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ABSTRACT

Many economic models in the fields of public finance, location theory, and choice under uncertainty involve characteristic nonconvexities in either preferences or production sets for some types of commodities. One useful way to attack such nonconvexities is to employ the convexifying effect of large numbers of agents on demand for a finite number of commodities. The alternative proposed here relies on the convexifying effect of large numbers of commodities rather than agents. Sufficient substitutability and a large number of commodities can be used to replace some convexity assumptions. Existence of an equilibrium and the first welfare theorem are proved using Zame's existence theorem and Lyapunov's theorem as the key tools.

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I. Introduction

In many economic contexts, nonconvexities in preferences or technology arise naturally. These nonconvexities create well – known problems with the existence of an equilibrium. One useful way to attack such problems is to use the convexifying effect of large numbers of agents when there is a finite number of commodities. However, this method might not be available in some situations. For instance, there might be an infinity of commodities generated by the location, time, or state – dependence of commodities and preferences, so that these convexification arguments might not work. Another possibility is that the number of commodities grows with the number of agents, for example when there are externalities or public goods and personalized commodity markets. Finally, there are the problems discussed in Berliant (1985b) with the foundations of models having a continuum of consumers and land. In this paper, we show that equilibria might exist in these various contexts due to the convexifying effect of a large number of commodities. In this sense, our work can be viewed as dual to the large number of agents story told in the work of Hildenbrand (1974).

We will allow nonconvexities in preferences of any consumer, or alternatively, any producer, but not in both the consumption and production sides of the economy simultaneously. For the sake of brevity, we discuss in this paper only nonconvexities in preferences; the case of nonconvexities in production can be covered in an analogous manner. Consider a consumer who has non–convex preferences (made precise in the next section) over a continuous set of commodities T. For these commodities, it is assumed that there is sufficient substitutability in both the preferences of this non–convex consumer and production of *some* producer. Standard consumers and producers satisfying convexity restrictions can also be accommodated in the same economy with the same commodity space. These assumptions are made precise by specializing the framework of Zame (1987). In particular, our assumptions on (non–convex) preferences and production sets have a hedonic interpretation. The commodities over which preferences are nonconvex are mapped into their characteristics. Preferences and technologies satisfy the usual convexity assumptions over these characteristics and the remaining (convex) commodities. The nonconvexities that we allow are introduced in the hedonic map from the nonconvex commodities to their characteristics. Heckman and Scheinkman (1987) is an example of a model with infinitely many inputs (in their case, labor supplied by different workers) where production depends on the characteristics of those inputs. They assume a linear map between inputs and characteristics. Our model allows for more general hedonic technologies.

For other examples of nonconvexities we refer to the public finance literature; see, for example, Baumol (1972), Starrett (1972), and Khan and Vohra (1987). Note that our results are more concerned with nonconvexities in isoquants and preferences than with increasing returns. However, our result does cover certain kinds of increasing returns and demonstrates the existence of a true competitive equilibrium. This is in contrast to the marginal cost pricing literature, where the notion of equilibrium merely requires firms to satisfy the first order conditions for profit maximization but not necessarily to choose a production plan that maximizes profits. The assumptions we use also have an expected utility interpretation, so that traders who prefer extreme over average lotteries can be accommodated.

The proofs of the existence of an equilibrium and of the first welfare theorem are straightforward applications of Zame's existence theorem and Lyapunov's theorem. For brevity, the only case considered here is that of nonconvexities in preferences. An analogous proof will work when there are similar nonconvexities in production. The existence proof begins by defining an artificial economy with piecewise linear utility for the nonconvex commodities. An extreme point of the set of equilibrium allocations is shown to be an equilibrium of the original economy. The moral of our story is that sufficient substitutability and a large number of commodities can be used to replace some convexity assumptions.

The next section describes our model. Section III contains the statement of our existence theorem and its proof. Section IV concludes with a discussion of our assumptions and the potential extensions of our results.

II. The Model

Our focus in this paper is not to generalize existence theorems in convex economies with topological vector spaces as commodity spaces, but rather to examine existence of equilibrium with nonconvexities in such economies. For this reason, we wish to specialize the framework so that we can prove theorems.

We use the basic framework of Zame (1987) with some further assumptions. First, we define $L \equiv (L^p)^k \times \hat{L}$, where $1 \leq p \leq \infty$, k is a finite integer, L^p has the norm topology, and $\hat{L}$ is a normed lattice. Then L is a normed lattice with non–negative orthant L_+, which is assumed to be the consumption set for each consumer. We use x, y and z to denote generic elements of L_+. When we wish to decompose $x \in L$ into its two basic components, we write $x = (x^A, x^B)$, where $x^A \in (L^p)^k$ and $x^B \in \hat{L}$. Let the domain of functions in $(L^p)^k$ be given by T, and let t be a generic element of T. T will represent the set of commodities over which preferences can be nonconvex. We postulate existence of a σ–algebra on T, call it $\mathcal{T}$, so that we can discuss measurability. The measures on T will be nonatomic. Let q be such that $1/p + 1/q = 1$.

There are several possible interpretations of the elements in the consumption set L_+. The set T could represent a set of differentiated commodities. Then x^A would give the consumption of each of these commodities, which could be different in k different locations, states of nature or time periods. A natural situation where nonconvexities occur would be when T represents a set of indivisible commodities. If k were equal to the number of consumers then x^A could represent personalized commodities in the context of externalities. Alternatively, T could represent continuous time, states of nature or locations. Then x^A can be interpreted as a continuous–time consumption path, the state–contingent consumption or the location–specific consumption of k ordinary commodities.

There are N consumers, indexed by i, where N is a finite integer. There are M firms, indexed by j, where $M \geq 1$ is a finite integer. Consumer i has an endowment $e_i \in L_+$ while the profit share of consumer i in firm j is given by θ_{ij} with $0 \leq \theta_{ij} \leq 1$ and $\sum_{i=1}^{N} \theta_{ij} = 1$ for all j. The preferences of consumer i will be given by a strict preference correspondence $P_i: L_+ \to 2^{L_+}$. The production set of firm j is called $Y_j \subseteq L$.

An *allocation* is given by an N + M – tuple $[(x_i),(y_j)]$ where $x_i \in L_+$ for all i and $y_j \in Y_j$ for all j; the allocation is *feasible* if $\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} e_i + \sum_{j=1}^{M} y_j$. An *equilibrium* is an N + M + 1 – tuple $[\pi,(x_i),(y_j)]$ where π is a non–zero continuous linear functional on L and $[(x_i),(y_j)]$ is a feasible allocation such that $\pi(y_j) = \max \{\pi(y_j') \mid y_j' \in Y_j\}$, $\pi(x_i) \leq \pi(e_i) + \sum_{j=1}^{M} \theta_{ij}\pi(y_j)$ for each i, and if $x' \in L_+$ with $x' \in P_i(x_i)$, then $\pi(x') > \pi(e_i) + \sum_{j=1}^{M} \theta_{ij}\pi(y_j)$. An *equilibrium allocation* is an allocation $[(x_i),(y_j)]$ such that there exists π that makes $[\pi,(x_i),(y_j)]$ an equilibrium. A *Pareto optimum* is a feasible allocation $[(x_i),(y_i)]$ such that there is no other feasible allocation $[(x_i'),(y_i')]$ such that $x_i' \in P_i(x_i)$ for all i.

The economy is said to be *irreducible* if whenever $i \neq i'$ and $[(x_i),(y_j)]$ is a feasible allocation, then there is a vector z in L such that $z \leq e_{i'}$, and $x_i + z \in P_i(x_i)$.

Next we proceed to list the assumptions that distinguish our framework from that of Zame (1987), aside from the particular form of L.

On the consumption side, we assume that for each consumer i, *either* P_i satisfies all of the assumptions of Zame (1987)[1], *or* i is a non–convex consumer, which means that consumer i has a complete preorder represented by a utility function of the

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[1]In particular, the Standard Assumptions on p. 1080 and the assumptions used in Theorem 1 on p. 1090.

form:

$$u_i(x_i) = f_i\left(\int_T h_1(x_i^A(t),t)\ dm_1,\ldots,\int_T h_C(x_i^A(t),t)\ dm_C,\ x_i^B\right)$$

where $f_i\colon \mathbb{R}^C \times \hat{L}_+ \to \mathbb{R}$ is continuous, quasi-concave, and non-decreasing in its first C arguments (given a fixed argument C+1); h_c is a measurable function mapping from $\mathbb{R}^k \times T$ into $\mathbb{R}$ that is continuous in its first k arguments for each fixed t and, for almost every t, satisfies $\sup\{\limsup_{n\to\infty} h_c(a_n,t)/\|a_n\| \mid \{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}_+^k,\ \lim_{n\to\infty} \|a_n\| = \infty\} < \infty$; and m_c is a nonatomic, positive measure on $(T,\mathcal{J})$ for each c. Notice that h_c *need not* possess any concavity properties. We assume that the preferences induced by u_i are locally non-satiated in $\hat{L}$: $\forall\ x = (x^A,x^B) \in L_+$, $\forall\ \epsilon > 0$, there exists $\bar{x}^B \in L_+$ with $u_i(\bar{x}) > u_i(x)$ and $\|x-\bar{x}\| < \epsilon$, where $\bar{x} = (x^A,\bar{x}^B)$.[2] Note that if we want the h_c functions to differ across consumers, we can simply include all of them in 1,...,C and let the dependence of the utility functions on some of the functions h_c be trivial.

When the h_c are convex these assumptions allow non-convex preferences. Extremes can be preferred to averages. So consumers may prefer to concentrate their consumption on particular commodities, at particular times, in particular states of nature or locations depending on the interpretation of T. The indifference curves describing trade-offs between commodities indexed by T, and between those commodities indexed by T and those in $\hat{L}$, can appear to be badly behaved. Note that the functional form of u_i can be given a hedonic interpretation, in which each integral aggregates some characteristic of a commodity bundle. The nonconvexities we allow arise in the hedonic map from commodities to characteristics and we require

[2]Alternatively, we could assume that f_i is increasing in its first C arguments and h_c is increasing in its first argument for each i and c. This would place assumptions on preferences over $(L^p)^k$ rather than preferences over $\hat{L}$. We could not simply assume local nonsatiation for preferences, since those preferences generated by the "artificial" economy below do not necessarily inherit this property.

characteristics to be additive across the index t. Finally, note that externalities and personalized commodities can be handled by having an h_c depend only on the coordinates of $x_i^A(t)$ that are associated with an externality (with k=N).

On the production side, we assume that if there is a non-convex consumer, the production set of at least one firm (firm 1) is of the form:

$$Y_1 = \{y \in L \mid (\int_T \beta_1(t)\cdot y^A(t)\ d\lambda_1,\ldots,\int_T \beta_G(t)\cdot y^A(t)\ d\lambda_G,\ y^B) \in Y_1^*\}$$

where Y_1^* is a given closed, convex subset of $\mathbb{R}^G x\hat{L}$ containing 0; $\beta_g \in (L^q)^k\ \forall$ g; and λ_g is a nonatomic measure for each g. For other producers (j > 1) we assume that Y_j is a convex subset of L containing 0.

The essential property of this assumption is that if there are commodities over which preferences are non-convex then there is a linear technology (Y_1) over these commodities. For example, suppose T is a set of differentiated commodities with G = k = 1 and $Y_1^* \equiv 0$. Then Y_1 represents a constant returns technology with $\beta_1(t)/\beta_1(t')$ being the marginal rate of transformation between commodities t and t'. Another way of interpreting the technology described by Y_1 is that production depends on G characteristics and the commodities in $\hat{L}$, where the characteristics are additive across the T-indexed non-convex commodities.

For example, suppose that housing density (per unit area of land) $y^A(t)$ can be produced at each point $t \in T$ with inputs y^B that must be allocated to G tasks (e.g. time spent building walls). The interpretation of $\beta_g(t)\cdot y^A(t)$ would be the amount of task g (in terms of density per unit area) that is needed to produce housing density (per unit area) $y^A(t)$. The densities are integrated over an area to obtain total inputs and outputs. The production set Y_1^* describes the technological relationship between the inputs y^B and the amount of each task that can be accomplished. For example, if y^B represents total labor time then Y_1^* might be defined by the constraint that the G integrals sum to this total time. On the consumer side of the model, we can employ a hedonic interpretation: x^A is the amount of housing per unit area (if $k > 1$ there could be several housing styles), and h maps x^A to characteristics, such as square feet of bedroom space per unit of housing area for the particular style. Other interpretations of the model can be constructed, but this should always be done carefully. See, for instance, the relevant examples of Jones (1984).

In the following, the roles of the special forms of production sets and utility functions can be interchanged without altering the proofs much. Hence, we could just as easily cover the case when the hedonic map for producers is nonconcave, and the hedonic map for consumers has a linear form.

As mentioned previously, our interest is in existence of equilibrium with nonconvexities. Thus, we alter Zame's (1987) Main Theorem in that we relax the convexity/concavity assumptions, but we strengthen other assumptions. In particular, the assumptions about the existence and form of a utility function as well as the form of production are strengthened. Since we have postulated little about the form of the utility function acting on the well–behaved commodities in $\hat{L}$ as well as how production acts on $\hat{L}$, we must impose the same assumptions as Zame does in order to prove that an equilibrium exists.

As in Zame (1987), aside from the norm topology, we need a topology τ on L that makes the feasible consumption and production sets compact. The most convenient topology to impose on L for this purpose depends on the particular space employed. For the space $(L^p)^k$, we shall explicitly use the Mackey topology for the topology τ_1. For the space $\hat{L}$, we shall leave the choice of the topology open for purposes of flexibility. Our prototypical model is when $\hat{L}$ is an $L^{\hat{p}}$ or Euclidean space, and the topologies that are most suited to these spaces are discussed in Zame (1987, section 4). As this issue is not our focus here we shall simply assume that there is an appropriate topology, τ_2, on $\hat{L}$. Obviously, we impose the product topology $\tau = \tau_1 x \tau_2$ on $L = (L^p)^k x \hat{L}$.

We use the *norm* topology on our commodity space L in order to be able to define the notions of extreme desirability and bounded marginal efficiency of production. They are needed for Zame's theorem. As these assumptions are not the focus of this paper, we do not dwell on this point.

III. Results

<u>Theorem:</u> Let $L \equiv (L^p)^k \times \hat{L}$ be a normed lattice satisfying the assumptions of Section II. Let τ be a Hausdorff vector space topology on L which is weaker than the norm topology, and such that: (1) the feasible consumption and production sets are compact in the topology τ, (2) each order interval in L is compact in the topology τ, (3) for each nonconvex consumer i[3], u_i is continuous in the norm topology and f_i is upper–semicontinuous in the Euclidean x τ_2 topology, (4) for each j the production set Y_j is closed in the topology τ.

In addition, assume that for each i there is a vector ν_i with $0 \leq \nu_i \leq \sum_{i=1}^{N} e_i$ and a relatively open subset W_i of L_+ which contains the feasible consumption set such that ν_i is extremely desirable (see Zame (1987, p. 1086)) for P_i on W_i. Assume also that the marginal efficiency of production is bounded (see Zame (1987, pp. 1088–1089)). Finally, if the economy is irreducible, then there exists an equilibrium.

<u>Proof:</u> We set up an artificial economy as follows. For $W \subseteq \mathbb{R}^{k+1}$ let co(W) denote the convex hull of W. Define:

$\text{Graph}_c(t) \equiv \{(a,b) \mid a \in \mathbb{R}^k,\ b \in \mathbb{R},\ b = h_c(a,t),\ a \geq 0\}$,

for $a \geq 0$, $\bar{h}_c(a,t) \equiv \sup \{r \in \mathbb{R} \mid (a,r) \in \text{co}(\text{Graph}_c(t))\}$,

$\bar{u}_i(x) = f_i(\int_T \bar{h}_1(x^A(t),t)\ dm_1, \ldots, \int_T \bar{h}_C(x^A(t),t)\ dm_C,\ x^B)$ for each non–convex consumer i.

Since h_c is continuous in its first argument and $\sup \{\limsup_{n\to\infty} h_c(a_n,t)/\|a_n\| \mid \{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^k_+,\ \lim_{n\to\infty} \|a_n\| = \infty\} < \infty$ a.s., $\bar{h}_c(a,t) < \infty$ a.s. Next we must check the standard assumptions of Zame (1987, p. 1080) for the artificial economy. First, L_+

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[3]See footnote 1 concerning other consumers.

is (norm) closed, convex, and $e_i \in L_+$ for each i. Second, it is easy to see that $\bar{h}_c(\cdot,t)$ is continuous a.s.(m_c), and consequently that $\bar{u}_i$ is continuous in the norm topology. By definition, $\bar{h}_c(\cdot,t)$ is concave for each c and t a.s., so since f_i is quasi–concave and nondecreasing in its first C components, $\bar{u}_i$ is quasi–concave. Local non–satiation has been assumed. The profit shares satisfy the usual conditions. Finally, Y_j is (norm) closed and convex, and $0 \in Y_j$.

Given our assumptions, it is easy to verify that the artificial economy satisfies the conditions of Zame (1987, Theorem 1) and therefore has an (artificial) equilibrium $[\bar{\pi},(\bar{x}_i),(\bar{y}_j)]$. The remainder of the proof proceeds in a fashion similar to Berliant (1985a). Fix an equilibrium $[\bar{\pi},(\bar{x}_i),(\bar{y}_j)]$, and let $E(\bar{\pi}) \equiv \{[(x_i^A),(y_i^A)] \in (L_+^p)^{kN}x(L^p)^{kM} \mid [\bar{\pi},(x_i^A,\bar{x}_i^B),(y_j^A,\bar{y}_j^B)]$ is an equilibrium.$\} \neq \emptyset$. Notice that $E(\bar{\pi})$ is a closed and therefore Mackey compact subset of $(L_+^p)^{kN}x(L^p)^{kM}$ since $\bar{\pi}$ is a continuous linear functional, Y_j is closed in the τ topology, and u_i is upper semi–continuous in the τ topology on L. It is easy to see that $E(\bar{\pi})$ is also a convex subset of $(L_+^p)^{kN}x(L^p)^{kM}$. The Krein–Milman theorem (see Rudin (1973, p. 70)) tells us that a compact, convex set in L is the closed, convex hull of its extreme points; so $E(\bar{\pi})$ has an extreme point. The next step is to show that any extreme point of $E(\bar{\pi})$ has the property that for all c, $h_c(\bar{x}_i^A(t),t) = \bar{h}_c(\bar{x}_i^A(t),t)$, so $u_i(\bar{x}_i) = \bar{u}_i(\bar{x}_i)$ for any consumer i.

If this is not the case, then there is a consumer i and a c such that $h_c(\bar{x}_i^A(t),t) < \bar{h}_c(\bar{x}_i^A(t),t)$ for $t \in D$, where $m_c(D) > 0$. The next step is to show that there exists a set $W \subseteq D$, $m_c(W) > 0$, and points $z_1,z_2 \in (L^p)^k$, $z_1 \neq z_2$, with $z_1(t) = z_2(t) = 0$ for $t \in T\backslash W$ a.s., $(1/2)\cdot z_1(t) + (1/2)\cdot z_2(t) = \bar{x}_i^A(t)$ for $t \in W$ a.s. and $\bar{h}_c(\bar{x}_i^A(t),t) = (1/2)\cdot\bar{h}_c(z_1(t),t) + (1/2)\cdot\bar{h}_c(z_2(t),t)$ for $t \in W$ a.s. By definition of $\bar{h}_c$, a.s.(t) there exist $\{(w_1^n,\ldots,w_{b_n}^n)\}_{n=1}^{\infty}$ (where $w_d^n \in (L_+^p)^k$ for all d and n) and $\{(\alpha_1^n,\ldots,\alpha_{b_n}^n)\}_{n=1}^{\infty}$ (where $\alpha_d^n \in L_+^{\infty}$ for all d and n, for all n $\sum_{d=1}^{b_n} \alpha_d^n = 1$) such that

$\bar{x}_i^A(t) = \sum_{d=1}^{b_n} \alpha_d^n \cdot w_d^n$, $\bar{h}_c(\bar{x}_i^A(t),t) - \sum_{d=1}^{b_n} \alpha_d^n \cdot h_c(w_d^n,t) < 1/n$. (Note: To simplify notation, we omit the t index in both w and α, as well as the "almost surely in $t \in D$" statements.) Define $\alpha_d^n = 0$ if $d > b_n$. Using the Banach–Alaoglu theorem, for each fixed d, $\{\alpha_d^n\}_{n=1}^{\infty}$ is contained in a weak* compact set in L_+^{∞}. Considering $\{(\alpha_1^n,\alpha_2^n,...)\}_{n=1}^{\infty}$ to be a sequence in the product space $(L_+^{\infty})^{\infty}$ (with the product topology), we can pass to a converging subsequence (without changing notation) with limit $(\alpha_1,\alpha_2,...)$. Now if for some d, $\alpha_d = 1$ on a set $S \subseteq D$ with $m_c(S) > 0$, then it must be the case that $h_c(\bar{x}_i^A(t),t) = \bar{h}_c(\bar{x}_i^A(t),t)$ for $t \in S$, a contradiction. Hence for each d, $\alpha_d = 1$ only on a set of measure zero. Also, $\sum_{d=1}^{\infty} \alpha_d = 1$, so there exists $\bar{d}$ and $V \in \mathscr{B}$ such that $\alpha_{\bar{d}}(t) > 0$ for $t \in V$, $m_c(V) > 0$. For some $\epsilon > 0$, there exists $W \subseteq V$, $m_c(W) > 0$, with $1-\epsilon > \alpha_{\bar{d}}(t) > \epsilon$ a.s. ($t \in W$). Let $\alpha \equiv \epsilon$, and let $z_1^n(t) \equiv \alpha_{\bar{d}}^n(t) \cdot w_{\bar{d}}^n(t)/\alpha$ for $t \in W$, $z_1^n(t) \equiv 0$ otherwise. Let $z_2^n(t) \equiv \sum_{d \neq \bar{d}} \alpha_d^n(t) \cdot w_d^n(t)/[1-\alpha]$ for $t \in W$, $z_2^n(t) \equiv 0$ otherwise. Now $\{z_1^n\}_{n=1}^{\infty}$ and $\{z_2^n\}_{n=1}^{\infty}$ are norm bounded sequences in $(L^p)^k$ (for otherwise $\|\bar{x}_i^A\|$ is infinite), so by the Banach Alaoglu theorem, we can pass to weak* convergent subsequences (without changing notation) with limits z_1 and z_2.

Without loss of generality, suppose $\alpha < 1/2$. Let $z_1' \equiv 2\alpha \cdot z_1 + (1-2\alpha) \cdot z_2$, and let $z_2' \equiv z_2$. Then $z_1' \neq z_2'$, with $z_1'(t) = z_2'(t) = 0$ for $t \in T \backslash W$ a.s., and $(1/2) \cdot z_1'(t) + (1/2) \cdot z_2'(t) = \bar{x}_i^A(t)$ for $t \in W$ a.s. and $\bar{h}_c(\bar{x}_i^A(t),t) = (1/2) \cdot \bar{h}_c(z_1'(t),t) + (1/2) \cdot \bar{h}_c(z_2'(t),t)$ for $t \in W$ a.s. A symmetric argument works for $\alpha > 1/2$. Then z_1' and z_2' have the desired properties.

Define a bounded, additive set function μ on $(T,\mathscr{J})$ by $\mu(Z) = \bar{\pi}(1_Z \cdot \bar{x}_i^A)$. Next we show that μ is a nonatomic measure on $(T,\mathscr{J})$.[4] The argument to show that μ is countably additive is parallel to the proof in Bewley (1972, Theorem 2). If μ is not

[4]This part of the proof need only be used if $p = \infty$.

countably additive on T, then there exists an increasing sequence of sets $F_j \in \mathcal{T}$ such that $\forall$ j, $\mu(F_j) < \mu(\bigcup_{s=1}^{\infty} F_s) - \epsilon$, where $\mu(T) > \epsilon > 0$. Let $E_j = F_j \cup (T \setminus \bigcup_{s=1}^{\infty} F_s)$, where $\bigcup_{s=1}^{\infty} E_s = T$ and $\setminus$ means set–theoretic subtraction. So $\forall$ j, $\mu(E_j) < \mu(T) - \epsilon$. Let $v_j^A(t) \equiv \bar{x}_i^A(t) \cdot 1_{E_j}(t) \in (L^p)^k$ and $v_j \equiv (v_j^A, \bar{x}_i^B) + \epsilon \cdot \nu_i / \bar{\pi}(\nu_i) \in L$, where ν_i is the extremely desirable vector. Then $\bar{\pi}(v_j) \leq \bar{\pi}(\bar{x}_i)$, and for large j, $\bar{u}_i(v_j) > \bar{u}_i(\bar{x}_i)$, a contradiction, so μ is a measure. Next, suppose that μ has an atom at $t^* \in T$. Hence $\mu(\{t^*\}) > 0$. Let $v^A(t) \equiv \bar{x}_i^A(t)$ for $t \neq t^*$, $v^A(t^*) \equiv 0$. Let $v \equiv (v^A, \bar{x}_i^B) + \mu(\{t^*\}) / \bar{\pi}(\nu_i) \cdot \nu_i$. Then $\bar{\pi}(v) = \bar{\pi}(\bar{x}_i)$ but $\bar{u}_i(v) > \bar{u}_i(\bar{x}_i)$, a contradiction.

Notice next that the integrals $\{\int \bar{h}_1(z_1'(t),t)\, dm_1, \ldots, \int \bar{h}_C(z_1'(t),t)\, dm_C; \int \bar{h}_1(z_2'(t),t)\, dm_1, \ldots, \int \bar{h}_C(z_2'(t),t)\, dm_C; \int \beta_1(t) \cdot z_1'(t)\, d\lambda_1, \ldots, \int \beta_G(t) \cdot z_1'(t)\, d\lambda_G; \int \beta_1(t) \cdot z_2'(t)\, d\lambda_1, \ldots, \int \beta_G(t) \cdot z_2'(t)\, d\lambda_G\}$, along with μ, form a finite set of nonatomic measures on $(T, \mathcal{T})$. In particular, they are nonatomic when restricted to the σ–algebra of measurable sets contained in W, $(W, \mathcal{W})$.

Using Lyapunov's theorem on the vector measure, there are disjoint sets $V, V' \in \mathcal{W}$ such that $\mu(V) = \mu(V') = (1/2) \cdot \mu(W)$, $\int_V \bar{h}_c(z_1'(t),t)\, dm_c = \int_{V'} \bar{h}_c(z_1'(t),t)\, dm_c = (1/2) \cdot \int_W \bar{h}_c(z_1'(t),t)\, dm_c$ for each c, $\int_V \bar{h}_c(z_2'(t),t)\, dm_c = \int_{V'} \bar{h}_c(z_2'(t),t)\, dm_c = (1/2) \cdot \int_W \bar{h}_c(z_2'(t),t)\, dm_c$ for each c, $\int_V \beta_g(t) \cdot z_1'(t)\, d\lambda_g = \int_{V'} \beta_g(t) \cdot z_1'(t)\, d\lambda_g = (1/2) \cdot \int_W \beta_g(t) \cdot z_1'(t)\, d\lambda_g$ for each g, $\int_V \beta_g(t) \cdot z_2'(t)\, d\lambda_g = \int_{V'} \beta_g(t) \cdot z_2'(t)\, d\lambda_g = (1/2) \cdot \int_W \beta_g(t) \cdot z_2'(t)\, d\lambda_g$ for each g. Now we can construct two allocations that yield the equilibrium allocation as a convex combination. Let $\hat{x}_i^A \in (L^p)^k$, $\hat{x}_i^A(t) = \bar{x}_i^A$ for $t \notin W$, $\hat{x}_i^A(t) = z_1'(t)$ if $t \in V$, $\hat{x}_i^A(t) = z_2'(t)$ if $t \in V'$. Let $\hat{x}_i \equiv (\hat{x}_i^A, \bar{x}_i^B)$. Let the new production plan of firm 1 be $\hat{y}_1 \equiv (\hat{y}_1^A, \bar{y}_1^B)$, where $\hat{y}_1^A \equiv \bar{y}_1^A + \hat{x}_i^A - \bar{x}_i^A$. All other agents get the same bundle or production plan in this new allocation

$[(\hat{x}_i),(\hat{y}_i)]$. Let $\tilde{x}_i^A \in (L^p)^k$, $\tilde{x}_i^A(t) = \bar{x}_i^A$ for $t \notin W$, $\tilde{x}_i^A(t) = z_2'(t)$ if $t \in V$, $\tilde{x}_i^A(t) = z_1'(t)$ if $t \in V'$. Let $\tilde{x}_i \equiv (\tilde{x}_i^A, \bar{x}_i^B)$. Let the new production plan of firm 1 be $\tilde{y}_1 \equiv (\tilde{y}_1^A, \bar{y}_1^B)$, where $\tilde{y}_1^A \equiv \bar{y}_1^A + \tilde{x}_i^A - \bar{x}_i^A$. All other agents get the same bundle or production plan in this new allocation $[(\tilde{x}_i),(\tilde{y}_i)]$ as in the equilibrium allocation $[(\bar{x}_i),(\bar{y}_j)]$. Given the form of firm 1's production set, both $\hat{y}_1$ and $\tilde{y}_1$ are in the production set since $\bar{y}_1$ is. Notice that for consumer i, $\bar{u}_i(\bar{x}_i) = \bar{u}_i(\hat{x}_i) = \bar{u}_i(\tilde{x}_i)$, $\bar{\pi}(\bar{x}_i) = \bar{\pi}(\hat{x}_i) = \bar{\pi}(\tilde{x}_i)$, and profits are the same under all 3 production plans for each producer. The material balance conditions also hold for all three allocations. Hence, all three are equilibrium allocations with respect to prices $\bar{\pi}$. Finally, $[(\bar{x}_i),(\bar{y}_j)] = 1/2\cdot[(\hat{x}_i),(\hat{y}_i)] + 1/2\cdot[(\tilde{x}_i),(\tilde{y}_i)]$, $[(\hat{x}_i),(\hat{y}_i)] \neq [(\tilde{x}_i),(\tilde{y}_i)]$. Hence $[(\bar{x}_i),(\bar{y}_j)]$ is not an extreme point of $E(\bar{\pi})$, a contradiction. So the hypothesis is false, and any extreme point of $E(\bar{\pi})$ has the property that for any consumer i, $u_i(\bar{x}_i) = \bar{u}_i(\bar{x}_i)$.

Finally, we claim that $[\bar{\pi},(\bar{x}_i),(\bar{y}_j)]$, where $[(\bar{x}_i^A),(\bar{y}_j^A)]$ is an extreme point of $E(\bar{\pi})$, is an equilibrium for the original economy. For each consumer i, any bundle x' with $u_i(x') > u_i(\bar{x}_i)$ also has $\bar{u}_i(x') \geq u_i(x') > u_i(\bar{x}_i) = \bar{u}_i(\bar{x}_i)$, so $\bar{\pi}(x') > \bar{\pi}(e_i) + \sum_{j=1}^{M} \theta_{ij}\bar{\pi}(\bar{y}_j)$. Feasibility and profit maximization follow from the fact that $[\bar{\pi},(\bar{x}_i),(\bar{y}_j)]$ is an equilibrium for the artificial economy.

Q.E.D.

Finally, notice that by standard arguments any equilibrium allocation of an economy satisfying the assumptions of section II can be shown to be Pareto optimal.

IV. Conclusion — Discussion and Potential Extensions

We have shown that sufficient substitutability and a large number of commodities can be used to alter or replace some convexity assumptions. This provides a method dual to the continuum (of agent) models for dealing with the problems created by nonconvexities in preferences and technologies. The hedonic interpretation of some of our assumptions might make our model useful for various kinds of applications.

It would be desirable to consider formally the large, finite economies that approximate our continuum of commodities model. This might yield more insight into how the convexifying effect of a large number of commodities works. We would expect that the Shapley–Folkman theorem could be used to obtain approximate equilibria for the finite economies.

Note that one case in which our assumption might be reasonable is when the set T represents a set of indivisible commodities. These will generally be the same for all agents. One would want each indivisible commodity to be consumed by exactly one person, and this happens in any equilibrium of such a model.

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