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Abstract

Consistent Extensions

Consistency is a property of allocation rules which says that what a solution recommends for any economy is always in agreement with what it recommends for associated "reduced economies". We propose here to evaluate the extent to which a solution may fail to be consistent by identifying its "minimal consistent extension," that is, the smallest consistent solution that contains it. We calculate the minimal consistent extensions of two solutions that have played a central role in the literature on the problem of fair division. We also propose the concept of "maximal consistent subsolution" and show how it can be used to relate several solutions that had been discussed separately in that literature.

JEL Classification Nos: D63, D71

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1. ***Introduction.*** A property of allocation rules, or ***solutions***, that has played a fundamental role in some recent literature is ***consistency***. A general statement is as follows: Consider a solution defined on some class of problems. Apply the solution to a problem in the class. Then *consistency* says that the restriction of any payoff vector chosen by the solution for that problem to any subgroup of agents is what the solution would recommend for the "reduced problem" obtained by imagining the departure of the members of the complementary group with their payoffs, and reevaluating the situation from the viewpoint of the subgroup.

When a solution is not *consistent*, one would like to know how serious the violations of *consistency* are. One way to evaluate these violations is to ask how much the solution would have to be modified in order to satisfy the property. In this note, we first propose to do that by minimally enlarging the solution. This is a well–defined operation since it follows from elementary considerations that there always is a *minimal consistent extension* of a given solution. We consider this notion in the context of fair allocation in classical economies and show that the *minimal consistent extensions* of two solutions that have often been discussed in the literature can be characterized in a simple way. Unfortunately, for these examples, the enlargement needed to obtain *consistency* is quite considerable. It remains to be determined whether this remains true in other models.

Another way to recover *consistency* when the solution in which one may be interested does not satisfy the property, is to subtract from, instead of adding to, what the solution recommends. Here, one would like to subtract as little as possible. This can be done in a meaningful way whenever the solution does have a *consistent* subsolution, since here too it follows from elementary considerations that such a solution has a *maximal consistent subsolution*. We show how this concept can be used to link several solutions that have been discussed separately in the literature.

2. ***Minimal consistent extensions.*** Although the principle of *consistency* has been applied in a wide variety of models (for a review of this literature, see Thomson, 1990, 1992a; also see Young, 1991, where it is a central theme), we will limit ourselves here to an examination of problems of fair division in classical economies and to the specific form of *consistency* that is appropriate for such a domain.

We consider economies with arbitrary finite numbers of agents. Let $\ell \in \mathbb{N}_{++}$ be the number of goods. Let $\mathbb{N}_{++} = \{1,2,...\}$ be the set of "potential" agents and $\mathcal{Q}$ be the class of all finite subsets of $\mathbb{N}_{++}$, with generic elements Q, Q′.... Each agent $i \in \mathbb{N}_{++}$ is equipped with a preference relation on $\mathbb{R}^{\ell}_{+}$, denoted R_i. Let P_i be the strict preference relation associated with R_i and I_i the indifference relation. Let $\mathcal{R}$ be the class of continuous, convex, and monotone ($z_i > z_i' \Rightarrow z_i P_i z_i'$)[1] preference relations. An ***economy*** is a pair $((R_i)_{i \in Q}, \Omega)$, or simply (R_Q, Ω), where $Q \in \mathcal{Q}$, for each $i \in Q$, $R_i \in \mathcal{R}$, and $\Omega \in \mathbb{R}^{\ell}_{++}$ is the endowment. Let $\mathcal{E}^Q$ be the class of economies so defined and $\mathcal{E} = \bigcup_{Q \in \mathcal{Q}} \mathcal{E}^Q$. Given $Q \in \mathcal{Q}$ and $e = (R_Q, \Omega) \in \mathcal{E}^Q$, the feasible set of e, denoted Z(e), is defined by $Z(e) = \{z \in \mathbb{R}^{\ell|Q|}_{+} \mid \sum_{i \in Q} z_i = \Omega\}$. A ***solution*** is a mapping defined on a class of economies which associates with each economy in the class a non–empty subset of its feasible set. We will consider solutions defined on $\mathcal{E}$ and the subdomain of $\mathcal{E}$ obtained by requiring preferences to be strictly monotone ($z_i \geq z_i' \Rightarrow z_i P_i z_i'$). If $z \in \varphi(e)$, we will say that z is ***φ–optimal for e***.

A solution is *consistent* if what it recommends for any economy is never "contradicted" by what it recommends for any associated "reduced" economy obtained by imagining the departure of some of the agents with their allotted consumptions, and reconsidering the problem of dividing the remaining resources among the remaining agents. Formally, let φ be a solution. Given $Q \in \mathcal{Q}$ and $e = (R_Q, \Omega) \in \mathcal{E}^Q$, let z be

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[1]Vector inequalities: given a, b $\in \mathbb{R}^{\ell}$, $a \geqq b$ means $a_k \geqq b_k$ for all k; $a \geq b$ means $a \geqq b$ and $a \neq b$; $a > b$ means $a_k > b_k$ for all k.

one of the recommendations made by φ for e, that is, let $z \in \varphi(e)$. Then, let some of the agents in Q leave the scene with their allotted consumptions, $Q' \subseteq Q$ designating the group of remaining agents. For the economy $(R_{Q'}, \sum_{i \in Q'} z_i)$, would the solution recommend that each agent receives the same bundle as before, so that $z_{Q'} \in \varphi(R_{Q'}, \sum_{i \in Q'} z_i)$?[2] If the answer is always yes, the solution is *consistent*. This property, and related ones, were studied by Thomson (1988).

Consistency. For all Q, $Q' \in \mathcal{Q}$ with $Q' \subseteq Q$, for all $e = (R_Q, \Omega) \in \mathcal{E}^Q$, and for all $z \in \varphi(e)$, $z_{Q'} \in \varphi(R_{Q'}, \sum_{i \in Q'} z_i)$.

The pair $(R_{Q'}, \sum_{i \in Q'} z_i)$ is the ***reduced economy of e with respect to Q′ and z.*** Let it be denoted $t^z_{Q'}(e)$.

A number of solutions are *consistent*. Examples are the pareto solution, which associates with each economy its set of efficient allocations, and the no–envy solution (Foley, 1967), which associates with each economy its set of allocations at which no agent would prefer someone else's consumption to his own:.

Pareto solution, P: Given $Q \in \mathcal{Q}$ and $e = (R_Q, \Omega) \in \mathcal{E}^Q$, $P(e) = \{z \in Z(e) |$ there is no $z' \in Z(e)$ such that $z'_i R_i z_i$ for all $i \in Q$ and $z'_i P_i z_i$ for some $i \in Q\}$.

No–envy solution, F (Foley, 1967): Given $Q \in \mathcal{Q}$ and $e = (R_Q, \Omega) \in \mathcal{E}^Q$, $F(e) = \{z \in Z(e) |$ for every pair $\{i,j\} \subseteq Q$, $z_i R_i z_j\}$.

However, some interesting solutions are not ***consistent***. Our objective here is to formulate a way of evaluating how far from being *consistent* a solution may be. We propose to enlarge it in a minimal way so as to recover the property. That this can be done is a consequence of the following observations: It follows directly from the definition of *consistency* that if all the members of a family Ψ of solutions with

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[2] Of course, if φ is multi–valued there may exist φ–optimal allocations other than $z_{Q'}$ in the reduced economy.

common domain and range are *consistent*, and the intersection $\overline{\varphi}(e) = \bigcap_{\psi \in \Psi} \psi(e)$ is non–empty for each economy e in the domain, then the well–defined solution $\overline{\varphi}$ also is *consistent*. Now, given a solution φ, let Ψ be the family of *consistent* solutions containing φ; that is, $\Psi = \{\psi | \psi \supseteq \varphi, \psi \text{ is } \textit{consistent}\}$. The solution that associates with each economy its whole feasible set is of course *consistent*. Therefore $\Psi \neq \emptyset$. Let $\overline{\varphi} = \bigcap_{\psi \in \Psi} \psi$. Since $\overline{\varphi} \supseteq \varphi$, $\overline{\varphi}$ is a well–defined solution. Therefore $\overline{\varphi}$ can be described as the *minimal consistent extension of* φ. The "size" of the difference $\overline{\varphi} \backslash \varphi$ is the price one has to pay to recover *consistency* if it is insisted upon that all the allocations picked by φ be included.

Minimal consistent extension. Given a solution φ, its ***minimal consistent extension***, ***mce(φ)***, is defined by $\text{mce}(\varphi) = \bigcap_{\psi \in \Psi} \psi$ where $\Psi = \{\psi | \psi \supseteq \varphi, \psi \text{ is } \textit{consistent}\}$.[3]

The next lemma relates the *minimal consistent extensions* of the union or the intersection of two solutions to the *minimal consistent extensions* of the components.

***Lemma* 1.** Given two solutions φ and φ', $\text{mce}(\varphi \cup \varphi') = \text{mce}(\varphi) \cup \text{mce}(\varphi')$. Also, if $\varphi \cap \varphi'$ is a well–defined solution, $\text{mce}(\varphi \cap \varphi') \subseteq \text{mce}(\varphi) \cap \text{mce}(\varphi')$; the inclusion may be strict.[4]

Proof. To prove the first statement, let, $\psi = \text{mce}(\varphi \cup \varphi')$. Then ψ belongs to the family $\{\mu | \mu \supseteq \varphi, \mu \text{ is } \textit{consistent}\}$, so that $\psi \supseteq \text{mce}(\varphi)$. Similarly, $\psi \supseteq \text{mce}(\varphi')$. Therefore $\text{mce}(\varphi \cup \varphi') \supseteq \text{mce}(\varphi) \cup \text{mce}(\varphi')$. To show the converse inclusion, let $\psi = \text{mce}(\varphi)$ and $\psi' = \text{mce}(\varphi')$. Since $\psi \supseteq \varphi$ and $\psi' \supseteq \varphi'$, $\psi \cup \psi' \supseteq \varphi \cup \varphi'$. Moreover, since *consistency* is preserved under union,[5] $\psi \cup \psi'$ is *consistent*. Therefore, $\psi \cup \psi'$ belongs to the family $\{\mu | \mu \supseteq \varphi \cup \varphi', \mu \text{ is } \textit{consistent}\}$ so that $\text{mce}(\varphi \cup \varphi') \subseteq \text{mce}(\varphi) \cup \text{mce}(\varphi')$.

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[3]The concept of a ***minimal monotonic extension*** of a correspondence, similarly defined, was proposed and analyzed by Sen (1987).

[4]The same statements hold true for arbitrary unions and intersections.

[5]This fact will be fully exploited in section 4.

To prove the second statement, let $\psi = \mathrm{mce}(\varphi)$ and $\psi' = \mathrm{mce}(\varphi')$. Clearly $\psi \supseteq \varphi \cap \varphi'$ and ψ is *consistent.* Therefore $\mathrm{mce}(\varphi \cap \varphi') \subseteq \psi$. Similarly $\mathrm{mce}(\varphi \cap \varphi') \subseteq \psi'$. Altogether we have $\mathrm{mce}(\varphi \cap \varphi') \subseteq \psi \cap \psi' = \mathrm{mce}(\varphi) \cap \mathrm{mce}(\varphi')$, as claimed.

To show that the inclusion in the second statement may be strict, consider the following example[6]: let φ be the solution that coincides with P for economies of cardinality 2 and with F otherwise, and let φ' be the solution that coincides with F for economies of cardinality 2 and with P otherwise.

We obtain that $\mathrm{mce}(\varphi)$ coincides with $F \cup P$ for economies of cardinality 2 and with F otherwise, and $\mathrm{mce}(\varphi')$ coincides with $F \cup P$ for economies of cardinality 2 and with P otherwise. Indeed, the solutions so defined are *consistent* and contain φ and φ' respectively. To show minimality for $\mathrm{mce}(\varphi)$, let $Q \in \mathcal{Q}$ with $|Q| = 2$, $e = (R_Q, \Omega) \in \mathcal{E}^Q$ and $z \in F(e)$, and let $i \in Q$. We enlarge Q by adding one agent – let him be indexed by k – with preferences identical to the preferences of agent i. Let $Q' = Q \cup \{k\}$, $\Omega' = \Omega + z_i$, $e' = (R_{Q'}, \Omega') \in \mathcal{E}^{Q'}$ and $z' \in Z(e')$ be such that $z'_Q = z$ and $z'_k = z_i$. It is immediate that $z' \in F(e')$ and since $\mathrm{mce}(\varphi) \supseteq F$ for any economy of cardinality 3, $z' \in \mathrm{mce}(\varphi)(e')$. Then, by *consistency* $z = z'_Q \in \mathrm{mce}(\varphi)(t_Q^{z'}(e')) = \mathrm{mce}(\varphi)(e)$. The argument for $\mathrm{mce}(\varphi')$ is similar. Also, $\varphi \cap \varphi' = F \cap P$, which is *consistent.* Since $\mathrm{mce}(\varphi) \cap \mathrm{mce}(\varphi')$ coincides with $F \cup P$ for economies of cardinality 2 and with $F \cap P$ otherwise, we have the strict inclusion $\mathrm{mce}(\varphi) \cap \mathrm{mce}(\varphi') \supset \mathrm{mce}(\varphi \cap \varphi')$.

Q.E.D.

In the next section, we identify the *minimal consistent extensions* of two solutions that are commonly discussed.

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[6]This is a variant of an example suggested to me by Steve Ching.

3. ***Two examples.*** An example of a solution that is not *consistent* is the solution that associates with each economy its set of allocations that pareto dominate equal division; these allocations are usually described as "individually rational from equal division." The individually rational solution from equal division is often advocated in the literature on the problem of fair division (see Thomson, 1992b, for a survey; also see Moulin, 1990, 1991). Many authors even take it as the definition of fairness.

Individually rational solution from equal division, I_{ed}: Given $Q \in \mathcal{Q}$ and $e = (R_Q,\Omega) \in \mathcal{E}^Q$, $I_{ed}(e) = \{z \in Z(e) | z_i R_i (\Omega/|Q|)$ for all $i \in Q\}$.

Another solution that is not *consistent* is the solution that picks the allocation(s) that all agents find indifferent to the same scale multiple of the aggregate bundle (Pazner and Schmeidler, 1978; Thomson, 1987; Moulin, 1991). This solution has played an important role in the literature as a resource–monotonic and population–monotonic selection from the individually rational from equal division and efficient solution.

An egalitarian solution,[7] ***E:*** Given $Q \in \mathcal{Q}$ and $e = (R_Q,\Omega) \in \mathcal{E}^Q$, $E(e) = \{z \in Z(e) |$ there exists $\lambda \in \mathbb{R}_+$ such that for all $i \in Q$, $z_i I_i (\lambda\Omega)\}$.

Note that the solution I_{ed} usually selects a continuum of non–pareto indifferent allocations, but that the solution E satisfies "pareto–indifference": if $z, z' \in E(R_Q,\Omega)$, then for all $i \in Q$, $z_i I_i z_i'$.

We would first like to calculate the *minimal consistent extension* of the intersection of I_{ed} with the pareto solution, but it turns out that it is a little more convenient to work with a subsolution of the pareto solution that is in fact very close to it. It is defined as follows: given $Q \in \mathcal{Q}$ and $e = (R_Q,\Omega) \in \mathcal{E}^Q$, let $P^*(e)$ be the subset of its efficient allocations admitting of supporting prices such that the values of all consumptions be positive (excluded in particular is any allocation at which one agent receives nothing). It is straightforward to verify that P^* is *consistent.*

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[7]We refer to this solution as ***an*** egalitarian solution since there are other ways of defining egalitarianism.

Theorem 1 says that if the point of departure is the distributional requirement of pareto domination of equal division, then in order to obtain *consistency,* this distributional objective has to be given up altogether.

***Theorem* 1.** On $\mathscr{E}$ the *minimal consistent extension* of $I_{ed} \cap P^*$ is P^*.

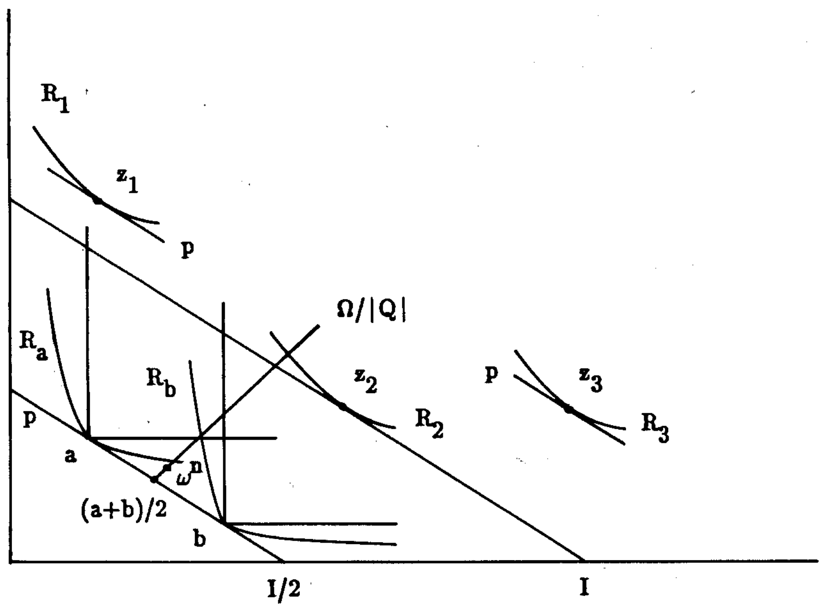

***Figure* 1.** The *minimal consistent extension* of $I_{ed} \cap P^*$ is P^* (Theorem 1). In this illustration $Q = \{1,2,3\}$.

Proof. (Figure 1). Let $Q \in \mathscr{Q}$, $e = (R_Q,\Omega) \in \mathscr{E}^Q$, and $z \in P^*(e)$ be given. Let $p \in \Delta^{\ell-1}$ be a price vector supporting z and such that $pz_i > 0$ for all $i \in Q$. Let $I = \min\{pz_i | i \in Q\}$. Note that $I > 0$. Let $a, b \in \mathbb{R}^\ell_+$ be such that $pa = pb = I/2$ and $a \neq b$. Let $n \in \mathbb{N}_{++}$ be large enough so that the point $\omega^n \in \mathbb{R}^\ell_+$ defined by $\omega^n = [\Omega+n(a+b)]/(|Q|+2n)$ be strictly below both $\{a\}+\mathbb{R}^\ell_+$ and $\{b\}+\mathbb{R}^\ell_+$. Let R_a and $R_b \in \mathscr{R}$ be two preference relations whose upper contour sets at a and b respectively admit p as a supporting price, and such that $aR_a\omega^n$ and $bR_b\omega^n$. Let e′ be the economy obtained from e by adding n agents with preferences R_a, n agents with preferences R_b, and the additional resources $n(a+b) \in \mathbb{R}^\ell_+$. Let $z' \in Z(e')$ be the allocation defined by $z'_Q = z$, $z'_i = a$ for each new agent i with preferences R_a, and $z'_i = b$ for each new agent i with preferences R_b. Note that $z' \in I_{ed} \cap P^*(e')$. Let $\psi = mce(I_{ed} \cap P^*)$.

Since $\psi \supseteq I_{ed} \cap P^*$, $z' \in \psi(e')$. Since ψ is *consistent*, $z = z'_Q \in \psi(R_Q, \sum_{i \in Q} z'_i) = \psi(e)$. Therefore $\psi \supseteq P^*$. Since P^* is *consistent*, we are done.

Q.E.D.

As a simple corollary of Theorem 1, we obtain the *minimal consistent extension* of another solution, the solution that associates with each economy its set of allocations at which each agent prefers what he receives to the average of what the others receive.

Average no–envy solution, A (Thomson, 1979, 1982; Baumol, 1986; Kolpin, 1991; Fluck, 1991): Given $Q \in \mathcal{Q}$ and $e = (R_Q, \Omega) \in \mathcal{E}^Q$, $A(e) = \{z \in Z(e) \mid$ for all $i \in Q$, $z_i R_i \left[\sum_{j \in Q \setminus \{i\}} z_j / (|Q|-1) \right] \}$ if $|Q| \geq 2$ and $A(e) = \{\Omega\}$ if $|Q| = 1$.

Corollary 1. On $\mathcal{E}$, the *minimal consistent extension* of $A \cap P^*$ is P^*.

Proof.[8] Given two solutions φ and φ' such that $\varphi \subseteq \varphi'$, if follows directly from the definition of a *minimal consistent extension* that if $\varphi' \subseteq \text{mce}(\varphi)$, then $\text{mce}(\varphi) = \text{mce}(\varphi')$. This result applies to $\varphi = I_{ed} \cap P^*$ and $\varphi' = A \cap P^*$. Indeed under convexity of preferences $I_{ed} \subseteq A$ (Thomson, 1982), and by Theorem 1, $A \cap P^* \subseteq \text{mce}(I_{ed} \cap P^*) = P^*$.

Q.E.D.

Next, we turn to the solution E. The *minimal consistent extension* of its intersection with the pareto solution can be described in terms of the egalitarian–equivalent solution, defined thus:

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[8]The proof can also be obtained by a simple modification of the proof of Theorem 1. The argument is the same until the choice of n, which should be made so that the points $\omega_a^n = [\Omega + (n-1)a + nb]/(|Q| + 2n - 1)$ and $\omega_b^n = [\Omega + na + (n-1)b]/(|Q| + 2n - 1)$ be strictly below $\{a\} + \mathbb{R}_+^\ell$ and $\{b\} + \mathbb{R}_+^\ell$ respectively. Then the preferences R_a and R_b are chosen so that the upper contour sets admit p as a supporting price at a and b respectively, $a R_a \omega_a^n$ and $b R_b \omega_b^n$. The proof continues as before.

Egalitarian–equivalent solution, E^* (Pazner and Schmeidler, 1978): Given $Q \in \mathcal{Q}$ and $e = (R_Q,\Omega) \in \mathcal{E}^Q$, $E^*(e) = \{z \in Z(e) |$ there exists $z_0 \in \mathbb{R}^\ell_+$ such that for all $i \in Q$, $z_i I_i z_0\}$.

It will be convenient here to slightly modify this definition. We will use instead the solution E^{**} defined by requiring the reference bundle z_0 to be positive. Let $\mathcal{R}' \subsetneq \mathcal{R}$ be the subclass of strictly monotone preferences. Given $Q \in \mathcal{Q}$, let $\mathcal{E}'^Q \subsetneq \mathcal{E}^Q$ be the subclass of economies (R_Q,Ω) where for each $i \in Q$, $R_i \in \mathcal{R}'$. Let $\mathcal{E}' = \bigcup_{Q \in \mathcal{Q}} \mathcal{E}'^Q$. Note that on $\mathcal{E}'$, the solution $E^{**} \cap P$ is *consistent.*[9]

***Theorem* 2.** On $\mathcal{E}'$, the *minimal consistent extension* of $E \cap P$ is $E^{**} \cap P$.

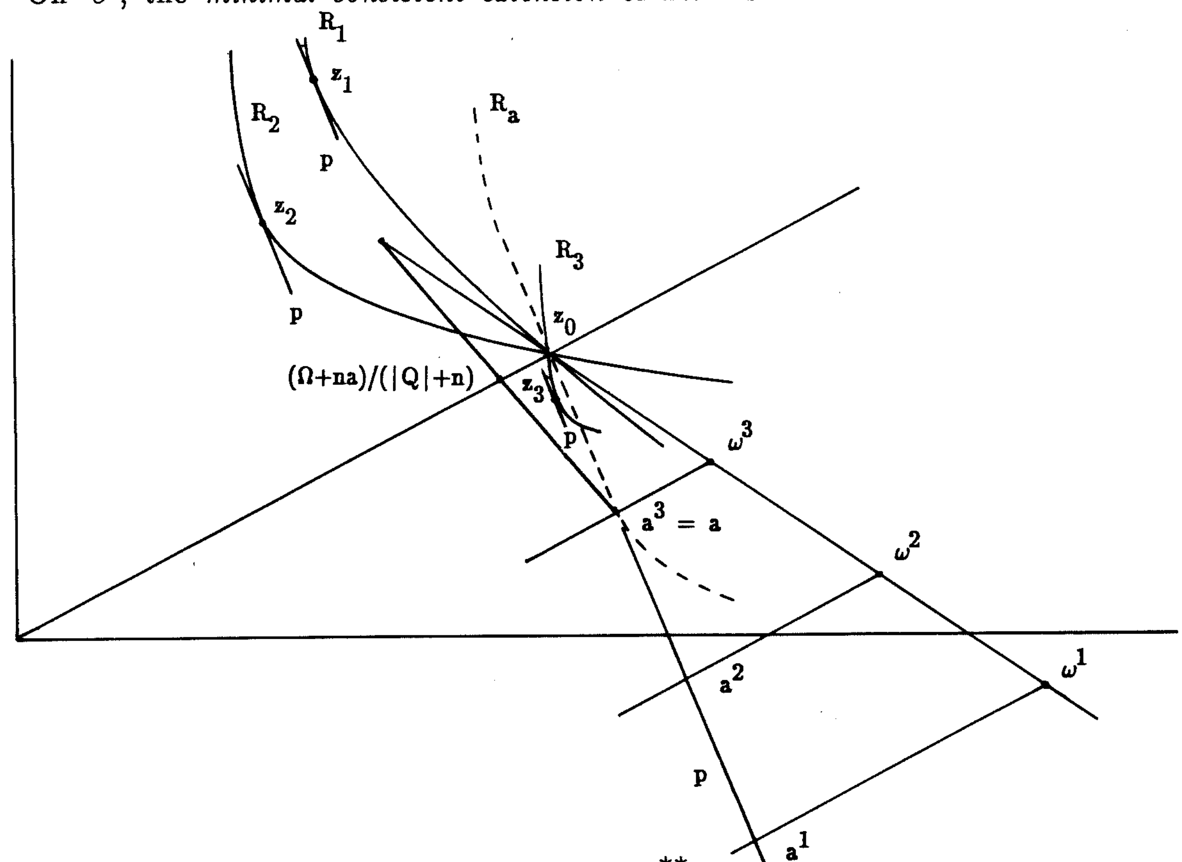

***Figure* 2.** The *minimal consistent extension* of $E \cap P$ is $E^{**} \cap P$ (Theorem 2). In the illustration, $Q = \{1,2,3\}$ and the smallest value of n for which $a^n \in \mathbb{R}^\ell_+$ is 3.

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[9]Alternatively, we could have considered the domain of preferences that are strictly monotone in $\mathbb{R}^\ell_{++}$ and such that for all $x_i \in \mathbb{R}^\ell_+ \backslash \mathbb{R}^\ell_{++}$, $x_i I_i 0$. On that domain, $E^{**} \cap P = E \cap P$.

Proof. (Figure 2) Let $Q \in \mathcal{Q}$, $e = (R_Q,\Omega) \in \mathcal{E}^Q$, and $z \in E^{**} \cap P(e)$ be given. Let $z_0 \in \mathbb{R}^{\ell}_{++}$ be such that $z_i I_i z_0$ for all $i \in Q$ and let $p \in \Delta^{\ell-1}$ be a supporting price vector for z. For each $n \in \mathbb{N}_{++}$, let $\omega^n = (1/n)[(|Q|+n)z_0-\Omega]$. Let $\lambda^n \in \mathbb{R}_+$ be such that the point $a^n = \omega^n-\lambda^n z_0$ satisfies $pa^n = pz_0$. As $n \to \infty$, $\omega^n \to z_0$. Therefore, and since $z_0 > 0$, there is $n \in \mathbb{N}_{++}$ such that $a^n \in \mathbb{R}^{\ell}_+$. Let n be so chosen and to simplify notation, write $a = a^n$. Then, let $R_a \in \mathcal{R}$ be a preference relation whose upper contour set at a is supported by the prices p and such that $aI_a z_0$. Let $Q' \supseteq Q$ be obtained by adding n agents such that for each $i \in Q' \backslash Q$, $R_i = R_a$, let $\Omega' = \Omega+na$, and $e' = (R_{Q'},\Omega') \in \mathcal{E}^{Q'}$. Let $z' \in Z(e')$ be defined by $z'_Q = z$ and $z'_i = a$ for all $i \in Q'\backslash Q$. Then $z' \in E\cap P(e')$ with reference bundle z_0 proportional to Ω' (indeed $z_0 = [|Q|+n(1-\lambda^n)]^{-1}\Omega'$) and supporting prices p. Let $\psi = \text{mce}(E\cap P)$. Since $\psi \supseteq E\cap P$, $z' \in \psi(e')$. Since ψ is *consistent*, $z = z'_Q \in \psi(R_Q,\sum_{i\in Q} z'_i) = \psi(e)$. Therefore $\psi \supseteq E^{**}\cap P$. Since $E^{**}\cap P$ is *consistent*, we are done.

Q.E.D.

4. ***Maximal consistent subsolutions.*** The procedure discussed in the preceding sections is certainly not the only way of evaluating the extent to which a solution φ may fail to be *consistent*. Alternatively we could delete from, instead of adding to, the φ–optimal set, and ask how much should be deleted to recover the property. This will work only if φ does contain a *consistent* subsolution, but this is the only precondition. Indeed, if all the members of a non–empty family Ψ of solutions are *consistent*, then so is the union $\underline{\varphi} = \bigcup_{\psi\in\Psi} \psi$. If $\psi \subseteq \varphi$ for all $\psi \in \Psi$ then of course $\underline{\varphi} \subseteq \varphi$, so that $\underline{\varphi}$ can be described as the *maximal consistent subsolution of* φ.

Maximal consistent subsolution. Given a solution φ containing a *consistent* subsolution, its ***maximal consistent subsolution, mcs(φ)***, is defined by $\text{mcs}(\varphi) = \bigcup_{\psi\in\Psi} \varphi$ where $\Psi = \{\psi | \psi \subseteq \varphi, \psi \text{ is } \textit{consistent}\}$.

Here too, we would like to know how different $\mathrm{mcs}(\varphi)$ is likely to be from φ. Note that $\mathrm{mcs}(\varphi)$ is equal to the solution $\underline{\varphi}$ defined, for each $Q \in \mathcal{Q}$ and $e = (R_Q,\Omega) \in \mathcal{E}^Q$, by

(*) $\qquad \underline{\varphi}(e) = \{z \in Z(e) \,|\, z_{Q'} \in \varphi(t^z_{Q'}(e)) \text{ for all } Q' \subseteq Q\}.$

Indeed, $\underline{\varphi}$ is *consistent*, and of course $\underline{\varphi} \subseteq \varphi$ (set $Q' = Q$ in the definition). Maximality follows from the fact that these conditions are necessary.

The next lemma relates the *maximal consistent subsolutions* of the union or the intersection of two solutions to the *maximal consistent subsolutions* of the components. To facilitate its proof, we introduce a property dual to *consistency*. It says that the φ–optimality of an allocation for some economy can be derived from the φ–optimality of its restrictions to all the associated 2–person reduced economies: the solution φ is ***conversely consistent*** if for all $Q \in \mathcal{Q}$, for all $e = (R_Q,\Omega) \in \mathcal{E}^Q$, and for all $z \in Z(e)$, if [for all $Q' \subseteq Q$ with $|Q'| = 2$, $z_{Q'} \in \varphi(t^z_{Q'}(e))$], then $z \in \varphi(e)$. The no–envy solution is trivially *conversely consistent* and under appropriate smoothness conditions on preferences, so is the pareto solution. (See Thomson 1992a for a discussion of these facts.)

Of course $\mathrm{mcs}(\varphi \cup \varphi')$ may be well–defined without either $\mathrm{mcs}(\varphi)$ or $\mathrm{mcs}(\varphi')$ being well–defined. For instance, if ψ is *consistent* but has no *consistent* proper subsolution, any pair $\{\varphi,\varphi'\}$ for which $\varphi \cup \varphi' = \psi$ will be such that $\mathrm{mcs}(\psi) = \psi$ while neither $\mathrm{mcs}(\varphi)$ or $\mathrm{mcs}(\varphi')$ is well–defined. If either φ or φ' has a *consistent* subsolution, then $\varphi \cup \varphi'$ also does and $\mathrm{mcs}(\varphi \cup \varphi')$ is well–defined.

***Lemma* 2.** Given two solutions φ and φ', each of which contains a *consistent* solution, $\mathrm{mcs}(\varphi \cup \varphi') \supseteq \mathrm{mcs}(\varphi) \cup \mathrm{mcs}(\varphi')$; the inclusion may be strict. Also, if $\varphi \cap \varphi'$ contains a *consistent* solution, $\mathrm{mcs}(\varphi \cap \varphi') = \mathrm{mcs}(\varphi) \cap \mathrm{mcs}(\varphi')$.[10]

[10]As in Lemma 1, the statements hold true for arbitrary unions and intersections.

Proof. To prove the first statement, let $\psi = \text{mcs}(\varphi)$ and $\psi' = \text{mcs}(\varphi')$. Then ψ belongs to the family $\{\mu | \mu \subseteq \varphi,\ \mu \text{ is } consistent\}$ and ψ' belongs to the family $\{\mu | \mu \subseteq \varphi',\ \mu \text{ is } consistent\}$, so that $\psi \cup \psi'$ belongs to the family $\{\mu | \mu \subseteq \varphi \cup \varphi',\ \mu \text{ is } consistent\}$. Therefore $\text{mcs}(\varphi \cup \varphi') \supseteq \psi \cup \psi' = \text{mcs}(\varphi) \cup \text{mcs}(\varphi')$.

The example to prove that the inclusion may be strict is the one we used in the proof of Lemma 1: consider a domain on which P is *conversely consistent* and let φ coincide with P for economies of cardinality 2 and with F otherwise and let φ' be defined in a symmetric way. Note that $\text{mcs}(\varphi)$ coincides with P for economies of cardinality 2 and with F∩P otherwise. Indeed, it is easy to check that the solution so defined is a *consistent* subsolution of φ. To show that it is maximal, given $Q \in \mathcal{Q}$ with $|Q| > 2$, $e = (R_Q, \Omega) \in \mathcal{E}^Q$ and $z \in \text{mcs}(\varphi)(e)$, note first that since $\text{mcs}(\varphi) \subseteq \varphi$, $z \in F(e)$. Also, since $\text{mcs}(\varphi)$ is *consistent* and $\text{mcs}(\varphi) \subseteq P$ for economies of cardinality 2, $z_{Q'} \in \text{mcs}(\varphi)(t^z_{Q'}(e)) \subseteq P(t^z_{Q'}(e))$ for all $Q' \subseteq Q$ with $|Q'| = 2$. Since on the domain under consideration, P is *conversely consistent*, it follows that $z \in P(e)$. Altogether $z \in F \cap P(e)$. Similarly, we deduce that $\text{mcs}(\varphi')$ coincides with F for economies of cardinality 2 and with F∩P otherwise. Also, $\varphi \cup \varphi' = F \cup P$, and since F∪P is *consistent*, $\varphi \cup \varphi' = \text{mcs}(\varphi \cup \varphi') = F \cup P$. Finally, we observe that $\text{mcs}(\varphi) \cup \text{mcs}(\varphi')$ coincides with F∪P for economies of cardinality 2 and with F∩P otherwise. Therefore, we have the strict inclusion $\text{mcs}(\varphi \cup \varphi') \supset \text{mcs}(\varphi) \cup \text{mcs}(\varphi')$.

To prove the second statement, note first of all, that if $\varphi \cap \varphi'$ contains a *consistent* subsolution, then so do both φ and φ', and therefore $\text{mcs}(\varphi \cap \varphi')$, $\text{mcs}(\varphi)$ and $\text{mcs}(\varphi')$ are all well–defined. Let $\psi = \text{mcs}(\varphi) \cap \text{mcs}(\varphi')$. Since $\psi \subseteq \varphi \cap \varphi'$ and ψ is *consistent*, $\text{mcs}(\varphi \cap \varphi') \supseteq \psi$. To show the converse inclusion, let $\psi' = \text{mcs}(\varphi \cap \varphi')$. Since $\psi' \subseteq \varphi$ and ψ' is *consistent*, $\psi' \subseteq \text{mcs}(\varphi)$. Similarly, $\psi' \subseteq \text{mcs}(\varphi')$. Therefore $\psi' \subseteq \text{mcs}(\varphi) \cap \text{mcs}(\varphi')$. Altogether, we have $\text{mcs}(\varphi \cap \varphi') = \text{mcs}(\varphi) \cap \text{mcs}(\varphi')$.

Q.E.D.

We will illustrate the notion of a *maximal consistent subsolution* by considering again the individually rational from equal division and efficient solution. Let $\varphi = \text{mcs}(I_{ed} \cap P)$. The existence of a *maximal consistent subsolution* of this solution follows from the fact that there is indeed a *consistent* subsolution of $I_{ed} \cap P$, namely the Walrasian solution from equal division. From formula (*), we obtain that for each economy e, $\varphi(e)$ is the set of allocations that pareto dominate equal division in e and whose restriction to any subgroup pareto dominates equal division in the associated reduced economy.

Consider now the following property of solutions:

Replication invariance. For all $Q, Q' \in \mathcal{Q}$, for all $e \in \mathcal{E}^Q$ and $e' \in \mathcal{E}^{Q'}$, for all $z \in \varphi(e)$ and $z' \in Z(e')$, for all $k \in \mathbb{N}_{++}$, if e′ is obtained from e by k–times replication and z′ is obtained from z by k–times replication, then $z' \in \varphi(e')$.

Note that *replication invariance* is also preserved under union so that the existence of a *maximal consistent* ***and replication invariant*** *subsolution* of a given solution will be guaranteed if the solution contains at least one subsolution with these properties. This is the case for the individually rational solution from equal division, since the Walrasian solution from equal division, which it contains, is *consistent*, as already noted, and it is also *replication invariant.* The next theorem says that its *maximal consistent and replication invariant subsolution*, coincides with a solution introduced by Kolm (1973, 1991) and defined, for each $Q \in \mathcal{Q}$ and $e = (R_Q,\Omega) \in \mathcal{E}^Q$, by:

$K(e) = \{z \in Z(e) |$ for all $i \in Q$, for all $z_0 \in \text{co}\{z_j | j \in Q\}$, $z_i R_i z_0\}$

where co{·} denotes the convex hull operator.

Theorem 3. The *maximal consistent and replication invariant subsolution* of I_{ed} is K.

Proof. First, note that K is *consistent and replication invariant.* Let φ be the *maximal consistent and replication invariant* subsolution of I_{ed}. Since $K \subseteq I_{ed}$, $K \subseteq \varphi$.

To show that $\varphi \subseteq K$, let $Q \in \mathcal{Q}$, $e = (R_Q,\Omega) \in \mathcal{E}^Q$ and $z \in \varphi(e)$ and suppose by contradiction, that there are $i \in Q$ and $z_0 \in co\{z_j | j \in Q\}$ such that $z_0 P_i z_i$. By continuity of preferences, there are positive integers $\{\lambda_j | j \in Q\}$ such that $y_0 = \sum_{j \in Q} \lambda_j z_j / (\sum_{j \in Q} \lambda_j)$ satisfies $y_0 P_i z_i$. Let $\lambda = \max\{\lambda_j | j \in Q\}$. Let e' and z' be obtained from e and z respectively by λ–times replication. By *replication invariance* of φ, $z' \in \varphi(e')$. Let Q" be a subgroup of agents in e' containing for each $j \in Q$, λ_j agents with preferences R_j. Let $e'' = t^{z'}_{Q''}(e')$. By *consistency* of φ, $z'_{Q''} \in \varphi(e'')$. However, equal division in e" is y_0 and since for all $k \in Q''$ identical to i, $y_0 P_i z'_k = z_i$, we have $z'_{Q''} \notin I_{ed}(e'')$ in contradiction with $\varphi \subseteq I_{ed}$.

Q.E.D.

A similar argument shows that the *maximal consistent and replication invariant* subsolution of $I_{ed} \cap P$ is $K \cap P$.

The concept of a *maximal consistent subsolution* can be used to reformulate a characterization of the Walrasian solution from equal division obtained in Thomson (1988): under the assumption of smoothness of preferences, any subsolution of the individually rational from equal division and efficient solution satisfying *consistency* and *replication invariance* is a subsolution of the Walrasian solution from equal division. Since the Walrasian solution from equal division satisfies all of these properties, it is the maximal such solution. Therefore, under smoothness of preferences, the *maximal consistent and replication invariant subsolution* of $I_{ed} \cap P$ is the Walrasian solution from equal division.

The concept can also help us establish a connection between two other notions. Recall the earlier definition of an average envy–free allocation as one such that each agent prefers his consumption to the average consumption of the others. Now, require that each agent prefers his consumption not only to the average of what the others receive but also to the average of what any ***subgroup*** of the others receive:

Strict no–envy solution, S, (Zhou, 1992): Given $Q \in \mathcal{Q}$ and $e = (R_Q,\Omega) \in \mathcal{E}^Q$, $S(e) = \{z \in Z(e) |$ for all $i \in Q$ and for all $Q' \subset Q$ with $Q' \neq \emptyset$ and $i \notin Q'$, $z_i R_i \left[\sum_{j \in Q'} z_j\right] / |Q'|\}$.

As noted, S is a subsolution of A. Also, A is not *consistent* but S is. Since an allocation is strictly envy–free if it is average envy–free and its restriction to any subgroup is average envy–free in the associated reduced economy, it follows directly that S is the *maximal consistent subsolution* of A. Consider now the solution that associates with each economy its set of efficient allocations whose k–replicas are strictly envy–free for the economy replicated k times, for all $k \in \mathbb{N}$. This solution is *consistent* as well as *replication invariant.* It is simply the *maximal consistent and replication invariant subsolution* of the average envy–free and efficient solution. Zhou (1992) shows that under smoothness of preferences it coincides with the Walrasian solution from equal division. This result also follows from the fact that on the domain of smooth preferences, any subsolution of the average envy–free and efficient solution satisfying *consistency* and *replication invariance* is a subsolution of the Walrasian solution from equal division (Thomson, 1988), and the fact that the Walrasian solution from equal division does satisfy *consistency* and *replication invariance.*

5. ***Concluding comments.*** We proposed a way of measuring the extent to which a solution may fail to be *consistent* by introducing the notion of its *minimal consistent extension* and we applied the notion to two examples concerning the problem of fair allocation in classical exchange economies. We also considered the notion of *maximal consistent subsolution* of a given solution and showed how it can help relate several concepts that have played a role in the literature on the problem of fair allocation.

These notions are certainly applicable to other domains where *consistency* has been found useful, such as bargaining theory, coalitional form games, and bankruptcy, and to other classes of allocation problems. We provide two additional examples of applications to allocation problems.

Tadenuma and Thomson (1991) analyze a model of fair allocation in economies with indivisible goods and show that there is no proper subsolution of the no–envy solution satisfying *consistency* and a condition of *neutrality.*[11] (The no–envy solution satisfies both properties). *Neutrality* also is preserved under intersection, and since the feasibility correspondence is *neutral*, the reasoning that led us to the concept of *minimal consistent extension* gives us the concept of *minimal consistent* ***and neutral*** *extension.* Thus, the result stated above can be rephrased as follows: all subsolutions of the no–envy solution have the same *minimal consistent and neutral extension*, which is the no–envy solution itself.

Sasaki and Toda (1992) consider the class ofmatching problems and search for subsolutions of the solution associating with each matching problem its set of matches that cannot be improved upon by any pair of agents; this solution coincides with the core. They show that there is no proper subsolution of the core satisfying *consistency.* Therefore, this result can be stated as: all subsolutions of the core have the *same minimal consistent extension*, which is the core itself.

The determination of the *minimal consistent extensions* and *maximal consistent subsolutions* of important solutions for other domains will be left to future research.

[11]A solution φ is *neutral* if whenever a φ–optimal allocation is such that permuting its components leave all agents indifferent, then the permuted allocation is also φ–optimal.

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