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THE REPLACEMENT PRINCIPLE IN PRIVATE GOOD ECONOMIES WITH SINGLE–PEAKED PREFERENCES

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Abstract

The objective of this paper is to understand the implications of the "replacement principle" for the fair allocation of an infinitely divisible commodity among agents with single–peaked preferences. A general formulation of the principle is as follows : when one of the components of the data entering the description of the problem to be solved changes, all of the relevant agents should be affected in the same direction. We apply it to situations when the preferences of one of the agents may change, under the name of *replacement–monotonicity.* We show that there is no *replacement–monotonic* selection from the envy–free and efficient solution. Then, we weaken the property by applying it only to situations in which the change in the preferences is not so disruptive that it turns the economy from one in which there is "too little" of the commodity to one in which there is "too much", or conversely. We show that there is only one selection from the envy–free and efficient solution satisfying this property of *one–sided replacement–monotonicity* and the standard condition of *replication–invariance.* It is a solution, known as the "uniform rule", that has played a central role in previous analyses of the problem.

Key–words. Replacement–monotonicity. Single–peaked preferences. No–envy. Uniform rule.

*1. **Introduction.*** The purpose of this paper is to study the implications for a simple resource allocation problem of a principle introduced in Thomson (1990) under the name of "replacement principle". The application is to the fair allocation of an infinitely divisible commodity in economies with single–peaked preferences, a problem whose axiomatic analysis was initiated by Sprumont (1991).

A decision problem is described in terms of certain "data". Some components of this data pertain to the individuals that are involved in the problem, such as their preferences and their endowments. Some components pertain to features of their general environment, such as the presence and the levels of public goods, the state of knowledge, or the production technology. With the specification of a class of problems to be analyzed comes the specification of the sets from which this data is taken. Essentially, the replacement principle states that changing the value of one of the components within its domain affects all "relevant" agents in the same direction : all gain or all lose as a result of the replacement. It expresses a strong form of solidarity among agents. Consider a change that is beneficial to the group as a whole, in the sense that a pareto improvement over the welfare levels initially chosen can be achieved, but suppose that no one in particular deserves any credit for the change. Then, the principle, when used in conjunction with efficiency, says that all relevant agents gain. On the other hand, consider a change that is hurtful to the group, in the sense that the welfare levels initially chosen are not jointly feasible anymore, and here suppose that no one in particular is to blame for the change. Then, the principle, again when used in conjunction with efficiency, implies that all relevant agents lose.

Here, we consider the replacement of the preferences of one agent with other preferences. We name the specific form taken by the principle in this application, "replacement–monotonicity". We investigate the implications of the condition for the allocation of an infinitely divisible private good in economies with single–peaked preferences. We are particularly interested in identifying efficient solutions to such problems satisfying the important distributional requirement of no–envy, which says that every agent should weakly prefer what he receives to what anyone else receives.

Our first results are mainly negative. First, of all of the solutions that have played a role in the literature on this topic, virtually none satisfies the property. More disapointing is our next result : there is no selection from the envy–free and efficient solution satisfying replacement–monotonicity.

We then observe that the violations of the property occur only when the change in the preferences causes the economy to turn from one in which there is "too much" of the commodity (that is, when the amount to divide is greater than the sum of the preferred consumptions), to one in which there is "too little" (this is when the sum of the preferred consumptions is greater than the amount to divide), or conversely. Of course, the relevance of this distinction is not surprising. It was pointed out by this author in previous analyses of this problem, devoted to the search for solutions satisfying certain monotonicity properties, with respect to the amount to divide on the one hand (Thomson, 1991a), and with respect to the number of agents on the other (Thomson, 1991b). Conditions ignoring the distinction were found to be quite demanding, but the weaker versions obtained by restricting their applications to changes in resources or populations that caused no such reversals could be met much more generally, while still retaining a wide range of relevance.

These observations motivate our introduction here of a weaker form of replacement–monotonicity defined by similarly limiting its application to situations in which such disruptions do not occur. We name this property "one–sided replacement–monotonicity". We discover a number of solutions satisfying the condition. Among them, however, only one meets the no–envy requirement. It is known in the literature as the "uniform rule". Then, we ask whether there are other one–sided replacement–monotonic selections from the envy–free and efficient solution other than the uniform rule. The answer is no if the mild requirement of replication–invariance is imposed too. This is the requirement that if an allocation is chosen for an economy, then, for any order of replication, the replica of that allocation is chosen for the replica economy. Our main result is a characterization of the uniform rule on the basis of these properties.

We also investigate the compatibility of one–sided replacement–monotonicity with the alternative distributional requirement of individual rationality from equal division, which says that each agent should weakly prefer what he receives to equal division. Here, we find that one–sided replacement–monotonicity is much less restrictive : a large number of efficient solutions are available. In fact, on an interesting subdomain of our primary domain, the requirement is also compatible with replacement–monotonicity itself.

Our analysis confirms the central role played by the uniform rule in solving the problem of fair division in economies with single–peaked preferences. It also shows that the replacement principle has enough power so that sometimes its implications can be fully described, and it illustrates the fact that one should be ready to formulate variants of the principle for each particular model. Finally, it invites its examination in other models.

2. The model. An amount $M \in \mathbb{R}_+$ of some infinitely divisible commodity has to be allocated among a set $N = \{1,\ldots,n\}$ of agents, indexed by i, each agent i being equipped with a continuous preference relation R_i defined over $[0,M]$. Let P_i denote the strict preference relation associated with R_i, and I_i the indifference relation. These preference relations are ***single–peaked***: for each R_i, there is a number $p(R_i) \in [0,M]$ such that for all $x_i, x_i' \in [0,M]$, if $x_i' < x_i \leqq p(R_i)$, or $p(R_i) \leqq x_i < x_i'$, then $x_i P_i x_i'$. The preference relation R_i can be described in terms of the function $\mathbf{r}_i$: $[0,M] \to [0,M]$ defined as follows: given $x_i \leqq p(R_i)$, $r_i(x_i) \geqq p(R_i)$ and $x_i I_i r_i(x_i)$ if such a number exists and $r_i(x_i) = M$ otherwise; given $x_i \geqq p(R_i)$, $r_i(x_i) \leqq p(R_i)$ and $x_i I_i r_i(x_i)$ if such a number exists and $r_i(x_i) = 0$ otherwise. (The number $r_i(x_i)$ is the consumption on the other side of agent i's preferred consumption that he finds indifferent to x_i, if such a consumption exists; it is 0 or M otherwise.). Let $\mathscr{R}$ be the class of all such preference relations. We write $R = (R_i)_{i \in N}$ and $p(R) = (p(R_i))_{i \in N}$. An ***economy*** is a pair $(R,M) \in \mathscr{R}^n \times \mathbb{R}_+$.

A ***feasible allocation for (R,M)*** $\in \mathscr{R}^n \times \mathbb{R}_+$ is a vector $x = (x_i)_{i \in N} \in \mathbb{R}^n_+$ such that $\Sigma x_i = M$.[1] Note that free disposal of the commodity is not assumed. Let ***X(M)*** be the set of feasible allocations of (R,M).

We wish to achieve equitable distributions. A ***solution*** is a mapping $\varphi : \mathscr{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n_+$ which associates with each economy $(R,M) \in \mathscr{R}^n \times \mathbb{R}_+$ a non–empty subset of X(M), denoted by $\varphi(R,M)$. Each of the alternatives in this subset is interpreted as a recommendation for that economy. We will consider multi–valued as well as single–valued solutions. Of course, we would prefer being able to identify well–behaved single–valued solutions since such solutions make precise recommendations.

Motivations for this model can be found in Sprumont (1991) and Thomson (1990) and we refer the reader to these papers for details. Its main application is to the problem of allocating a task among the members of a team; in many relevant cases, it is very natural to assume that the enjoyment of an activity increases up to a "satiation point" and decreases after that point. When the activity has to be completed, how should it be divided ?

Contrarily to many other models, it is relatively easy here to define appealing single–valued solutions. However, we start by introducing two fundamental multi–valued solutions, one of which embodies the standard notion of efficiency and the other an important concept of fairness.

First is the solution that associates with each economy its set of feasible allocations x such that there is no other feasible allocation that all agents weakly prefer to x, and at least one agent strictly prefers to x.

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[1]When the bounds of summation are not indicated, the summation should be understood to be carried out over the entire set of agents.

Pareto solution, P: $x \in P(R,M)$ if $x \in X(M)$ and there is no $x' \in X(M)$ with $x_i' R_i x_i$ for all $i \in N$ and $x_i' P_i x_i$ for some $i \in N$.

It is easy to see, as noted by Sprumont (1991), that $x \in P(R,M)$ if and only if (i) $x_i \leqq p(R_i)$ for all $i \in N$ when $M \leqq \Sigma p(R_i)$, and (ii) $x_i \geqq p(R_i)$ for all $i \in N$ when $M \geqq \Sigma p(R_i)$.

From the viewpoint of distribution, the pareto solution is of course very unsatisfactory. We will be interested in solutions satisfying some distributional requirement. Our main such requirement is that every agent weakly prefers what he receives to what any other agent receives. We refer to the solution that associates with each economy its set of such "envy-free" allocations as the "the no-envy solution".

No-envy solution, F (Foley, 1967) : $x \in F(R,M)$ if $x \in X(M)$ and for all $i, j \in N$, $x_i R_i x_j$.

We will also makes some remarks on the requirement that every agent weakly prefers what he receives to equal division.

Individually rational solution from equal division, I_{ed}: $x \in I_{ed}(R,M)$ if $x \in X(M)$ and $x_i R_i (M/n)$ for all $i \in N$.

These two distributional requirements are compatible. An example of a subsolution of the pareto solution satisfying both is the uniform rule (Benassy, 1982).

Uniform rule, U: $x = U(R,M)$ if $x \in X(M)$ and (i) when $M \leqq \Sigma p(R_i)$, $x_i = \min\{p(R_i),\lambda\}$ for all $i \in N$, where λ solves $\Sigma\min\{p(R_i),\lambda\} = M$, and (ii) when $M > \Sigma p(R_i)$, $x_i = \max\{p(R_i),\lambda\}$ for all i, where λ solves $\Sigma\max\{p(R_i),\lambda\} = M$.

The uniform rule has been characterized by Sprumont (1991) on the basis of strategy-proofness (see also Ching, 1992), and by Thomson (1990, 1991a,b) on the basis of a variety of consistency and monotonicity conditions, conditions to which we will come back, as they turn out to have relevance to the analysis undertaken here.[2]

[2]For other studies in which the uniform rule is analyzed, see Gensemer, Hong and Kelly (1992a,b).

*3. **Replacement–monotonicity***. To introduce the replacement principle, it is best to start from the various monotonicity conditions that have been recently the object of much attention. These conditions usually pertain to two economies that differ only in the specification of one of the parameters entering in their description, this parameter being taken from a space endowed with an order structure. The condition involves comparing the recommendations made by the solution for two values of the parameter that can be compared according to the order. In many applications, the change in the parameter unambiguously leads to an "expansion of opportunities" and the monotonicity requirement is that all "relevant" agents benefit from the change (the qualification "relevant" is explained later); in some other situations, it causes a "restriction of opportunities" and the requirement is that all relevant agents lose.

Examples of such conditions are (i) in the context of abstract bargaining theory, "strong monotonicity" (Kalai, 1977), the requirement that if the feasible set expands, all agents gain (the parameter here is the feasible set itself, which is specified as a subset of utility space, feasible sets being ordered by set inclusion), (ii) in the context of a variety of abstract game theoretical models and concrete allocation models, "population–monotonicity" (Thomson, 1983), which says that the arrival of additional agents should cause all agents initially present to lose (here, the parameter is the number of agents, and the order is the natural ordering of $\mathbb{N}$), (iii) in the context of concrete allocation models, "resource–monotonicity" (Roemer, 1986, Chun and Thomson, 1988; Moulin and Thomson, 1988), which says that all agents benefit from an increase in the resources available (here, the parameter is the vector of resources and the order is the vector ordering of an ℓ–dimensional commodity space), (iv) in the context of production economies, "technological–monotonicity" (Moulin, 1986; Moulin and Roemer, 1986), which says that all agents benefit from an improvement in the technology (technologies are specified as subsets of an ℓ–dimensional commodity space and they are ordered by set inclusion).

We argue in Thomson (1992b) in the context of a variable population, but the idea is applicable beyond that context, that it is really because we have in mind efficiency that, when we know that a pareto improvement is possible, we insist that all agents gain, and when we know that a pareto improvement is not possible, that they all lose. However, in order to keep considerations of efficiency separate from considerations of monotonicity, we suggest there that solidarity among agents always be expressed in the more general form, that "agents be affected in the same direction". Together with efficiency, whether agents gain or lose will often be unambiguous.

In more general situations, the change in the parameter may lead to an expansion of opportunities but it may alternatively produce a restriction of opportunities, and the natural monotonicity condition is that all relevant agents be affected "in the same direction". In bargaining theory, such a condition was considered by Thomson and Myerson (1980) when the feasible set changes, and in the theory of quasi–linear choice problems, it was used by Chun (1986) when population varies. Other contributions include Tadenuma and Thomson (1990) and Fleurbaey (1992), also in models with variable populations.

Our proposal here is very simple. We formulate a general requirement of monotonicity on the way the outcome is evaluated by agents even if the space from which the parameter that changes is drawn is not equipped with a natural order structure; or when it is, we propose to just ignore the order. In those general cases, it will usually be impossible to known how opportunities would be affected by the change, so we ask that all relevant agents be affected in the same direction.

Above, we spoke of "relevant" agents. This is because we want to be able to handle situations in which the number of agents itself changes, (one of the examples mentioned above is population–monotonicity : when new agents arrive, the relevant agents are the agents initially present; when some agents leave, the relevant agents are the remaining agents). In the present paper, the parameter on which we focus is the preferences of one of the agents; then the relevant agents are "all the others".

To summarize, the replacement principle will take the following form in the present study: when the preferences of one agent change, then all other agents are affected in the same direction.

The effects of a change in the preferences of one agent on the others has been studied in bargaining theory (Kihlstrom, Roth and Schmeidler, 1980), where preferences are ordered in terms of their risk aversion (one of the questions addressed by these authors is whether an increase in an agent's risk aversion necessarily benefits all other agents). In the context of resource allocation, we also note Fleurbaey (1992), whose main objective is the identification of useful orders on spaces of preferences on which to base the formulation of monotonicity conditions. Here, we do not attempt to define orders on the space of admissible preferences, although we recognize that in situations where the replacement principle might be too strong, weakenings of the requirement obtained by restricting its applications to replacements that can be evaluated according to some order might be quite useful.

Finally, we should observe that in cases when the parameter is taken from a one–dimensional euclidean space, and once the decision has been made to separate out efficiency considerations from monotonicity considerations, a monotonicity condition becomes equivalent to a replacement condition. For instance, in the context of the present paper, we studied the property that a change in the amount to divide affects all agents in the same direction (Thomson, 1991a). Using the terminology developed here, such a condition could be called "endowment replacement–monotonicity", and the condition that is the main subject of the present paper "preference replacement–monotonicity". In order to simplify our language, we will shorten the latter phrase to "replacement–monotonicity".

However, we would like to emphasize the wide applicability of the replacement operation and use the phrase "replacement principle" for the general idea, keeping in mind that in most applications, the form that should be given to the principle will probably have to take into account the special features of the model, giving rise to specific "replacement–monotonicity" conditions that would not be meaningful in other models. Also, variants of these conditions for each particular model might be worth formulating. In fact, for the current model, we will be led to introducing one such variant.

We are now ready for a formal introduction of the particular condition of replacement–monotonicity analyzed here. Given $R \in \mathscr{R}^n$, and $i \in N$, the notation R_{-i} designates the list obtained from R after the deletion of its i^{th} component R_i. Similarly, given $x \in \mathbb{R}^n$, the notation x_{-i} designates the vector obtained from x by deleting its i^{th} coordinate x_i.

Replacement–monotonicity. For all (R,M) and $(R',M) \in \mathscr{R}^n \times \mathbb{R}_+$, for all $i \in N$, if $R_{-i}=R'_{-i}$, then either $[\varphi_j(R,M) R_j \varphi_j(R',M)$ for all $j \in N\backslash\{i\}]$ or $[\varphi_j(R',M) R_j \varphi_j(R,M)$ for all $j \in N\backslash\{i\}]$.

A stronger form of the requirement is obtained by (i) imagining changes in the preferences of more than one agent, the conclusion being that all of the others be affected in the same direction. Another strengthening results by (ii) requiring that if strict preference holds for one agent in $N\backslash\{i\}$, it holds for all of them. (In the current model, the proviso that no agent be allocated zero would be necessary for the strengthening to have a chance of being satisfied). (iii) The two strengthenings could be imposed together.

For $n \leqq 2$, *replacement–monotonicity* has of course no force, and therefore, from here on, we assume that $n \geqq 3$. What are its implications ? First observe that the uniform rule does not satisfy it. The following very simple example illustrates this fact.

Example 1. Let N = {1,2,3} and $R \in \mathscr{R}^3$ be such that p(R) =(0,4,0). Let M = 6. Then U(R,M) = (1,4,1). Now, let $R' \in \mathscr{R}^3$ be such that $R'_{-3} = R_{-3}$, and $p(R'_3) = 4$. Then U(R′,M) = (0,3,3). As agent 3's preferences change from R_3 to R'_3, agent 1 gains and agent 2 loses, in violation of *replacement–monotonicity.*

Unfortunately, such violations extend much beyond the uniform rule. No efficient solution satisfying the no–envy requirement is immune.

Proposition 1. There is no *replacement–monotonic* selection from the envy–free and efficient solution.

Proof. Let $\varphi \subsetneq FP$. Let $N = \{1,2,3,4,5\}$ and $R \in \mathscr{R}^5$ be such that $x_1 I_1 (2-x_1)$ for all $x_1 \in [0,1]$, $p(R_2) = 1$ and $.9 I_2 20$, $x_3 I_3 (14-x_3)$ for all $x_3 \in [0,7]$, $p(R_4) = 7$ and $0 I_4 7.1$, $p(R_5) = 6$ and $5.9 I_5 20$. Let $M = 20$.

Let $x = \varphi(R,M)$. Since $\varphi \subsetneq P$ and $M < 22 = \Sigma p(R_i)$, it follows that (i) $x_i \leqq p(R_i)$ for all $i \in N$. By feasibility, for at least one $i \in N$, $x_i > 1$. Since $\varphi \subsetneq F$, agent 2 does not envy agent i at x, so that $x_2 \geqq .9$. From (i), the equality $p(R_1) = p(R_2)$, and the fact that agent 1 does not envy agent 2 at x, we obtain $x_1 = x_2$, so that $x_1 \geqq .9$ as well. Also, from (i), $x_1 \leqq 1$ and $x_2 \leqq 1$ so that there is at least $20 - 2\times 1 = 18$ to divide among the other three agents. Since by (i), $x_5 \leqq 6$, there is $i \in \{3,4\}$ such that $x_i \geqq 6$. Since $x_5 \leqq 6$ and agent 5 does not envy agent i at x, we obtain $x_5 \geqq 5.9$. Therefore, there is at most $20 - 2(.9) - 5.9 = 12.3$ to divide between agents 3 and 4. From (i), the equality $p(R_3) = p(R_4)$, and the fact that neither agent 3 nor agent 4 envies the other at x, we obtain $x_3 = x_4 \leqq 6.15$.

Now let $R' \in \mathscr{R}^5$ be such that $p(R'_5) = 2$ and $0 I'_5 2.1$, and $R'_{-5} = R_{-5}$. Let $y = \varphi(R',M)$. Since $\varphi \subsetneq P$ and $M > 18 = \Sigma p(R'_i)$, it follows that (ii) $y_i \geqq p(R'_i)$ for all $i \in N$. By feasibility, for at least one $i \in N$, $y_i < 7$. Since $\varphi \subsetneq F$, agent 4 does not envy agent i at y, so that $y_4 \leqq 7.1$. From (ii), the equality $p(R_3) = p(R_4)$, and the fact that agent 3 does not envy agent 4 at y, we obtain $y_3 = y_4$, so that $y_3 \leqq 7.1$ as well. Also, from (ii) $y_3 \geqq 7$ and $y_4 \geqq 7$ so that there is at most $20 - 2\times 7 = 6$ to divide among the other three agents. Since by (ii) $y_5 \geqq 2$, there is $i \in \{1,2\}$ such that $y_i \leqq 2$. Since $y_5 \geqq 2$ and agent 5 does not envy agent i at y, we obtain $y_5 \leqq 2.1$. Therefore, there is at least $20 - 2\times(7.1) - 2\times 1 = 3.7$ to divide between agents 1 and 2. From (ii), the equality $p(R_1) = p(R_2)$ and the fact that neither agent 1 nor agent 2 envies the other at y, we obtain $y_1 = y_2 \geqq 1.85$.

Now, it remains to observe that $x_1 P_1 y_1$ and $y_3 P_3 x_3$. As agent 5's preferences change from R_5 to R'_5, agent 1 loses and agent 2 gains, in violation of *replacement–monotonicity.*

Q.E.D.

The previous proposition shows that *replacement–monotonicity* is quite a strong condition when used in conjunction with efficiency and no–envy. What if no–envy were dropped ? A number of appealing solutions that do not satisfy the condition (Thomson, 1990) have been discussed in the literature, including the "proportional solution", which divides the good proportionally to the preferred consumptions, and the "equal–distance solution", which equates departures from preferred consumptions not proportionally, but unit per unit.

Proportional solution, Pro: $x = \text{Pro}(R,M)$ if $x \in X(M)$ and there exists $\lambda \in \mathbb{R}_+$ such that $x_i = \lambda p(R_i)$ for all $i \in N$; if no such λ exists, $x = (M/n,\ldots,M/n)$.

Equal–distance solution, Dis: $x = \text{Dis}(R,M)$ if $x \in X(M)$ and (i) when $M \leqq \Sigma p(R_i)$, there exists $d \geqq 0$ such that $x_i = \max\{0, p(R_i)-d\}$ for all $i \in N$, and (ii) when $M \geqq \Sigma p(R_i)$, there exists $d \geqq 0$ such that $x_i = p(R_i)+d$ for all $i \in N$.

The fact that these solutions are not *replacement–monotonic* is established by the following examples.

Example 2. Let $N = \{1,2,3\}$ and $R \in \mathscr{R}^3$ be such that $p(R_1) = 2$ and $r_1(1) = 3$, $p(R_2) = 6$ and $r_2(5) = 9$ and $p(R_3) = 12$. Let $M = 15$. Then $\text{Pro}(R,M) = (1.5,4.5,9)$. Now, let $R' \in \mathscr{R}^3$ be such that $R'_{-3} = R_{-3}$, and $p(R'_3) = 2$. Then $\text{Pro}(R',M) = (3,9,3)$. As agent 3's preferences change from R_3 to R'_3, agent 1 loses and agent 2 gains, in violation of *replacement–monotonicity.*

Example 3. Let $N = \{1,2,3\}$ and $R \in \mathscr{R}^3$ be such that $p(R_1) = 2$ and $r_1(1) = 4$, $p(R_2) = 4$ and $r_2(3) = 4.5$, and $p(R_3) = 6$. Let $M = 9$. Then $\text{Dis}(R,M) = (1,3,5)$. Now, let $R' \in \mathscr{R}^3$ be such that $R'_{-3} = R_{-3}$, and $p(R'_3) = 0$. Then $\text{Dis}(R',M) =$

(3,5,1). As agent 3's preferences change from R_3 to R_3', agent 1 gains and agent 2 loses, in violation of *replacement–monotonicity.*

An explanation for these results is given by the following general impossibility. It uses the requirements that the solution depend only on preferred consumptions (a requirement satisfied by many solutions, including the uniform rule, the proportional and equal–distance solutions) and "anonymity" (a requirement that is also satisfied by all of these solutions), which says that the "names of agents not matter". For a formal statement, let Π^n be the class of permutations of order n. Given $\pi \in \Pi^n$, and given $(R,M) \in \mathscr{R}^n \times \mathbb{R}_+$, let $\pi(R)$ be the profile obtained from R by subjecting its components to the permutation π, and similarly, given $x \in \mathbb{R}_+$, let $\pi(x)$ be obtained from x by subjecting its components to the permutation π. We write the requirement for correspondences.

Anonymity. For all let $(R,M) \in \mathscr{R}^n \times \mathbb{R}_+$ and for all $x \in \varphi(R,M)$, for all $\pi \in \Pi^n$, $\pi(x) \in \varphi(\pi(R),M)$.

Proposition 2. There is no *replacement–monotonic* and *anonymous* selection from the pareto solution that depends only on preferred consumptions.

Proof. Let $\varphi \subsetneq P$ be a *replacement–monotonic* and *anonymous* solution that depends only on preferred consumptions. Let $N = \{1,2,3,4\}$ and $R \in \mathscr{R}^4$ be such that $p(R_1) = 0$, $p(R_2) = 5$, $6P_24$, $R_2 = R_3 = R_4$, and $M = 12$.

Let $x = \varphi(R,M)$. We have $\Sigma p(R_i) = 15 > M$ and since $\varphi \subsetneq P$, it follows that $x_i \leqq p(R_i)$ for all $i \in N$. In particular, $x_1 = 0$. Then, by *anonymity,* $x_2 = x_3 = x_4 = 4$. Altogether, $x = (0,4,4,4)$.

Now, let $R' \in \mathscr{R}^4$ be such that $p(R_4') = 0$ and $R'_{-4} = R_{-4}$. Let $y = \varphi(R',M)$. We have $\Sigma p(R_i') = 10 < M$, and since $\varphi \subsetneq P$, it follows that $y_i \geqq p(R_i')$ for all $i \in N$. By *anonymity,* $y_1 = y_4$ and $y_2 = y_3$. Therefore, y is of the form (2t, 6–2t, 6–2t, 2t) for $t \in [0,1/2]$. First, suppose $t > 0$. Then, as agent 4's preferences change from R_4 to R_4', agent 1 loses and agent 2 gains, in violation of *replacement–monotonicity.* If $t = 0$, consider the economies $\tilde{R}$ and $\tilde{R}'$, obtained from R and R′ by replacing R_3 by $\tilde{R}_3$ such that $4\tilde{P}_3 6$. Since φ depends only on preferred consumptions, $x = \varphi(\tilde{R})$ and $y = \varphi(\tilde{R}')$. When agent 4's preferences change from R_4 to R_4', however, agent 2 gains and agent 3 loses, in violation of *replacement–monotonicity.*

Q.E.D.

Some selections from the pareto solution do satisfy the property though provided the domain is appropriately restricted. Consider for example the following "egalitarian type" solution.

Equal–sacrifice solution, Sac: $x \in Sac(R,M)$ if $x \in X(M)$ and (i) when $M \leqq \Sigma p(R_i)$, there exists $\sigma \geqq 0$ such that $r_i(x_i) - x_i \leqq \sigma$ for all $i \in N$, strict inequality holding only if $x_i = 0$, and (ii) when $M \geqq \Sigma p(R_i)$, there exists $\sigma \geqq 0$ such that $x_i - r_i(x_i) = \sigma$ for all $i \in N$.

To see the need for a domain restriction, consider the next example.

Example 4. Let $N = \{1,2,3\}$ and $R \in \mathscr{R}^3$ be such that $0I_1 2$, $4I_2 8$ and $4.5I_2 7$, and $8I_3 12$. Let $M = 12$. Then $Sac(R,M) = (0,4,8)$ (at that allocation, agent 1's sacrifice is strictly smaller than agents 2 and 3's common sacrifice, but equality cannot be obtained since agent 1's consumption is zero.) Now, let $R' \in \mathscr{R}^3$ be such that $R'_{-3} = R_{-3}$ and $p(R_3') = 0$. Then, $Sac(R',M) = (2.5,7,2.5)$. As agent 3's preferences change from R_3 to R_3', agent 1 loses and agent 2 gains, in violation of *replacement–monotonicity.*

Now, consider the domain of economies such that $0I_i M$ for all $i \in N$. On this domain, the equal–sacrifice solution is *replacement–monotonic.* This should not be surprising since under that domain restriction, the solution happens to satisfy other useful monotonicity conditions, even when all other standard solutions fail to do so. The condition that is the most relevant in the present context is the condition mentioned earlier that all agents be affected in the same direction by changes in the amount to divide, a condition studied for this model by Thomson (1991a) under the name of *resource–monotonicity*. In the discussion of this property in the paragraphs below, we will assume, as it is convenient, that preferences are defined over $\mathbb{R}_+$, and instead of the domain restriction $0I_iM$ for all $i \in N$ we will require that $\lim_{x_i \to \infty} r_i(x_i) = 0$ for all $i \in N$. Moreover, the equal–sacrifice solution satisfies the property of *consistency* (Thomson, 1990). This property pertains to solutions defined on domains of economies of arbitrary cardinality. It says that if an allocation is chosen for some economy, then the restriction of that allocation to a subgroup would be chosen for the economy obtained by imagining all of the members of the complementary group to leave with their alloted consumptions. We show next that if a solution defined on a domain of economies of arbitrary cardinality is both *resource–monotonic* and *consistent*, then it is *replacement–monotonic*. Note that efficiency is not needed to derive this conclusion.

Lemma 1. If a solution defined on a domain of economies of arbitrary cardinality is *resource–monotonic* and *consistent*, then it is *replacement–monotonic*.

Proof. Let φ be a *resource–monotonic* and *consistent* solution defined on a domain of economies of arbitrary cardinality. Let $N = \{1,2,...,n\}$, $i \in N$, and $R, R' \in \mathscr{R}^n$ with $R_{-i} = R'_{-i}$. Finally, let $M \in \mathbb{R}_+$. Let $x = \varphi(R,M)$ and $x' = \varphi(R',M)$. We want to show that either $x_jR_jx'_j$ for all $j \in N\backslash\{i\}$ or $x'_jR_jx_j$ for all $j \in N\backslash\{i\}$. Since φ is *consistent*, $x_{-i} = \varphi(R_{-i}, \Sigma_{j\in N\backslash\{i\}} x_j)$ and $x'_{-i} = \varphi(R'_{-i}, \Sigma_{j\in N\backslash\{i\}} x'_j) = \varphi(R_{-i}, \Sigma_{j\in N\backslash\{i\}} x'_j)$ The two economies $(R_{-i}, \Sigma_{j\in N\backslash\{i\}} x_j)$ and $(R_{-i}, \Sigma_{j\in N\backslash\{i\}} x'_j)$ differ only in the amount to divide, and since φ is *resource–monotonic*, the desired conclusion follows.

Q.E.D.

Lemma 1 implies that the equal–sacrifice solution is not the only selection from the pareto solution to be *replacement–monotonic*. Indeed, if follows from Thomson (1990, 1991a) that there is a large class of solutions satisfying the two properties of *resource–monotonicity* and *consistency* provided appropriate domain restrictions are made, for instance that $\lim_{x_i \to \infty} r_i(x_i) = 0$ for all $i \in N$, as discussed above.

It is interesting to note, in order to gain further understanding of the logical relation between all of the conditions mentioned above that the equal–division rule, which is neither *resource–monotonic* nor *population–monotonic* (see Thomson, 1991a,b, for simple examples illustrating these facts), is obviously *replacement–monotonic* (and equally obviously, not efficient). This example also shows that the requirement that the solution be a subsolution of the pareto solution cannot be dispensed with in the impossibilities of Propositions 1 and 2. This is in contrast with our earlier studies where we established the incompatibility of *resource–monotonicity* and *population–monotonicity* with the no–envy requirement itself (these incompatibilities hold even if efficiency is not imposed).

The distributional requirement of individual rationality from equal division is less restrictive than no–envy. A particularly simple example of a *replacement–monotonic* selection from the individually rational from equal division and efficient solution is obtained on the domain of economies $(R,M) \in \mathscr{R}^n \times \mathbb{R}_+$ such that for all $i \in N$, if $p(R_i) \leqq M/n$, then $M/n R_i 0$ and if $p(R_i) \geqq M/n$, then $M/n R_i M$. Then, if $M < \Sigma p(R_i)$ or if $\Sigma p(R_i) < M$, select $x \in I_{ed}P(R,M)$ such that the ratios $s_i(x_i) = [r_i(x_i) - x_i]/[M/n - x_i]$ be the same for all $i \in N$; if $M = \Sigma p(R_i)$, select $x = p(R)$.

*4. **One–sided replacement–monotonicity.*** In this section we propose a weakening of *replacement–monotonicity*, motivated by the observation that in the examples used to establish the negative results of Propositions 1 and 2, the change in the preferences that is considered has the effect of turning the economy from one in which there is too much of the commodity to one in which there is too little. In our earlier studies of the compatibility of *resource–monotonicity* and *population–monotonicity* with the basic distributional requirements of no–envy and individual rationality from equal division, we established negative results that could be attributed to exactly the same possibility, namely that variations in the parameter under investigation, the amount to divide on the one hand and the number of agents on the other, could turn the economy from one in which there is too much to divide to one in which there is too little. In each of these studies, we were naturally led to the weaker monotonicity condition obtained by limiting its scope to situations in which no such disruptions occurred. It is very appealing here to investigate a similar reformulation. We therefore propose to limit the range of application of *replacement–monotonicity* to cases where the amount to divide stays "on the same side" of the sum of the preferred consumptions, resulting in the following condition, again written for single–valued solutions.

One–sided replacement–monotonicity. For all (R,M) and (R′,M) $\in \mathcal{R}^n \times \mathbb{R}$, for all i $\in$ N, if $R_{-i}=R'_{-i}$, and either [$M \geqq \sum_{k\in N} p(R_k)$ and $M \geqq \sum_{k\in N} p(R'_k)$] or [$M \leqq \sum_{k\in N} p(R_k)$ and $M \leqq \sum_{k\in N} p(R'_k)$] then either [$\varphi_j(R,M) R_j \varphi_j(R',M)$ for all j $\in$ N\{i}] or [$\varphi_j(R',M) R_j \varphi_j(R,M)$ for all j $\in$ N\{i}].

Note that here too, we could formulate strenghtenings of the property by either (i) allowing groups of agents to change their preferences and, provided the direction of the inequality is not reversed, requiring all members of the complementary group to be affected in the same direction, or (ii) by requiring that if strict preference holds for one agent, then so it does for all agents (with the same proviso stated earlier in our discussion of a similar strengthening of *replacement–monotonicity*), or (iii) combining both strenghtenings.

It is easy to see that the uniform rule, which as noted earlier is not *replacement–monotonic*, does satisfy the weaker property of *one–sided replacement–monotonicity.* In fact, the requirement is now satisfied by many solutions, in particular by the other solutions introduced in the previous section, such as the proportional and equal–distance solutions. The equal–sacrifice solution is another example, without the domain restriction that we found necessary to guarantee its *replacement–monotonicity.* However, among all of them, only one satisfies the no–envy requirement, the uniform rule. Is the uniform rule the only selection from the envy–free and efficient solution to be *one–sided replacement–monotonic* ?

The answer is almost yes. It is not a simple "yes" because the characterization presented below also uses the requirement of "replication invariance", which says that if an allocation is recommended for some economy, then for any order of replication, the replicated allocation is also recommended for the replicated economy. But it is "almost" yes because this requirement is very mild, being satisfied by all of the solutions that we have mentioned so far. (The condition is also used by Thomson, 1991b, in a characterization of the uniform rule on the basis of *population–monotonicity*.). For a formal statement, we need an extra piece of notation. Given an economy $(R,M) \in \mathscr{R}^n \times \mathbb{R}_+$, and $k \in \mathbb{N}$, we denote by $k*(R,M)$ the economy made up of k agents identical to agent i, for each $i \in N$, and in which the amount to divide is kM. Given $x \in X(M)$, we similarly denote by $k*x$ the allocation obtained by k–times replication of x. In the formulation below, and because we want to emphasize the fact that it is not only the single–valued solutions discussed above that are *replication–invariant*, we revert to a consideration of multi–valued solutions. Also, note that for the condition to make sense, we need to generalize the notion of a solution by requiring that it be defined on the replicated domain.

Replication–invariance. For all $(R,M) \in \mathscr{R}^n \times \mathbb{R}_+$, for all $k \in \mathbb{N}$, for all $x \in \varphi(R,M)$, $k*x \in \varphi(k*(R,M))$.

We are now ready for the main characterization of this paper.

Theorem 1. The uniform rule is the only *replication–invariant* and *replacement–monotonic* selection from the envy–free and efficient solution.

Proof. Let $\varphi \subseteq FP$ be a *replication–invariant* and *replacement–monotonic* solution. Let $(R,M) \in \mathscr{R}^n \times \mathbb{R}_+$ and $x = \varphi(R,M)$. Suppose, by contradiction, that $x \neq U(R,M)$, and without loss of generality, that $M \leqq \Sigma p(R_i)$. Since $\varphi \subseteq P$, this is possible only if $M < \Sigma p(R_i)$. Since in fact, $\varphi \subseteq FP$, there are two agents in N – let them be denoted i and j – such that $x_i < r_i(x_i) \leqq x_j \leqq p(R_j)$.

Let $k \in \mathbb{N}$ be such that $k(\Sigma p(R_i)-M) \geqq M$ and $k((p(R_i)-x_i)/2) > M$. Let e be the economy obtained by k times replication of (R,M). By *replication–invariance*, $k*x =$

$\varphi(e)$. Let $\ell \in N\backslash\{i,j\}$. We now select one of the agents of type ℓ – let him be be indexed by ℓ^* – and we change his preferences from $R_{\ell^*} = R_\ell$ to $R'_{\ell^*} \in \mathcal{R}$ with the following properties:

(i) $p(R'_{\ell^*}) = p(R_i)$

(ii) $(x_i+p(R_i))/2I'_{\ell^*}p(R_j)$

Let e′ be the economy obtained from e by replacing R_{ℓ^*} by R'_{ℓ^*}. Note that by the choice of k,

$$k \sum_{m \in N\backslash\{\ell\}} p(R_m) + (k-1)p(R_\ell) + p(R'_{\ell^*}) \geqq kM,$$

so that *one–sided replacement–monotonicity* applies to the pair {e,e′}.

Let $y = \varphi(e')$. Since $\varphi \subsetneq FP$, any two agents of the same type receive the same amount at y. With a slight abuse of notation, for all $m \in N\backslash\{\ell\}$, we denote by y_m the common consumption of all agents of type m. Note that there are only k–1 identical agents of type ℓ. Their common consumption is denoted by y_ℓ. Agent ℓ^*'s consumption in e′ is denoted by y_{ℓ^*}.

Next, we claim that $y_j \geqq r_i(x_i)$. Suppose not. We will distinguish two cases. (a) If $y_j \in \,]x_i,r_i(x_i)[$, it follows that as agent ℓ^*'s preferences change from R_{ℓ^*} to R'_{ℓ^*}, all agents of type j lose, and by *one–sided replacement–monotonicity*, the agents of type i should lose too. This, and the inclusion $\varphi \subsetneq P$, implies that $y_i \leqq x_i$. But then, the agents of type i envy the agents of type j at y, in contradiction with $y \in F(e')$.

(b) The second case is $y_j \leqq x_i$. As agent ℓ^*'s preferences change, all agents of type j lose and as before, by *one–sided replacement–monotonicity*, apart from agent ℓ^*, so should all other agents. Therefore, and since $\varphi \subsetneq P$, apart from agent ℓ^*, all agents consume less at y than at x, and in particular the consumption of each of the k agents of type j decreases by at least $r_i(x_i)-x_i$. This means that agent ℓ^*'s consumption increases by at least $k(r_i(x_i)-x_i)$. Then, by the choice of k, we obtain $y_{\ell^*} \geqq x_{\ell^*} + k(r_i(x_i) - x_i) \geqq k(r_i(x_i) - x_i) > p(R'_{\ell^*}) = p(R_i)$, in violation of $\varphi \subsetneq P$, which requires $y_{\ell^*} \leqq p(R'_{\ell^*})$.

We have now concluded that $y_j \geqq r_i(x_i) \geqq p(R'_{\ell^*})$. Since $\varphi \subsetneq P$, $y_j \leqq p(R_j)$, so that altogether we obtain $y_j \in [p(R'_{\ell^*}),p(R_j)]$. Again, since $\varphi \subsetneq P$, $y_{\ell^*} \leqq p(R'_{\ell^*})$. Since R'_{ℓ^*} satisfies (ii), and $\varphi \subsetneq F$, agent ℓ^* does not envy the agents of type j at y only if $y_{\ell^*} \in [(x_i+p(R_i))/2,p(R_i)]$.

Consequently, and since $\varphi \subsetneq FP$, we obtain $y_i \in [(x_i+p(R_i))/2,p(R_i)]$ also. This shows that as agent ℓ^*'s preferences change from R_{ℓ^*} to R'_{ℓ^*}, the agents of type i gain. By *one–sided replacement–monotonicity*, apart from agent ℓ^*, so should all other agents. Since $\varphi \subsetneq P$, it follows that apart from agent ℓ^*, every agent consumes more as a result of the change. Moreover, the consumption of each of the k agents of type i increases by at least $(p(R_i)-x_i)/2$. This means that agent ℓ^*'s consumption decreases by at least $k(p(R_i)-x_i)/2$. Then, by the choice of k, we obtain $y_{\ell^*} \leqq x_{\ell^*} - k(p(R_i)-x_i)/2 \leqq M - k(p(R_i)-x_i)/2 < 0$, which is impossible.

Q.E.D.

An open question is whether *replication–invariance* can be dispensed with in this characterization.[3] If no–envy is dropped, many solutions can be obtained by adapting Lemma 1. Indeed, it can be proved in virtually identical terms that any subsolution of the pareto solution defined on a class of economies of arbitrary cardinality that is *one–sided resource–monotonic* (this is the property obtained from *resource–monotonicity* by limiting its range of application to changes in resources that do not reverse the direction of the inequality between the amount to divide and the sum of the preferred amounts) and *consistent*, is in fact *one–sided replacement–monotonic*. (But note that efficiency was not needed in Lemma 1).

[3]This question may not be easy to answer. In our characterization of the uniform rule on the basis of *population–monotonicity*, we also used the condition without being able to show that it was independent of the other axioms. However, as we pointed out earlier, the axiom is indeed very weak.

Lemma 2. If a subsolution of the pareto solution defined on a domain of economies of varying cardinality is *one–sided resource–monotonic* and *consistent*, then it is *one–sided replacement–monotonic.*

We close with a discussion of whether there are subsolutions of the pareto solution other than the uniform rule satisfying *one–sided replacement–monotonicity* and individual rationality from equal division (instead of no–envy). Simple examples can be constructed to show that the proportional, equal–distance, and equal–sacrifice solutions do not satisfy individual rationality from equal division (Thomson, 1990), and they are readily disqualified. However the uniform rule does select individually rational allocations from equal division and it is *one–sided replacement–monotonic.* Therefore, a legitimate question is whether there are other solutions with these properties.

The answer is yes. Indeed, a large class of such solutions exist. Given any *one–sided replacement–monotonic* and *one–sided resource–monotonic* selection from the pareto solution, ψ, let φ be defined as follows. Let $(R,M) \in \mathcal{R}^n \times \mathbb{R}_+$ and suppose first that $M \leqq \Sigma p(R_i)$. Then, given $i \in N$, let $x_i^* = r_i(M/n)$ if $p(R_i) \leqq M/n$ and $x_i^* = M/n$ otherwise. Then, let $\varphi_i(R,M) = x_i^* + \psi(R',M - \Sigma x_i^*)$ for all $i \in N$, where R' is a profile of preferences such that for all $i \in N$, R_i' coincides over the interval $[0,M - x_i^*]$ with R_i translated to the left by the amount x_i^*; The construction is extended to the case $M > \Sigma p(R_i)$ by symmetry, by first giving each agent am amount x_i^* similarly defined and then subtracting quantities calculated by applying ψ so as to obtain feasibility.

This result confirms the fact, already noted in our earlier search for *one–sided resource–monotonic* and *one–sided population–monotonic* subsolutions of the pareto solution that the no–envy requirement gives us less leeway than individual rationality from equal division (in the case of *consistency*, we found these two distributional requirements to have the same consequences).

4. Conclusion. We investigated the implications of a condition of *replacement–monotonicity* for the problem of fair division when preferences are single–peaked. We found this condition to be quite restrictive and in particular not to be satisfied by any selection from what is perhaps the main solution to this problem, the no–envy solution. We were then led to formulating a weaker version of the condition, tailored to the specific features of the model. Together with the basic requirement of efficiency and no–envy, we discovered this weaker condition of *one–sided replacement–monotonicity* to be satisfied by only one *replication–invariant* selection from the pareto solution, the uniform rule. Since this solution has come up repeatedly in earlier studies of the problem, this is of course cause for celebration.

Moreover, and perhaps more importantly, the current paper shows that the replacement principle is an operationally useful condition and suggests that it may be worthwhile to study it in other contexts. In a companion paper (Thomson, 1992a) pertaining to the optimal choice of a public good level, we have shown that the implications of the principle can also be fully described. We are currently engaged in its study in other models.

Our present results are summarized in the following table.

	Individual rationality from equal division	No–envy	*Replacement–monotonicity*	*One–sided replacement–monotonicity*
Uniform rule	yes	yes	no (Ex 1, Prop 1) Prop 2	yes
Proportional solution	no	no	no (Ex 2)	yes
Equal–distance solution	no	no	no (Ex 3)	yes
Equal–sacrifice solution	no	no	yes*	yes
$\varphi \subsetneq I_{ed}P$	yes (by Def)	may or may not	yes**	such solutions exist
$\varphi \subsetneq FP$	may or may not	yes (by Def)	no (Prop 1)	only if φ=U(Th 1)***

*This positive result holds under appropriate domain restrictions. Otherwise, a negative result obtains (Example 4).

**This positive result holds on the domains of economies such that for each i, if $p(R_i) \leqq M/n$ then $M/n R_i 0$ and if $p(R_i) \geqq M/n$, then $M/n R_i M$.

***This characterization is proved under the additional assumption that the solution also satisfies *replication–invariance.*

Table 1.

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