambda_1$ implies that the term in the square bracket, $\xi_1(\lambda_2)$, is positive. Hence, the inequality follows. The proof is identical for $\lambda_1 > \lambda_2$ (in that case $\hat{y} = \infty$). □

Proof of Theorem 3.4 i) and ii): Suppose momentarily that there is an optimal switchpoint strategy. Let us show that $\hat{y} = 0$ iff (3.4) holds. Suppose that the optimal switchpoint is zero. This implies

$$\xi_1(\lambda_2)e^{-\lambda_2 y} \geq \frac{\delta(U_1 - U_2)}{U_2}, \text{ for } y \geq 0 \tag{A.1.3}$$

(3.4) follows immediately. Conversely, suppose (3.4) holds and hence clearly so does (A.1.3). However (A.1.3) is precisely all that needs to be checked in order for the optimality equation (A.1.1) to hold. Hence $\hat{y} = 0$ is optimal.

There are two steps to showing that there is an optimal switchpoint strategy. If $\xi_1(\lambda_2) \geq \frac{\delta(U_1 - U_2)}{U_2}$ then we are done since we have shown that $\hat{y} = 0$ is an optimal strategy. So suppose instead that $\xi_1(\lambda_2) < \frac{\delta(U_1 - U_2)}{U_2}$.

Lemma 1 There is $\alpha_1 < 0$, $b_1 > 0$, $\alpha_2 < 0$ and $\hat{y} > 0$ such that

$$g(y) = \{\alpha_1 e^{-\lambda_1 y} + b_1 e^{\Theta_1 y} + U_1\}I[0, \hat{y}) + \{\alpha_2 e^{-\lambda_2 y} + U_2\}I[\hat{y},\infty) \tag{A.1.4}$$

$$\alpha_1 e^{-\lambda_1 \hat{y}} + b_1 e^{\Theta_1 \hat{y}} + U_1 = \alpha_2 e^{-\lambda_2 \hat{y}} + U_2 \tag{A.1.5}$$

$$\alpha_1(-\lambda_1)\, e^{-\lambda_1 \hat{y}} + b_1 \Theta_1 e^{\Theta_1 \hat{y}} = \alpha_2(-\lambda_2)\, e^{-\lambda_2 \hat{y}} \tag{A.1.6}$$

$$\alpha_1(-\lambda_1)^2 e^{-\lambda_1 \hat{y}} + b_1 \Theta_1^2\, e^{\Theta_1 \hat{y}} = \alpha_2(-\lambda_2)^2\, e^{-\lambda_2 \hat{y}} \tag{A.1.7}$$

$$\alpha_1 + b_1 + U_1 = 0 \tag{A.1.8}$$

From the proposition above it follows that the return to any switchpoint policy is of the form (A.1.4). If this return additionally satisfies (A.1.5)-(A.1.7) then the function is continuous and so are its derivatives at $\hat{y}$. Finally, (A.1.8) is clearly the boundary condition $g(0) = 0$. In other words this lemma will establish that there is a switchpoint whose returns are C^2 and satisfy the appropriate boundary conditions. Using the signs of the coefficients that we establish in this lemma we shall then prove (in Lemma 2) that the

Bellman equation (A.1.1) holds.

Proof of Lemma 1: Since $1/2v_1g'' + m_1g' - \delta g + \delta U_1 = 0$ on $[0,\hat{y})$ it follows by simple substitution that, in the presence of (A.1.5), (A.1.6) and (A.1.8), (A.1.7) implies that $\xi_1(\lambda_2)\alpha_2e^{-\lambda_2\hat{y}} - \delta(U_2-U_1) = 0$. But it should also be clear that this condition implies, in the presence of those three equations that the second derivative is continuous, i.e. that (A.1.7) holds. So we will in fact show that there is α_1, b_2, α_2 and $\hat{y}$ such that (A.1.5), (A.1.6), (A.1.8) and the following hold:

$$\xi_1(\lambda_2)\alpha_2e^{-\lambda_2\hat{y}} - \delta(U_2-U_1) = 0 \tag{A.1.9}$$

(A.1.8) substituted succesively in (A.1.5) and (A.1.6) yields

$$\alpha_1(\hat{y}) = \frac{(U_1 - U_2) - U_1\left[1 + \frac{\Theta_1}{\lambda_2}\right]e^{\Theta_1\hat{y}}}{\left[1 + \frac{\Theta_1}{\lambda_2}\right]e^{\Theta_1\hat{y}} + \left[\frac{\lambda_1}{\lambda_2} - 1\right]e^{-\lambda_1\hat{y}}} \tag{A.1.10}$$

which upon substitution gives the following

$$\alpha_2(\hat{y})e^{-\lambda_2\hat{y}} = \alpha_1(\hat{y})\left[\frac{\lambda_1}{\lambda_2}e^{-\lambda_1\hat{y}} + \frac{\theta_1}{\lambda_2}e^{\theta_1\hat{y}}\right] + U_1\frac{\theta_1}{\lambda_2}e^{\theta_1\hat{y}} \tag{A.1.11}$$

Denote the left-hand side of (A.1.9) $h(\hat{y})$. After substituting for (A.1.10) and (A.1.11) we clearly have an equation in the single variable $\hat{y}$. It remains to show that $h(\hat{y}) = 0$ has a positive solution. Note that upon taking limits we get $h(0) = -\xi_1(\lambda_2)U_2 - \delta(U_2-U_1) > 0$, by hypothesis. Similarly, $h(\infty) = -[\xi_1(\lambda_2)\frac{\theta_1}{\lambda_2+\theta_1} + \delta]\ (U_2-U_1)$. From the definitions we have that $\xi_1(\lambda_2)\theta_1 = [1/2v_1\lambda_2^2 + m_1(-\lambda_2) - \delta]\theta_1$ which upon substituting $m_1\theta_1 = -1/2v_1\theta_1^2 + \delta$ yields $\xi_1(\lambda_2)\theta_1 = [1/2v_1\lambda_2\theta_1 - \delta](\lambda_2 + \theta_1)$. Hence $h(\infty) = -[1/2v_1\lambda_2\theta_1]$ $(U_2-U_1) < 0$. The function h is clearly continuous and hence by the intermediate value theorem has at least one value of $\hat{y} > 0$ at which $h(\hat{y}) = 0$.

Since the return to a switchpoint policy must be increasing it follows that $\alpha_2 < 0$. From (A.1.6) and (A.1.7) it follows that $\alpha_1 < 0$. Substituting (A.1.6) in (A.1.7) we get

$$\alpha_1 e^{-\lambda_1 \hat{y}} \lambda_1 (\lambda_1 - \lambda_2) = -b_1 e^{\Theta_1 \hat{y}} \Theta_1(\Theta_1 + \lambda_2) \tag{A.1.12}$$

By hypothesis, $\lambda_1 > \lambda_2$. It follows that $b_1 > 0$.■

Lemma 2. Let $H(y) \equiv \frac{1}{2}(v_1 - v_2)\, g''(y) + (m_1 - m_2)\, g'(y) + \delta(U_1 - U_2)$. Then, $H(y) \gtreqless 0$, as $y \lesseqgtr \hat{y}$.

Proof: For $y > \hat{y}$,

$$H(y) = \xi_1(\lambda_2)\, \alpha_2 e^{-\lambda_2 y} + \delta(U_1 - U_2) < 0$$

The inequality follows from the fact that $\alpha_2 < 0$ and $H(\hat{y}) = 0$. For $y < \hat{y}$

$$H(y) = -\xi_2(\lambda_1)\, \alpha_1 e^{-\lambda_1 y} - \xi_2(\Theta_1)\, b_1 e^{\Theta_1 y} + \delta(U_1 - U_2)$$

Substituting from (A.1.5) - (A.1.7) straightforward algebra gives

$$H(y) = \alpha_1 e^{-\lambda_1 \hat{y}} [\xi_2(\lambda_1) e^{\lambda_1(\hat{y}-y)} + \kappa \xi_2(\theta_1) e^{-\theta_1(\hat{y}-y)} + \gamma]$$

where $\kappa = \dfrac{\lambda_1(\lambda_2 - \lambda_1)}{\theta_1(\lambda_2 + \theta_1)}$ and $\gamma = \delta(\dfrac{\lambda_2 - \lambda_1}{\lambda_2})(\dfrac{\lambda_1 - \theta_1}{\theta_1})$. Recall that $\lambda_1 > \lambda_2$ and hence we have $\xi_2(\lambda_1) > 0$. By hypothesis $\kappa < 0$ and $\gamma > 0$. $H(\hat{y}) = 0$ then implies that $\xi_2(\theta_1) < 0$. In turn this implies that $H(y) > 0$ for $y < \hat{y}$. The proof of Theorem 3.4 is complete. □

Remark: A little algebra in (A.1.12) reveals that as $\delta \to 0$, $\hat{y} \uparrow \infty$; i.e. as the agent becomes more and more patient he asymptotically employs the action that the principal prefers.

Appendix 2

In this appendix we prove the results of Sections 5 and 6.

Proof of Proposition 5.1. Consider any sample path with agent compensation $w(\omega,\cdot)$. Write $\bar{w}(\omega) = \delta \int_0^\infty e^{-\delta s} w(\omega,s)ds$, the "mean wage" for the measure induced by the discount rate $\delta e^{-\delta s}$. By Jensen's inequality,

$$u[\bar{w}(\omega)] > \delta \int_0^\infty e^{-\delta s}\ u[w(\omega,s)]\ ds.$$

Denote $\bar{w} = E\bar{w}(\omega)$, taking the expectation now with respect to the measure induced by the given control strategy. Since u is concave, again, by Jensen's inequality,

$$u(\bar{w}) \geq Eu[\bar{w}(\omega)] \geq E\delta \int_0^\infty e^{-\delta s} u[w(\omega,s)]ds.$$

Since the principal discounts the future at the same rate as the agent and is risk-neutral, along any sample path the principal's returns are identical under time varying compensation $w(\omega,\cdot)$ or mean wage $\bar{w}(\omega)$. From risk-neutrality it follows that the principal is indifferent between environment-varying compensation $\bar{w}(\omega)$ or a constant compensation $\bar{w}$. □

Proof of Proposition 5.2. **The weighted first-best maximand is**

$$(1 - \lambda)\left\{E\delta \int_0^\infty e^{-\delta s}[dY + kds] - w\right\} + \lambda\delta\int_0^\infty q(s)ds + u(w)\Big\} \qquad \text{(A.2.1)}$$

and a strategy is of course the choice of instantaneous controls (m,v) for every time instant and environment, and a constant compensation level w. From (A.2.1) it is clear that the two choices can be made independently. Further, the maximand for w is $\lambda u(w) - (1 - \lambda)w$ and this is clearly maximized for $\tilde{w}$ s.t. $u'(\tilde{w}) = \frac{1 - \lambda}{\lambda}$ when $\tilde{w} > 0$ or at $\tilde{w} = 0$, e.g., when $\lambda = 0$. It is further clear from (A.2.1), that $E\delta \left\{\int_0^\infty e^{-\delta s}\ [dY(1 - \lambda) + q(m,v)\lambda]ds\right\}$ is maximized by the constant use of the control which maximizes $m(1 - \lambda) + q(m,v)\lambda$. ■

Proof of Proposition 6.1. One way in which one could estimate the rate of approach to efficiency would be to directly analyze the asymptotic behavior of the optimal bankruptcy scheme $(w^*(\delta),k^*(\delta),y^*(\delta))$, as $\delta \downarrow 0$. Since explicit expressions for these parameters cannot be obtained, such a direct line of attack is not very fruitful. Instead, we concentrate on finding a particular set of schemes $(w(\delta),k(\delta),y(\delta))$ for which the rate of convergence stated in Proposition 6.1 is valid. Clearly, such a rate is therefore a lower bound for the rate implied by $(w^*(\delta),k^*(\delta),y^*(\delta))$, which in turn is a lower bound for the general class of all admissible compensation schemes.

Suppose the first-best constant control is $(\tilde{m},\tilde{v})$ and the associated wage is $\tilde{w}$. Consider, $k(\delta) = \tilde{m}$, $w(\delta) = \tilde{w}$, for all δ. We will specify $y(\delta)$ shortly. For any $y \geq 0$, let $\beta^*(\tilde{m},\tilde{w},\delta)$ denote the stationary Markovian optimal best response policy of the agent. Define

$$\tilde{U} \equiv U(\tilde{w},\tilde{m},\tilde{v})$$

$$T_\delta(y) = \min \{t > 0: \; Y(t) = 0 | \; Y(0) = y, \; \beta^*(\tilde{m},\tilde{w},\delta)\}$$

$$\tilde{T}(y) = \min \{t > 0: \; Y(t) = 0 | \; Y(0) = y, \; \pi \equiv (\tilde{m}, \; \tilde{v})\}$$

Clearly,

$$V_\delta(y) \leq \bar{U}(\tilde{w}) \; (1 - E^{-\delta T_\delta(y)}) \tag{A.2.2}$$

$$V_\delta(y) \geq \tilde{U} \; (1 - E^{-\delta \tilde{T}_\delta(y)}) \tag{A.2.3}$$

(A.2.2) and (A.2.3) imply that

$$\begin{aligned} 1 - Ee^{-\delta T_\delta(y)} &\geq \frac{\tilde{U}}{\bar{U}(\tilde{w})} (1 - Ee^{-\delta \tilde{T}(y)}) \\ &\equiv b(1 - Ee^{-\delta \tilde{T}(y)}) \end{aligned}$$

Now, by standard arguments (e.g. Dutta (1990)) it follows that

$$1 - Ee^{-\delta \tilde{T}(y)} = 1 - e^{-\lambda y} \tag{A.2.4}$$

where

$$\lambda = \left[\sqrt{\frac{2}{\tilde{v}}} \right] \sqrt{\delta} \; \equiv \; a\sqrt{\delta}$$

From (A.2.4) it follows that

$$H_\delta(y) \geq (\tilde{m} - \tilde{w}) - \frac{\delta y(\delta)}{b(1 - e^{-\lambda y(\delta)})} \tag{A.2.5}$$

Collecting (A.2.4) and (A.2.5) together we have

$$\frac{H_\delta(y)}{\tilde{m} - \tilde{w}} \geq 1 - c \; \frac{\delta y(\delta)}{1 - e^{-\lambda y(\delta)}}$$

$$\frac{V_\delta(y)}{\tilde{U}} \geq 1 - e^{-\tilde{\lambda}y(\delta)}$$

where $c^{-1} \equiv (\tilde{m} - \tilde{w})b$.

The remainder of the proof will be as follows. We shall demonstrate the existence of $y(\delta)$ such that i) $\delta y(\delta) = O(\sqrt{\delta}\ \ln\delta)$, ii) $e^{-\tilde{\lambda}y(\delta)} = O(\sqrt{\delta}\ \ln\delta)$. Clearly, the proof will then be complete.

For an arbitrary integer $n \geq 1$ define

$$y(\delta) \equiv \frac{-n\ \ln\ \delta}{\sqrt{\delta}}$$

It follows that $\delta y(\delta) = -n\sqrt{\delta}\ \ln\delta = O(\sqrt{\delta}\ \ln\delta)$. Further, $-\tilde{\lambda}y(\delta) = an\ \ln\delta$ and so $e^{-\lambda y(\delta)} = \delta^{an}$. If n is chosen such that $an \geq \frac{1}{2}$, then δ^{an} goes to zero faster than $\sqrt{\delta}\ \ln\ \delta$. Hence, for $n \geq \frac{1}{2a}$, $e^{-\tilde{\lambda}y(\delta)} = O(\sqrt{\delta}\ \ln\delta)$, and the proof of Proposition 6.1 is complete. □