Games of Fair Division

Tadenuma, Koichi and William Thomson

Working Paper No. 351
July 1993

Games of Fair Division

Koichi Tadenuma and William Thomson

Rochester Center for Economic Research
Working Paper No. 351

GAMES OF FAIR DIVISION

by

Koichi Tadenuma*

and

William Thomson**

May 1991
Revised, June 2, 1993

*Department of Economics, Hitotsubashi University, 2–1 Naka, Kunitachi, Tokyo 186, JAPAN

**Department of Economics, University of Rochester, Rochester, NY 14627, USA.
Thomson gratefully acknowledges the NSF for its support under grant #8809822. The comments of S. Nakamura and Y. Tatamitani are gratefully acknowledged. We are particularly grateful to B. Dutta and a referee for their detailed comments.

GAMES OF FAIR DIVISION

Abstract

We consider the problem of fairly allocating an indivisible good to one of several agents equally entitled to it when monetary compensations to the others are possible. Our primary normative concept is no–envy. First, we show that there is no non–manipulable selection from the no–envy solution. Then we study the direct revelation games associated with subsolutions of the no–envy solution. The set of equilibrium allocations of any one of them coincides with the set of envy–free allocations for the true preferences.

Keywords: Fair allocation. Indivisible goods. No–envy solution. Manipulation games. Nash implementation.

Koichi–Tadenuma
Department of Economics
Hitotsubashi University
2–1 Naka
Kunitachi
Tokyo 186 JAPAN

William Thomson
Department of Economics
University of Rochester
202 Harkness Hall
Rochester, NY 14627–0156

1. ***Introduction.*** It is now well understood that on many economic domains of interest, any allocation rule satisfying minimal efficiency and distributional requirements is subject to manipulation: configuration of preferences exist such that, assuming that all but one agent announce their preferences truthfully, the last agent can gain by misrepresenting his. A great variety of such results have been established.

However, the natural follow-up question is rarely asked: if a rule is manipulable, how manipulable is it? Assuming now all agents to behave strategically, how far will the rule take us from the allocations it would have selected under truthful behavior? It is widely believed that manipulation will often cause efficiency to fail, even if the rule was designed to select efficient allocations, as most rules of interest do, and that in addition the distributional objective the rule embodies will not be attained. But few studies of these issues have actually been undertaken. A notable exception is Hurwicz (1978), who studies the manipulability of the Walrasian rule. Thomson (1984, 1987, 1988) analyzes the manipulability of classes of rules satisfying efficiency and individual rationality, or equity, requirements. An important literature devoted to the manipulability of planning procedures, most of which focusing on the equilibria of "instantaneous" games played at each step of the procedure, should also be mentioned, as well as several contributions to the manipulability of solutions to the matching problem; references here are Roth (1984), Gale and Sotomayor (1985), and Zhou (1991b). The object of the present paper is to pursue this line of research.

Specifically, we consider the problem of fairly allocating a single indivisible good to one of several agents assumed to have equal rights on this good, monetary transfers being available to compensate the agents that do not receive the good. Our central normative concept is no-envy (Foley, 1967): an allocation is envy-free if no agent prefers the bundle of anyone else to his own. Since the set of envy-free allocations is often large, (just like the set of individually-rational allocations, in situations where an initial allocation is given) a problem of selection arises. This problem was addressed by Alkan, Demange and Gale (1991), Tadenuma and Thomson (1991a,b, 1993), and Aragones Alabart (1990) who proposed a variety of selections, advocating some of them on the basis of intuitive considerations of fairness, and deriving others from axiomatic considerations. Instead of having to study the manipulability of each possible selection, we will be able, however, to establish results that hold for ***all selections***.

Indeed, after showing that ***no subsolution of the no–envy solution is immune to manipulation***, we ask how manipulable such allocation rules are likely to be. We prove that all of them are ***equivalent under manipulation***: given any two such solutions, the sets of equilibrium allocations of their associated manipulation games are identical. Moreover, this (common) set of equilibrium allocations ***is the set of envy–free allocations for the true preferences!***

One negative conclusion to be drawn from our analysis is that the search for appealing selections from the no–envy solution is rendered totally moot by strategic behavior.

But on the positive side, our results indicate the need to go beyond the impossibility theorems that are common in the literature on strategy–proofness. These theorems only state that the list of truthful announcements is not an equilibrium of the direct revelation game associated with a given solution. When we ask, What are the equilibria, then? we find that in some situations, they can be identified and that the associated equilibrium allocations still bear an interesting relation to the allocations that were intended. Some, in fact the main, properties of the solution may well be preserved.

In order to establish these results, we have to develop equilibrium notions that are appropriate for games with an outcome correspondence (as opposed to function). In general, equilibrium can be defined in several ways for such games and one should expect the results to depend on which equilibrium notion is used. However, in the present situation, no ambiguity arises from having to work with an outcome correspondence.

2. ***The Model*** There are two goods. One is indivisible (job, house, contract) and can be attributed to only one agent. The other is infinitely divisible and can be used for compensations. We refer to it as "money". There is a set $N = \{1,...,n\}$ of agents. For each $i \in N$, the consumption space of agent i is the set of pairs $(\delta_i,m_i) \in \{0,1\}\times\mathbb{R}$: $(0,m_i)$ is the bundle containing only m_i units of money and $(1,m_i)$ is the bundle made up of the object together with m_i units of money. Note that no restriction in sign is imposed on the consumption of money. For each $i \in N$, agent i's preference ordering R_i (with associated indifference and strict preference relations denoted by I_i and P_i respectively) is assumed to be continuous and strictly monotonic in money and to satisfy the following "compensation assumption": for all δ_i, $\delta_i' \in \{0,1\}$, $\delta_i \neq \delta_i'$ and for all $m_i \in \mathbb{R}$, there is m_i' such that $(\delta_i,m_i)I_i(\delta_i',m_i')$. Let $\mathscr{R}$ be the class of all such preference orderings. Let M be the amount of money available. We assume M to be known and fixed. Therefore, an economy is completely specified by a list $R \in \mathscr{R}^n$. A ***feasible allocation*** is a list $z = (z_i)_{i\in N} = (\delta_i,m_i)_{i\in N} \in [\{0,1\}\times\mathbb{R}]^n$ such that $\sum_N (\delta_i,m_i) = (1,M)$: $\delta_i = 1$ if agent i is the "winner" and $\delta_i = 0$ if agent i is a "loser".[1] Let Z be the set of feasible allocations. ***A solution*** is a correspondence $\varphi: \mathscr{R}^n \to Z$ that associates with each $R \in \mathscr{R}^n$ a non-empty subset of Z. Each of the points in $\varphi(R)$ is interpreted as one desirable way of allocating the resources.

Let P be the ***Pareto solution***: $P(R) = \{z \in Z | \nexists\ z' \in Z \text{ with } [\forall\ i \in N,\ z_i'R_iz_i] \text{ and } [\exists\ i \in N \text{ s.t. } z_i'P_iz_i]\}$.

[1]This model has been considered by Luce and Raiffa (1957), Kolm (1972), Crawford and Heller (1979), van Damme (1987), Moulin (1989), and Tadenuma and Thomson (1993). A related model is obtained by assuming that there are arbitrary numbers of indivisible goods and agents. That model was analyzed by Svensson (1983, 1988), Maskin (1987), Alkan, Demange and Gale (1991), Tadenuma and Thomson (1991a,b), and Aragones Alabart (1990).

The fairness notion that we consider is well–known (Foley, 1967): $z \in Z$ is ***envy–free for*** $R \in \mathscr{R}^n$ if for all i, $j \in N$, $z_i R_i z_j$. Let $F(R)$ be the set of envy–free allocations of R.

Under the assumptions on preferences made above, there always exist envy–free allocations. (This follows from Alkan, Demange and Gale, 1991; a direct proof is given in Tadenuma and Thomson, 1993). Also, it is easy to see that any envy–free allocation is efficient. (This follows from Svensson, 1983). The following useful facts concerning the structure of the set of envy–free allocations are discussed in Tadenuma and Thomson (1993): this set can essentially (i.e. up to permutations of bundles leaving all agents indifferent) be parameterized by how much money the winner receives. There is indeed an interval $[\underline{w},\overline{w}]$ of amounts of money received by the winner at allocations in F(R). The allocation attributing $(1,\underline{w})$ to him and $(0,(M-\underline{w})/(n-1))$ to each of the losers is the worst for him and the best for the losers in that set, whereas the opposite holds for the allocation attributing $(1,\overline{w})$ to him and $(0,(M-\overline{w})/(n-1))$ to each of the losers. The former is obtained when the winner is indifferent between what he receives and what each of the losers receives. At the latter, (at least) one of the losers is indifferent between what he receives and what the winner receives. Given $z \in F(R)$, we sometimes designate by m_w the amount of money received by the winner at z, and by m_ℓ the amount of money received by each of the losers; we also write $z_w = (1,m_w)$ and $z_\ell = (0,m_\ell)$.

It follows from the above paragraph that for most economies, there is a continuum of (non Pareto–indifferent) envy–free allocations, and the problem of selection arises. A variety of approaches to this problem can be taken. As mentioned earlier, several authors have defined selections from the no–envy solution and motivated them on the basis of elementary considerations of fairness. In Tadenuma and Thomson (1993), we followed an axiomatic approach and looked for selections satisfying alternative sets of conditions. These conditions led to characterizations of a certain single–valued solution that will be discussed again in section 5. Here, however, we do not commit ourselves to a particular solution. Instead, we establish results that hold for arbitrary selections of the no–envy solution.

3. *The manipulability of solution correspondences.* Let φ: $\mathscr{R}^n \to Z$ be a solution. If φ is manipulable, we would like to ascertain the extent of its manipulability. We propose to do this by identifying the set of equilibrium allocations of a "manipulation game" associated with φ in a natural way. In this game, strategies are preference announcements and the outcome corresponding to a given list of announcements is obtained by applying φ itself. Such games are often called "direct revelation games". Our objective is to identify strategy combinations satisfying the usual best response property and to describe the resulting outcomes in terms of the true preferences. However, a difficulty immediately comes up: φ may not be single–valued. In fact, most solutions in economics are not single–valued. How should the notion of a "best response" be defined then?[2] There are of course some cases where φ is, if not single–valued, at least "essentially" single–valued: if two allocations are φ–optimal for some economy i.e. $z, z' \in \varphi(R)$ for some $R \in \mathscr{R}^n$, then they are Pareto–indifferent: $z_i I_i z_i'$ for all $i \in N$. However, Pareto–indifference of the elements of $\varphi(R)$ holds for the list of announced preferences R, but not in general for the list of true preferences. So, even then, the problem of multi–valuedness has to be confronted. Thomson (1984, 1987, 1988) studies the manipulability of solutions in classical exchange economies and proposes several extensions of the notion of a Nash equilibrium to deal with this problem. It turns out that in some interesting situations, only one equilibrium notion survives a natural test described in the next paragraph.[3] This will be the case here as

[2]Note that for simplicity, we still use the term "game" in spite of the multivaluedness of φ.

[3]Such a situation is described in Thomson, 1988.

well. Therefore we will define only that notion and provide a justification for its use in a lemma (Lemma 3).

Let $(R,z) \in \mathscr{R}^n \times Z$ with $z \in \varphi(R)$ be given. Given $R \in \mathscr{R}^n$, $i \in N$, and $R_i' \in \mathscr{R}$, the notation (R_i',R_{-i}) designates the list R after the replacement of its i^{th} component R_i by R_i'. Say that ***(R,z) is a Nash equilibrium of the game $(\mathscr{R}^n,\varphi)$ played in $R^0 \in \mathscr{R}^n$*** if for all $i \in N$, for all $R_i' \in \mathscr{R}$ with $R_i' \neq R_i$, and for all $z' \in \varphi(R_i',R_{-i})$, $z_iR_i^0z_i'$. A possible difficulty with this definition is that it may be too restrictive. It disqualifies any pair (R,z) such that for some $z' \in \varphi(R_i',R_{-i})$, $z_i'P_i^0z_i$, although there might be many other allocations $z'' \in \varphi(R_i',R_{-i})$ such that $z_iP_i^0z_i''$. Then, one should perhaps not expect agent i to think that R_i' is a better response than R_i against R_{-i}. Of course, this will depend on how these comparisons of sets are made. However, we need not be concerned about this possibility here: indeed, if a strategy $R_i' \in \mathscr{R}$ is available to agent $i \in N$ so that for some $z' \in \varphi(R_i',R_{-i})$, $z_i'P_i^0z_i$, then in fact some other strategy $R_i'' \in \mathscr{R}$ is also available to him such that for all $z'' \in \varphi(R_i'',R_{-i})$, $z_i''P_i^0z_i$ (this is the content of Lemma 3). So R_i could certainly not be considered to be a best response, and our equilibrium notion is indeed justified.

4. ***The main results.*** Say that a solution φ on $\mathscr{R}^n$ is ***manipulable*** if for some economy $R^0 \in \mathscr{R}^n$, and for all allocations $z \in \varphi(R^0)$, some agent $i \in N$ has available a strategy $R_i \in \mathscr{R}$ such that he would strictly prefer all allocations $z' \in \varphi(R_i,R_{-i}^0)$ to z according to his true preferences R_i^0. This is the definition used by Hurwicz (1972).

The first main result of this paper (Theorem 1) is an impossibility of a form with which many readers will be familiar: on our domain, there is no non-manipulable subsolution of the no-envy solution. A special case of this result for the two-agent case was established by Alkan, Demange and Gale (1991). For economic environments, the first theorem of that kind was established by Hurwicz (1972): in 2-person classical exchange economies, there is no non-manipulable subsolution of the individually rational and efficient solution. Of particular relevance here is the following result: in 2–person classical exchange economies there is no non–manipulable subsolution of the envy–free and efficient solution (Thomson, 1987). A recent result, also for 2–person classical exchange economies, is due to Zhou (1991a). It implies both Hurwicz's and Thomson's impossibilities. The most general result of this kind is due to Barbera and Jackson (1991).

The second main result of this paper (Theorem 2) is that all selections from the no–envy solution are equivalent from the viewpoint of manipulation. For each economy, the set of equilibrium allocations of the manipulation game associated with any such selection played in that economy coincides with its set of true envy–free allocations.

Lemma 1 and Lemma 2 below describe simple but very useful implications of no–envy. For each $R_i \in \mathscr{R}^n$, let $m_w(R_i)$, $m_\ell(R_i) \in \mathbb{R}$ be such that $(1,m_w(R_i))I_i(0,m_\ell(R_i))$ and $m_w(R_i)+(n-1)m_\ell(R_i) = M$.[4] Since each R_i is continuous and strictly monotone with respect to the consumption of the divisible good and satisfies the compensation assumption, $m_w(R_i)$ and $m_\ell(R_i)$ exist and are unique.

Lemma 1. For all $R \in \mathscr{R}^n$, for all $i \in N$, and for all $z \in F(R)$, either $z_i = (0,m_i)$ for $m_i \geqq m_\ell(R_i)$ or $z_i = (1,m_i)$ for $m_i \geqq m_w(R_i)$.

Proof. Let $R \in \mathscr{R}^n$, $z \in F(R)$ and $i \in N$ be given. Recall that for the losers not to envy each other, they should receive the same amount of money. Suppose now that $z_i = (0,m_i)$ for $m_i < m_\ell(R_i)$. Then, the winner receives $m_w = M-(n-1)m_i > m_w(R_i)$ so that $(1,m_w)P_i(1,m_w(R_i))I_i(0,m_\ell(R_i))P_i(0,m_i)$ and agent i envies him.

[4]For readers familiar with Kolm's (1972) method of identifying the set of envy–free allocations in 2–person classical exchange economies, the set $\{(1,m_w(R_i)), (0,m_\ell(R_i))\}$ is the counterpart of agent i's "envy boundary".

Suppose next that $z_i = (1,m_i)$ for $m_i < m_w(R_i)$. Then, any loser $j \in N\backslash\{i\}$ receives an amount $m_j > m_\ell(R_i)$ so that $(0,m_j)P_i(0,m_\ell(R_i))I_i(1,m_w(R_i))P_i(1,m_i)$, and agent i envies him.

Q.E.D.

Lemma 2. Let $R \in \mathscr{R}^n$ and $z \in F(R)$. Let $i \in N$ be the winner at z. Then

(i) $m_w(R_i) = \min_{j \in N} m_w(R_j)$ and $m_w(R_i) \leqq m_i \leqq \min_{j \neq i} m_w(R_j)$ and

(ii) $\max_{j \neq i} m_\ell(R_j) \leqq m_\ell \leqq m_\ell(R_i)$

Proof: Suppose that $m_w(R_i) > \min_{j \in N} m_w(R_j)$. Then there is $k \in N$, $k \neq i$, such that $m_w(R_k) < m_w(R_i)$ and $z_k = (0,m_\ell)$. By Lemma 1, $m_i \geqq m_w(R_i)$ and $m_\ell \geqq m_\ell(R_k) > m_\ell(R_i)$. But then z is not feasible. Thus, $m_w(R_i) = \min_{j \in N} m_w(R_j)$.

Let $h \in N$, $h \neq i$ be an agent such that $m_w(R_h) = \min_{j \neq i} m_w(R_j)$. if $m_i > m_w(R_h)$, then $m_h < m_\ell(R_h)$, contradicting Lemma 1.

By Lemma 1, $m_\ell(R_j) \leqq m_\ell$ for all $j \in N$, $j \neq i$. Hence, $\max_{j \neq i} m_\ell(R_j) \leqq m_\ell$. If $m_\ell > m_\ell(R_i)$, then $m_i < m_w(R_i)$, which contradicts Lemma 1.

Q.E.D.

Theorem 1. There is no non–manipulable subsolution of the no–envy solution.

Proof. Let $\varphi \subseteq F$ be given and let $R^0 \in \mathscr{R}^n$ be such that there is a non–degenerate interval of amounts of money received by the winner at points of $F(R^0)$ (that is, $\underline{w} < \overline{w}$). We show that at any $z \in \varphi(R^0)$, there is at least one agent who can gain by cheating, assuming all others tell the truth. Let $i \in N$ be the winner at z.

Case 1. ***For all*** $j \in N$, $j \neq i$, $z_j P_j^0 z_i$. Let $\overline{m}_w = \min_{j \neq i} m_w(R_j^0)$. Note that $\overline{m}_w > m_w$. Let $R_i \in \mathscr{R}$ be such that $m_w(R_i) = (m_w + \overline{m}_w)/2$. By Lemma 2, for all $z' \in F(R_i, R_{-i}^0)$, $z_i' = (1,m_i')$ for $m_i' \geqq m_w(R_i) > m_w$, and $z_i' P_i^0 z_i$. Therefore agent i can benefit from cheating.

Case 2. There is $j \in N$ such that $z_j I_j^0 z_i$. Since $F(R^0)$ is non degenerate, $z_i P_i^0 z_j$ and $m_w(R_i^0) < m_w$. Let $R_j \in \mathscr{R}$ be such that $m_w(R_j) = (m_w + m_w(R_i^0))/2$ (and $m_\ell(R_j) = (m_\ell + m_\ell(R_i^0))/2$). By Lemma 2, for all $z' \in F(R_j, R_{-j}^0)$, $z_j' = (0, m_j')$ for $m_j' \geqq m_\ell(R_j) > m_\ell$ and $z_j' P_j^0 z_j$. Therefore agent j can benefit from cheating.

Q.E.D.

The next lemma provides the justification for the equilibrium notion that we will use in the analysis of the manipulation games.

Lemma 3. Let $R \in \mathscr{R}^n$, $z \in F(R)$ and $i \in N$ be given. If there are $R_i' \in \mathscr{R}$ and $z_i' \in F(R_i', R_{-i})$ such that $z_i' P_i^0 z_i$, then there is $R_i'' \in \mathscr{R}$ such that for all $z'' \in F(R_i'', R_{-i})$, $z_i'' P_i^0 z_i$.

Proof. We distinguish two main cases.

Case 1. Agent i is a loser at z.

Subcase 1a. Agent i is the winner at z': $z_i' = (1, m_i')$ for some m_i'. Let agent j be the winner at z. Then $m_w(R_j) \leqq m_w$. By Lemma 2, we have $m_i' \leqq m_w(R_j)$. Therefore $m_i' \leqq m_w$. Since $(1, m_w) R_i^0 (1, m_i') P_i^0 (0, m_i)$, it follows that $m_\ell(R_i^0) > m_i$. Let $R_i'' = R_i^0$. Then by Lemma 1, for all $z'' \in F(R_i'', R_{-i})$, either $z_i'' = (0, m_i'')$ for $m_i'' \geqq m_\ell(R_i^0) > m_i$, or $z_i'' = (1, m_i'')$ for $m_i'' \geqq m_w(R_i^0)$. In either case $z_i'' P_i^0 z_i$.

Subcase 1b. Agent i is a loser at z'. $z_i' = (0, m_i')$ for $m_i' > m_i$. Let agent j be the winner at z'. Then $m_w(R_j) \leqq m_j' < m_w$. Let $R_i'' \in \mathscr{R}$ be such that $m_w(R_i'') = (m_j' + m_w)/2 > m_w(R_j)$. It follows from Lemma 2 that for all $z'' \in F(R_i'', R_{-i})$, $z_i'' = (0, m_i'')$ for $m_i'' \geqq m_\ell(R_i'') > m_i$ and $z_i'' P_i^0 z_i$.

Case 2: Agent i is the winner at z'. The analysis of this case being parallel to that of Case 1 is relegated to the appendix.

Q.E.D.

Let $E_{\varphi}(R^0) = \{(R,z) \in \mathscr{R}^n \times Z \mid z \in \varphi(R)$ and $\forall_i$, $\forall R_i' \in \mathscr{R}$ with $R_i' \neq R_i$, $\forall\, z' \in \varphi(R_i',R_{-i})$, $z_i R_i^0 z_i'\}$ be the set of Nash equilibria of the game $(\mathscr{R}^n,\varphi)$ played in R^0, and $E^*_{\varphi}(R^0) = \{z \in Z \mid \exists\ R \in \mathscr{R}^n$ such that $(R,z) \in E_{\varphi}(R^0)\}$ be the corresponding set of Nash equilibrium allocations.

Lemma 2 shows that an agent who has the lowest $m_w(R_i)$ is the winner at an envy–free allocation. If there are more than one such agent, then we call the winner the tie–break winner. In this case, if agent i is the tie–break winner, the amount of money he receives must be equal to $m_w(R_i)$ by Lemma 2.

The next property of solutions is that any agent can be the tie–break winner for some preference profile at any level of money.

Non–discrimination. For all $i \in N$, for all $m_0 \in \mathbb{R}$, there exist $R \in \mathscr{R}^n$, $j \in N$, $j \neq i$ and $z \in \varphi(R)$ such that $m_0 = m_w(R_i) = m_w(R_j) = \min\limits_{k \in N} m_w(R_k)$, and $z_i = (1,m_0)$[5].

We are now ready for our second main result, stating the equivalence under manipulation of all subsolutions of the no–envy solution satisfying *non discrimination.*

Theorem 2. Let φ: $\mathscr{R}^n \to Z$ be a subsolution of the no–envy solution satisfying *non–discrimination.* Then the set of Nash equilibrium allocations of the direct revelation game associated with φ played in each economy in $\mathscr{R}^n$ coincides with the set of envy–free allocations of that economy.

Proof: Let $R^0 \in \mathscr{R}^n$ and $z \in F(R^0)$. Let $i \in N$ be the winner at z. By *non–discrimination,* there exist $R \in \mathscr{R}^n$, and $j \in N$, $j \neq i$ such that $m_i = m_w(R_i) = m_w(R_j) = \min\limits_{k \in N} m_w(R_k)$ and $z \in \varphi(R)$.

(i) Let $R_i' \in \mathscr{R}$. If $m_w(R_i') < m_i = m_w(R_j)$, then by Lemma 2, for all $z' \in \varphi(R_i',R_{-i}) \subseteq F(R_i',R_{-i})$, $z_i' = (1,m_i')$ for some $m_i' \leqq m_w(R_j) = m_i$. Thus $z_i R_i^0 z_i'$. If

---

[5]This property is weaker than complete indifference defined in the next section. Combined with no–envy, it is also weaker than the following property:
Neutrality: If $z \in \varphi(R)$ and z' is obtained from z by a permutation of the bundles that leaves all the agents just as well as before(we call it an "indifferent permutation"), then $z' \in \varphi(R)$. Neutrality is defined and used in Tadenuma and Thomson (1991a).

$m_w(R_i') \geqq m_i$, then again by Lemma 2, for all $z' \in \varphi(R_i',R_{-i}) \subseteq F(R_i',R_{-i})$, either $z_i' = z_i$ or $z_i' = (0,m_i')$ for some $m_i' \leqq m_\ell(R_j) = m_\ell$. In the latter case, since $z \in F(R^0)$, $z_i R_i^0 (0,m_\ell) R_i^0 (0,m_i')$. Therefore, agent i cannot gain by switching from R_i to R_i'.

(ii) Let $k \in N$, $k \neq i$, and $R_k' \in \mathscr{R}$. If $m_w(R_k') \leqq m_i$, then for all $z' \in \varphi(R_k',R_{-k}) \subseteq F(R_k',R_{-k})$, either $z_k' = z_k$ or $z_k' = (1,m_k')$ for some $m_k' \leqq m_w(R_i) = m_i$. In the latter case, since $z \in F(R^0)$, $z_k R_k^0 z_i R_k^0 z_k'$. If $m_w(R_k') > m_i$, then for all $z' \in \varphi(R_k',R_{-k}) \subseteq F(R_k',R_{-k})$, $z_k' = (0,m_k')$ for some $m_k' \leqq m_\ell(R_i) = m_\ell$. Thus $z_k R_k^0 z_k'$. Therefore, agent k cannot gain by switching from R_k to R_k'.

Conversely, let $R^0 \in \mathscr{R}^n$ and $(R,z) \in E_\varphi(R^0)$. Suppose that $z \notin F(R^0)$. Since $z \in \varphi(R) \subseteq F(R)$, the losers at z receive the same bundles and therefore cannot envy each other according to their true preferences. Suppose now that some agent $i \in N$ R_i^0–envies some agent $j \in N$. Note that $(1,m_w(R_i^0)) I_i^0 (0,m_\ell(R_i^0)) P_i^0 z_i$. By Lemma 1, if $z' \in \varphi(R_i^0,R_{-i}) \subseteq F(R_i^0,R_{-i})$, either $z_i' = (0,m_i')$ for some $m_i' \geqq m_\ell(R_i^0)$ or $z_i' = (1,m_i')$ for some $m_i' \geqq m_w(R_i^0)$. Therefore, by switching from R_i to R_i^0, agent i can guarantee an allocation that he strictly prefers to z_i according to R_i^0, which contradicts $(R,z) \in E_\varphi(R^0)$.

Q.E.D.

Note that since $F(R^0) \neq \emptyset$ for all $R^0 \in \mathscr{R}^n$, the first part of the proof also establishes the existence of a Nash equilibrium of the direct revelation game associated with φ.

In fact, we will show next that no group of agents can profitably deviate. A pair ***$(R,z) \in \mathscr{R}^n \times Z$ with $z \in \varphi(R)$ is a strong Nash equilibrium of the game $(\mathscr{R}^n,\varphi)$ played in $R^0 \in \mathscr{R}^n$*** if there is no $S \subseteq N$, $S \neq \emptyset$, such that for some $R_S' = (R_i')_{i \in S}$ with $R_S' \neq R_S$, for some $z' \in \varphi(R_S',R_{-S})$ and for all $i \in S$, $z_i' P_i^0 z_i$.

Theorem 3. Same as Theorem 2 with strong Nash equilibrium replacing Nash equilibrium.

Proof. We claim that the pair (R,z) considered in the first part of the proof of Theorem 2 is also a strong Nash equilibrium of the direct revelation game associated with φ.[6] To see this, suppose that there is $S \subseteq N$ with $|S| \geqq 2$ such that for some $R'_S = (R'_j)_{j\in S}$ with $R'_S \neq R_S$, for some $z' \in \varphi(R'_S,R_{-S})$ and for all $j \in S$, $z'_j P^0_j z_j$. Let $k \in N$ be the winner at z'. Since $|S| \geqq 2$, at least one agent in S, say agent h, must be a loser at z'. Because $z \in F(R^0)$ and $z'_h P^0_h z_h$, we have $m'_h = m'_\ell > m_\ell$. Then, by feasibility, $m'_k = m'_w < m_w$. Since $z \in F(R^0)$, $z_k P^0_k z'_k$. Thus $k \notin S$. By Lemma 1, $m'_k \geqq m_w(R_k) \geqq m_w(R_i) = m_w$, a contradiction.

Q.E.D.

Thus the set of strong Nash equilibrium allocations of the game also coincides with the set of envy–free allocations. For a general study of such "double implementation" in Nash and strong Nash equilibrium, we refer the reader to Suh (1993).

5. ***Implementing the no–envy solution.*** In this section, we establish a useful implication of Theorem 2 for the implementation of the no–envy solution.

The following property of solutions has been central in the study of implementation. A solution φ is ***monotonic*** if whenever an allocation is φ–optimal for some economy,[7] it remains φ–optimal for the economy obtained by changing preferences in such a way that the allocation does not fall in anybody's estimation. Let Z_i be the set of consumptions that are the i^{th} component of some feasible allocation (here $Z_i = \{0,1\} \times \mathbb{R}$). Given $z \in Z$, $R' \in \mathcal{R}^n$ ***is obtained from*** $R \in \mathcal{R}^n$ ***by a monotonic transformation at*** z if for all i, $\{z'_i \in Z_i |\ z_i R'_i z'_i\} \supseteq \{z'_i \in Z_i |\ z_i R_i z'_i\}$.

Monotonicity (Maskin, 1977): For all R, $R' \in \mathcal{R}^n$, for all $z \in \varphi(R)$, if R' is obtained from R by a monotonic transformation at z, then $z \in \varphi(R')$.

---

[6]We would like to thank a referee for pointing this out to us. Tatamitami (1992) independently made the same observation.

[7]We say that "z is φ–optimal for R" if $z \in \varphi(R)$.

It is easy to see that the no–envy solution is *monotonic* (and that under our assumptions so is the Pareto solution). We show next that there is essentially no proper subsolution of the no–envy solution that is. We write "essentially" because this result also involves the following very natural and mild condition of *complete indifference* (it is met by all the solutions that have been discussed in the literature): if an allocation is such that all agents are indifferent between all of its components, then it is φ–optimal.[8]

Complete indifference: For all $R \in \mathcal{R}^n$, for all $z \in Z$, if for all $i, j \in N$, $z_i I_i z_j$, then $z \in \varphi(R)$.

Lemma 4. If a solution satisfies *monotonicity* and *complete indifference*, then it contains the no–envy solution.

Proof. Let φ: $\mathcal{R}^n \to Z$ be a correspondence satisfying *monotonicity* and *complete indifference.* Let $R \in \mathcal{R}^n$ and $z \in F(R)$ be given. Let $R' \in \mathcal{R}^n$ be such that for all $i, j \in N$, $z_i I_i' z_j$. By *complete indifference*, $z \in \varphi(R')$. Also, R is obtained from R' by a monotonic transformation at z. By *monotonicity*, $z \in \varphi(R)$.

Q.E.D.

Let φ: $\mathcal{R}^n \to Z$ be a solution. Say that φ ***is implementable*** (in Nash equilibrium) if there exists a ***game form*** (S,h), where $S = S_1 \times \ldots \times S_n$ and h: $S \to Z$ such that for each economy $R \in \mathcal{R}^n$, the set of Nash equilibrium allocations of (S,h) played in R coincides with $\varphi(R)$. Maskin (1977) showed that if φ is *implementable*, then φ is *monotonic.* He also showed that *monotonicity*, together with a certain condition of *no veto power* (if an alternative is at the top of the preferences of all but possibly one agents, then it is φ–optimal), are sufficient conditions for φ to be *implementable.* His proof is constructive. He provided an algorithm that produces for each *implementable*

[8]Note that such an allocation, being envy–free, is efficient.

solution, a game that does implement it. In the present context, *no veto power* is vacuously satisfied, since there is no alternative satisfying the hypothesis of the condition. Also, as already pointed out, the no–envy solution is *monotonic*. Therefore, it is *implementable*. However, Lemma 4 implies that essentially no proper subsolution of the no–envy solution is *implementable*.

In order to implement the no–envy solution, we could of course use the game that results by operating Maskin's algorithm. Unfortunately, in this game, strategies are complicated since they include a complete description of the economy. Several authors (Saijo, 1988, McKelvey, 1989)[9] have succeeded in reducing the complexity of Maskin's strategy spaces but the strategy spaces they use remain complex. We show next how the no–envy solution can in our context be implemented by a game in which each agent announces a single number. This notion of implementation is of course based on the variant of the concept of Nash equilibrium used in the previous analysis.

This simple implementation is obtained as a corollary of Theorem 2, by noting that it applies to a particular selection from the no–envy solution whose computation requires the knowledge of only one number for each agent. This solution was obtained by Tadenuma and Thomson (1993) in an axiomatic study of the problem of selection. It can be characterized on the basis of certain *consistency* and *population–monotonicity* conditions.[10] It is defined by systematically picking the envy–free allocation at which the amount of money of the winning bundle is smaller than at any other envy–free allocation.

Definition. Given $R \in \mathscr{R}^n$, $\varphi^*(R) = \{z \in F(R) |\ m_W \leqq m'_W \text{ for all } z' \in F(R)\}$.

---

[9]Saijo (1988) has agents announce the preferences of two agents, an allocation, and an integer. In McKelvey (1989)'s simplest game, each agent announces a lower contour set, an allocation, and an integer.

[10]*Consistency* says that the departure of some of the agents with their allotted bundles does not affect the desirability of the distribution of the remaining resources among the remaining agents. *Population–monotonicity* says that the arrival of additional agents with equally valid claims on the existing resources as those of the agents originally present affect all of these agents in the same direction: they all lose or they all gain.

Alternatively, $z \in \varphi^*(R)$ if $z_i = (1, m_w(R_i))$ for $i \in N$ such that $m_w(R_i) \leqq m_w(R_j)$ for all $j \in N$ and $z_j = (0, m_\ell(R_i))$ for all $j \in N$, $j \neq i$. In order to define φ^*, only one number is required of each agent $i \in N$, namely $m_w(R_i)$.[11] Given all the desirable properties satisfied by φ^*, it is, of course, a great disappointment that it should be manipulable. However, it permits the following very simple implementation of the no–envy solution.

Corollary to Theorem 2. The no–envy solution is implemented by the direct revelation game associated with φ^*. In that game, each agent is required to announce a single number.[12]

6. ***Concluding comments.*** We have shown that under manipulation, all selections from the no–envy solution behave identically. The negative aspect of this conclusion is that the problem of selection has no solution immune to manipulation. The positive aspect is that the allocations obtained at equilibrium do satisfy the no–envy condition for the true preferences. As a result, our basic distributional objective of no–envy is attained and since here no–envy implies efficiency, manipulation does not lead to violations of efficiency, contrarily to what might have been expected.

These conclusions should reinforce our initial statements. Most of the literature on the problem of manipulation has consisted of general results of the kind: there is no

---

[11]The allocations selected by φ^* are "egalitarian–equivalent" allocations in the sense of Pazner and Schmeidler (1978).

[12]The solution φ^* is single–valued only up to indifferent permutations. The following selection is a single–valued solution satisfying non–discrimination when $n \geq 3$. Given $R \in \mathscr{R}^n$, let $S(R) = \{i \in N \mid m_w(R_i) = \min_{k \in N} m_w(R_k)\}$. If $S(R) = \{i\}$ for some $i \in N$, let $\varphi^{**}(R)$ be the unique allocation in $\varphi^*(R)$ at which agent i is the winner. If $S(R) = \{1,n\}$, let $\varphi^{**}(R)$ be the unique allocation in $\varphi^*(R)$ at which agent 1 is the winner. In all other cases, let $\varphi^{**}(R)$ be the unique allocation in $\varphi^*(R)$ at which the agent with the largest index in $S(R)$ is the winner.

solution such that in its induced direct revelation game, it is always in the interest of each agent to tell the truth if all other agents already do. The next question, What happens if all agents manipulate? is rarely addressed. The answer to this question is sometimes much less disappointing. This is certainly the case in the model examined here.

APPENDIX

2. *Completion of the proof of Lemma 2.*

Case 2: Agent i is the winner at z.

Subcase 2a: Agent i is a loser at z′, i.e. $z_i' = (0, m_i')$ for some m_i'. Let agent j be the winner at z′. By Lemma 2, $m_i' \leqq m_\ell(R_j)$. Since agent j is a loser at z, $m_\ell \geqq m_\ell(R_j)$. Therefore, $m_i' \leqq m_\ell$. Since $(0, m_\ell) R_i^0 (0, m_i') P_i^0 (1, m_i)$, then $m_w(R_i^0) > m_i$. Let $R_i'' = R_i^0$ and $z'' \in F(R_i'', R_{-i})$. By lemma 1, if $z_i'' = (1, m_i'')$, then $m_i'' \geqq m_w(R_i^0)$ and if $z_i'' = (0, m_i'')$, then $m_i'' \geqq m_\ell(R_i^0)$. In either case, $z_i'' P_i^0 z_i$.

Subcase 2b: Agent i is the winner at z′, i.e. $z_i' = (1, m_i')$ for $m_i' > m_i$. By Lemma 2, $m_i' \leqq m_w(R_j)$ for all $j \in N$, $j \neq i$. Let $R_i'' \in \mathscr{R}$ be such that $m_w(R_i'') = (m_i + m_i')/2$. Then, $m_i < m_w(R_i'') < m_i' \leqq m_w(R_j)$ for all $j \in N$, $j \neq i$. If follows from Lemma 2 that for all $z'' \in F(R_i'', R_{-i})$, $z_i'' = (1, m_i'')$ for $m_i'' \geqq m_w(R_i'') > m_i$, and $z_i'' \; P_i^0 z_i$.

Q.E.D.

REFERENCES

Alkan, A., G. Demange, and D. Gale (1991), "Fair allocation of indivisible goods and criteria of justice," ***Econometrica 59***, 1023–1039.

Aragones Alabart E. (1990), "Assignacions sense enveja en economies amb indivisibilitats," Universitat Autonoma de Barcelona mimeo.

Barbera, S. and M. Jackson (1991), "Strategy–proof exchange", mimeo.

Crawford, V. and W. Heller (1979), "Fair division with indivisible commodities," ***Journal of Economic Theory*** 21, 10–27.

van Damme, E. (1987), ***Stability and Perfection of Nash Equilibria***, Springer–Verlag, Berlin.

Foley, D. (1967), "Resource allocation and the public sector," ***Yale Economic Essays*** 7, 45–98.

Gale, D. and M. Sotomayor (1985), "Ms. Macchiavelli and the stable matching problem", ***American Mathematical Monthly***, 92, 261–268.

Hurwicz, L. (1972), "On informationally decentralized systems," in ***Decision and Organization***, (C.B. McGuire and R. Radner, eds.), Amsterdam: North Holland Publishing Co., 297–336.

Hurwicz, L. (1978) "On the interaction between information and incentives in organizations," in ***Communication and Control in Society*** (K. Krippendorf, ed.), Scientific Publishers, Inc. NY, 123–147.

Kolm, S.C. (1972), ***Justice et Equite***, Editions du CNRS, Paris.

Luce, R.D. and H. Raiffa (1957), ***Games and Decisions***, Wiley, New York.

Maskin, E. (1977), "Social choice and welfare optimality," MIT mimeo.

Maskin, E. (1987), "On the fair allocation of indivisible goods," in ***Arrow and the Foundations of the Theory of Economic Policy***, (G. Feiwel, ed.) London, Macmillan, 341–349.

McKelvey, R.D. (1989), "Game forms for Nash implementation," ***Social choice and welfare*** 6, 139–156.

Moulin, H. (1989), "Fair division with money: two interpretations of common ownership," Duke University mimeo (May 1990).

Pazner, E., and D. Schmeidler (1978), "Egalitarian equivalent allocations: A new concept of economic equity, ***Quarterly Journal of Economics*** 92, 671–687.

Roth, A. E. (1984), "Misrepresentation ans stability in the marriage problem", ***Journal of Economic Theory 34***, 383–387.

Saijo, T. (1988), "Strategy space reduction in Maskin's theorem: Sufficient condition for Nash implementation," ***Econometrica*** **56**, 693–700.

Suh, S. (1993), "Implementation with coalition formation", University of Rochester mimeo.

Svensson, L.G. (1983), "Large indivisibles: An analysis with respect to price equilibrium and fairness," ***Econometrica*** **51**, 939–954.

Svensson, L.G. (1988), "Fair wages where individual choice sets are incomplete – An application of a model with indivisibilities," ***Scandinavian Journal of Economics*** **90**, 563–573

Tadenuma, K. and W. Thomson (1991a), "No–envy and consistency in economies with indivisible goods," ***Econometrica*** ***59***, 1755–1767.

Tadenuma, K. and W. Thomson (1991b), "Solutions to the problem of fair allocation in economies with indivisible goods", University of Rochester mimeo.

Tadenuma, K. and W. Thomson (1993), "The fair allocation of an indivisible good when monetary compensations are possible," ***Mathematical Social Sciences 25***, 117–132.

Tatamitani, Y. (1992), "Coalitional formation in games of fair division", University of Tsukuba mimeo.

Thomson, W. (1984), "The manipulability of resource allocation mechanisms," ***Review of Economic Studies*** **51**, 447–460.

Thomson, W. (1987) "The vulnerability to manipulative behavior of economic mechanisms designed to select equitable and efficient outcomes," in ***Information, Incentives and Economic Mechanisms***, (T. Groves, R. Radner, and S. Reiter, eds.), University of Minnesota Press, 375–396.

Thomson, W. (1988), "The manipulability of the Shapley value," ***International Journal of Game Theory*** **17**, 101–127.

Thomson, W. (1993), "Fair allocation rules," University of Rochester mimeo, forthcoming.

Zhou, L. (1991a), "Inefficiency of strategy–proof allocation mechanisms in pure exchange economies," ***Social Choice and Welfare 8***, 247–254.

Zhou, L. (1991b), "Stable matchings and equilibrium outcomes of the Gale–Shapley algorithm for the marriage problem", ***Economics Letters 36***, 25–29.