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POPULATION–MONOTONIC SOLUTIONS TO THE PROBLEM OF FAIR DIVISION WHEN PREFERENCES ARE SINGLE–PEAKED

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Abstract

POPULATION–MONOTONIC SOLUTIONS TO THE PROBLEM OF FAIR DIVISION WHEN PREFERENCES ARE SINGLE–PEAKED

We consider the problem of fairly allocating an infinitely divisible commodity among agents with single–peaked preferences. We search for methods of performing this division satisfying the following property pertaining to changes in the number of agents. Consider changes that are not so large, in the sense that if initially there is not enough to bring all agents to their satiation points, then this still is the case after the change, and if initially there is so much that agents have to be brought beyond their satiation points, then again, this remains the case. The requirement is that such changes affect all agents that are present before and after the change in the same direction. Our main result is that there is essentially only one selection from the envy–free and efficient solution satisfying this property. It is the "uniform" rule. The stronger requirement that any change in the population affect all agents that are present before and after the change in the same direction is met by no selection from the no–envy solution.

Keywords. Fair division. Single–peaked preferences. Population–monotonicity. Uniform rule.

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1. ***Introduction.*** We consider the problem of fairly allocating an infinitely divisible commodity among agents with single–peaked preferences. By this, we mean that each agent has a preferred consumption; the further he moves away from it, in either direction, the worse off he is. We search for desirable methods, or ***solutions***, of performing the division. If the amount to be allocated were equal to the sum of the preferred consumptions, the problem would simply be solved by giving each agent his preferred amount. This special case neatly separates the situations where the amount to be divided is greater than the sum of the preferred consumptions from the situations where the opposite inequality holds. What should be done in general?

This model has recently been analyzed by Sprumont (1991) and Thomson (1990, 1991a). It can be given several interpretations: rationing at disequilibrium prices in a two–good economy; allocation of a task among a group of workers paid an hourly wage and whose disutility of labor is a convex function of labor supplied; allocation of a commodity when preferences become satiated at some point and free disposal is not allowed.

As in these earlier studies, our approach is axiomatic. Here, we look for solutions that behave well when the number of agents changes, in the following sense: suppose that initially there is not enough of the commodity to give each agent his preferred consumption; then, if new agents come in, there still will not be enough. Conversely, suppose that initially there is too much of the commodity to give each agent his preferred consumption; then, if some of the agents leave, there still will be too much. In any one of these situations, we ask that all agents present before and after the change be affected in the same direction. When combined with efficiency, this requirement says that if there are more agents to share the commodity when initially less of it would be socially undesirable — then, after the arrival of the new agents, less of it would still be socially undesirable — all agents initially present are made worse off. If there are fewer agents to share the commodity when initially more of it would be socially undesirable — then after the arrival of the new agents, more of it would still be socially undesirable — all agents initially present are made worse off. We name this property *one–sided population–monotonicity.*

Our primary distributional requirement is no–envy: an allocation is envy–free if no agent would prefer someone else's consumption to his own. This is an intuitively appealing requirement but it has the drawback of typically being satisfied by too many allocations. We ask whether the requirement of *one–sided population–monotonicity* is compatible with no–envy and of course efficiency. The answer is yes, but our main result is that essentially there is only one *one–sided population–monotonic* selection from the envy–free and efficient solution satisfying a mild additional condition that is satisfied by most solutions. It is the uniform rule, a solution that has been central to the earlier analyses of the problem. This solution is also a selection from the solution that associates with each economy its set of efficient allocations that pareto–dominate equal division, the individually rational solution from equal division.

We also consider the following stronger property of *two–sided population–monotonicity*: ***any*** change in the population affects all agents present before and after the change in the same direction. We show that there are selections from the pareto solution satisfying this property. However, there are none from the individually–rational solution from equal division, and there are none from the no–envy solution.

A lesson to be drawn from this paper is that, although only a handful a general principles (consistency; monotonicity with respect to how many options are available; monotonicity with respect to the size of the population; strategy–proofness) underlie virtually all of the developments that have recently taken place in the fast growing literature devoted to the axiomatic study of allocation rules, special forms of these principles, tailored to the specific features of whatever model is being considered, are sometimes necessary for a successful analysis.

2. ***The model.*** There is an infinite population of "potential agents," indexed by the positive integers, $\mathbb{N}$. Each agent $i \in \mathbb{N}$ is equipped with a continuous preference relation R_i defined over $\mathbb{R}_+$. This preference relation is ***single–peaked***: there is $x_i^* \in \mathbb{R}_+$ such that for all $x_i, x_i' \in \mathbb{R}_+$, if $x_i' < x_i \leqq x_i^*$, or if $x_i^* \leqq x_i < x_i'$, then $x_i P_i x_i'$ (P_i denotes the strict preference relation associated with R_i, and I_i the indifference relation). Let $\boldsymbol{p(R_i)} \in \mathbb{R}_+$ be the preferred consumption according to R_i. The preference relation R_i can be described in terms of the function $r_i \colon \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ defined as follows: given $x_i \leqq p(R_i)$, $r_i(x_i) \geqq p(R_i)$ and $x_i I_i r_i(x_i)$ if this is possible, and $r_i(x_i) = \infty$ otherwise; given $x_i \geqq p(R_i)$, $r_i(x_i) \leqq p(R_i)$ and $x_i I_i r_i(x_i)$ if this is possible, and $r_i(x_i) = 0$ otherwise. Let $\mathscr{R}$ be the class of all such preference relations and $\tilde{\mathscr{R}}$ be the subclass of preference relations R_i whose associated r_i is bounded. Let $\mathscr{P}$ be the class of finite subsets of $\mathbb{N}$.

An ***economy*** is a pair $e = ((R_i)_{i \in Q}, M) \in \mathscr{R}^{|Q|} \times \mathbb{R}_+$, or simply (R_Q, M), where $Q \in \mathscr{P}$ and $R_i \in \mathscr{R}$ for all $i \in Q$. A ***feasible allocation for e = (R_Q, M)*** is a list $x = (x_i)_{i \in Q} \in \mathbb{R}_+^{|Q|}$ such that $\sum_{i \in Q} x_i = M$. Let ***X(e)*** be the set of feasible allocations of e. Let $p(R_Q) = (p(R_i))_{i \in Q}$.

A ***solution*** is a mapping φ which associates with every $Q \in \mathscr{P}$ and every $e = (R_Q, M) \in \mathscr{R}^{|Q|} \times \mathbb{R}_+$, a non–empty subset of X(e). Each of the points in $\varphi(e)$ is interpreted as one desirable way of allocating the commodity. We will discuss solution functions as well as solution correspondences. An allocation $x \in \varphi(e)$ is ***φ–optimal for e.*** When a solution φ is single–valued and $\{x\} = \varphi(e)$, we slightly abuse notation and write $x = \varphi(e)$.

The requirement of efficiency is the usual one. We will impose it throughout most of our analysis,[1] that is, search for subsolutions of the following solution:

Pareto solution, P: $x \in P(e)$, where $e = (R_Q, M)$, if $x \in X(e)$ and there is no $x' \in X(e)$ with $x_i' R_i x_i$ for all $i \in Q$ and $x_i' P_i x_i$ for some $i \in Q$.

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[1]Some of our impossibility results, in section 4, are proved without this requirement.

It is easy to check that the efficient allocations of (R_Q, M) are characterized by the property that each agent consumes less than his preferred amount if $\sum_{i \in Q} p(R_i) \geqq M$, and more than his preferred amount if $\sum_{i \in Q} p(R_i) \leqq M$.

From the viewpoint of distribution, the pareto solution is of course very unsatisfactory. We will want to complement it with some requirement with distributional content. The concept that has played the most important role in the recent literature on fair allocation is the concept of an envy–free allocation: at such an allocation, no agent would prefer switching bundles with anyone else (see Thomson, 1991b, for a review of this literature). An attractive feature of the no–envy concept is that it directly corresponds to the sort of mental operations that the man on the street routinely performs in order to evaluate the fairness of a situation. Also, it is an ordinal concept, that is, it depends only on preferences and not on notions of utility. Unfortunately, there typically are many envy–free and efficient allocations (in fact, a continuum) and the need to make more precise recommendations arises.

No–envy solution, F (Foley, 1967): $x \in F(e)$, where $e = (R_Q, M)$, if $x \in X(e)$ and there is no pair $\{i,j\} \subseteq Q$ such that $x_j P_i x_i$.

In "classical" problems of fair division, when preferences are monotone, it is often suggested that equal division be taken as a point of departure; in fact, many writers find it natural to assume that agents are "entitled" to equal division. This idea can be extended to the non–classical problems discussed here where preferences are not monotone, and situations where there might be too much of the commodity, by then giving agents equal "responsibilities". In either case, the requirement that all agents end up at least as well off as at equal division follows quite naturally. Just like the no–envy test, there typically are many efficient allocations, (again, a continuum) passing this test and the question of selection also has to be addressed.

Individually rational solution from equal division, I_{ed}: $x \in I_{ed}(e)$, where $e = (R_Q,M)$, if $x \in X(e)$ and $x_i R_i (M/|Q|)$ for all $i \in Q$.

The following two solutions do make very precise recommendations, since in fact they always select a single point. For the first one, consumptions are chosen proportional to the preferred consumptions. (In the rare case where all preferred consumptions are zero — perferences are then identical — it makes sense to choose equal division). This is a very natural application of the general idea of proportionality, which is a central tenet of the theory of economic justice. Instead of evaluating sacrifices proportionately, however, one can simply take the distance between his actual consumption and his preferred consumption as a measure of how well an agent is treated, and then select allocations at which these measures are equal across agents (some adjustment is needed to guarantee that no agent receives a negative consumption).

Proportional solution, Pro: $x \in \mathrm{Pro}(e)$, where $e = (R_Q,M)$, if $x \in X(e)$ and there is $\lambda \in \mathbb{R}_+$ such that for all $i \in Q$, $x_i = \lambda p(R_i)$; otherwise, $\mathrm{Pro}(e) = (M/|Q|,\ldots,M/|Q|)$.

Equal-distance solution, Dis: $x \in \mathrm{Dis}(e)$, where $e = (R_Q,M)$, if $x \in X(e)$ and (i) when $\sum_{i \in Q} p(R_i) \geqq M$, there is $d \geqq 0$ such that $x_i = \max\{0, p(R_i)-d\}$ for all i, and (ii) when $\sum_{i \in Q} p(R_i) \leqq M$, there is $d \geqq 0$ such that $x_i = p(R_i)+d$ for all i.

Another single-valued solution, which has played the most important role in previous analyses of the problem (Sprumont, 1991; Thomson, 1990, 1991a) is the uniform rule:

Uniform rule, U: $x \in U(e)$, where $e = (R_Q,M)$, if $x \in X(e)$ and (i) when $\sum_{i \in Q} p(R_i) \geqq M$, $x_i = \min\{p(R_i),\lambda\}$ where λ solves $\sum_{i \in Q} \min\{p(R_i),\lambda\} = M$, and (ii) when $\sum_{i \in Q} p(R_i) \leqq M$, $x_i = \max\{p(R_i),\lambda\}$ where λ solves $\sum_{i \in Q} \max\{p(R_i),\lambda\} = M$.

The uniform allocation is both envy-free and individually rational from equal division. On the other hand, the proportional and equal-distance solutions do not necessarily select allocations with these properties, although they do treat identical agents identically (see Thomson, 1991a for a discussion of these facts).

3. ***One-sided population-monotonicity.*** We now formulate a property of single-valued solutions pertaining to changes in the number of agents. In order better to understand this condition, it is useful to take as point of reference the classical domain of private good economies. Consider such an economy, in which some bundle of goods has to be divided among some group of agents with equal rights on the goods. Perform this division by applying some solution. Then, imagine new agents to arrive with claims as valid as those of the agents initially present, resources being kept fixed. We submit that it is natural to require that when the solution is applied to the new, enlarged, problem, all agents initially present be negatively affected.[2]

An abstract version of this requirement was considered in the context of bargaining by Thomson (1983). The requirement was also studied in classical economies by Chichilnisky and Thomson (1987) and Chun and Thomson (1988), on domains of economies with indivisible goods by Alkan (1989) and Tadenuma and Thomson (1990), and on domains of economies with both private and public goods by Moulin (1990a,c). A related condition was examined for coalition form games by Sprumont (1990) and Moulin (1990b). In the present context, the requirement can legitimately be imposed if in the initial economy, there is not enough of the commodity to give each agent his preferred consumption. On the other hand, if there is so much of the commodity that each agent initially consumes more than his preferred amount, the arrival of additional agents may help and it becomes natural to require that if one of the agents initially present gains, then they all do. If initially, no agent consumes much more than his preferred amount

[2]In this and the following paragraphs we initially assume that the rule is already required to pick efficient allocations, as is natural, but issues of monotonicity can be meaningfully discussed independently of issues of efficiency, and below we have chosen a formulation of the axioms that allows for this separation.

and/or if too many additional agents arrive, then again at least one of the agents initially present will have to lose and the requirement will be that all lose. In general, therefore, the relevant requirement here is that all agents be affected in the same direction, as Chun (1986) first proposed in the context of quasi–linear social choice. This requirement will be formally introduced in section 4.

It turns out that in our context, it is quite strong, as will be evidenced by a number of negative results that we will offer (Section 4). The reason for these impossibilities is closely related to the reason for the negative results that we established in an earlier study of the way solutions respond to changes in the amount to be divided (Thomson, 1991a). There, we asked whether it was reasonable to require that agents always be affected in the same direction by such changes. The answer was negative if the distributional requirements of either no–envy, or individual rationality from equal division, were imposed too. These results led us to propose a weaker monotonicity condition in the formulation of which the distinction was made between two kinds of changes: (i) changes that switch the situation from the classical one in which preferences are monotone and an additional unit of the commodity is always socially desirable, to the situation that is special to this model, when there is already "too much of the commodity" and an additional unit is socially undesirable, or conversely, (ii) less disruptive changes that do not cause such a switch. Here, we will similarly be led to distinguishing between changes in the numbers of agents that cause switches, and changes that do not cause switches.

The issue is whether the direction of the inequality between the sum of the preferred consumptions and the amount to be divided is reversed by the changes in the number of agents or the amount to be divided. We will only consider changes in the number of agents that do not reverse the inequality and require that all agents that are present before and after the change be affected in the same direction. This requirement can be met, and it is compatible with both no–envy and individual rationality from equal division.

Moreover, it can also provide the basis for a characterization of the uniform rule. This characterization is the main result of this section (Theorem 1).

We now give a formal statement of the property:

One–sided population–monotonicity. For all Q, $Q' \in \mathcal{P}$ with $Q' \subset Q$, for all $(R_Q, M) \in \mathcal{R}^{|Q|} \times \mathbb{R}_+$, if $\sum_{i \in Q'} p(R_i) \geqq M$ or if $\sum_{i \in Q} p(R_i) \leqq M$, then $\varphi_i(R_{Q'}, M) R_i \varphi_i(R_Q, M)$ for all $i \in Q'$, or $\varphi_i(R_Q, M) R_i \varphi_i(R_{Q'}, M)$ for all $i \in Q'$.

Note that this formulation makes no presumption of efficiency since we only require that the agents that are present before and after the change in the population be affected "in the same direction". When efficiency is imposed as well, the direction of the change is unambiguous: if $\sum_{i \in Q'} p(R_i) \geqq M$, the arrival of $Q \backslash Q'$ is bad news for the group Q' and the requirement is that all of its members lose; if $\sum_{i \in Q} p(R_i) \leqq M$, it is the departure of $Q \backslash Q'$ that is bad news for the group Q' and again the requirement is that all of its members lose.

An alternative formulation of the axiom that would make the most sense for, but would not be limited to, solutions already required to be efficient would be that if $\sum_{i \in Q'} p(R_i) \geqq M$, all agents $i \in Q'$ are made worse off by the arrival of the group $Q \backslash Q'$, and if $\sum_{i \in Q} p(R_i) \leqq M$, all agents in Q' are made better off by the arrival of the group $Q \backslash Q'$.[3]

In the following lemma, we record the fact that all of the single–valued solutions defined earlier satisfy the property. (We omit the proofs of all the lemmas stating positive results.)

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[3]Note that the corresponding property that has been used in the analysis of classical economies (all agents initially present are made worse off by the arrival of newcomers, resources being kept fixed) is most appropriate for, (again, it is not limited to), efficient solutions. In order to emphasize the conceptual distinction between efficiency and monotonicity, we have chosen in this paper the more general formulation.

Lemma 1. The proportional and equal–distance solutions as well as the uniform rule satisfy *one–sided population–monotonicity.*

Another condition will be useful too: informally, the desirability of an allocation is preserved under replication. We formulate it for solution correspondences. If an allocation is φ–optimal for some economy, then the replicated allocation is φ–optimal for the replicated economy. Note that this allows allocations in the replicated economy to be φ–optimal without being replicated allocations; the individually rational solution from equal division illustrates this possibility. On the other hand, in our model, if a subsolution of the pareto solution satisfies "equal treatment of equals"[4] and this property, then all φ–optimal allocations of a replicated economy are obtained by replicating a φ–optimal allocation of the model economy. Here, an illustration is the envy–free and efficient solution.

Replication–expansion: For all Q, $Q' \in \mathcal{P}$, for all $k \in \mathbb{N}$, for all $(R_Q,M) \in \mathcal{R}^{|Q|} \times \mathbb{R}_+$, for all $R'_{Q'} \in \mathcal{R}^{|Q'|}$, for all $x \in \varphi(R_Q,M)$, for all $x' \in \mathbb{R}^{k|Q|}$, if Q' can be partitioned into $|Q|$ groups of k agents, $(Q^i)_{i\in Q}$, such that for all $i \in Q$ and for all $j \in Q^i$, $R'_j = R_i$ and $x'_j = x_i$, then $x' \in \varphi(R'_{Q'},kM)$.

This condition is very mild, as evidenced by the next lemma.

Lemma 2. The Pareto solution, the no–envy solution, and the individually rational solution from equal division, the proportional and equal–distance solutions, and the uniform rule all satisfy *replication–expansion.*

In our model *replication–expansion*, together with *single–valuedness*, a property that will be imposed here, imply ***replication–invariance***: the φ–optimal allocation of a replicated economy is the replica of the φ–optimal allocation of the model economy.

We are now ready for the main theorem. Since *replication–invariance* is needed only for half of the cases the theorem covers, we state the theorem in two parts.

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[4]A formalization of the idea of "equal treatment of equals" appears in section 4.

***Theorem* 1.** On the domain of economies $(R_Q,M) \in \mathscr{R}^{|Q|} \times \mathbb{R}_+$, $Q \in \mathscr{P}$, with $\sum_{i \in Q} p(R_i) \geqq M$, the uniform rule is the only *one–sided population–monotonic* selection from the envy–free and efficient solution. On the domain of economies $(R_Q,M) \in \tilde{\mathscr{R}}^{|Q|} \times \mathbb{R}_+$, $Q \in \mathscr{P}$, with $\sum_{i \in Q} p(R_i) < M$, the uniform rule is the only *one–sided population–monotonic* and *replication–invariant* selection from the envy–free and efficient solution.

An informal argument might be helpful before going into the details of the proof. We argue by contradiction. Consider first an economy in which there is not enough of the commodity and suppose that it is not allocated according to the uniform rule. This implies the existence of two agents, indexed 1 and 2 for simplicity, such that $x_1 < r_1(x_1) \leqq x_2$. The presence of the "gap" $[x_1, r_1(x_1)]$, in which the consumption of agent 2 should not be, — otherwise agent 1 would envy him — constitutes the basis for the proof. Let $g = r_1(x_1) - x_1$. New agents, whose preferred consumption is g/2, are now introduced in succession. The proof is by induction on the number of new agents. By efficiency, no such agent should ever consume more than g/2. By *one–sided population–monotonicity*, upon the arrival of a new agent, every agent initially present should be made worse off, i.e. its consumption should decrease, so that in fact the consumption of none of them can decrease by more than g/2. In particular agent 2's consumption cannot decrease enough for it to "jump over the gap". Since agent 1's consumption decreases too, the gap can only increase. The preferences of the new agents are also chosen so that g/4 is indifferent to M. Since agent 2's consumption remains greater than $r_1(x_1) \geqq g/2$, for the new agents not to envy agent 2, each of them should consume at least g/4. These configurations of consumptions are preserved as more and more agents are introduced. This implies that the new agents together receive an amount that is unbounded above, in contradiction with feasibility.

Next, consider an economy in which there is too much of the commodity and suppose that it is not allocated according to the uniform rule. This implies the existence of two agents, indexed 1 and 2 for simplicity, such that $x_1 \leqq r_2(x_2) < x_2$. Here it is the presence of the gap $[r_2(x_2),x_2]$ that will constitute the basis of the proof. Let $g = x_2 - r_2(x_2)$. First the economy is replicated k times (the choice of k is explained later on). By *replication–invariance*, we know what to do with the replicated economy. Then all the agents of type $j \neq 1$ are deleted, one at a time, the last deleted agent being an agent of type 2. By *one–sided population–monotonicity*, whenever any such agent leaves, all remaining agents should be made worse off, which here means that each should receive a greater amount. By no–envy, all (remaining) agents of a given type should consume the same amount, that is, should receive the same increment. We will note later that no agent ever consumes more than some amount $\bar{x}$, so that if k is chosen large enough, we can ensure that even if the agents of type 1 were to receive the whole amount that is freed, their common consumption would not increase sufficiently to jump over the gap, that is, it would remain below $r_2(x_2)$. (To achieve this, k should satisfy $\bar{x}/k \leqq g$.) What guarantees that no agent ever consumes more than some maximal amount $\bar{x}$ is the assumption that $r_i(0) < \infty$ for all i. It implies that if an agent's consumption were to increase too much, at some point he would become envious of the agents of type 1, whose consumption is bounded above by $r_2(x_2)$. As the economy shrinks, the (common) consumption of the remaining agent(s) of type 2 increases, but this can only increase the gap. At the end of the process of elimination, when only all the agents of type 1 and one agent of type 2 are left, the common consumption of the agents of type 1 is still at most $r_2(x_2) \leqq M-g$. Upon the departure of the last agent of type 2, it has to become M. But this is impossible, since again, by the choice of k, it cannot increase by more than g.

Proof of Theorem 1. Let $\varphi \subseteq FP$ be given. Suppose, by contradiction, that for some $Q \in \mathcal{P}$ and $e = (R_Q,M) \in \mathcal{R}^{|Q|} \times \mathbb{R}_+$, $x = \varphi(R_Q,M) \neq U(R_Q,M)$.

***Case* 1.** $\sum_{i \in Q} p(R_i) \geqq M$. Since $\varphi \subseteq P$, $x \neq U(R_Q,M)$ means that there are i, j ∈ Q with x_i

$< p(R_i)$ and $x_i < x_j$. To simplify the notation, suppose that $i = 1$ and $j = 2$. Since $\varphi \subsetneq F$, agent 2 does not envy agent 1 at x, so that $x_1 < p(R_1) < r_1(x_1) \leqq x_2$. Let $g = r_1(x_1)-x_1$. Let $R_0 \in \mathscr{R}$ be a preference relation such that $p(R_0) = g/2$ and $r_0(g/4) = M$. We construct a sequence of economies by adding to the initial economy agents with preference relation R_0: for each $k \in \mathbb{N}$, let $\tilde{Q}^k$ be the set consisting of the first k such agents indexed by $i_1,\ldots,i_k$, $Q^k = Q \cup \tilde{Q}^k$, and $e^k = (R_{Q^k},M)$. Also, let $e^0 = e$.

Note that $\sum_{j\in Q^k} p(R_j) \geqq \sum_{j\in Q} p(R_j) \geqq M$, so that since $\varphi \subsetneq P$, if $x^k = \varphi(e^k)$, then $x_j^k \leqq p(R_j)$ for all $j \in Q^k$, in particular, $x_{i_k}^k \leqq p(R_0) = g/2$.

We will show by induction that for all k, $x_2^k \geqq r_1(x_1)$. Note first that this is true for $k = 0$. At stage $k \geqq 1$, upon the arrival of agent i_k, it follows from *one–sided population–monotonicity* that all agents in Q^{k-1} lose. Since $\varphi \subsetneq P$, this means that for all $i \in Q^{k-1}$, $x_i^k \leqq x_i^{k-1}$, and since $\sum_{i\in Q^{k-1}}(x_i^{k-1}-x_i^k) = x_{i_k}^k \leqq p(R_0) = g/2$, we have that for all $i \in Q^{k-1}$, $x_i^{k-1}-x_i^k \leqq g/2$, in particular $x_2^{k-1}-x_2^k \leqq g/2$. For agent 1 not to envy agent 2 at x^k, we need either $x_2^k \leqq x_1^k$ or $x_2^k \geqq r_1(x_1^k) \geqq r_1(x_1)$. Since $x_2^{k-1} \geqq r_1(x_1)$ by the induction hypothesis, $x_2^{k-1}-x_2^k \leqq g/2$, and $g = r_1(x_1)-x_1$, we have $x_2^k \geqq r_1(x_1)$, as claimed.

Now, given the specification of R_0, for each $j \in \tilde{Q}^k$ not to envy agent 2 at x^k, we need $x_j^k \geqq g/4$, so that $\sum_{j\in\tilde{Q}^k} x_j^k \geqq kg/4$. For k large enough, $kg/4 > M$, in contradiction with feasibility.

Case 2. $\sum_{i\in Q} p(R_i) < M$. Since $\varphi \subsetneq P$, $x \neq U(R_Q,M)$ means that there are $i, j \in Q$ with $x_i < x_j$ and $p(R_j) < x_j$. To simplify the notation, suppose that $i = 1$ and $j = 2$. Since $\varphi \subsetneq F$, agent 2 does not envy agent 1 at z, so that $x_1 \leqq r_2(x_2) < p(R_2) < x_2$. Let $g = x_2-r_2(x_2)$. Now, for each $j \neq 1$, let $\bar{x}_j = r_j(x_1)$, $\bar{x} = \max_{j\neq 1} \bar{x}_j$ and $k^* > \bar{x}/g$. Let e^{k^*} be obtained by k*–times replication of e and $x^* = \varphi(e^{k^*})$. By *replication–invariance*, x^{k^*} is obtained by k^*–times replication of x.

Starting from e^{k^*}, we successively delete all the agents of type $j \neq 1$, the last deleted agent being an agent of type 2. Let $e^0 = e^{k^*}$. At stage $k = 1,\ldots,k^*(|Q|-1)$ of this process, let i_k be the agent that is deleted. Let Q^k be the remaining group of agents, $e^k = (R_{Q^k},k^*M)$, and $x^k = \varphi(e^k)$. Note that $\Sigma_{j\in Q^k} p(R_j) \leqq k^* \Sigma_{j\in Q} p(R_j) \leqq k^*M$, so that since $\varphi \subsetneq P$, $p(R_j) \leqq x_j^k$ for all $j \in Q^k$. Since $\varphi \subsetneq FP$, for all k and for all agents i, j of type 1, $x_i^k = x_j^k$, and for all agents i, j of type 2, $x_i^k = x_j^k$. Let x_1^k and x_2^k denote these common consumptions. We will show by induction on k that $x_1^k \leqq r_2(x_2^k) < p(R_2) < x_2^k$. This is obviously true for $k = 0$. Given $k \geqq 1$, let us suppose that these inequalities hold for k–1.

By *one–sided population–monotonicity*, and since $\varphi \subsetneq P$, for all $j \in Q^k$, $x_j^{k-1} \leqq x_j^k$. The inequalities $x_1^{k-1} \leqq p(R_2)$ (from the induction hypothesis) and $x_1 \leqq x_1^{k-1}$, and the requirement that at x^{k-1} agent i_k does not envy agent 1, imply that if $x_1^{k-1} \leqq p(R_{i_k})$ (this certainly will be the case if i_k is an agent of type 2) then $x_{i_k}^{k-1} \leqq r_{i_k}(x_1^{k-1}) \leqq r_{i_k}(x_1)$, and if $p(R_{i_k}) \leqq x_1^{k-1}$ then $x_{i_k}^{k-1} \leqq x_1^{k-1} \leqq p(R_2) \leqq r_2(x_1)$. In either case $x_{i_k}^{k-1} \leqq \bar{x}$. Therefore, and using the inequalities derived above from *one–sided population–monotonicity*, $x_1^k - x_1^{k-1} \leqq \bar{x}/k^* < g$. Since $r_2(x_2^k) \leqq r_2(x_2^{k-1}) < p(R_2) < x_2 \leqq x_2^k$, for the agent(s) of type 2 not to envy the agents of type 1 at x^k, we need $x_1^k \leqq r_2(x_2^k)$ or $x_1^k \geqq x_2^k$. The inequalities $x_1^k - x_1^{k-1} < g$ and $x_1^{k-1} \leqq r_2(x_2^{k-1})$ (from the induction hypothesis) imply that the second case is not possible, so that $x_1^k \leqq r_2(x_2^k)$. The process of elimination continues for $k^*(|Q|-1)$ steps until all agents of type $j \neq 1$ have been taken care of.

At the last step, apart from the k^* agents of type 1, there remains only one agent of type 2. Upon his removal, the common consumption of the agents of type 1 becomes

M. For any $k < k^*(|Q|-1)$, $x_1^k+g \leqq M$. Since $x_1^{k^*(|Q|-1)}-x_1^{k^*(|Q|-1)-1} < g$, we obtain a contradiction.

Q.E.D.

Replication–invariance is used in the second part of the theorem, and the proof for that case only requires having access to new agents with preferences identical to those of the agents initially present. Such a proof can therefore be described as a "fixed–type" proof. On the other hand, the proof for the first part involves no replication axiom but it relies on the availability of new agents whose preferences may be different from the preferences of the agents initially present. A fixed–type proof based on *replication–invariance* is however possible for the first part too.[5] (We gave the argument for the case $\sum_{i \in Q} p(R_i) \leqq M$ to show explicitly how it can be ensured that the consumptions of all agents remain bounded above. The case $\sum_{i \in Q} p(R_i) \geqq M$ is a little simpler since consumptions, which decrease, are bounded below by zero.) Whether a proof relying only on the availability of agents with arbitrary preferences instead of *replication–invariance* can be devised for the second part is an open question.

A second open question pertains to selections from the individually rational from equal division and efficient solution: Is the uniform rule the only selection satisfying *one–sided population–monotonicity* and *replication–invariance*? The following facts, whose proofs are similar to those of the steps of Theorem 1, provide the beginning of an answer: if the amount to be divided is smaller than the sum of the preferred consumptions, any such solution gives his preferred consumption to each agent whose preferred consumption is smaller than equal division, as the uniform rule does. If the amount to be divided is greater than the sum of the preferred consumptions, the solution gives his preferred

[5]The argument would differ in that, after replicating the initial economy k^* times, where k^* satisfies $x_2/k^* \leqq g$, we would keep ***adding*** agents of type 2.

consumption to each agent whose preferred consumption is greater than equal division; again, this is as the uniform rule does. These results imply that in 2–person economies, the solution coincides with the uniform rule.[6]

4. ***Two–sided population–monotonicity.*** We now turn to the stronger monotonicity condition discussed in the introduction: ***any*** change in the number of agents affects all agents that are present before and after the change in the same direction. Here, no restriction is imposed to prevent the change from being too disruptive.

Two–sided population–monotonicity: For all Q, $Q' \in \mathcal{P}$ with $Q' \subset Q$, for all $(R_Q,M) \in \mathcal{R}^{|Q|} \times \mathbb{R}_+$, either $\varphi_i(R_Q,M)R_i\varphi_i(R_{Q'},M)$ for all $i \in Q'$, or $\varphi_i(R_{Q'},M)R_i\varphi_i(R_Q,M)$ for all $i \in Q'$. ***Strict two–sided population–monotonicity*** holds if in addition, in each of the previous cases, whenever one of the preferences is strict, then they all are.

Most of the results of this section are negative. We start with an examination of three solutions examined earlier.

***Lemma* 3.** The uniform rule, the proportional solution, and the equal–distance solution do not satisfy *two–sided population–monotonicity.*

Proof. Let Q = {1,2,3} and Q′ = {1,2}. (i) Let $p(R_Q) = (1,3,2)$, and M = 5. Then, $U(R_{Q'},M) = (2,3)$ and $U(R_Q,M) = (1,2,2)$. Agent 1 gains from agent 3's arrival, whereas agent 2 loses. This proves the result for U. (ii) Let $p(R_Q) = (1,2,9)$, $r_1(.5) = 1.5$, $r_2(1) = 5$, and M = 6. Then, $Pro(R_{Q'},M) = (2,4)$ and $Pro(R_Q,M) = (.5,1,4.5)$. Agent 1 gains from agent 3's arrival, whereas agent 2 loses. This proves the result for Pro. (iii) Let $p(R_Q) = (1,2,3.5)$, $r_1(.5) = 1.5$, $r_2(1.5) = 4$, and M = 5. Then, $Dis(R_{Q'},M) = (2,3)$ and $Dis(R_Q,M) = (.5,1.5,3)$. Agent 1 gains from agent 3's arrival,

[6]Prior experience with this model suggests that the answer may very well be negative: indeed, if the uniform rule is the only selection from the envy–free and efficient solution to satisfy the property of *one–sided resource–monotonicity* informally described in the paragraphs preceding our introduction of *one–sided population–monotonicity*, it is not the only selection from the individually rational from equal division and efficient solution to satisfy *one–sided resource–monotonicity* (Thomson, 1991a).

whereas agent 2 loses. This proves the result for Dis.

Q.E.D.

In fact, Lemma 3 is a consequence of a more general result which involves two requirements. The first one is that the solution depend only on preferred consumptions. This is a requirement of simplicity which happens to be satisfied by several of the solutions that we have examined. Moreover, it plays an essential role in ensuring strategyproofness, as noted by Sprumont (1991). The second requirement is that identical agents receive indifferent (according to their common preference relation) amounts. We formulate it for single–valued solutions. It is satisfied by all of the solutions discussed earlier.

Symmetry. For all $e = (R_Q,M) \in \mathscr{R}^{|Q|} \times \mathbb{R}_+$ with $|Q| \geqq 2$, for all i, j $\in$ Q, if $R_i = R_j$, then $\varphi_i(e) I_i \varphi_j(e)$.

A version of the requirement for multi–valued solutions is that for each economy in which two agents have identical preferences, the allocation obtained from any one of the recommendations made by the solution by exchanging the coordinates pertaining to these two agents is also one of the recommendations. Therefore, the ***set*** of recommendations exhibits a symmetry reflecting the symmetry that exists in the preferences. This more general requirement is satisfied by all of the multi–valued solutions introduced above. Turning again our attention to single–valued solutions, note that in conjunction with efficiency, *symmetry* says that the consumptions of two identical agents are not only indifferent but in fact identical. In the proof of the following result, which makes no use of efficiency, identical agents receive indifferent, but sometimes distinct, consumptions.

Lemma 4. There is no *two–sided population–monotonic* and *symmetric* solution that depends only on preferred consumptions.[7]

[7]I am grateful to T. Yamato for suggesting that the requirement $\varphi \subsetneq P$ imposed in an earlier version of this paper might be unnecessary.

Proof. The proof is by way of an example. Let φ be a *symmetric* solution that depends only on preferred consumptions. Let $Q = \{1,2,3\}$, $Q' = \{1,2\}$, $R_1 = R_2 = R_3$ with $p(R_1) = 5$, $r_1(4.1) = 7.9$, r_1 being linear in the interval $[0,5]$, and $M = 12$. Let $e = (R_Q,M)$ and $e' = (R_{Q'},M)$. Let $x = \varphi(e)$. If $x_i \leqq 5$ for all $i \in Q$, then by *symmetry* $x = (4,4,4)$. If $x_i \geqq 5$ for exactly one $i \in Q$, say $x_3 \geqq 5$, we obtain by *symmetry* $x_1 = x_2$, so that $x_3 = 12 - 2x_1$ and again by *symmetry* $x_1 I_1 (12{-}2x_1)$. Since $x_3 \geqq 5$, we need $x_1 \leqq 3.5$ but then by the choice of r_1, $(12{-}2x_1)P_1x_1$. Finally, suppose that $x_i \leqq 5$ for exactly one $i \in Q$, say $x_1 \leqq 5$. Then, by *symmetry* $x_2 = x_3$, and $x_1 I_1 (12{-}x_1)/2$. Since for $x_2 = x_3 \geqq 5$, we need $x_1 \leqq 1$, by the choice of r_1, we obtain $[(12{-}x_1)/2]P_1x_1$. Therefore the only possibility is $x = (4,4,4)$.

Let $y = \varphi(e')$. By *symmetry* $y = (6,6)$, $y = (4.1,7.9)$, or $y = (7.9,4.1)$.

If $y = (6,6)$ or $y = (4.1,7.9)$, let $R_1' = R_3' = R_1$ and R_2' be such that $p(R_2') = p(R_2)$ and $r_2(4) = 5.5$. Since φ depends only on preferred consumptions, $\varphi(R_Q',M) = \varphi(R_Q,M) = x$ and $\varphi(R_{Q'}',M) = \varphi(R_{Q'},M) = y$. After the change in preferences, as Q' enlarges to Q, agent 1 loses and agent 2 gains, in contradiction with *two–sided population–monotonicity.* If $y = (7.9,4.1)$, we repeat the argument by changing agent 1's prefrences in the way we just changed agent 2's preferences.

Q.E.D.

Do there exist *two–sided population–monotonic*, or perhaps *strictly two–sided population–monotonic*, selections from the individually rational from equal division and efficient solution? As far as the stronger property is concerned, the answer could not be an unqualified yes since if an agent's preferred consumption is zero, then he gets 0 at any efficient allocation in any economy (R_Q,M) such that $\sum_{i\in Q} p(R_i) \geqq M$ of which he is a member. The inequality is preserved by the addition of new agents. But even the weaker property cannot be met, as follows from the next theorem.

***Theorem* 2.** There is no selection from the individually rational from equal division solution satisfying *two–sided population–monotonicity.*[8]

Proof. Let $\varphi \subseteq I_{ed}$ be given. Let $Q = \{1,2,3\}$, $Q' = \{1,2\}$, $p(R_Q) = (2,3,2)$, and $M = 6$. Let $e = (R_Q,M)$ and $x \in I_{ed}(e)$. Since $p(R_1) = p(R_3) = M/|Q| = 2$, $x_1 = x_3 = 2$, and therefore $x_2 = 2$. Let $e' = (R_{Q'},M)$ and $y \in I_{ed}(e')$. Since $p(R_2) = M/|Q'| = 3$, $y_2 = 3$, and therefore $y_1 = 3$.

So $\varphi(e) = (2,2,2)$ and $\varphi(e') = (3,3)$. Upon the arrival of agent 3, agent 1 gains and agent 2 loses, in contradiction with *two–sided population–monotonicity.*

Q.E.D.

A direct implication of Theorem 2 is that the solution that associates with each economy its equal division allocation is not *two–sided population–monotonic.* But in fact, this solution is not even *one–sided population–monotonic.* To see this, let $Q = \{1,2,3\}$, $Q' = \{1,2\}$, $p(R_Q) = (2,3,0)$ and $M = 6$. Then, the solution picks (2,2,2) for (R_Q,M) and (3,3) for $(R_{Q'},M)$. We have $\sum_{i\in Q'} p(R_i) = \sum_{i\in Q} p(R_i) \leqq M$ so that the hypothesis of *one–sided population–monotonicity* is met. However, upon the arrival of agent 3, agent 1 gains and agent 2 loses. The fact that the equal division solution does not satisfy our weaker monotonicity property should be contrasted with what we know of classical domains. There, this solution satisfies the form of population–monotonicity that is natural for that domain (all agents initially present are made weakly worse off by the arrival of additional agents) but not efficiency, and it is a simple example illustrating the tradeoffs between monotonicity and efficiency.

If individual rationality from equal division is replaced by no–envy, another impossibility obtains:

[8]A related impossibility is given in the appendix.

***Theorem* 3.** There is no selection from the no–envy solution satisfying *two–sided population–monotonicity.*

Proof. Let $\varphi \subseteq F$ be given. Let $Q = \{1,2,3,4\}$, $Q' = \{1,2\}$, $p(R_1) = 0$, $p(R_2) = 2$, $r_2(0) = 3$, r_2 being linear on the interval $[0,2]$, $R_2 = R_3 = R_4$, and $M = 4$. Let $e = (R_Q,M)$ and $x \in F(e)$. First, we note that $x_2 \neq 2$. Otherwise, for neither agent 3 nor 4 to envy agent 2, we need $x_3 = x_4 = 2$, in contradiction with feasibility. Next, we observe that $x_1 < 2$. Indeed, if $x_1 \geqq 2$, then for some $i \in \{2,3,4\}$, $x_i < x_1$ and agent 1 envies agent i at x. Let $e' = (R_{Q'},M)$ and $y \in F(e')$. For agent 1 not to envy agent 2 at y, we need $y_1 \leqq 2$ and for agent 2 not to envy agent 1 at y, we need $y_2 \leqq 2$. Therefore, $y = (2,2)$.

So $\varphi_1(e) < 2$, $\varphi_2(e) \neq 2$ and $\varphi(e') = (2,2)$. Upon the arrival of agents 3 and 4, agent 1 gains and agent 2 loses, in contradiction with *two–sided population–monotonicity.*

Q.E.D.

Faced with these impossibilities[9], we will limit ourselves to selections from the pareto solution and ask whether such selections satisfying *two–sided population–monotonicity* exist. The answer is yes. The solution defined next will provide the point of departure for this positive answer. It involves evaluating an allocation $x \in X(e)$ on the basis of the differences $c_i(x_i)=|r_i(x_i)-x_i|$. This is appealing because the number $c_i(x)$, which is the size of agent i's upper contour set at x_i, can be interpreted as a measure of his "sacrifice at x". Selecting efficient allocations at which sacrifices are equal across agents is of course tempting but such allocations do not always exist, because of boundary problems.[10] As second best, we recommend the allocation at which sacrifices are as "equal as possible".

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[9]The fact that the uniform rule does not satisfy *two–sided population–monotonicity* is a consequence of Theorem 2 as well as of Theorem 3, since this rule is a selection from I_{ed} as well as a selection from F. We also gave a direct and elementary proof in Lemma 3.

[10]We encountered a similar problem in our definition of the equal–distance solution.

Also, in order to avoid the difficulties that occur with economies for which these sacrifices are infinite for some agents at some allocations, we limit ourselves at first to the domain of economies for which for all i, $r_i(x_i) < \infty$ for all $x_i \in \mathbb{R}_{++}$. (This is almost the domain restriction of Theorem 1.)

Equal-sacrifice solution, Sac: $x = \text{Sac}(e)$, where $e = (R_Q,M)$, if $x \in X(e)$ and (i) when $\sum_{i\in Q} p(R_i) \geqq M$, there exists $\sigma \geqq 0$ such that $r_i(x_i)-x_i \leqq \sigma$ for all i, strict inequality holding only if $x_i = 0$, and (ii) when $\sum_{i\in Q} p(R_i) \leqq M$, there exists $\sigma \geqq 0$ such that $x_i-r_i(x_i) = \sigma$ for all i.

It is easy to check that this solution is single-valued, produces efficient allocations, and satisfies *symmetry* and *replication-invariance*. We also have:

Lemma **5**: On the domain of economies for which it selects interior allocations, the equal-sacrifice solution is a selection from the pareto solution satisfying *two-sided population-monotonicity.*

The relevance of the interiority restriction in Lemma 5 is illustrated by the following example: $Q = \{1,2,3\}$, $Q' = \{1,2\}$, $p(R_Q) = (1,8,8)$, $r_1(0) = 2$, $r_2(7) = 9.5$, $R_2 = R_3$, and $M = 12$. Let $e = (R_Q,M)$ and $e' = (R_{Q'},M)$. Then $\text{Sac}(e) = (0,6,6) = x$ (then $r_1(x_1)-x_1 < 2.5$ and $r_2(x_2)-x_2 = r_3(x_3)-x_3 > 2.5$) and $\text{Sac}(e') = (2.5,9.5) = y$ (here $y_1-r_1(y_1) = y_2-r_2(y_2) = 2.5$). Upon the arrival of agent 3, agent 1 gains and agent 2 loses.

To be able to work on a wider domain, we now define a variant of the equal-sacrifice solution. For each R_i with $p(R_i) > 0$, let $t_{R_i}:[0,r_i(0)[\to \mathbb{R}$ be a continuous increasing function such that $t_{R_i}(0) = 0$ and $t_{R_i}(x_i) \to \infty$ as $x_i \to r_i(0)$. Now, given any $e = (R_Q,M)$ such that $\sum_{i\in Q} p(R_i) \geqq M$, let $x = \varphi(e)$ be such that $x \in P(e)$, $x_i = 0$ if $p(R_i) = 0$ and $t_{R_i}(r_i(x_i)-x_i) = t_{R_j}(r_j(x_j)-x_j))$ for all i, $j \in Q$. As $M \to \sum_{i\in Q} r_i(0)$, the resulting x is such that $x \to (r_i(0))_{i\in Q}$. Then, set $\varphi(R_Q,\sum_{i\in Q} r_i(0)) = (r_i(0))_{i\in Q}$. Finally, given any $e = (R_Q,M)$ such that $\sum_{i \in Q} r_i(0) < M$, simply choose $\varphi(e) = x \in X(e)$ so that $x_i - r_i(0) = x_j - r_j(0)$ for all $i, j \in Q$. To obtain a *symmetric* solution, specify the t_i so that $t_{R_i} = t_{R_j}$ whenever $R_i = R_j$. Any solution so defined satisfies *two-sided population monotonicity.*

The fact that the equal–sacrifice solution is not a selection from the individually rational solution from equal division is a consequence of Lemma 5 and Theorem 2, since in the example used to prove that theorem, the equal–sacrifice allocation happens to be interior. A direct proof is given by the following simple example: let $Q = \{1,2\}$ and $e = (R_Q,M)$ be such that $r_1(2) = 4$, $r_2(4) = 6$, and $M = 6$. Then $\mathrm{Sac}(e) = (2,4)$. Since $M/2 = 3$ and $3P_1 2$, $\mathrm{Sac}(e) \notin I_{ed}(e)$. Here is a proof that equal–sacrifice solution is not a selection from the no–envy solution: let $Q = \{1,2\}$, and $e = (R_Q,M)$ be such that $r_1(2) = 4$, $r_2(3) = 5$, and $M = 5$. Then $\mathrm{Sac}(e) = (2,3)$. Since $3P_1 2$, $\mathrm{Sac}(e) \notin F(e)$.

In order to avoid the domain restriction $r_i(0) < \infty$ for all i, let $f: \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function with $f > 0$ and $\int_0^\infty f(u)du < \infty$. Then, given $e = (R_Q,M)$, let $\mathrm{Sac}^f(e)$ be the allocation $x \in P(e)$ at which the "f–weighted" sacrifices $\int_{x_i}^{r_i(x_i)} f(u)du$ are equal across agents (or again, as equal as possible). On the domains of economies for which they select interior allocations, the resulting solutions are *two-sided population–monotonic.* The same modification as above can be used to avoid the interiority restriction.

All the members of the following family, which are well defined under the same domain restriction as the equal–sacrifice solution, (the restriction that each r_i be bounded,) also satisfy *two-sided population–monotonicity* under the same interiority condition.

Definition. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be two continuous and strictly increasing functions such that $f(0) = g(0) = 0$ and $f(\mathbb{R}) = g(\mathbb{R}) = \mathbb{R}$. Given any $x_i \in \mathbb{R}_+$, let $a_i(x_i) = p(R_i) - x_i$ and $b_i(x_i) = r_i(x_i) - p(R_i)$. Then, given $e = (R_Q,M)$, let $x \in P(e)$ be such that $f(a_i(x_i)) + g(b_i(x_i)) = f(a_j(x_j)) + g(b_j(x_j))$ for all $i, j \in Q$ if this is possible. Otherwise, pick

$x \in P(e)$ so that these expressions are as equal as possible as in the definition of the equal–sacrifice solution.

If $f = g$, the resulting solutions treat units of the good above preferred consumptions and units of the good below preferred consumptions symmetrically. The equal–sacrifice solution is obtained by taking f and g to be the identity functions.

These examples show that quite a few selections from the pareto solution satisfy *two–sided population–monotonicity.*

5. ***Conclusion.*** We considered the problem of fair division of a commodity in economies with single–peaked preferences, and formulated two properties of solutions pertaining to possible changes in the number of agents that are tailored for this domain. We used the weaker property to provide a characterization of the uniform rule. The uniform rule does not satisfy the stronger property but no selection from the individually rational solution from equal division or from the no–envy solution does. However, selections from the pareto solution satisfying this property do exist.

The uniform rule has been shown to be essentially the only *strategy–proof* rule (Sprumont, 1991), the only selection from the envy–free and efficient solution to be *consistent* (Thomson, 1990), or to satisfy a certain property of *one–sided resource–monotonicity* (Thomson, 1991). The present paper confirms its importance.

Our results are summarized in the following matrix. A "yes" in cell (a,b) means that the solution in row a satisfies the condition in column b. A "no" means the opposite.

	Pareto–domination of equal division	No–envy	*Replication invariance*	*One–sided population–monotonicity*	*Two–sided population–monotonicity*
Uniform rule	yes	yes	yes (Lem 2)	yes (Lem 1)	no (Lem 3) (Th 2, Th 3)
Proportional solution	no	no	yes (Lem 2)	yes (Lem 1)	no (Lem 3)
Equal–distance solution	no	no	yes (Lem 2)	yes (Lem 1)	no (Lem 3)
Equal–sacrifice solution	no	no	yes	yes	yes* (Lem 5)
$\varphi \subseteq I_{ed}P$	yes (by Def)	may or may not	may or may not	$\varphi = U$ is an example	no*** (Th 2)
$\varphi \subseteq FP$	may or may not	yes (by Def)	may or may not	only if $\varphi = U$ (Th 1)**	no*** (Th 3)

* A domain restriction is needed for this positive result. Under the same restriction, other *two–sided population–monotonic* selections from the pareto solution can be constructed.

** This characterization holds on the domain of economies for which each r_i is finite. and under the assumption that φ also satisfies *replication–invariance.*

*** This non–existence result holds even if no efficiency requirement is imposed.

Table 1

Appendix

In this appendix, we briefly discuss a concept intermediate in spirit between no–envy and individual rationality from equal division. It simply says that every agent prefers his consumption ***to what the others receive on average.*** This concept is discussed in Thomson, (1979, 1982), Baumol, (1986), and Kolpin (1991), and we refer the reader to these sources for motivation and applications to other domains.

Average no–envy solution, A: $x \in A(e)$, where $e = (R_Q,M)$, if $x \in X(e)$ and $x_i R_i (\sum_{j \neq i} x_j/(|Q|-1))$, for all $i \in Q$.

The following facts are easily established: $I_{ed} \subsetneq A$; there is no containment relation between A and F (unless of course $n = 2$, in which case $A = F$). The proportional, equal–distance, and equal–sacrifice solutions are not selections from A.

Here we show that there is no *two–sided population–monotonic* selection from the average envy–free solution. Because of the inclusion $I_{ed} \subsetneq A$, this impossibility implies Theorem 2.

***Theorem* 4.** There is no selection from the average no–envy solution satisfying *two–sided population–monotonicity.*

Proof. The example used to prove this result is almost the same as the example used to prove Theorem 3. Let $\varphi \subsetneq A$ be given. Let $Q = \{1,2,3,4,5\}$, $Q' = \{1,2\}$, $p(R_1) = 0$, $p(R_2) = 2$, $r_2(0) = 3$, r_2 being linear on the interval $[0,2]$, $R_2 = \ldots = R_5$, and $M = 4$. Let $e = (R_Q,M)$ and $x \in A(e)$. First, we note that $x_2 \neq 2$. Otherwise, $x_1+x_3+x_4+x_5 = 2$. Let then $x_j = \min\{x_3,x_4,x_5\}$. Clearly $x_j \leqq 2/3$. Then $x_j < (\sum_{\ell \neq j} x_\ell)/4 \leqq p(R_j)$ and agent j would prefer the average consumption of the other agents at x. Next, we observe that $x_1 < 2$. Indeed, if $x_1 \geqq 2$, then $p(R_1) \leqq (\sum_{j \neq 1} x_j)/4 < x_1$ and agent 1 would prefer the average consumption of the other agents at x. Let $e' = (R_{Q'},M)$ and $y \in A(e')$. Since $|Q'| = 2$ and in that case $F = A$, we obtain as in the proof of Theorem 3 that $y = (2,2)$. The proof concludes as in that Theorem. Q.E.D.

References

Alkan, A., "Monotonicity and fair assignments," Bogazici University Discussion paper ISS/AD 89–04 (1989).

Baumol, W., ***Superfairness***, M.I.T. Press 1986.

Chichilnisky, G. and W. Thomson, "The Walrasian mechanism from equal division is not monotonic with respect to variations in the number of agents," ***Journal of Public Economics*** 32 (1987), 119–124.

Chun, Y., "The solidarity axiom for quasi–linear social choice problems," ***Social Choice and Welfare*** 3 (1986), 297–320.

________ and W. Thomson, "Monotonicity properties of bargaining solutions when applied to economics," ***Mathematical Social Sciences*** 15 (1988), 11–27.

Foley, D., "Resource allocation and the public sector," ***Yale Economic Essays*** 7 (1967), 45–98.

Kolpin, V., "Equity and the core," ***Mathematical Social Sciences***, forthcoming (1990).

Moulin, H., "Interpreting common ownership," ***Recherches èconomiques de Louvain***, 56 (1990a), 3–4.

__________, "Cores and large cores when population varies," ***International Journal of Game Theory*** 19 (1990b) 219–252.

________________, "Fair division with money: two interpretations of common ownership," Duke University mimeo (1990c).

Sprumont, Y., "The division problem with single–peaked preferences: A characterization of the uniform allocation rule," ***Econometrica*** 59 (1991), 509–519.

____________, "Population monotonic allocation schemes for cooperative games with transferable utility," ***Games and Economic Behavior*** 2 (1990), 378–394.

Tadenuma, K. and W. Thomson, "The fair allocation of an indivisible good when monetary compensations are possible," ***Mathematical Social Sciences***, forthcoming (1990).

Thomson, W., "On allocations attainable through Nash equilibrium; A comment," ***Aggregation and revelation of preferences***, (J. J. Laffont, ed) Amsterdam, North–Holland, (1979), 420–431.

__________________, "An informationally efficient equity criterion," ***Journal of Public Economics***, 18 (1982), 243–263.

__________________, "The fair division of a fixed supply among a growing population," ***Mathematics of Operations Research*** 8 (1983) 319–326.

__________________, "Monotonic allocation mechanisms," University of Rochester DP, 1987.

__________________, "Consistent solutions to the problem of fair division when preferences are single–peaked," University of Rochester mimeo, 1990, ***Journal of Economic Theory,*** forthcoming.

__________________, "Resource–monotonic solutions to the problem of fair division when preferences are single–peaked," University of Rochester mimeo, 1991a.

__________________, "Equity concepts in economics," University of Rochester mimeo, 1991b.