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Abstract

For more than two decades Yaari's (1965) classical work on uncertain lifetime has inspired a great deal of research in economics. Despite the massive literature that builds on Yaari's models, I show that there is a critical deficiency in his analysis that has gone unnoticed for almost three decades. The deficiency has generated many misleading results and erroneous claims in the literature. I prove that without a bequest motive, Yaari's models cannot have an interior solution that lasts until the maximum lifetime. Contrary to the basic life cycle hypothesis, it is shown that saving must be depleted earlier than the maximum lifetime. A reinvestigation of Yaari's models produces several new testable implications for the life cycle hypothesis, offers a different interpretation for Hurd's (1989) results, and provides some support for the existence of a bequest motive.

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I have benefited from discussions with Laurence Kotlikoff and Changyong Rhee.

## I. Introduction

Although uncertainty of survival has long been an integral part of life, systematic economic analysis of uncertain lifetime has remained dormant until the publication of Yaari's seminal article in 1965. In sharp contrast to previous work on the subject, Yaari (1965) develops a series of dynamic models to examine the impact of uncertain lifetime on consumer allocation over time, taking into account whether life insurance is available and whether a bequest motive is present. Two major innovations in the analysis are the modeling of a wealth constraint and the introduction of the notion of an actuarial note. For more than two decades Yaari's investigation has inspired a great deal of work in economics. His models have virtually been regarded as the standard models of uncertain lifetime and they have been employed to study a wide variety of problems in both micro and macro economics. Applications and extensions of Yaari's results can now be broadly divided into four different categories.

The first category concerns with welfare comparisons between certain and uncertain lifetimes. Barro and Friedman (1977) apply Yaari's model to derive the rather surprising result that, under certain conditions, the expected utility under an uncertain lifetime exceeds the expected utility under a known lifetime. Further investigation along this line has been made by Katz (1979), Pelzman and Rousslang (1982), and Chang (1991).

The second set of applications is found in the value of life literature. Most government regulations on safety and health matters involve cost-benefit analysis. Among the prime considerations in the benefits or costs are the number and the value of lives saved. Earlier work on the valuation of life has been cast in a static atemporal context. Since the value of life depends on how long a person lives, it becomes clear that uncertain lifetime cannot be dismissed.[1] Yaari's models have been used to determine how much an individual is willing to pay for an increase in survival probability (or for a reduction in the risk of death) at each age. Examples are Arthur (1981), Shepard and Zeckhauser (1982, 1984), Rosen (1988), Cropper and Sussman (1988), and Cropper and Portney (1990).

While the previous two sets of applications are concerned with microeconomic questions, Yaari's models have recently been applied to address macroeconomic problems. Many issues in macroeconomics rely crucially on assumptions on the life horizon of consumers; Ricardian equivalence is perhaps the most obvious example (Evans (1992)). Building on Yaari's work, Blanchard (1984, 1985) develops a model to investigate the impact of finite and uncertain lifetimes on the determination of interest rate, government debt, and government deficits. Blanchard's model is further extended in the work of Weil (1985), Buiter (1988), and Evans (1991).[2]

The fourth and the most extensive use of Yaari's results is in the literature on the life cycle hypothesis (LCH) of saving. Assuming a deterministic length of life and the absence of a bequest motive, the basic life cycle hypothesis predicts that individuals would dissave in old age and their assets would decline to zero at death. In other words, the age-wealth profile would exhibit a humped shape. There has been a lot of empirical work on this subject and the evidence is extremely mixed. One group of

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1 Cropper and Portney (1990) illustrate the importance of life cycle factors and lifetime uncertainty on some recent environmental policy debates.

2 Buiter (1988, p.292) predicts that "the Yaari-Blanchard-Weil model may well become the workhorse of the late eighties for analytical macroeconomic research and teaching, because of its simplicity and flexibility."

studies find that households do dissave in advanced ages,[3] while another group reveals that the age-wealth profile does not decline at all and people continue to accumulate assets even after retirement.[4] On the whole, the evidence that the age-wealth profile is not humped shape appears to dominate the literature. Even after correcting for various sources of bias, Modigliani (1986, 1988a), one of the founders and advocates of the life cycle hypothesis, concedes that the decumulation reported in the first group of studies seems to be too slow to be explained by the basic life cycle hypothesis. One simple and popular explanation of the slow rate of dissaving among the elderly is that people have a bequest motive (Bernheim (1987), Kotlikoff (1988)). Instead of invoking a bequest motive, Davies (1981) offers an alternative explanation by utilizing Yaari's models to demonstrate that uncertainty about the length of life can exert a significant negative impact on consumption at all ages and the effect is more pronounced at old ages. He argues that uncertain lifetime may be the major force behind the slow asset decumulation of the retired.[5]

Despite the massive literature that builds on Yaari's pioneering contribution, I show that there is a critical deficiency in his analysis that has gone unnoticed for almost three decades. The deficiency has generated many misleading results and erroneous claims in the literature. I

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3 See, e.g., Shorrocks (1975), Diamond and Hausman (1984), King and Dicks-Mireaux (1982), and Hurd (1987).

4 See, e.g., Lydall (1955), Atkinson (1971), Mirer (1979, 1980), Kotlikoff and Summers (1981), Danziger et al. (1982), Menchik and David (1983), Ando and Kennickell (1987), Bernheim (1987), and Kotlikoff (1988). Burbidge and Robb (1985) and Masson (1986) find mixed results: the humped-shape age-wealth profile is detected for some groups of people (such as blue-collar households) but not for others (such as white-collar households).

5 Davies' explanation has been well received; see e.g., Masson (1986).

prove that without a bequest motive, Yaari's models cannot have an interior solution that lasts until the maximum lifetime. Contrary to the basic life cycle hypothesis, it is shown that saving must be depleted earlier than the maximum lifetime. Correcting for the deficiency and reinvestigating Yaari's models, I show that many new and useful results emerge. For instance, the analysis produces several new testable implications for the life cycle hypothesis, offers a different interpretation for Hurd's (1989) results, and provides some support for the existence of a bequest motive.

The key to these results is that all previous studies have either ignored or mistreated the wealth constraint in solving the optimal control problem in Yaari's first model (Case A in his exposition). The wealth constraint is a state variable inequality constraint. Although there is an extensive literature in optimal control theory that deals with state variable inequality constraints (Hartl (1984)), few substantive applications have been made in economics (Kamien and Schwartz (1991), Leonard and Long (1992)). This paper applies the theory to solve Yaari's optimal control problem and illustrates the problems that arise when the state variable inequality constraint is not handled properly. Hence, in addition to rectifying and enriching the applications of Yaari's models of uncertain lifetime, this paper contributes to highlight the theoretical importance of state variable inequality constraints in solving optimal control problems.

The plan of the rest of the paper is as follows. Section II points out the deficiency in Yaari's analysis and discusses the confusions and errors in the literature. Section III demonstrates that correcting for the deficiency will create many new and useful results. Section IV concludes the paper.

## II. Yaari's Models of Uncertain Lifetime

Of the four models formulated in Yaari, the one without bequest and annuity market (Case A) is the most widely used. I will focus mainly on this model because it is the most problematic one. The other three models will also be briefly discussed. To enhance comparisons, Yaari's symbols will be adopted here.

Assume that a rational consumer chooses a consumption plan over an uncertain life horizon T. Let $\pi(T)$ be the probability density function of T on $[0,\bar{T}]$, where $\bar{T}$ (a positive finite number) is the maximum possible lifetime. At each time instant t, the consumer derives instantaneous utility $g[c(t)]$ from consumption expenditures $c(t)$. Future utilities are telescoped with a discount function $\alpha(t)$.[6] Saving (wealth, assets) at time t, $S(t)$, follows the law of motion

$$S'(t) = j(t)S(t) + m(t) - c(t), \tag{1}$$

where $j(t)$ and $m(t)$ denote the interest rate and earnings at time t, respectively. Let $\Omega(t) = \int_t^{\bar{T}} \pi(\tau)d\tau$ be the probability that the consumer will be alive at time t and $\pi_t(\tau) = \pi(\tau)/\Omega(t)$ be the conditional probability density at time $\tau$ given that the consumer is alive at time t. Hence $\pi_t(t)$ is the hazard rate of death at time t. The following is a list of assumptions:

Assumption 1: $\Omega(0) = 1$, $\Omega(\bar{T}) = 0$, $\pi(0) \geq 0$, $\pi(T) \geq 0$, and $\pi(t) > 0$ for $t \in (0,\bar{T})$.

Assumption 2: $\alpha(t) > 0$ and $\alpha'(t)$ is continuous for $t \in [0,\bar{T}]$.

Assumption 3: $g'(c) > 0$, $g''(c) < 0$, and $g''(c)$ is continuous for $c \in [0,\infty)$.

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6 Time consistency requires that $\alpha(t) = \exp[-\sigma t]$, where $\sigma$ is a positive constant (Strotz (1956)).

The consumer chooses $c(t)$, $t \in [0,T]$, to solve the optimization problem

(P1)

$$\max_{c(.)} \int_0^{\bar{T}} \Omega(t)\alpha(t)g[c(t)]dt \tag{2}$$

$$\text{subject to: (i) } c(t) \geq 0 \text{ for all } t, \tag{3}$$

$$\text{(ii) } c(t) \leq m(t) \text{ whenever } S(t) = 0, \tag{4}$$

$$\text{(iii) } S(\bar{T}) = 0. \tag{5}$$

The second constraint, which is equivalent to the constraint that $S(t) \geq 0$ (the wealth constraint), is justified in detail in Yaari. He states that the solution $c^*$ of problem (P1), if it exists, will be composed of three types of segments. The first type is $c^*(t) = 0$ when constraint (i) is binding, and the second type is $c^*(t) = m(t)$ when constraint (ii) is binding. The third type is $c^*$ interior (neither constraint is binding), in which case it must satisfy the differential equation

$$c^{*\prime}(t) = -\left\{ j(t) + \frac{\alpha'(t)}{\alpha(t)} - \pi_t(t) \right\} \frac{g'[c^*(t)]}{g''[c^*(t)]}. \tag{6}$$

Yaari offers a proof of (6) in a footnote, which goes as follows. Assume that $c^*$ is interior throughout and use (1) to express c in terms of S, problem (P1) becomes a standard variational problem

(P2)

$$\max \int_0^{\bar{T}} \Omega(t)\alpha(t)g[m(t)-S'(t)+j(t)S(t)]dt \tag{7}$$

$$\text{subject to } S(0) = S(\bar{T}) = 0.$$

Yaari argues that the Euler equation corresponding to problem (P2) is precisely equation (6). To see the deficiency in Yaari's argument, I consider two cases: (C1) $g'(c) < \infty$ for all c, and (C2) $g'(0) = \infty$. The following propositions show that whether problem (P2) has an interior solution throughout $[0,\overline{T}]$ depends crucially on whether condition (C1) or (C2) is satisfied.

IIA. $g'(c) < \infty$ for all c.

When the derivative of the instantaneous utility function is bounded everywhere, the following proposition can be established.

*Proposition 1:* If $g'(c) < \infty$ for all c, then problem (P2) does not have an interior solution throughout $[0,\overline{T}]$.

*Proof:* Assume that (P2) has an interior solution throughout $[0,\overline{T}]$. Let $Q(t,S(t),S'(t)) = \Omega(t)\alpha(t)g[m(t)-S'(t)+j(t)S(t)]$, then $Q_S(t) = \partial Q(t,S(t),S'(t))/\partial S(t) = j(t)\Omega(t)\alpha(t)g'[m(t)-S'(t)+j(t)S(t)]$, and $Q_{S'}(t) = \partial Q(t,S(t),S'(t))/\partial S'(t) = -\Omega(t)\alpha(t)g'[m(t)-S'(t)+j(t)S(t)]$. It is clear that $Q_S(t) = -j(t)Q_{S'}(t)$. This equality, together with the Euler equation $Q_S(t) = dQ_{S'}(t)/dt$, implies that $dQ_{S'}(t)/dt = -j(t)Q_{S'}(t)$. Solving this differential equation,

$$Q_{S'}(t) = Q_{S'}(0)\exp[-\textstyle\int_0^t j(s)ds] \text{ for } t \in [0,\overline{T}]. \tag{8}$$

As $\Omega(\overline{T}) = 0$ and $g'(c) < \infty$, $Q_{S'}(\overline{T}) = 0$. Since $\exp[-\int_0^{\overline{T}} j(s)ds] > 0$, (8) implies that $Q_{S'}(0) = 0$, and consequently $Q_{S'}(t) = 0$ for all $t \in [0,\overline{T}]$. Because $\Omega(t) > 0$ for $t \in [0,\overline{T})$ and $\alpha(t) > 0$ for $t \in [0,\overline{T}]$, it follows that $g'[c(t)] = 0$ for all $t \in [0,\overline{T})$. This contradicts with the non-satiation assumption that $g'(c) > 0$. Hence, the variational problem (P2) cannot have an interior solution throughout $[0,\overline{T}]$. Q.E.D.

One obvious way to remedy the problem is to relax the non-satiation assumption that $g'(c) > 0$. Suppose $g'(c) = 0$ at $c = c^{\#}$, thus $c^{\#} = \text{argmax}_c\, g(c)$ (since $g''(c) < 0$). The proof of Proposition 1 demonstrates that $g'[c(t)]$ must equal zero for all t, therefore $c^*(t) = c^{\#}$ for all t and the consumer is satiated at each time instant. As $c^{\#}$ is a constant, $c^*(t) = c^{\#}$ cannot satisfy the differential equation (6). Accordingly, all the intertemporal aspects of the decision problem are irrelevant since $dc^*(t)/dt = 0$. Another problem with this remedy is that the solution $c*(t) = c^{\#}$ is feasible only if the earnings m(t) are large enough to finance $c^{\#}$ for all $t \in [0,\overline{T}]$. In any case, this solution is clearly not a satisfactory one.

One can see from the proof of Proposition 1 that the non-existence of an interior solution throughout $[0,\overline{T}]$ is mainly caused by the assumption that $\Omega(\overline{T}) = 0$, i.e., lifetime is uncertain and finite. This subtle point has been overlooked in all previous work. For instance, Kamien and Schwartz (1981, 1991), Davies (1981), Shepard and Zeckhauser (1982, 1984), Hurd (1989), and many others erroneously claim that it is possible for problem (P1) to have an interior solution throughout $[0,\overline{T}]$.

The non-existence problem also occurs in Yaari's two other models (Cases C and D), the proofs are provided in the Appendix. It is clear that if lifetime is certain, then the problem of the non-existence of an interior solution throughout the life horizon will disappear. The problem also does not appear in the case where there is a bequest motive, even if lifetime is uncertain. To see this, consider Case B in Yaari:

$$\max_{c(.)} \int_0^{\overline{T}} \{\Omega(t)\alpha(t)g[c(t)] + \pi(t)\beta(t)\varphi[S(t)]\}dt \tag{9}$$

(P3) subject to $c(t) \geq 0$ for all t, where $\beta(t)$ is a subjective weighing function for bequests and $\varphi[S(t)]$ is the utility derived from leaving a bequest of $S(t)$. The following result is immediate:

*Proposition 2:* It is possible that problem (P3) has a solution $c^*$ that is interior throughout $[0,\bar{T}]$.

*Proof:* Let $M(t,S(t),S'(t)) = \Omega(t)\alpha(t)g[m(t)-S'(t)+j(t)S(t)] + \pi(t)\beta(t)\varphi[S(t)]$, then $M_S(t) = -j(t)M_{S'}(t) + \pi(t)\beta(t)\varphi'[S(t)]$. Together with the Euler equation, this implies that $dM_{S'}(t)/dt + j(t)M_{S'}(t) = \pi(t)\beta(t)\varphi'[S(t)]$. Solving this differential equation, $M_{S'}(t) = \{M_{S'}(0) + \int_0^t \pi(z)\beta(z)\varphi'[S(z)]\exp[\int_0^z j(x)dx]dz\}\exp[-\int_0^t j(s)ds]$. The fact that $\Omega(\bar{T}) = 0$ (and hence $M_{S'}(\bar{T}) = 0$) no longer implies that $M_{S'}(0) = 0$, for $M_{S'}(0) = -\int_0^{\bar{T}} \pi(z)\beta(z)\varphi'[S(z)]\exp[\int_0^z j(x)dx]dz$ in this case. Consequently, problem (P3) can have a solution $c^*$ that is interior throughout $[0,\bar{T}]$. Q.E.D.

The importance of bequests will further be discussed in section III below. Here it suffices to mention that the presence of a bequest motive plays an important role in solving Yaari's optimal control problems.[7] It should also be emphasized that adding a bequest motive does not always solve the non-existence problem. For example, even though there is a bequest motive in Yaari's Case D, the availability of life insurance alters the

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7 King's (1985, p.282) assertion that "whatever the optimal level of bequests ..., a potential donor will still allocate consumption over his or her life cycle using the same criteria as analyzed in preceding sections. The existence of bequests does not, therefore, alter the first-order conditions determining the rate of change of consumption over the life cycle" is clearly erroneous. Adding a bequest motive does not only change the first-order conditions but also alters more fundamentally the existence of an interior solution for the problem.

problem in such a way that the non-existence problem cannot be eliminated. The proof of this is contained in the Appendix. Of the four models in Yaari, the only one that does not suffer from the non-existence problem is the one in which there is a bequest motive and life insurance is not available, i.e., Case B.

It is obvious from the proof of Proposition 2 that as long as the utility function depends on $S(t)$, then the non-existence problem will not emerge. In other words, even if a consumer does not have a bequest motive ($\beta(t) = 0$), the problem mentioned in Proposition 1 will not arise if he derives utility from his wealth holdings (i.e., $g[c(t),S(t)]$).

Proposition 1 shows that problem (P1) cannot have an interior solution throughout $[0,\bar{T}]$. It suggests that if the problem has a solution, then one of the two constraints must be binding at some point in time. This intuition is verified in Proposition 3 below. The key is to recognize that there is a state variable inequality constraint in problem (P1) (the wealth constraint, i.e., constraint (ii)) that should be explicitly incorporated into the optimal control problem. To see this, it is more convenient to analyze the problem as an optimal control problem. The direct adjoining approach (Jacobson, Lele, and Speyer (1971), Hartl (1984)) will be employed to deal with the wealth constraint $S(t) \geq 0$.[8] Let the Hamiltonian be

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8 There is an extensive literature on optimal control problems with state variable inequality constraints; see, e.g., the survey and references in Hartl (1984). There are two main ways to attach the state inequality constraints into the Hamiltonian. For the model here, the first way (direct adjoining approach) adjoins the constraint $S(t)$ directly to the Hamiltonian while the second one (indirect adjoining approach) adjoins the control constraint $S'(t) \geq 0$ if $S(t) = 0$. The two approaches produce slightly different necessary conditions. Kreindler (1982) shows that the conditions derived from the direct adjoining approach are sharper and simpler than those of the indirect adjoining approach.

$$H = \Omega(t)\alpha(t)g[c(t)] + \lambda(t)[j(t)S(t)+m(t)-c(t)] + \mu(t)S(t), \tag{10}$$

where $\lambda(t)$ and $\mu(t)$ are multipliers. The first-order conditions are given by

$$\partial H/\partial c(t) = \Omega(t)\alpha(t)g'[c(t)] - \lambda(t) = 0, \tag{11}$$

$$\partial H/\partial S(t) = -\lambda'(t) = j(t)\lambda(t) + \mu(t), \tag{12}$$

$$\mu(t)S(t) = 0, \tag{13}$$

$$\mu(t) \geq 0, \tag{14}$$

$$\lambda(\overline{T}) = \gamma + p, \tag{15}$$

$$\gamma \geq 0, \ \gamma S(\overline{T}) = 0, \ p \text{ unrestricted in sign.} \tag{16}$$

Solving the differential equation (12),

$$\lambda(t) = \lambda(0)e^{-\int_0^t j(s)ds} - \int_0^t e^{-\int_x^t j(s)ds} \mu(x)dx. \tag{17}$$

If problem (P1) has an interior solution, then the following proposition must hold.

*Proposition 3:* If $g'(c) < \infty$, then there must be a $t^* < \overline{T}$ such that the state variable inequality constraint $S(t) \geq 0$ is binding on the entire interval $[t^*,\overline{T}]$.

*Proof:* If $S(t) > 0$ for $t \in [0,\overline{T}]$, then (13) implies that $\mu(t) = 0$ for all t so that $\lambda(t) = \lambda(0)\exp[-\int_0^t j(s)ds]$ (from (17)). Since $\Omega(\overline{T}) = 0$ and $g'(c) < \infty$, (11) implies that $\lambda(\overline{T}) = 0$, hence $\lambda(0) = 0$. Consequently, $\lambda(t) = 0$ and therefore $g'[c(t)] = 0$ for all $t \in [0,\overline{T})$, i.e., no interior solution is possible. To avoid the result $\lambda(0) = 0$, it is necessary that there exists a $t^* < \overline{T}$ such that $S(t) = 0$ for $t \in [t^*,\overline{T}]$. Then $\mu(t) > 0$ for $t \in [t^*,\overline{T}]$ and $\lambda(0) = \int_0^{\overline{T}} \mu(x)\exp[-\int_0^x j(s)ds]dx \neq 0$. As a result, $\lambda(t)$, as well as $g'[c(t)]$, will not be forced to zero for all t. Q.E.D.

Proposition 3 demonstrates that there must be a constrained segment lasting till $\overline{T}$. As $S(t) = 0$ for $t \in [t^*,\overline{T}]$, $S'(t) = 0$ and hence $c^*(t) = m(t)$ for $t \in [t^*,\overline{T}]$. Since the optimal solution entails at least one constrained segment, it would be useful to examine whether $c^*(t)$ has any jump discontinuities. Yaari believes that jumps are possible, for he remarks that the solution "$c^*$ is everywhere continuous, except possibly if S becomes equal to zero at some t where $c^*(t) > m(t)$, in which case a downward jump will occur" (Yaari (1965, p.143)).

*Proposition 4:* The solution $c^*(t)$ and the multiplier $\lambda(t)$ are continuous everywhere in $[0,\overline{T}]$.

*Proof:* Let $\theta_i$ denote the ith junction time at which the state variable S(t) enters into, or exits from, a constrained segment ($i = 1,2,\ldots$). Since $g''(c) < 0$, the $c^*$ that maximizes H is unique. It is obvious from (1) that the state variable inequality constraint is of first-order. (A state variable inequality constraint is said to be of p-th order if the p-th time-derivative of the constraint is the first to contain the control variable explicitly.) It follows from McIntyre and Paiewonsky (1967) (see also Jacobson, Lele, and Speyer (1971)) that $c^*(t)$ and $\lambda(t)$ must be continuous at every junction time $\theta_i$. The proof is completed since $c^*(t)$ and $\lambda(t)$ are continuous everywhere else in $[0,\overline{T}]$. Q.E.D.

Proposition 4 shows that neither the control variable $c^*(t)$ nor the multiplier $\lambda(t)$ exhibits any jump discontinuity at the junction times. Hence Yaari's remark that jumps are possible is fallacious. Notice that the proposition does not depend on whether $g'(c) < \infty$ or $g'(0) = \infty$. Propositions

3 and 4 illustrate the usefulness of the theory of state variable inequality constraints in solving Yaari's optimal control problem and in rectifying the errors found in previous studies.

IIB. $g'(0) = \infty$.

If $g'(0) = \infty$, then problem (P1) can have an interior solution throughout $[0,\overline{T}]$. This is because from (11), $g'[c(t)] = \lambda(t)/[\Omega(t)\alpha(t)]$, so that $\Omega(\overline{T}) = 0$ implies that $g'[c(\overline{T})] = \infty$. As a result, $c(\overline{T}) = 0$ and $\lambda(t)$ does not have to be forced to zero. The non-existence problem is avoided. Whether $c^*(t)$ will decline to zero as t approaches $\overline{T}$ hinges on the assumption on the function m(t), as revealed by the following proposition.

*Proposition 5:* If $g'(0) = \infty$ and $m(\overline{T}) > 0$, then there must be a $t^* < \overline{T}$ such that the state variable inequality constraint $S(t) \geq 0$ is binding on the entire interval $[t^*,\overline{T}]$. Hence, $c^*(t) = m(t)$ on the interval $[t^*,\overline{T}]$.

*Proof:* Suppose that $S(t) > 0$ for $t \in [t^{\#},\overline{T}]$, where $0 \leq t^{\#} < \overline{T}$. Thus (13) yields $\mu(t) = 0$ for $t \in [t^{\#},\overline{T}]$, and it follows from (11) and $\Omega(\overline{T}) = 0$ that $g'[c(\overline{T})] = \infty$, i.e., $c^*(\overline{T}) = 0$. This, together with $S(\overline{T}) = 0$, $m(\overline{T}) > 0$, and (1), implies that $S'(\overline{T}-) > 0$, where $S'(\overline{T}-)$ denotes the left-hand limit of $S'(\overline{T})$. On the other hand, $S(t) \geq 0$ for all t and $S(\overline{T}) = 0$ imply that $S'(\overline{T}-) \leq 0$. Contradiction. Q.E.D.

Proposition 5 shows that if $m(\overline{T}) > 0$, then $c^*(t) = m(t)$ on the interval $[t^*,\overline{T}]$ and $c^*(t)$ will never decline to zero.[9] In some of their models where

9 Ulph and Hemming (1980) obtain a similar result in a different model. By assuming that g(c) is CRRA (constant relative risk aversion), they obtain a closed form solution for $c^*(t)$ and directly characterize its time path.

$m(\bar{T}) > 0$, Davies (1981, p.567) and Hurd (1989, p.783) implicitly claim that the interior segment of the solution $c^*(t)$ will last until $\bar{T}$. Proposition 5 proves that these claims are all wrong. Since there is a constrained segment on $[t^*,\bar{T}]$, the interior segment of $c^*(t)$ cannot last until $\bar{T}$.

Several authors treat problem (P1) as a special case of (P3) because (2) can be obtained from (9) when $\varphi(S) = 0$ (as in Kamien and Schwartz (1981, p.59; 1991, p.63), Hurd (1989, p.799)) or when $\varphi'(S) = 0$ (as in Hurd (1989, p.794)). Therefore they believe that the solution of problem (P1) can be obtained by just setting $\varphi(S) = 0$ or $\varphi'(S) = 0$ in the solution of (P3). Propositions 2, 3, and 5 demonstrate that this approach is inappropriate for there is a fundamental difference between the solutions of problems (P1) and (P3).[10] For problem (P1), if $g'(c) < \infty$ or $m(\bar{T}) > 0$, the wealth constraint must begin to be binding at some time before the maximum lifetime. No such restriction is found in problem (P3).

Blanchard's (1984, 1985) model does not suffer from the non-existence problem because he assumes that there does not exist a finite $\bar{T}$ such that $\Omega(\bar{T}) = 0$. Since $\lambda(t)$ and $\Omega(t)$ both tend to zero as t tends to $\infty$, the first-order condition (11) holds without contradiction. Nevertheless, Blanchard's assumption $\lim_{t\to\infty} \Omega(t) = 0$ means that his model of uncertain lifetime is actually indistinguishable from an infinite horizon model of certain lifetime.

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Proposition 5 is much more general than their result because it neither relies on any specific functional form for $g(c)$ nor requires a closed form solution for $c^*(t)$. Furthermore, several features in their model (such as the decisions on the retirement date and the amount of pension fund) obscure the essential elements of the problem. The proof here is simpler, more general, and precisely exposes the source of the underlying problem.

10 It is easy to check that Proposition 2 also holds when $g'(0) = \infty$, since the proof does not rely on $g'(c)$.

Both Propositions 3 and 5 show that $S(t)$ will decline to zero even before $\bar{T}$ and will remain zero thereafter. It is easy to see that this result does not depend on the size of the initial endowment $S(0)$. Therefore, Hurd's (1989, p.786) claim that "If initial wealth is low, the bequeathable wealth constraint is binding at some time $T < N$ [the maximum lifetime, $\bar{T}$ in the symbols here]" is misleading because the fact that $S(t) = 0$ before $\bar{T}$ does not rely on the level of initial wealth $S(0)$ at all.

If $m(\bar{T}) = 0$, the contradiction mentioned in the proof of Proposition 5 may still arise. In this case, even though $c(\bar{T}) = m(\bar{T}) = 0$ is feasible, it is possible that $S'(\bar{T}-) > 0$ as long as $c(t) < m(t)$ for those t in the neighborhood of $\bar{T}$. Therefore, the condition $S'(t) < 0$ in the neighborhood of $\bar{T}$ is required to guarantee that $c(\bar{T}) = 0$. Accordingly, $c^*(t)$ can stay in the interior segment until the maximum lifetime.[11] This result is summarized in the following proposition.

*Proposition 6:* If $g'(0) = \infty$ and $m(\bar{T}) = 0$, then $c(\bar{T}) = 0$ and $c^*(t)$ will decline to zero as t approaches $\bar{T}$, provided that $S'(t) < 0$ in the neighborhood of $\bar{T}$.

This result contrasts sharply with the one obtained when lifetime is certain. When there is no uncertainty about $\bar{T}$ (i.e., $\Omega(t) = 1$ for $t \in [0,\bar{T})$), the optimal solution $c^*(t)$ for problem (P1) satisfies the differential equation $c^{*\prime}(t) = -\{j(t) + [\alpha'(t)/\alpha(t)]\}\{g'[c^*(t)]/g''[c^*(t)]\}$.

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11 This shows that Masson's (1986, p.187) assertion that "Another blocked interval is bound to take place at the end of maximum lifetime, a fact apparently overlooked by Davies" is false. A constrained segment at the end of maximum lifetime is bounded to occur only if the conditions given in Propositions 3 or 5 are satisfied.

It is well known that whether c(t) rises or falls over time hinges on the sign of $j(t) + \alpha'(t)/\alpha(t)$. Even if c(t) falls over time, the first-order condition $\alpha(\overline{T})g'[c(\overline{T})] = \lambda(\overline{T})$ precludes the possibility that $c(\overline{T}) = 0$ because $\alpha(\overline{T})$ and $\lambda(\overline{T})$ are bounded functions. Hence consumption can never reach zero in a model of certain and finite lifetime.

## III. Interpretations and Implications

It is useful to summarize the theoretical results obtained in the previous section. Here again the focus is on Yaari's Case A where life insurance is unavailable and the consumer has an uncertain lifetime but no bequest motive.

*R1. If the marginal utility of consumption is bounded everywhere, then positive saving throughout the life horizon is impossible. Saving must be exhausted at some time before the maximum lifetime. During this period of zero saving, consumption at each time instant will be equal to whatever non-interest income the consumer has.*

*R2. If the marginal utility of consumption is bounded everywhere except at zero consumption, then positive saving throughout the life horizon is impossible if the consumer receives positive non-interest income (from social security, pension funds, or other sources) lasting through the maximum lifetime. Saving must be exhausted at some time before the maximum lifetime. During this period of zero saving, consumption at each time instant will be equal to whatever non-interest income the consumer has.*

*R3. If the marginal utility of consumption is bounded everywhere except at zero consumption, then positive saving throughout the life horizon is possible if the consumer does not have non-interest income lasting through the maximum lifetime. If saving declines to zero at the maximum lifetime, then consumption will also fall to zero at that time.*

These three main results reveal that the marginal utility of consumption and the non-interest income play a crucial role in determining the time path of optimal consumption. If non-interest income stays positive throughout the life horizon, then R1 and R2 together show that saving must be depleted before the maximum lifetime, regardless of whether the marginal utility of consumption is bounded. If non-interest income is zero at the maximum lifetime, then the time paths of saving and optimal consumption rely on whether the marginal utility of consumption is bounded.

An intuitive explanation for these results can be offered as follows. When the optimal consumption $c^*$ is interior, the first-order condition (11) means that the expected discounted marginal utility of consumption $\Omega(t)\alpha(t)g'[c(t)]$ is equal to the marginal value of saving $\lambda(t)$. This can be rearranged to yield $g'[c(t)] = \lambda(t)/[\Omega(t)\alpha(t)]$ for $t < \overline{T}$. Since the multiplier $\lambda(t)$ and the discount function $\alpha(t)$ are bounded for $t < \overline{T}$, the fact that $\Omega(t)$ approaches zero as t tends to $\overline{T}$ implies that the marginal utility of consumption has to go unbounded at $\overline{T}$. This cannot happen if the utility function is assumed to be bounded everywhere. Even if the utility function is allowed to be unbounded at zero consumption, it is impossible for consumption to reach zero if there is positive non-interest income at $\overline{T}$, because zero consumption means that the consumer deliberately discards all those non-interest income. To avoid these problems, the first-order condition must not hold, i.e., the solution $c^*$ must cease to be interior at some time before $\bar{T}$. Hence there must be a boundary segment lasting through the maximum lifetime.

The three results R1, R2, and R3 have all been overlooked in the extensive literature that builds on Yaari's classical model of uncertain lifetime. In addition to their contribution to pure theory, what is the significance of these theoretical results? The answer is immediate. They provide a new set of testable implications, especially for the life cycle hypothesis of saving. In contrast to the argument found in all previous studies on the LCH and uncertain lifetime that saving will decline to zero at the maximum lifetime, both R1 and R2 show that saving will be depleted even earlier. If it does not, then R3 establishes that consumption will decline to zero at the maximum lifetime.

How do these implications square with evidence? Consider R3 first. Many countries have some sort of social security programs and R3 would not be applicable to these countries. For example, the social security benefits in the U.S. are positive and do not decline with age.[12] In addition, there is no evidence that consumption declines and reaches zero at the maximum lifetime. In fact, most findings show that consumption steadily increases with age (Kotlikoff and Summers (1981, 1988), Danziger et al. (1982)) and does not fall even after retirement.

Both R1 and R2 produce similar implications. In most applications of

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[12] In fact, the current law stipulates that there is generally a cost-of-living increase in social security benefits each year based on the Consumer Price Index for Urban Wage Earners and Clerical Workers and some other indices (Social Security Administration, 1991). Hurd (1989, p.797) also assumes constant annuities.

Yaari's model (e.g., Davies (1981), Shepard and Zeckhauser (1982, 1984), Hurd (1989)), a constant relative risk aversion (CRRA) utility function ($g(c) = c^{\gamma}/\gamma$, $\gamma < 1$) is assumed because it can generate tractable and closed form solutions for $c^*(t)$ and $S(t)$. Since $g'(0) = \infty$ for the CRRA utility function, R2 is applicable to these studies. Is there any evidence that saving is depleted even before the maximum lifetime?

As pointed out in the Introduction, most studies in the literature find that saving stays unchanged or continues to rise after retirement. Although a few studies find that the aged dissave, the decumulation is too slow to be consistent with theory. Therefore there appears to be very little empirical support for the model. The only exception to date is the study by Hurd (1989) who finds a very high decumulation rate from the Longitudinal Retirement History Survey. His parameter estimates imply such a steep consumption trajectory that most people will exhaust their wealth in only a few years and will consume their annuity income thereafter. For instance, one set of estimates show that the mean time and the median time (after age 65) to depletion of wealth is 7 years and 5.3 years, respectively. The distribution of the depletion time is very skewed: 10 percent of his sample depletes its wealth in less than 1.3 years and 90 percent in less than 14.3 years. The results that wealth be exhausted before the maximum lifetime and consumption be equal to non-interest income thereafter are exactly the ones predicted by the model (R1 and R2).[13] However, Hurd (1989, p. 791) argues

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13 Since Hurd's (1989) model contains a bequest motive, one may wonder whether R1 - R3 are applicable to explain his findings because there is no bequest motive in my model. This does not present a problem because he finds that the estimate of the marginal utility of bequests is very small and is not statistically significantly different from zero. As he finds no evidence for a bequest motive, results R1 - R3 are applicable.

that <u>low initial wealth generates these results</u>. He fails to recognize R1 and R2 that early wealth depletion follows directly from the optimal control problem, regardless of the size of the initial wealth. Hence, whether low initial wealth causes early wealth depletion remains to be established. To accomplish this, it is necessary to devise some new test to isolate the low-initial-wealth effect (if it exists) from the early-wealth-depletion effect that is an integral part of the solution of (P1). In any case, results R1 and R2 offer a reasonable and rigorous alternative interpretation for Hurd's findings.

Results R1 and R2 also offer some practical implications for empirical work. For instance, it would be useful to examine the robustness of Hurd's results by incorporating the restriction $S(t) = 0$ for $t \in [t^*, \bar{T}]$ explicitly into his estimation procedure, where $t^*$ is a new parameter to be estimated along with the other parameters of the model. It would also be desirable to analyze why Hurd's (1989) findings (high decumulation rate) are so different from the rest of the literature.[14]

In King and Dicks-Mireaux's (1982) study of Canadian households and Diamond and Hausman's (1984) analysis of the U.S. National Longitudinal Survey of Mature Men, they both find that a sizeable minority (about 20 percent) of the population in their sample has very low net worth. King and Dicks-Mireaux (1982, p.249-251) contend that this finding is suggestive of the fact that this group of people "does not save in accordance with the life-cycle view of 'rational' behavior" because they "may not plan for the future (are 'backward-looking' rather than 'forward-looking'), may simply be unable to manage their own financial affairs." Results R1 - R3 show that

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14 Bernheim (1991) raises two objections to Hurd's (1989) analysis.

there is no need to invoke irrational behavior to explain low wealth accumulation. A rational model of saving and uncertain lifetime, like the one considered in this paper, demonstrates that people will exhaust their wealth early so that it is not surprising to observe a significant percentage of people holding very low wealth.

Given that almost all the studies in the literature do not find evidence of fast wealth decumulation, several explanations have been advanced to reconcile the discrepancy between theory and evidence. Davies' (1981) attempt is perhaps the best known and has been well received especially among the proponents of the LCH of saving (Masson (1986), Modigliani (1986, 1988a)). However, Davies (1981) also fails to recognize results R1 - R3 that saving must be depleted before the maximum lifetime in some of his models. Intuitively, decumulation would be slower when saving is required to reach zero <u>only at</u> the maximum lifetime than when it is required to reach zero <u>before</u> the maximum lifetime. If this is true, then Davies' explanation would be weakened because he has under-estimated the true decumulation rate. It would therefore be useful to investigate the robustness of his results by explicitly incorporating the wealth depletion restriction into the simulations.[15]

Another explanation, albeit a straightforward one, for the slow decumulation is to assume that consumers have a bequest motive. Thus, wealth will not decline at old age because consumers desire to leave bequests for

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15 Shepard and Zeckhauser (1982, 1984) also fail to recognize results R1 - R3 in their application of Yaari's models to estimate the value of life. It would therefore be useful to examine how their value of life estimates will change if the wealth depletion restriction is explicitly incorporated into the simulations.

their offsprings or relatives.[16] Similar to the empirical work on whether there is wealth decumulation among the elderly, the evidence on the existence of a bequest motive is also extremely mixed. While most studies find a significant bequest motive (e.g., Kotlikoff and Summers (1981, 1988), Menchik and David (1983), Hamermesh and Menchik (1987), Kotlikoff (1988), Bernheim (1991)), some claim that there is little evidence of bequest motive (Modigliani (1988b), Hurd (1987, 1989), Hurd and Mundaca (1989)). Results R1 and R2 tend to support the existence of a bequest motive because with the exception of Hurd (1989), the prediction that wealth be exhausted before the maximum lifetime is simply not observed for the majority of households and is not supported by the prevailing evidence of slow decumulation.

## IV. Conclusion

For more than two decades Yaari's classical work on uncertain lifetime has inspired a great deal of research in economics. His models, which have been employed to study a variety of problems in both microeconomics and macroeconomics, have become <u>the</u> standard models of uncertain lifetime. Despite the massive literature that builds on Yaari's pioneering contribution, I show that there is a critical deficiency in his analysis that has gone unnoticed for almost three decades. The deficiency has generated many mistakes and false claims in the literature. To illustrate the problems, I discuss the models in the context of the life cycle hypothesis of savings because of its importance and its reliance on Yaari's

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[16] Mirer (1979) argues that a life cycle theory that includes other motives for saving, such as bequests, emergencies, power, or status, would be so inclusive that it would be difficult to test because no particular age-wealth profile could contradict the theory. This is a common objection to the use of bequest to explain the slow decumulation.

results.

My major results are as follows. Under realistic and mild conditions, I prove that Yaari's models cannot have an interior solution that lasts until the maximum lifetime. While previous authors believe that saving is depleted only at the maximum lifetime, I show that saving must be depleted even earlier. These results generate several new testable implications for the life cycle hypothesis of saving. They also offer a nice alternative interpretation for Hurd's (1989) results. Unlike King and Dicks-Mireaux (1982) who suggest that households with low wealth holdings behave irrationally, I argue that their behavior is consistent with a rational model of saving. Finally, the results provide some support for the existence of a bequest motive.

Since this paper is primarily a theoretical investigation, the next step in the research agenda would be to assess quantitatively and empirically the importance of the theoretical results established in the analysis.

## Appendix

This appendix proves that if $g'(c) < \infty$, the problem of the non-existence of an interior solution throughout $[0,\bar{T}]$ also appears in the other two models (Case C and Case D) formulated in Yaari.

Case C: Perfect Annuity Market

In this case, there is a perfect annuity market and the consumer can purchase life insurance in the form of actuarial notes. The optimal control problem is given by (Yaari, 1965, p.146):

$$\underset{c(.)}{\text{Max}} \int_0^{\bar{T}} \Omega(t)\alpha(t)g[c(t)]dt$$

subject to: (i) $c(t) \geq 0$ for all t,

$$\text{(ii)} \quad \int_0^{\bar{T}} e^{-\int_0^t r(x)dx} [m(t)-c(t)]dt = 0.$$

Let $Y = \int_0^{\bar{T}} \{\exp[\int_0^t r(x)dx]\}m(t)dt$ and $W(t) = \int_0^t \{\exp[\int_0^z r(x)dx]\}c(z)dz$, then constraint (ii) is equivalent to the set of constraints

(iii) $W'(t) = \{\exp[\int_0^t r(x)dx]\}c(t)$, $W(0) = 0$, and $W(\bar{T}) = Y$.

Let $\theta(t)$ be the multiplier, then the Hamiltonian for this problem is

$$H^C = \Omega(t)\alpha(t)g[c(t)] + \theta(t)\{\exp[\int_0^t r(x)dx]\}c(t).$$

The first-order conditions are given by

$$\partial H^C/\partial c = \Omega(t)\alpha(t)g'[c(t)] + \theta(t)\exp[\int_0^t r(x)dx] = 0, \quad \text{(E1)}$$

$$\partial H^C/\partial W = -\theta'(t) = 0, \quad \text{(E2)}$$

It follows from (E2) that $\theta(t)$ is a constant. Since $\Omega(\bar{T}) = 0$, $g'(c) < \infty$, and $\exp[\int_0^{\bar{T}} r(x)dx] > 0$, thus $\theta(t) = 0$. Consequently, (E1) dictates that $g'[c(t)] = 0$ for all t.

Case D: Perfect Annuity Market with Bequests

This case is similar to Case C except that the consumer has a bequest motive. Therefore, the consumer's assets consist of both regular notes (saving) and actuarial notes, and he has to choose the optimal portfolio mix between two different types of notes. The optimal control problem is given by (Yaari, 1965, p.149):

$$\underset{c(.),S(.)}{\text{Max}} \int_0^T \{\Omega(t)\alpha(t)g[c(t)] + \pi(t)\beta(t)\varphi[S(t)]\}dt$$

subject to: (i) $c(t) \geq 0$ for all t,

$$\text{(ii)} \quad \int_0^T e^{-\int_0^t r(x)dx}[m(t)-c(t)-\pi_t(t)S(t)]dt = 0$$

Let $A(t) = \int_0^t \{(\exp[\int_0^z r(x)dx]\}[c(z)+\pi_z(z)S(z)]\}dz$, then constraint (ii) can be replaced by the set of constraints,

(iii) $A'(t) = \{\exp[\int_0^t r(x)dx]\}[c(t)+\pi_t(t)S(t)]$, $A(0) = 0$, and $A(\overline{T}) = Y$,

where $Y = \int_0^{\overline{T}} \{\exp[\int_0^t r(x)dx]\}m(t)dt$. Let $\phi(t)$ be the multiplier, then the Hamiltonian for this problem is

$$H^D = \Omega(t)\alpha(t)g[c(t)] + \pi(t)\beta(t)\varphi[S(t)] + \phi(t)\{\exp[\textstyle\int_0^t r(x)dx]\}[c(t)+\pi_t(t)S(t)].$$

The first-order conditions are given by

$$\partial H^D/\partial c = \Omega(t)\alpha(t)g'[c(t)] + \phi(t)\exp[\textstyle\int_0^t r(x)dx] = 0, \quad \text{(E3)}$$

$$\partial H^D/\partial S = \pi(t)\beta(t)\varphi'[S(t)] + \phi(t)\exp[\textstyle\int_0^t r(x)dx]\pi_t(t) = 0, \quad \text{(E4)}$$

$$\partial H^D/\partial A = -\phi'(t) = 0. \quad \text{(E5)}$$

It follows from (E5) that $\phi(t)$ is a constant. Since $\Omega(\overline{T}) = 0$, $g'(c) < \infty$, and $\exp[\int_0^{\overline{T}} r(x)dx] > 0$, thus (E3) implies that $u(t) = 0$ and consequently, $g'[c(t)] = 0$ for all t.

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