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RESOURCE–MONOTONIC SOLUTIONS TO THE PROBLEM OF FAIR DIVISION WHEN PREFERENCES ARE SINGLE–PEAKED

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Abstract

RESOURCE MONOTONIC SOLUTIONS TO THE PROBLEM OF FAIR DIVISION WHEN PREFERENCES ARE SINGLE–PEAKED

We consider the problem of fairly allocating an infinitely divisible commodity among a group of agents with single–peaked preferences. We search for methods, or ***solutions***, of performing this division, that satisfy the following property pertaining to changes in the amount to be divided: suppose that the change is not "too large" in that if initially there is not enough to bring all agents to their satiation points, then after the change, there still is not enough; or if initially there is so much that all agents have to be brought beyond their satiation points, then after the change, there still is too much. In any of these circumstances, we require all agents to be affected in the same direction. We show that essentially, there is a unique selection from the solution that associates with each economy its set of envy–free and efficient allocations satisfying this property: it is the uniform rule, a solution that has played a central role in previous analyses of the problem.

Key words. Fair division. Single–peaked preferences. Resource–monotonicity. Uniform rule.

1. ***Introduction.*** We consider the problem of allocating an infinitely divisible commodity among a group of agents who have single–peaked preferences. Agents are assumed to have equal rights on whatever amount is available and the question is how to achieve a fair division.[1] We search for desirable methods, or ***solutions***, of performing this division. The ideal situation is when the amount to be divided is equal to the sum of the preferred consumptions. Then, every agent can be given his preferred consumption. If the amount to be divided is less than the sum of the preferred consumptions, the situation is the usual one since having less of the commodity will be socially undesirable: there is "not enough" (to bring everyone to his satiation point). If the amount to be divided is greater than the sum of the preferred consumptions, the opposite holds and it is having more of the commodity that will be socially undesirable. Then, one can say that there is "too much". What should be done in these cases?

This model was recently considered by Sprumont (1991) who searched for strategy–proof solutions, and by Thomson (1990) who looked for solutions satisfying a certain property of consistency pertaining to economies of variable size. It can be given a variety of interpretations: rationing in a two–good economy in which prices are in disequilibrium; allocation of a task among the members of a team, paid an hourly wage and whose disutility of labor is a convex function of labor supplied; allocation of a commodity when preferences are satiated at some point and free disposal is not allowed.

Our purpose here is a normative analysis of monotonicity issues. We imagine changes in the amount to be divided and ask of solutions that if the change is "not so large" in the sense that if there is not enough initially, there still is not enough after the change, and if there is too much initially, there still is too much after the change,

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[1]Alternatively, we could imagine agents to be endowed with possibly different amounts of the commodity. The problem then would be to reallocate it in some equitable way.

then all agents be affected in the same direction. Combined with efficiency, this says that if there is more of the commodity when more of it is socially desirable, all agents gain; if there is more of the commodity when such an increase is socially undesirable, they all lose. We name this property *one-sided resource-monotonicity.*

On "classical" domains, where preferences are monotone, the property that has been considered is that all agents benefit from an increase in the amount to divide. Obviously, if at some point the commodity becomes less desirable, this requirement does not make sense. A weaker one which is natural in such situations is that all agents be affected in the same direction: all gain together or lose together as a result of an increase in the amount to be divided (in fact, as a result of any change in the amount to be divided). We name this property *two-sided resource-monotonicity.* It turns out that this condition, which of course implies *one-sided resource-monotonicity* is actually quite strong, because it forces comparisons between situations when increases are desirable and situations when increases are undesirable. It can be met, but when complemented with efficiency and one of several alternative distributional requirements, it cannot. The weaker requirement of *one-sided resource-monotonicity* is compatible with these other requirements and there is a sense in which it is the most that one can obtain.

Indeed, our results are as follows. First, we characterize the class of *one-sided resource-monotonic* selections from the pareto solution. We introduce a variety of examples in the class, motivated by intuitive considerations of fairness; these solutions are based on the special features of the model and would not be well defined in more general models. There exists an infinite class of *one-sided resource-monotonic* selections from the intersection of the pareto solution with the solution that selects for each economy its set of allocations that pareto dominate equal division, the "individually rational solution from equal division," but all such selections coincide over whole intervals. Our main result is that, on a large subdomain of our principal domain, there is a unique *one-sided resource-monotonic* selection from the intersection of the pareto solution with the solution that selects for each economy its set of envy-free allocations, the "no-envy" solution. This solution is the uniform rule, a solution that plays a prominent role in the Sprumont and Thomson papers. We also offer a characterization without the domain restriction. Then uniqueness is lost but all admissible solutions are obtained by "piecing together" uniform allocations of certain subeconomies.

Next, we strengthen the monotonicity requirement and demand that all agents be affected in the same direction by any change in the amount to be divided. We find that no *two-sided resource-monotonic* selection from either the individually rational solution from equal division or the no-envy solution, exists. However, there are symmetric and *two-sided resource-monotonic* selections from the pareto solution and we give a characterization of all such selections.

These results confirm the central role played by the uniform rule, already established in previous studies, in solving the problem of fair division in economies with single-peaked preferences. They should also be seen in the wider context of the recent literature on the design of allocation rules, whose main objective is to identify the tradeoffs one faces in this design. The paper is intended as a contribution to this larger program.

2. ***The model.*** An amount $M \in \mathbb{R}_+$ of some infinitely divisible commodity has to be allocated among a set $N = \{1,\ldots,n\}$ of agents, indexed by i, each agent i being equipped with a continuous preference relation R_i defined over $\mathbb{R}_+$. These preference relations are ***single-peaked***: for each i, there is $x_i^* \in \mathbb{R}_+$ such that for all $x_i, x_i' \in \mathbb{R}_+$, if $x_i' < x_i \leqq x_i^*$, or $x_i^* \leqq x_i < x_i'$, then $x_i P_i x_i'$ (P_i denotes the strict preference relation associated with R_i, and I_i the indifference relation). Let $\boldsymbol{p(R_i)} \in \mathbb{R}_+$ be the preferred consumption according to R_i. The preference relation R_i can be described in

terms of the function r_i: $\mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ defined as follows: given $x_i \leqq p(R_i)$, $r_i(x_i) \geqq p(R_i)$ and $x_i I_i r_i(x_i)$ if this is possible, and $r_i(x_i) = \infty$ otherwise; given $x_i \geqq p(R_i)$, $r_i(x_i) \leqq p(R_i)$ and $x_i I_i r_i(x_i)$ if this is possible, and $r_i(x_i) = 0$ otherwise. (The number $r_i(x_i)$ is the consumption on the other side of agent i's preferred consumption that he finds indifferent to x_i, if such a consumption exists; it is 0 or ∞ otherwise.) We define $r_i(\infty) = \lim_{x_i \to \infty} r_i(x_i)$. Let $\mathscr{R}$ be the class of all such preference relations. We write $R = (R_i)_{i \in N}$ and $p(R) = (p(R_i))_{i \in N}$. An ***economy*** is a pair $(R,M) \in \mathscr{R}^n \times \mathbb{R}_+$.

A ***feasible allocation for (R,M)*** $\in \mathscr{R}^n \times \mathbb{R}_+$ is a vector $x = (x_i)_{i \in N} \in \mathbb{R}^n_+$ such that $\Sigma x_i = M$. Note that free disposal of the commodity is not assumed. Let ***X(M)*** be the set of feasible allocations of (R,M).

Our objective is to distribute the amount M equitably. A ***solution*** is a mapping $\varphi: \mathscr{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n_+$ which associates with every economy $(R,M) \in \mathscr{R}^n \times \mathbb{R}_+$ a non–empty subset of X(M), $\varphi(R,M)$. Each of the points in $\varphi(R,M)$ is interpreted as one possible recommendation. Apart from several solutions derived from standard economic notions, we will introduce others that are specific to the model. A number of them are single–valued, a desirable property that is difficult to obtain on "classical" domains, that is, domains of economies with infinitely divisible goods and continuous, convex, and monotonic preferences.

Note that in our formulation solutions may be required to depend only on the restriction of each R_i to [0,M],[2] or they may be allowed to depend on the whole of each R_i. With only small changes in the exposition, we could have assumed preferences to be defined over some fixed interval $[0,M_0]$, where M_0 is possibly different from M. Such a specification of the domain would be particularly appropriate in the

[2]In his analysis, where M is fixed, Sprumont assumes preferences only to be defined on the interval [0,M].

case of dividing a task among workers, M_0 being the maximal amount of time each worker can work.

The intersection of two solutions φ and φ' is denoted $\varphi\varphi'$. If φ is a single–valued solution and $\{x\} = \varphi(R,M)$, we slightly abuse notation and write $x = \varphi(R,M)$.

3. ***Basic solutions.*** In this section, we present the two solutions that have been most often advocated in the literature on the problem of fair division, and variants of them. Unfortunately, these solutions typically are not very discriminating. One of our objectives later on will be to identify selection procedures that are well behaved from the viewpoint of monotonicity. In the course of this investigation, we will be led to introducing a variety of solutions whose definitions will make use of the particular features of the model under study. First, however, we take care of efficiency.

Pareto solution, P: $x \in P(R,M)$ if $x \in X(M)$ and there is no $x' \in X(M)$ with $x_i' R_i x_i$ for all i and $x_i' P_i x_i$ for some i.

It is easy to check that at an efficient allocation, all agents consume less than their preferred amounts if $\Sigma p(R_i) \geqq M$, and more than their preferred amounts if $\Sigma p(R_i) \leqq M$.

Our first fundamental concept with meaningful distributional implications is pareto domination of equal division. It is often suggested that in order to solve problems of fair division, each agent be given ownership rights on an equal share of the available resources; in the present model, in order to allow for cases when there is too much of the commodity so that it has become a burden, we should say "equal rights or equal responsibilities." From that idea, the requirement that all agents end up better off than at equal division follows naturally.

Individually rational solution from equal division, I_{ed}: $x \in I_{ed}(R,M)$ if $x \in X(M)$ and $x_i R_i (M/n)$ for all i.

A refinement of that notion is that no group of agents on its own be able to make all of its members better off, assuming each initially receives M/n. The set of allocations passing this test is "the core from equal division".

Our second fundamental concept says that no agent should prefer anyone else's consumption to his own.[3] This idea has played the central role in recent developments in the theory of fair allocation. In addition to its direct intuitive appeal, it has the advantage of being meaningful in situations where indivisible goods are present (assignment of jobs), and when equal division is not well defined (division of a heterogeneous good such as land or time).

No-envy solution, F (Foley, 1967): $x \in F(R,M)$ if $x \in X(M)$ and for all i, j, $x_iR_ix_j$.

We could additionally require that no group be able to make all of its members better off by redistributing among them the resources received by any other group of the same cardinality. An allocation passing this test is "group envy-free".

We close with a concept intermediate in spirit between no-envy and individual rationality from equal division. It simply says that every agent prefers his consumption to what the others receive on average.

Average no-envy solution, A (Thomson, 1979, 1982; Baumol, 1986): $x \in A(R,M)$, if $x \in X(M)$ and $x_iR_i(\sum_{j\neq i} x_j/(n-1))$ for all i.

A general discussion of these criteria and of the way they relate to each other appears in Thomson (1991). When applied to the present model, a few important facts are (Thomson, 1990): envy-free and efficient allocations, and individually rational from

[3]The following definition due to Pazner and Schmeidler (1978), as well as variants and extensions of it, have been very useful in other contexts: the allocation $x \in X(M)$ is ***egalitarian-equivalent for (R,M)*** if there exists a reference amount x_0 such that $x_iI_ix_0$ for all i. Let $\boldsymbol{E}^*(R,M)$ be the set of these allocations. Here, this concept will not be useful, since $E^*P(R,M)$ will typically be empty. Suppose for instance that $\Sigma p(R_i) = M$. Then, $\{(p(R_1),\ldots,p(R_n))\} = P(R,M)$. As a result, if for at least one pair $\{i,j\}$, $p(R_i) \neq p(R_j)$, then $E^*P(R,M) = \emptyset$.

equal division and efficient allocations, always exist; the "uniform allocation," introduced later on, enjoys all of these properties. Any allocation that is individually rational from equal division is average envy–free. There is no containment relation between the sets of envy–free allocations and of average envy–free allocations (unless of course $n = 2$, in which case the two concepts coincide). There typically is a large set of allocations (in fact a continuum) satisfying each of these conditions and the need arises then for more precise selections. Since the set of group envy–free allocations and the core from equal division may be empty, these two concepts do not help in this regard. They will not be discussed any further. Moreover, we would like our selections to be well behaved from the viewpoint of monotonicity.

4. ***One–sided resource–monotonicity.*** We now consider changes in the amount to be divided, limiting our attention to single–valued solutions.

Suppose that the amount to be divided increases. Given that all agents are assumed to have equal rights on the goods, if they had monotone preferences, it would be desirable that all gain. This property of "resource monotonicity" of solutions has been the object of much attention recently in the context of classical economies (Roemer, 1986; Chun and Thomson, 1988; Moulin and Thomson, 1988). This is also what we would like to require here if initially there is not too much of the commodity and after the increase, there is still not too much of it, so that each agent initially receives less than his preferred consumption and still does after the increase, as efficiency requires. If there is initially too much of the commodity, so that all agents have already passed their preferred consumptions — again, this is as efficiency requires — we would like to ask that all agents should lose as a result of a further increase: if $M \leqq M' \leqq \Sigma p(R_i)$ or $\Sigma p(R_i) \leqq M' \leqq M$, then $\varphi_i(R,M')R_i\varphi_i(R,M)$ for all i.

However, we believe that it is important fully to separate out considerations of monotonicity from considerations of efficiency, so that we will use the following alternative condition:

One–sided resource monotonicty. For all $R \in \mathscr{R}^n$, for all $M, M' \in \mathbb{R}_+$, if $M \leqq M' \leqq \Sigma p(R_i)$ or if $\Sigma p(R_i) \leqq M' \leqq M$, then either $\varphi_i(R,M')R_i\varphi_i(R,M)$ for all i or $\varphi_i(R,M)R_i\varphi_i(R,M')$ for all i. ***Strict one–sided resource monotonicity*** holds if, in addition, whenever one of the preferences is strict, then they all are.

The general requirement that all agents be affected in the same direction as their environment changes is the essence of solidarity. An application of this idea to the theory of bargaining appears in Thomson and Myerson (1980).[4] Its usefulness in the theory of quasi–linear social choice with a variable population was observed by Chun (1986).

Our first example of a solution satisfying *one–sided resource–monotonicity* is the "proportional solution". This solution (mentioned by Sprumont) is the natural expression on our domain of a fundamental principle which underlies much of the theory of allocative fairness (Young, 1988, quotes Aristotle: "What is just ... is what is proportional and what is unjust is what violates the proportion").

Proportional solution, Pro: $x = \text{Pro}(R,M)$ if $x \in X(M)$ and there exists $\lambda \in \mathbb{R}_+$ such that $x_i = \lambda p(R_i)$ for all i; if no such λ exists, $x = (M/n,\ldots,M/n)$.

Note that λ exists as soon as the preferred consumption of at least one agent is positive. In the rare case when all preferred consumptions are zero, we propose equal division. Since in that case, all agents have identical preferences, this choice is quite natural. Unfortunately, it causes the rule to be discontinuous with respect to preferences.[5] Unless $\Sigma p(R_i) = M$, at a proportional allocation, no agent with a positive

[4]"Any change in the feasible set affects all agents in the same diretion."

[5]Sprumont (1991) discusses issues of topologies. Note that continuity with respect to the amount to be divided would hold.

preferred consumption reaches it. Clearly, a proportional allocation is necessarily efficient.

The following variant of the proportional solution is continuous with respect to preferences. Moreover it treats units of the good above or below the preferred consumptions symmetrically (as do all of the other solutions that we will discuss). This is desirable for some of the interpretations of the model, such as rationing.[6] Of course, it remains efficient.

Symmetrically proportional solution, Pro*: $x = \mathrm{Pro}^*(R,M)$ if $x \in X(M)$ and (i) when $\Sigma p(R_i) \geqq M$, there exists $\lambda \in \mathbb{R}_+$ such that $x_i = \lambda p(R_i)$ for all i, and (ii) when $\Sigma p(R_i) \leqq M$, there exists $\lambda \in \mathbb{R}_+$ such that $M - x_i = \lambda[M - p(R_i)]$ for all i.

The symmetrically proportional solution is also *one-sided resource-monotonic*. Both the proportional solution and its symmetricized version are *strictly one-sided resource-monotonic* if all preferred consumptions are positive.

A version of the proportional solution that depends only on the restrictions of the preference relations to the interval [0,M] is obtained by requiring proportionality to the "constrained" preferred consumptions: $p_M(R_i) = p(R_i)$ if $M \geqq p(R_i)$ and $p_M(R_i) = M$ otherwise. The resulting solution as well as the solution obtained in a similar way from the symmetrically proportional solution are *one-sided resource-monotonic*, and strictly so if all preferred consumptions are positive.

These properties should be compared with those of the next solution. When it is not possible to give every agent his preferred consumption, and sacrifices have to be imposed on them, it is natural that all agents be required to sacrifice and the proportional solution achieves this (again, except in the special case when some of the preferred consumptions are equal to zero). But the idea of ***proportional*** sacrifice is not

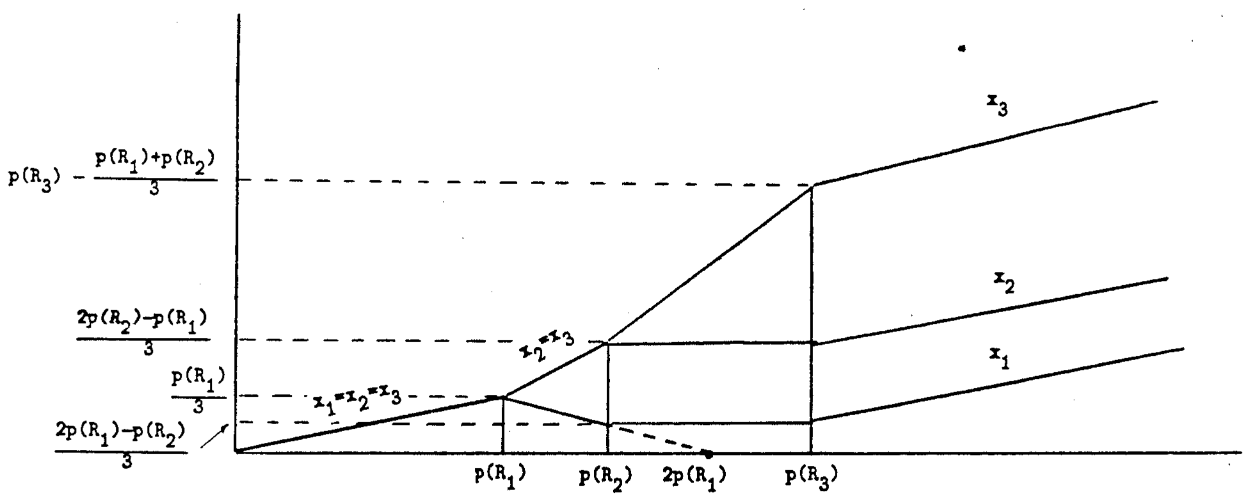
the amount to be divided would hold.

[6]Aumann and Maschler (1985) use a similar requirement in their study of bankruptcy problems.

the only one that merits attention. The next solution is based on comparing distances from preferred consumptions ***unit for unit*** as opposed to proportionally. It selects the allocation at which all agents are equally far from their preferred consumptions, except when boundary problems occur, in which case, those agents whose consumptions would be negative are given zero instead.

Equal-distance solution, Dis: $x = \text{Dis}(R,M)$ if $x \in X(M)$ and (i) when $\Sigma p(R_i) \geqq M$, there exists $d \geqq 0$ such that $x_i = \max\{0, p(R_i)-d\}$ for all i, and (ii) when $\Sigma p(R_i) \leqq M$, there exists $d \geqq 0$ such that $x_i = p(R_i)+d$ for all i.

The equal-distance solution is single-valued and produces efficient allocations. It is *one-sided resource-monotonic.* However, the variant of the solution obtained by using the restrictions of the preference relations to the interval [0,M] instead of the whole preference relations is not (except for the two-agent case). This is illustrated in Figure 1 which depicts the amount received by each of three agents as a function of the amount to be divided. Note that as M increases from the smallest preferred consumption to the second smallest preferred consumption, agent 1 is made progressively worse off whereas agents 2 and 3 are made better off.



An illustration of the equal-distance solution in the three-agent case.

Figure 1

The propositions below, which should not be surprising, relate the above rules to our primary distributional criteria of individual rationality from equal division and no–envy.[7]

***Proposition* 1.** Neither the proportional solution nor the equal–distance solution necessarily selects individually rational from equal division allocations.

Proof. Let $N = \{1,2\}$. (i) Let $p(R) = (3,6)$ and $M = 6$. Then $\mathrm{Pro}(R,M) = (2,4)$. Since $M/2 = 3$ and $3P_1 2$, $\mathrm{Pro}(R,M) \notin I_{ed}(R,M)$. (ii) Let $p(R) = (3,6)$, $r_1(2) = 4$, and $M = 7$. Then $\mathrm{Dis}(R,M) = (2,5)$. Since $M/2 = 3.5$ and $3.5P_1 2$, $\mathrm{Dis}(R,M) \notin I_{ed}(R,M)$. Q.E.D.

***Proposition* 2.** Neither the proportional solution nor the equal–distance solution necessarily selects envy–free allocations.

Proof. Let $N = \{1,2\}$. (i) Let $p(R) = (2,4)$, $r_1(1.5) = 4$, and $M = 4.5$. Then $\mathrm{Pro}(R,M) = (1.5,3)$. Since $3P_1 1.5$, $\mathrm{Pro}(R,M) \notin F(R,M)$. (ii) Let $p(R) = (3,4)$, and $M = 5$. Then $\mathrm{Dis}(R,M) = (2,3)$. Since $3P_1 2$, $\mathrm{Dis}(R,M) \notin F(R,M)$.

Q.E.D.

In light of Propositions 1 and 2, it is natural to attempt redefining the two solutions so as to recover either individual rationality from equal division or no–envy. An appealing way to do this is to perform lexicographic operations. Such operations are standard in game theory and social choice. Taking the differences $d_i(x) = |x_i - p(R_i)|$ as a point of departure, for example, we would look for allocations x in $I_{ed}P(R,M)$ or in $FP(R,M)$ at which these differences are not equal, as in the definition of the equal–distance solution, but as equal as possible, according to the lexicographic ordering.

[7]The results also hold for the versions of the solutions obtained by using the restrictions of the preferences selection to the interval $[0,M]$.

Formally, given $t \in \mathbb{R}^n$, let $\tilde{t}$ be obtained by rewriting the coordinates of t in decreasing order. Given t and $t' \in \mathbb{R}^n$, say that ***t is lexicographically greater than t'***, written $t >_L t'$, if $[\tilde{t}_1 > \tilde{t}'_1]$, or $[\tilde{t}_1 = \tilde{t}'_1$ and $\tilde{t}_2 > \tilde{t}'_2]$, or ... Also, let $d(x) = (d_1(x),\dots,d_n(x))$.

Then, we define selections from $I_{ed}P$ and FP as follows:

Definition. $Dis^L_{I_{ed}}(R,M) = \{x \in I_{ed}P(R,M) \mid d(y) \geq_L d(x)$ for all $y \in I_{ed}P(R,M)\}$

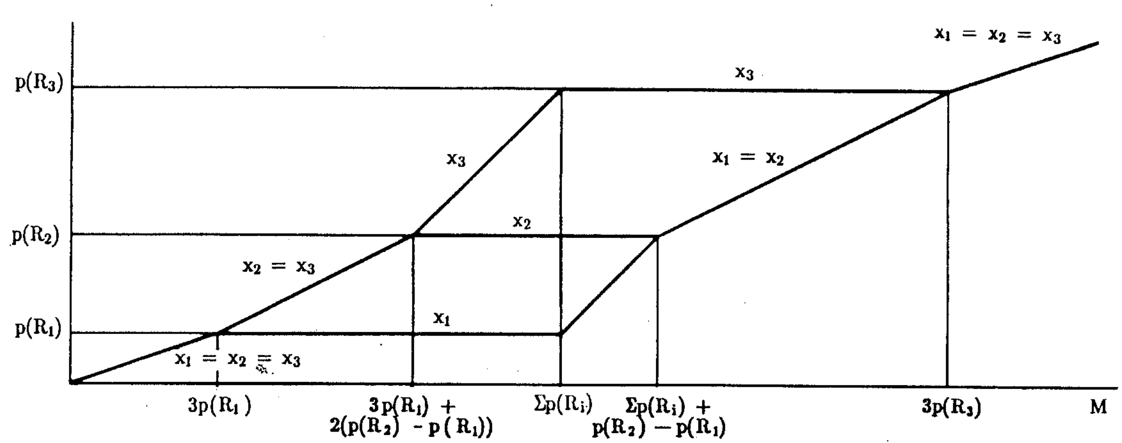
Definition. $Dis^L_F(R,M) = \{x \in FP(R,M) \mid d(y) \geq_L d(x)$ for all $y \in FP(R,M)\}$.

Similarly, we could define selections from $I_{ed}P$ and FP in the spirit of the proportional solution by considering the vector $(\frac{x_1}{p(R_1)},\dots,\frac{x_n}{p(R_n)})$ (where $0/0 = \infty$ by convention), and looking for allocations in $I_{ed}P(R,M)$ and FP(R,M) whose associated vectors are lexicographically minimal in $I_{ed}P(R,M)$ and FP(R,M).

Unfortunately, so adapting the solutions will cost us *one–sided resource–monotonicity.* It is therefore with some relief that we encounter our next solution, which does satisfy *one–sided resource–monotonicity* and always selects an allocation that is efficient, envy–free ***and*** individually rational from equal division. This solution, called the uniform rule, already played an important role in Sprumont (1991) and Thomson (1990), who characterized it on the basis of strategy–proofness and consistency respectively.

Uniform rule, U: $x = U(R,M)$ if $x \in X(M)$ and (i) when $\Sigma p(R_i) \geqq M$, $x_i = \min\{p(R_i),\lambda\}$ for all i, where λ solves $\Sigma\min\{p(R_i),\lambda\} = M$, and (ii) when $\Sigma p(R_i) < M$, $x_i = \max\{p(R_i),\lambda\}$ for all i, where λ solves $\Sigma\max\{p(R_i),\lambda\} = M$.

If $\Sigma p(R_i) \geqq M$, the uniform allocation is obtained by successively making the agents who receive the least as well–off as possible, and if $\Sigma p(R_i) < M$, by successively making the agents who receive the most as well off as possible. Here are the payments as a function of M (Figure 2 illustrates the rule for n=3). For M small, all agents receive the same amount; this holds until all have received an amount equal to the smallest preferred consumption. Then, the agent with the smallest preferred consumption does not receive anything for a while. Instead, any increase in M is divided equally among the remaining agents until each of them has received an amount equal to the second smallest preferred consumption. Then, the agent with the second smallest preferred consumption does not receive anything for a while... This process continues until each agent has received his preferred consumption. Any increase beyond $\Sigma p(R_i)$ goes first to the agent with the smallest preferred consumption until he has received an amount equal to the second smallest preferred consumption. A further increase is divided equally among the agents with the two smallest preferred consumptions until they have received an amount equal to the third smallest preferred consumption ... This goes on until all agents have reached the largest preferred consumption. Afterwards, they share equally any further increase.[8]



An illustration of the uniform rule in the three–agent case.

Figure 2

[8]The similarity between the uniform rule and the rule proposed by Maimonides for the adjudication of conflicting claims (O'Neill, 1982; Aumann and Maschler, 1985) should be noted. Indeed, the algorithm describing that solution is identical up to the point where each agent has received his preferred consumption, by replacing the vector of preferred consumptions by the vectors of claims.

As noted above, the uniform allocation is efficient, envy–free, and individually rational from equal division. Also, it depends only on preferred consumptions. Equipped with this new solution, we now understand that Proposition 2 was simply an illustration of a more general fact: the uniform rule is the only selection from the envy–free and efficient solution to depend only on preferred consumptions. Since the proportional and equal–distance solutions are efficient solutions that depend only on preferred consumptions, they have to violate no–envy. To prove the fact, let φ be such a selection, and suppose by way of contradiction that for some $(R,M) \in \mathcal{R}^n \times \mathbb{R}_+$, $x = \varphi(R,M) \neq U(R,M)$. Suppose, without loss of generality, that $\Sigma p(R_i) \geqq M$. Then, for some i, j with $x_i < x_j$, $x_i \neq p(R_i)$. Since $\varphi \subseteq P$, $x_i < p(R_i)$. Now, let R_i' be such that $p(R_i') = p(R_i)$ but $x_j P_i' x_i$. Since φ depends only on preferred consumptions, $\varphi(R_i',R_{-i},M) = \varphi(R,M)$. But $x \notin F(R_i',R_{-i},M)$ since agent i now envies agent j at x.[9] (It is not true however that the uniform rule is the only selection from the individually rational from equal division and efficient solution to depend only on preferred consumptions).

The fact that there exists a *one–sided resource–monotonic* selection from the envy–free and efficient solution (the uniform rule) should be emphasized. On classical domains, there is no such selection satisfying the counterpart of this property, as shown by Moulin and Thomson (1988). Next, we ask whether there are *one–sided resource–monotonic* selections from the envy–free and efficient solution other than the uniform rule. The answer is no, provided a fairly weak domain restriction is imposed, namely, that for each agent, there be a finite consumption indifferent to 0: if there is too much of the commodity, it eventually becomes a "bad". (Technically, this restriction prevents the consumption of the agent with the largest preferred consumption

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[9]Note that the argument would apply to any solution simply required to be such that the consumption received by any agent depends on his own preference relation only through his preferred consumption.

to become infinite while the consumptions of the others remain finite, without his becoming envious of them.) We describe after the theorem all the additional solutions that would become admissible if the domain restriction were removed.

***Theorem* 1.** Let $\tilde{\mathscr{R}}^n$ be the domain of preference profiles such that each r_i is bounded. On the domain $\tilde{\mathscr{R}}^n \times \mathbb{R}_+$, the uniform rule is the only *one–sided resource–monotonic* selection from the envy–free and efficient solution.

Proof. Let $\varphi\colon \tilde{\mathscr{R}}^n \times \mathbb{R}_+ \to \mathbb{R}^n_+$ be a *one–sided resource–monotonic* selection from FP.

(i) ***φ is continuous with respect to M.*** Suppose not. Then, there exist sequences $\{M^\nu\}$ in $\mathbb{R}_+$ and $\{x^\nu\}$ in $\mathbb{R}^n_+$, $M^0 \in \mathbb{R}_+$, $x^0 \in \mathbb{R}_+$ such that $M^\nu \to M^0$, $x^\nu \to x^0$, $x^\nu = \varphi(R,M^\nu)$ for all ν and $x^0 \neq \varphi(R,M^0)$. Let $y = \varphi(R,M^0)$. Since $\varphi \subseteq P$, $M^0 \neq \Sigma p(R_i)$ (otherwise $y = p(R)$ and $x^0 = \varphi(R,M^0)$). Suppose, without loss of generality, that $M^0 < \Sigma p(R_i)$. There exists $\bar{\nu}$ such that for all $\nu \geqq \bar{\nu}$, $M^\nu \leqq \Sigma p(R_i)$. Then, since $x^0 \neq y$, and since $\varphi \subseteq P$, there are i, j such that $x^0_i < y_i \leqq p(R_i)$ and $y_j < x^0_j \leqq p(R_j)$. There is $\bar{\bar{\nu}}$ so that for all $\nu \geqq \bar{\bar{\nu}}$, $x^\nu_i < y_i$ and $y_j < x^\nu_j$. Let $\nu \geqq \max\{\bar{\nu},\bar{\bar{\nu}}\}$. In the change from M^ν to M^0, agent i gains and agent j loses, in contradiction with *one–sided resource–monotonicity.*

(ii) ***For all M such that $\Sigma p(R_i) \geqq M$, $\varphi(R,M) = U(R,M)$.*** Suppose, by way of contradiction, that for some M with $\Sigma p(R_i) \geqq M$, $x = \varphi(R,M) \neq U(R,M)$. Since $\varphi \subseteq P$, $x_i \leqq p(R_i)$ for all i. Then there are i, j such that $x_i < p(R_i)$ and $x_i < x_j$. Since $\varphi(R,M') \geqq 0$ for all M', $\varphi_{j_0}(R,M') \to 0$ as $M' \to 0$. Since $\varphi \subseteq F$, $p(R_i) < r_i(x_i) \leqq x_j$. Therefore, by (i), there is $\overline{M} \leqq M$ such that $\varphi_j(R,\overline{M}) = p(R_i)$. Since $\varphi \subseteq F$, for agent i not to envy agent j at $\varphi(R,\overline{M})$, we need $\varphi_i(R,\overline{M}) = p(R_i)$. Therefore, in the change from M to $\overline{M}$, agent i gains and agent j loses, in contradiction with *one–sided resource–monotonicity.*

(iii) ***For all i, $\varphi_i(R,M) \to \infty$ as $M \to \infty$.*** If not, there exists i such that $\bar{x}_i = \sup\{\varphi_i(R,M) \mid M \in \mathbb{R}_+\} < \infty$. Also, since $\varphi(R,M) \geqq 0$ for all M, there exists j such that $\varphi_j(R,M) \to \infty$ as $M \to \infty$. Let $\bar{x}_j = \max\{r_j(0), \bar{x}_i\}$. Then for M large enough, $\varphi_j(R,M) > \bar{x}_j$. We have $\varphi_i(R,M) P_j \varphi_j(R,M)$ so that agent j envies agent i at $\varphi(R,M)$, in contradiction with $\varphi \subsetneq F$.

(iv) ***For all M such that*** $\Sigma p(R_i) < M$, $\varphi(R,M) = U(R,M)$. Suppose, by way of contradiction that for some M with $\Sigma p(R_i) < M$, $x = \varphi(R,M) \neq U(R,M)$. Since $\varphi \subsetneq P$, $x_i \geqq p(R_i)$ for all i. Then there are i, j such that $x_i > p(R_i)$ and $x_i > x_j$. By (iii), $\varphi_j(R,M) \to \infty$ as $M \to \infty$. Since $\varphi \subsetneq F$, $x_j \leqq r_i(x_i) \leqq p(R_i)$. Therefore by (i), there is $\overline{M}$ such that $\varphi_j(R,\overline{M}) = p(R_i)$. Since $\varphi \subsetneq F$, for agent i not to envy agent j at $\varphi(R,\overline{M})$, we need $\varphi_i(R,\overline{M}) = p(R_i)$. Therefore, in the change from M to $\overline{M}$, agent i gains and agent j loses, in contradiction with *one–sided resource–monotonicity.*

Q.E.D.

Theorem 1 would actually hold if the domain were restricted by the weaker condition that, assuming agents numbered so that $p(R_1) \leqq \cdots \leqq p(R_n)$, $r_n(\infty) < p(R_1)$. The only change in the proof would occur in step (iii), which would continue after the second sentence as follows: "We claim that this implies that $\varphi_n(R,M) \to \infty$ as $M \to \infty$. Indeed, either $i = n$, and then we are done; or if $i \neq n$, for every $B > p(R_n)$, there is M large enough so that $p(R_i) \leqq p(R_n) \leqq B < \varphi_i(R,M) \leqq \varphi_n(R,M)$, where the last inequality follows from the requirement that agent i not envy agent n at $\varphi(R,M)$. Now, since $r_n(\infty) < p(R_1)$, there is M large enough so that $r_n(\varphi_n(R,M)) < p(R_1) \leqq \varphi_1(R,M) < \varphi_n(R,M)$ where the middle inequality follows from $\varphi \subsetneq P$, and agent n envies agent 1 at $\varphi(R,M)$, in contradiction with $\varphi \subsetneq F$."

The following example shows that Theorem 1 would not be true if the domain restriction were dropped altogether. Let $N = \{1,2\}$, $p(R) = (1,3)$ and $r_2(\infty) = 2$. Let φ be such that $\varphi(R,M) = U(R,M)$ for all $M \leqq \Sigma p(R_i)$ and $\varphi(R,M) = (p_1(R), M - p_1(R))$ for all $M \geqq \Sigma p(R_i)$. It is easy to check that $\varphi \subsetneq FP$ and that φ is *one–sided resource–monotonic.* It is also possible to modify the definition of φ so that the amount received by agent 1 would actually increase beyond $p(R_1)$ (the main thing is that it should not increase beyond $r_2(\infty)$).

A complete characterization without the domain restriction can be obtained by generalizing this example. First of all, it follows from steps (i) – (ii) of the proof of Theorem 1, that even without it, any $\varphi \subseteq FP$ satisfying *one–sided resource–monotonicity* is such that $\varphi(R,M) = U(R,M)$ whenever $M \leqq \Sigma p(R_i)$. If M increases beyond $\Sigma p(R_i)$, $\varphi(R,M)$ is obtained by "juxtaposing" the uniform allocations of certain subeconomies, as follows (We omit the proof of these assertions, most of which are patterned after the various steps of Theorem 1).

(i) The agents are partitioned into groups, each group consisting of agents with successive preferred consumptions, two agents with the same preferred consumption belonging to the same group. (Therefore, there is a natural order for the groups, the preferred consumptions of all the members of any group being strictly lower than the preferred consumptions of all the members of the next highest group). For any i such that $\sup\{\varphi_i(R,M) | M \in \mathbb{R}_+\} < \infty$, let s_i be this supremum. If $\varphi_i(R,M) \to \infty$ as $M \to \infty$, let $s_i = \infty$.

(ii) The consumptions of all the members of the highest group become infinite with M. For any agent i in that group, $s_i = \infty$.

(iii) The consumptions of the members of any other group have a common finite supremum. For any i in such a group and for any j in the next highest group, $s_i < r_j(s_j)$. (Therefore, there will be only one group if for all i, $r_i(\infty)$ is less than the smallest preferred consumption, as explained in the paragraph following Theorem 1; also, the consumptions of all the members of any group other than the highest group remain finite).

(iv) What each group receives as a whole is allocated between its members by applying the uniform rule.

(v) What each group receives as a whole is a continuous increasing function of M. These functions add up to M.

A direct consequence of Theorem 1 is that on $\tilde{\mathscr{R}}^n \times \mathbb{R}_+$, there exists no selection from the envy–free and efficient solution satisfying *strict one–sided resource–monotonicity.*[10] This is because the uniform rule does not satisfy this stronger property, as can be seen from Figure 2.

What if no–envy were replaced by individual rationality from equal division? Since, when $n = 2$, no–envy is a weaker condition (Thomson, 1990), it follows from Theorem 1 that, if $\varphi \subseteq I_{ed}P \subseteq FP$ is *one–sided resource–monotonic*, then $\varphi = U$ on $\tilde{\mathscr{R}}^2 \times \mathbb{R}$. However, if $n > 2$, the uniform rule does not remain the only *one–sided resource–monotonic* selection from $I_{ed}P$ on $\tilde{\mathscr{R}}^n \times \mathbb{R}$.

It is possible completely to characterize all these selections but they constitute an infinite–dimensional family. In order to give the reader an idea of what they look like, we only indicate the features shared by all of them for $n = 3$, and, after numbering agents so that $p(R_1) \leqq p(R_2) \leqq p(R_3)$, under the assumption that $2p(R_2) \geqq p(R_1)+p(R_2)$.

Then any *one–sided resource–monotonic* $\varphi \subseteq I_{ed}P$ is such that for all $M \in [0,3p(R_1)] \cup [3p(R_3),\infty[$, $\varphi(R,M) = (M/3,M/3,M/3) = U(R,M)$; for all $M \in [3p(R_1),\Sigma p(R_i)]$, $\varphi_1(R,M) = p(R_1)$; for all $M \in [\Sigma p(R_i),3p(R_2)]$, $\varphi(R,M) = (M-p(R_2)-p(R_3),p(R_2),p(R_3)) = U(R,M)$; for all $M \in [3p(R_2),\infty[$, $\varphi_3(R,M) = p(R_3)$. Otherwise, some indeterminacy exists. However, precise bounds can be established. (Details are available from the author.)

It follows from these remarks and the inclusion $I_{ed} \subseteq A$ that the uniform rule is not the only *one–sided resource–monotonic* selection from AP. However, at least for

[10]This is true even if all preferred consumptions are positive. (Recall that this restriction had helped in the case of the proportional solution.)

n = 3, the class of *one–sided resource–monotonic* selections from AP is not any larger than the class of *one–sided resource–monotonic* selections from $I_{ed}P$.

We conclude this section with a characterization of all *one–sided resource–monotonic* selections from the pareto solution: for each $R \in \mathscr{R}^n$ simply choose n non–decreasing functions $\varphi_i(R,\cdot):\mathbb{R}_+ \to \mathbb{R}_+$ adding up to M and such that $\varphi_i(R,\Sigma p(R_i)) = p(R_i)$ for all i.

*5. **Two–sided resource–monotonicity.*** We now turn to an analysis of the following stronger monotonicity condition: ***any*** change in resources affects all agents in the same direction, whether or not the two amounts remain on the same side of the sum of the preferred consumptions.

***Two–sided resource–monotonicity*:** For all $R \in \mathscr{R}^n$, for all M, $M' \in \mathbb{R}_+$, either $\varphi_i(R,M')R_i\varphi_i(R,M)$ for all i, or $\varphi_i(R,M)R_i\varphi(R,M')$ for all i. (Here too, a further strengthening could be obtained by requiring that in addition, if one of the preferences is strict, then they all are.)

The non–existence of *two–sided resource–monotonic* selections from the individually rational from equal division and efficient solution follows from (i) Theorem 1, (ii) the fact that the uniform rule does not satisfy this property, and (iii) the fact used earlier that for n = 2, any allocation that is individually rational from equal division is envy–free (Thomson, 1990). The next proposition states an even stronger impossibility since it makes no use of efficiency. It also provides a proof of (ii), as it involves economies for which there is a unique allocation that is individually rational from equal division. This allocation is therefore the uniform allocation.

***Proposition* 3:** There is no selection from the individually rational solution from equal division satisfying *two–sided resource–monotonicity.*

Proof. Let $\varphi \subseteq I_{ed}$. Let N = {1,2}, p(R) = (1,2), M = 2 and M′ = 4. Let $x \in I_{ed}(R,M)$. Since M/2 = 1 = $p(R_1)$, x = (1,1). Let $y \in I_{ed}(R,M')$. Since M′/2 = 2

$= p(R_2)$, $y = (2,2)$. Since $\varphi \subsetneq I_{ed}$, $x = \varphi(R,M)$ and $y = \varphi(R,M')$. In the change from M to M′, agent 1 loses and agent 2 gains, in contradiction with *two-sided resource monotonicity.*

Q.E.D.

It also follows from Theorem 1 and the fact that the uniform rule does not satisfy *two-sided resource-monotonicity* that there is no *two-sided resource-monotonic* selection from the envy-free and efficient solution. This impossibility is also a consequence of the next proposition, in which again no use is made of efficiency.

Proposition 4: There is no selection from the envy-free solution satisfying *two-sided resource-monotonicity.*

Proof. Let $\varphi \subsetneq F$. Let $N = \{1,2\}$, $p(R) = (2,7)$, $r_1(1) = 4$, $r_2(5) = 8$, with r_1 linear in the interval [0,2] and r_2 linear in the interval [0,7], $M = 5$ and $M' = 13$. Let $x = \varphi(R,M)$. For agent 1 not to envy agent 2 at x, we need $1 \leqq x_1 \leqq 2.5$ or $4 \leqq x_1 \leqq 5$. Since agent 2's preferences are monotone in the interval [0,5], for him not to envy agent 1 at x, we need $x_1 \leqq x_2$. Altogether, we have $1 \leqq x_1 \leqq 2.5$.

Now, let $y = \varphi(R,M')$. For agent 2 not to envy agent 1 at y, we need $0 \leqq y_2 \leqq 5$ or $6.5 \leqq y_2 \leqq 8$. For any y_2 in the second interval, $p(R_1) \leqq y_1 \leqq y_2$ so that agent 1 does not envy agent 2. But for any y_2 in the first interval, $y_1 \geqq 8$, and agent 1 envies agent 2 at y.

We conclude by noting that any $x \in F(R,M)$ is preferred by agent 1 to any $y \in F(R,M')$ whereas the reverse holds for agent 2, in contradiction with *two-sided resource monotonicity.*

Q.E.D.

When $n = 2$, no–envy and average no–envy coincide. Therefore, Proposition 4 also establishes the non–existence of *two–sided resource–monotonic* selections from the average envy–free solution.

What about simply looking for *two–sided resource–monotonic* selections from the pareto solution? The next proposition says that neither the proportional nor equal–distance solutions satisfy this property.

***Proposition* 5.** Neither the proportional solution nor the equal–distance solution satisfies *two–sided resource–monotonicity.*

Proof. Let $N = \{1,2\}$. (i) Let $p(R) = (2,4)$, $r_1(1) = 2.5$, $r_2(2) = 7$, $M = 3$, $M' = 9$. Then $Pro(R,M) = (1,2)$, $Pro(R,M') = (3,6)$. In the change from M to M′, agent 1 loses and agent 2 gains. (ii) Let $p(R) = (1,2)$, $r_1(0) = 1.5$, $r_2(1) = 4$, $M = 1$, $M' = 5$. Then $Dis(R,M) = (0,1)$, $Dis(R,M') = (2,3)$. In the change from M to M′, agent 1 loses and agent 2 gains.

Q.E.D.

A noteworthy feature of several of the rules that we have examined is that they depend only on preferred consumptions. This property plays an important role in obtaining strategy–proofness (Sprumont, 1991). Unfortunately, it is this aspect of the proportional and equal–distance solutions that is largely responsible for their violating *two–sided resource–monotonicity* as follows from the next result, which also involves the very mild requirement that identical agents be treated identically. We state the requirement for solution functions.

Symmetry: For all $(R,M) \in \mathscr{R}^n{}_x\mathbb{R}$, for all i,j, if $R_i = R_j$, then $\varphi_i(R,M) I_i \varphi_j(R,M)$.

Note that efficiency plays no role in the next result.

***Proposition* 6.** There is no *two–sided resource–monotonic* and *symmetric* solution that depends only on preferred consumptions.

Proof. Let φ be a *symmetric* solution that depends only on preferred consumptions. Let $N = \{1,2\}$, $R \in \mathscr{R}^2$ be such that $p(R_1) = 2$, $r_1(1.5) = 4.5$, r_1 being linear on the segment $[0,2]$, $R_1 = R_2$, $M = 2$, and $M' = 6$. Let $x = \varphi(R,M)$. By *symmetry*, $x_1 I_1 x_2$. This is possible only if $x = (1,1)$. Let $y = \varphi(R,M')$. Again, by *symmetry*, $y_1 I_1 y_2$. By the choice of r_1, this is possible only if $y = (3,3)$, $y = (1.5,4.5)$ or $y = (4.5,1.5)$. If $y = (3,3)$ or $y = (1.5,4.5)$, let $R' \in \mathscr{R}^2$ be such that $R_1' = R_1$, $p(R_2') = 2$ and $r_2(1) = 2.5$. Since φ depends only on preferred consumptions, $\varphi(R,M) = \varphi(R',M)$ and $\varphi(R,M') = \varphi(R',M')$. If preferences are R', as M increases to M', agent 1 gains and agent 2 loses. If $y = (4.5,1.5)$, we repeat the same argument by changing agent 1's preferences in the same way we just changed agent 2's preferences. In either case, we obtain a violation of *two-sided resource-monotonicity*.

Q.E.D.

Fortunately, *two-sided resource-monotonic* selections from the pareto solution do exist. The class of all such solutions can even be characterized, in spite of the fact that it is large. This characterization is given in Theorem 2 below. This theorem essentially says that any of these solutions can be obtained as follows: first fix $R \in \mathscr{R}^n$. Then choose any n non-decreasing functions $\varphi_i(R,\cdot):[0,\Sigma p(R_i)]\to\mathbb{R}_+$ such that $\Sigma\varphi_i(R,M) = M$ for all M in that interval and $\varphi_i(R,\Sigma p(R_i)) = p(R_i)$ for all i. Once this choice is made, the solution is also specified on $[\Sigma p(R_i),\Sigma r_i(0)[$. This is because with any M' in that interval can be associated a unique $M \in [0,\Sigma p(R_i)]$ and $x' \in P(R,M')$ such that $x_i' I_i \varphi_i(R,M)$ for all i. *Two-sided resource-monotonicity* requires that $x' = \varphi(R,M')$. This completes the construction if $\Sigma r_i(0) = \infty$. If not, choose any n non-decreasing functions $\varphi_i(R,\cdot):[\Sigma r_i(0),\infty[\to\mathbb{R}_+$ such that $\Sigma\varphi_i(R,M) = M$ for all M in $[\Sigma r_i(0),\infty[$ and $\varphi_i(R,\Sigma r_i(0)) = r_i(0)$ for all i. The reason for this freedom of choice in that interval is that by giving to each agent more than $r_i(0)$, we ensure that each is worse off than at any amount to be divided that would require that each receives less than his preferred consumption.

Putting these observations together, we have:

***Theorem* 2.** A subsolution φ of the pareto solution satisfies *two-sided resource-monotonicity* if and only for each $R \in \mathscr{R}^n$,

(i) for each i, $\varphi_i(R,\cdot) : [0,\Sigma p(R_i)] \rightarrow \mathbb{R}_+$ is non-decreasing with $\varphi_i(R,\Sigma p(R_i)) = p(R_i)$, and the list $(\varphi_i(R,\cdot))_{i\in N}$ satisfies $\Sigma\varphi_i(R,M) = M$ for all M in that interval,

(iia) if $\Sigma r_i(0) < \infty$,

(*) for all $M' \in [\Sigma p(R_i),\Sigma r_i(0)]$, let $M \in [0,\Sigma p(R_i)]$ be such that for some $x' \in X(M')$, $x_i' I_i \varphi_i(R,M)$ for all i. The value M exists uniquely. Then $\varphi(R,M') = x'$.

(**) for each i, $\varphi_i(R,\cdot)$: $[\Sigma r_i(0),\infty[\rightarrow \mathbb{R}_+$ is non-decreasing with $\varphi_i(R,\Sigma r_i(0)) = r_i(0)$ and the list $(\varphi_i(R,\cdot))_{i\in N}$ satisfies $\Sigma\varphi_i(R,M) = M$ for all M in that interval.

(iib) if $\Sigma r_i(0) = \infty$, φ is extended to the interval $[\Sigma p(R_i),\infty[$ by the operation described in (*).

It is important to note that the specification of φ in steps (i) and (**) can be made to depend on other features of preferences than the preferred consumptions.

We conclude with the discussion of a solution that is based on the idea that the size of agent i's upper contour set at x_i can meaningfully be taken as a measure of the sacrifice imposed on him at x. Then the solution equates sacrifices across agents, an adjustment being made to guarantee that all consumptions are non-negative.

Equal-sacrifice solution, Sac: $x \in Sac(R,M)$ if $x \in X(M)$ and (i) when $\Sigma p(R_i) \geqq M$, there exists $\sigma \geqq 0$ such that $r_i(x_i)-x_i \leqq \sigma$ for all i, strict inequality holding only if $x_i = 0$, and (ii) when $\Sigma p(R_i) \leqq M$, there exists $\sigma \geqq 0$ such that $x_i-r_i(x_i) = \sigma$ for all i.

The existence of allocations satisfying this definition is guaranteed under the domain restriction used in Theorem 1 (that each r_i be finite).[11] It is easy to construct

[11]The version of the equal-sacrifice solutions obtained by using the restriction of the preferences to the interval [0,M] (equating "constrained sacrifices") is *one-sided*

examples showing that this solution satisfies neither individual rationality from equal division nor no–envy. It is *one–sided resource–monotonic* but not *two–sided resource–monotonic*, as revealed by the following example: $N = \{1,2\}$, $r_1(0) = 1$, $r_2(1) = 3$, $M = 0$, $M' = 5$. Then $Sac(R,M) = (0,0)$ and $Sac(R,M') = (2,3)$. As M increases to M', agent 1 loses and agent 2 gains.

If the domain restriction is not imposed, generalizations of the idea can be obtained as follows: For each i, let $\bar{x}_i = \inf_{x_i \to \infty} r_i(x_i)$. Then, let $\varphi_i(R_i,\cdot):[0,\Sigma\bar{x}_i] \to \mathbb{R}_+$ be n non–decreasing functions such that $\Sigma\varphi_i(R_i,M) = M$ for all $M \in [0,\Sigma\bar{x}_i]$ and $\varphi_i(R_i,\Sigma\bar{x}_i) = \bar{x}_i$ for all i. For each $M \in]\Sigma\bar{x}_i,\infty[$, let $x \in X(M)$ be such that $r_i(x_i)-x_i = r_j(x_j)-x_j$ for all $i, j \in N$. If $r_i(0) = \infty$ for all i, this natural adaptation of the equal–sacrifice solution satisfies *two–sided resource–monotonicity*.

An alternative generalization, which under the same domain restriction has the same properties is defined as follows: given a continuous function $f:\mathbb{R}_+ \to \mathbb{R}_{++}$ such that $\int_0^\infty f(u)du < \infty$, pick $x \in P(R,M)$ such that the "f–weighted sacrifices" $\int_{x_i}^{r_i(x_i)} f(u)du$ be equal across agents (again, since boundary problems may occur, some agents may have to be given 0; this is as in the definition of the equal–sacrifice solution).

6. ***Conclusion.*** It is instructive to compare the conclusions we obtained here for the problem of fairly dividing a commodity among agents with single–peaked preferences to what is known for classical problems of fair division.

The condition of monotonicity with respect to resources that has been analyzed in classical domains (all agents benefit from an increase in the available resources) would be unnatural for the current model. However, conditions which take into account the

resource–monotonic if $n = 2$, but not if $n > 2$. It is not *two–sided resource–monotonic* for any n.

special features of the model can be formulated. We proposed two such conditions, which we named *one–sided,* and *two–sided, resource–monotonicities.* Although it is easy to define *one–sided resource–monotonic* selections from the pareto solution, and many *one–sided resource–monotonic* selections from the individually rational from equal division and efficient solution exist, there is a unique *one–sided resource–monotonic* selection from the envy–free and efficient solution on a large subdomain of our primary domain. There are *two–sided resource–monotonic* selections from the pareto solution, but none from either the individually rational solution from equal division or the no–envy solution.

Although not all of these results are positive, they are in sharp contrast with the results pertaining to the classical domain. There, resource monotonicity is satisfied by no selection from the envy–free and efficient solution, nor from the individually rational from equal division and efficient solution. (See Moulin and Thomson, 1988.)[12]

Our results are summarized in the following table. A "yes" in cell (a,b) means that the solution in row a satisfies the property in column b. A "no" means the opposite.

[12]In fact, the property is incompatible with considerably weakened distributional requirements, as is established in that paper.

	Individual rationality from equal division	No–envy	*One–sided resource–monotonicity*	*Two–sided resource–monotonicity*
Uniform rule	yes	yes	yes	no (Prop 3,4)
Proportional solution*	no (Prop 1)	no (Prop 2)	yes	no (Prop 5)
Equal–distance solution	no (Prop 1)	no (Prop 2)	yes**	no (Prop 5)
Equal–sacrifice solution****	no	no	yes**	yes**
$\varphi \subsetneq I_{ed}$	yes (by Def)	may or may not	may or may not	no (prop 3)
$\varphi \subsetneq F$	may or may not	yes (by Def)	only if φ=U(Th1)***	no (Prop 4)

* The same properties hold for the symmetrically proportional solution.

** This result holds on the domain of economies for which $r_i(0) = \infty$ for all i. It does not hold for the version of the solution obtained by using the restrictions of preferences to the interval [0,M], unless there are only 2 agents.

*** This characterization of the uniform rule also involves the requirement $\varphi \subsetneq P$. It holds on the domain of economies for which each r_i is bounded. Otherwise, variants of the rule are obtained.

**** This solution is well–defined on the domain of economies for which each r_i is bounded. Variants of this solution that have the same properties can be defined for the whole domain.

Table 1

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