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1. Introduction

This paper examines a class of games in which a single infinitely–lived principal must decide in each period the agent to be hired for that period, where agents live for only two consecutive periods. If the "incumbent" agent (i.e. the agent employed by the principal in the previous period) has one period of life left, the choice the principal then faces is between retaining the incumbent and replacing him with a hitherto untried agent; otherwise the principal simply selects an untried agent. Agents generate rewards for the principal when employed by her. The distribution of these rewards in any period depends on the action taken by the incumbent agent that period; further, this distribution may differ depending on whether the agent is in his first or second period of life. Agents may be one of a finite number of "types", where an agent's type influence his payoff from taking any particular action; the actual agent types are determined as the realizations of i.i.d. draws on the space of possible types. We presume both moral hazard and adverse selection exist in our model: neither an agent's true type, nor his actions when employed are observable by the principal. Rather, the principal's only source of information concerning these lies in inferences she can draw from observed rewards and knowledge of the distribution determining agents' types.

Models such as this wherein a principal and agent potentially interact over a number of periods have been the focus of considerable attention in recent years.[1] The model considered here differs from the "standard" repeated principal–agent model in numerous aspects, the most obvious of which is that, whereas in most models the parties have coincident live spans, here the principal lives forever while agents only live for two periods. Our motivation for this assumption is the preponderance of instances in which a single "promotion" or "retention" decision by an employer is paramount in certain

[1]Examples include Radner (1981, 1985, 1986), Rubinstein and Yaari (1983), Rogerson (1985), Fudenberg, *et al* (1990), Baron and Besanko (1987), Dewatripont (1989), Laffont and Tirole (1988, 1990), Dutta and Radner (1992), and Banks and Sundaram (1992) among many others.

on–going relationships. Thus for example we can think of the principal as a firm, and the agent as a worker who, at some pre–specified time, is either (say) promoted to management or else is let go; or the principal is a university and the agent a faculty member, with the retention decision then corresponding to the tenure decision; or the principal is the (median) voter of some constituency and the agent their political representative, who by law is limited to two terms in office (eg. the US president).

A further difference relates to the analytical approach of the model. In most of the standard literature, the predominant aim has been the design and characterization of "optimal" revenue–sharing agreements among the principal and agent, i.e., second–best contractual arrangements which minimize efficiency losses subject to incentive constraints.[2] This has lead to important insights into the nature of agency problems, and the difficulties associated with obtaining efficiency in agency contracts. However, it is widely acknowledged that, even in the one–shot model, the contracts predicted by this theory are quite complex when compared to observable "real world" contracts. The distinction becomes more acute when we consider second–best contracts of repeated principal–agent models, for such contracts must necessarily involve non–trivial dependence on history, and indeed, past observations can matter in intricate ways (see, e.g., Rogerson, 1985).

Thus our second point of departure from the "standard" approach is to suppress any consideration of optimal contracts. Rather, an agent's payoff in his first period of employment is simply some function of his chosen action that period and his type, whereas his second period payoff is a (potentially different) function of his second period action, type, and (possibly) the reward generated in the first period. Similarly, the principal's payoff from a first period employee is some increasing function of the reward generated, and likewise for a second period employee (where the latter may depend on the first period reward as well). As we explain in detail in section 2, where we describe preferences

[2]See, e.g., Radner (1981, 1985), Rubinstein and Yaari (1981), or Rogerson (1985).

formally, these utility functions can be viewed as "reduced form" functions under some a known and fixed revenue sharing agreement, i.e., as specifying the expected utility from various actions. However, an advantage of this reduced form approach is that a number of other interpretations are also possible. In particular, we do not rule out the possibility that the principal and agent have coincident interests.[3] A second advantage is that it allows us to focus attention on the principal's retention decision, and the consequent incentives for the agents to take actions she (the principal) finds desirable.[4]

Finally, a majority of the papers in the Principal–Agent literature take a "Stackelberg" approach to equilibrium. Namely, it is assumed that the agent has a reservation utility level, below which he will not enter into a contract with the principal. Optimal contracts are solved for by maximizing the principal's payoffs subject to the agent's incentive constraints and the condition that the agent receive, in expected utility terms, at least his reservation level. In contrast, by explicitly modeling the presence of alternatives to the incumbent agents, we give the *principal* a reservation level of payoffs below which she finds it worthwhile to sever her connections to the incumbent.[5] In addition, we adopt a pure "Nash" equilibrium approach, where agents and principals play best–responses to each others' strategies; in particular, the retention decision by the principal is required to be sequentially rational.

Our first result (Theorem 1) establishes, under certain conditions, the existence of an *anonymous* equilibrium in the model, in which the principal treats all agents identically

[3]Even in this case, the principal's decision problem is non–trivial, since the reward consequences to the principal of the same action by different types may be different, i.e., the agent's type might still matter.

[4]Thus, our approach bears some similarity to that of Radner (1986), who studies the efficiency of outcomes when the principal uses "review strategies" (essentially, up–or–out rules where the decision to retain an incumbent is made once every n periods for some fixed $n \geq 1$). It is important to note that the restriction to "up or out" rules for the principal does not imply that equilibria are neccessarily "simple" in any way. The conditions under which incumbent agents are retained (or, equivalently the set of histories on which they are replaced) could, in principle, take on extremely complex forms.

[5]Of course, the value of this reservation level will be determined by the equilibrium in play.

with respect to her retention decision and all agents adopt identical strategies up to their type. This equilibrium possesses strong properties: agent strategies are ordered in type in each period, in the sense that "better" types of agents take weakly "better" actions. Further, the principal employs a *cut–off strategy,* i.e., an incumbent who has completed one period is retained for a second if, and only if, the first–period reward exceeds a critical cut–off amount.[6]

In Theorem 2 we establish upper–semicontinuity of the above equilibrium correspondence in π, the prior belief regarding the type of a generic untried agent. As a special case, this result establishes that as π converges to point–mass on some agent–type (i.e., to a *pure moral hazard problem*), the corresponding sequence of equilibria converges to an equilibrium of the game with the limit degenerate belief.

Further interesting characteristics of the equilibria are hard to come by in such a general model, and so in section 4 we consider properties of these equilibria in what we refer to as a "flat" environment, where an agent's first and second period payoff functions are the same, as are the reward distributions. Theorem 3 then identifies a *temporal monotonicity* property of the equilibria: agent strategies are monotone across periods in that each given type takes (weakly) higher actions in the first period of employment than in the second period; under a strengthening of the assumptions, these inequalities can be made strict (Corollary 1). We can think of this temporal monotonicity as a "performance effect" of the principal–agent interaction, in that the principal's retention decision provides the incentive for agents to work harder in their first period than they would otherwise (i.e. myopically). It is perhaps worth emphasizing one feature of this equilibrium, viz., that even though all agents take lower actions in their second periods, the principal nonetheless

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[6]Cut–off strategies have been used by many authors (for instance, Barro, 1973, or Reed, 1991), but have not (as far as we know) been derived as *equilibrium* strategies when there are no restrictions on choosing the set of acceptable rewards. The review strategies of Radner (1986) also fall into the class of cut–off strategies if reviews are conducted every period, but are otherwise a weaker class of strategies.

finds it worthwhile to retain incumbents who generated sufficiently large rewards in their first period, since her beliefs shift towards the better types.[7] This we can think of as the "selection effect" of the interaction.[8]

We then turn to a study of the case of pure moral hazard in the flat environment. Our main result here (Corollary 2) shows that all anonymous equilibria of the pure moral hazard case must be "trivial"; that is, in every equilibrium of this game, all agents play a *myopic* action (an action that maximizes their one–period utility) in every period. Therefore, in contrast to the earlier results, when there is no uncertainty about agent preferences there exists no equilibrium performance effect: the principal cannot credibly provide the incentive for agents to work hard in the first period.

The combination of Corollaries 1 and 2 raises an interesting possibility. The former result showed that, under certain circumstances, there exist equilibria in which agents' strategies strictly separate over time, with agents of all types taking higher actions in their first period than their last. Since all agents evidently take myopic actions in their last period (the future is now irrelevant to them), the question naturally arises whether there are conditions under which the principal can actually be made *better* off by introducing adverse selection into the model.[9] Indeed, it appears plausible that under suitable circumstances, the principal's payoffs could be improved even if the new types of agent–types introduced into a pure moral hazard problem are all "worse" (i.e., less efficient at generating rewards) than the single agent–type of the original problem.[10]

Perhaps unfortunately, this intuition turns out to be misguided. In Theorem 4, we

[7]This offers a possible explanation of why tenure arrangements might be equilibrium outcomes, even if it were true that all people work less after they get tenure.

[8]The terms "performance effect" and "selection effect" are due to Reed (1991).

[9]This question is remniscent of the motivation in Kreps, *et al* (1984), who show that admitting a "small" probability that one of the players in a finitely–repeated prisoners' dilemma is "irrational" (say, is an automaton) could create equilibria under which both players are better off.

[10]For instance, the probability of the old type could be arbitrarily close to unity, while the new types may be only marginally worse.

show that the principal can *never* benefit from the introduction of "worse" types of agents (and, perhaps less surprisingly, can also never lose from the introduction of "better" types of agents). Thus the performance effects alluded to above are never sufficiently strong to overcome the loss to the principal from having "worse" types in the population, and at times selecting these worse types for second period employment.

2. The Model

We consider a discrete–time infinite horizon model with periods indexed by $t = 0,1,2,\dots$ In each period, an infinitely–lived *principal* must decide on the choice of *agent* to be hired that period, where agents live for two consecutive periods: the choice the principal faces is then between rehiring the current incumbent agent (i.e., the agent who was hired by the principal in the previous period) and replacing him with a hitherto untried agent if the incumbent agent has one period of life left; otherwise the principal simply selects an untried agent. All untried agents at any date are *ex ante* identical to the principal; we assume, without loss, therefore, that at each date there is only a single untried agent available.[11] This agent will be described by his date of appearance t.

Agents in the model are distinguished by their *type*, where each agent may be one of a finite number of possible types $\omega \in \Omega \equiv \{\omega_1,\dots,\omega_n\}$. These types are the realizations of i.i.d. draws from a common distribution $\pi = (\pi_1,\dots,\pi_n) \in P(\Omega)$,[12] and π is assumed to be common knowledge.

Agents generate *rewards* $r \in \mathbb{R}$ for the principal while employed. The distribution of these rewards in any period depends on the action the agent takes in that period, where the

[11]There is an implicit assumption here that agents who are not hired in the period of their appearance are no longer available to the principal, and neither are incumbents who are replaced at any point. This assumption is without loss of generality for the class of equilibria we consider below.

[12]P(X) will represent the set of all probability distributions on any finite set X. The assumption that agents are i.i.d. draws from a common distribution ensures that they are, in fact, *a priori* identical.

set A of actions available to any agent in any period is assumed to be a compact subset of $\mathbb{R}$. We allow these reward distributions to vary according to whether an agent is in his first or second period of employment by the principal; thus let $F_i(.|a)$ denote the distribution, parameterized by the action $a \in A$, for an agent in their i–th period of employment.

We admit both moral hazard and adverse selection in our framework. Namely, we assume that the actions taken by the agents while employed are not observable by the principal; and that an agent's "true" type ω is private information to the agent and therefore is not known to the principal. Rather, the principal must use her knowledge of the prior distribution π together with observed rewards to infer the agents' behavior.

A final point is important. We do *not* require $\pi_k > 0$ for all $k = 1,\dots,n$. Indeed, since we wish to examine, among other things, the convergence of equilibria under adverse selection to those arising in the pure moral hazard case (i.e., when $\pi_k = 1$ for some k), it is essential to allow for the possibility that some of the components of π could be 0.

Preferences:

The utility of an incumbent agent in their first period of employment depends on the action taken by the agent that period and the agent's type, and is denoted $u_1(a,\omega)$, whereas their second period payoff is a function of both their type, their second period action, and also (possibly) on the reward generated in the first period, and is denoted $u_2(r,a,\omega)$. The utility level of agents while unemployed is normalized to 0. To make the choice problem faced by the agent non–trivial we shall assume throughout that for each (r,ω) there exists actions $a_1(\omega)$, $a_2(r,\omega)$ satisfying $u_1(a_1(\omega),\omega) > 0$ and $u_2(r,a_2(r,\omega),\omega) > 0$. All agents discount future utilities by the common factor $\delta \in [0,1]$.

The principal's preferences take on a simpler form: her utility in the first period of an agent's employment depends solely on the reward r generated by the incumbent agent that period, and is given by $v_1(r)$; while her second period utility depends on the reward generated in the second period as well as (possibly) on that of the first period, and is denoted $v_2(r;\hat{r})$, where $\hat{r}$ is the realized reward from the first period. The principal's discount factor is given by $\alpha \in [0,1)$.

A comment or two on the generality of this specification of preferences is, perhaps, in order here. One possible interpretation of the functions u_i, v_i is as the "reduced form" expected utility functions for the principal and agent under a pair of known reward–sharing rules in the standard principal–agent framework. For instance, suppose that the agent receives a base wage of w_1 and a bonus of $\alpha_1(r)$ in the first period, and in the second period receives a base wage of w_2 and a bonus of $\alpha_2(r)$; let α denote the tuple $(w_1,\alpha_1,w_2,\alpha_2)$. Suppose also that the "cost" (i.e., disutility) of action to a type–ω agent in each period is given by $c(a;\omega)$, and that the agent's preferences over income and cost is given by the "quasi–linear" form $y - c(a;\omega)$. Then, the agent's total utility in the first period of employment will be $w_1+\alpha_1(r)-c(a;\omega)$; and the *ex ante* expected utility the agent obtains is

$$E_1(\alpha)(a,\omega) = w_1+\int \alpha_1(.)dF_1(.|a)-c(a;\omega).$$

Similarly, the agent's second period expected payoff would be

$$E_2(\alpha)(a,\omega) = w_2+\int \alpha_2(.)dF_2(.|a) - c(a;\omega).$$

Thus setting $u_1(a,\omega)$ equal to $E_1(\alpha)(a,\omega)$ and $u_2(r,a,\omega)$ equal to $E_2(\alpha)(a,\omega)$, and letting the principal's per–period payoffs be the residual rewards $v_i(r) = r-\alpha_i(r)-w_i$, $i = 1,2$, we have the preference structure described above.[13]

Under this scenario the first period reward generated by the agent does not affect either participant's second period utility function; however we could add such an influence by, for example, imagining a "human capital" component to the rewards, and have the first period reward affect the agent's second period cost function. Alternatively, the first period reward could enter into the second period sharing rule.

Suppose instead that we have $u_i(.) = \omega - c(a)$ and $v_i(.) = r$, $i = 1,2$; these are the

[13]In particular, our formulation admits as a special case the situation envisaged in, for example, Radner (1986), Dutta and Radner (1987), and Banks and Sundaram (1992), where the agent recives a fixed wage w in each period of employment.

preferences assumed in the models of choosing a political representative found in, among others, Ferejohn (1986) and Austen–Smith and Banks (1989), where here the agent operates under a two–term limit.[14] In this set–up ω gives the value of holding office apart from any effort or policy considerations; hence uncertainty surrounds the intrinsic value different potential representatives place on being in office, information which is of value to the principal (in the role of say a median voter) if different levels of ω select different actions.

However, many other interpretations are also possible for the principal's and agents' preferences. In particular, our formulation leaves open the possibility that principal and agents have coincident underlying preferences over rewards, for example $u_i(a,\omega) = \int v_i(r)dF_i(r|a) - c(a,\omega)$. In this set–up, then, the principal and agent agree on what are good and bad outcomes; the heterogeneity comes about by the presumption that it is the agent who generates these outcomes.

Histories and Strategies:

In general, a *t–history* for the game is a list of the agents who have been employed in periods 1,...,t–1, the rewards they generated, and the current incumbent agent; a *strategy* ν for the principal is a sequence of measurable maps $\{\nu_t\}$, where for each t, ν_t specifies, as a function of the t–history to date, whether the current incumbent agent is to be replaced; and a *strategy* λ for agent t is a sequence of measurable maps $\{\lambda_{t\tau}\}$, where $\lambda_{t\tau}$ specifies an action for agent t in period $(t+\tau)$ given the $(t+\tau)$–history to date, $\tau = 0,1$. In this paper we focus on a class of strategies of special importance that we call *anonymous strategies* wherein all agents are treated symmetrically by the principal.

A *personal history* for a given agent is simply a reward $r \in \mathbb{R}$ generated by the agent in her first period of employment. Since the principal must replace an agent if that agent

[14]Ferejohn (1986) and Austen–Smith and Banks (1989) consider models where only moral hazard is present.

is at the end of his two periods of life, we can define an *anonymous strategy* σ for the principal as simply the retention rule employed by the principal between an agent's first and second periods. That is, an anonymous strategy is a measurable map $\sigma:\mathbb{R} \to \{0,1\}$, where 0 denotes the action of replacing, and 1 denotes the action of continuing with, the first period incumbent. Let Σ denote the set of all anonymous strategies for the principal. An *anonymous strategy* for a generic agent will then simply be a first period action as a function of the agent's type, as well as a second period action as a function of type and the reward generated in the first period. Since (as we shall see) allowing the agent to mix in his choice of first period action is essential in guaranteeing existence, we will denote an anonymous agent strategy by the pair $(\mu,\gamma) = (\{\mu_k\},\{\gamma_k\})$, where $\mu_k \in P(A)$ is a mixed action for the agent of type ω_k in the first period, while $\gamma_k:\mathbb{R}\to A$ is a function specifying a type–ω_k's second period action as a function of the realized first period reward. Let Γ denote the set of all anonymous strategies available to the agents.

Two further definitions will be useful in the sequel. First, an anonymous strategy σ for the principal will be called a *cut–off strategy* if there exists an $r^* \in \mathbb{R}$ such that $\sigma(r) = 1$ if and only if $r \geq r^*$. That is, a cut–off strategy is such that an incumbent agent is retained for a second period if and only if the reward generated by the agent in the first period exceeds some critical level. Second, an anonymous strategy (μ,γ) for an agent will be called *type–monotone* if (i) the agents' first period actions $\mu_k \in P(A)$ are "ordered" in the sense that there exist intervals $[a_k,b_k] \subset \mathbb{R}$ such that $b_k \leq a_{k+1}$ for all k, and $\mu_k([a_k,b_k]) = 1$, and (ii) the agents' second period actions are "ordered" in the sense that for all $r\in\mathbb{R}$, $\gamma_k(r) \leq \gamma_{k+1}(r)$ for all k. Thus type–monotone strategies have higher types adopting (weakly) higher actions in both periods.

Best–Responses and Equilibrium:

Since our focus in this paper is only on equilibrium in anonymous strategies, we avoid spurious generality and define equilibrium for that case alone. We begin with a description of the principal's best–response problem.

The principal starts with a prior π regarding an agent's true type, and updates her beliefs using the observed rewards and a conjectured (anonymous) strategy for the agent. Formally, let the conjectured agent strategy be $(\mu,\gamma) \in \Gamma$, and suppose that the current incumbent has generated a reward of r in her first period of employment. Now for $\mu \in P(A)$, we have

$$\varphi(r|\mu) = \int_A f_1(r|a)\mu(da)$$

as the density associated with the mixed action μ. Applying Bayes' Rule, then, the principal's updated belief that the agent is of type k, denoted $\beta_k(r)$, is

$$\beta_k(r) = \frac{\pi_k\varphi(r;\mu_k)}{\Sigma_{j=1}^n \pi_j\varphi(r;\mu_j)}.$$

Given the updating rule, each strategy σ for the principal defines in the obvious (if notationally dense) manner, a t–th period expected utility against (μ,γ) denoted $r_t(\sigma;\mu,\gamma)$. The *worth* of the profile (σ,μ,γ) to the principal is then given by $W(\sigma,\mu,\gamma) = \Sigma_{t=0}^{\infty} \alpha^t r_t(\sigma;\mu,\gamma)$. A strategy σ^* is said to be *optimal against* (μ,γ) for the principal if $W(\sigma^*,\mu,\gamma) \geq W(\sigma,\mu,\gamma)$ for all $\sigma \in \Sigma$.

The agent's best response problem is easier to define: if an agent is retained for a second period an optimal action will clearly require $\gamma_k(r) \in \text{argmax}\ \{u_2(r,a,\omega_k) |\ a \in A\}$ for all $r \in \mathbb{R}$. Let $V(r,\omega) = \max\ \{u_2(r,a,\omega) | a \in A\}$, and for a given anonymous strategy σ for the principal, let $R_\sigma = \{r \in \mathbb{R}\colon \sigma(r) = 1\}$ denote the set of first–period rewards for which the agent is retained. Then a first–period strategy μ is said to be *optimal against* σ if for all k $= 1,\ldots,n$, $\mu_k(\hat{a}) > 0$ only if

$$\hat{a} \in \text{argmax}\ \{\ u_1(a,\omega_k) + \delta \int_{R_\sigma} V(r,\omega)dF_1(r|a) |\ a \in A\}.$$

Finally, we say that the profile (σ,μ,γ) constitutes an *equilibrium* for this model if σ and

(μ,γ) are optimal against each other.

It is important to note, in closing, that while we require best–responses to be chosen from the class of anonymous strategies, this is not really a restriction. If the principal bases the criteria on which a particular agent t is retained in office only on t's performance, t's optimization problem is unaffected by the history of previous play; thus, if at all it admits any solution, it must admit a solution in which the optimal strategy is based solely on t's own history. Conversely, if all agents use anonymous strategies, the principal cannot benefit by conditioning her response against any particular agent on the performance of any other agent. It follows that equilibria in anonymous strategies remain equilibria even when some or all players may condition their actions on the entire history of play.

3. The Existence of Equilibrium

In this section we establish the existence of an equilibrium in anonymous strategies to the model of section 2, under some technical assumptions on the reward distributions and on the (reduced form) utility functions. Further, this equilibrium will have the principal adopting a cut–off strategy, and the agents adopting type–monotone strategies.

We begin with the maintained assumptions on F_i, $i = 1,2$:

<u>Assumption 1</u>: For each $a \in A$, $F_i(.|a)$ admits a density $f_i(.|a)$ with respect to the Lesbegue measure.

<u>Assumption 2</u>: The set $\{r | f_i(r|a) > 0\}$ is a bounded interval in $\mathbb{R}$ that is independent of $a \in A$; the closure of this interval in denoted S_i.

<u>Assumption 3</u>: $f_i(.|.)$ is jointly continuous on $S_i \times A$.

<u>Assumption 4</u>: $f_i(.|.)$ satisfies the Monotone Likelihood Ratio Property (MLRP), i.e. for all $a,\hat{a} \in A$ with $a > \hat{a}$, the ratio $f_i(r|a)/f_i(r|\hat{a})$ is increasing in r.

Assumptions' 1 and 2 are self–explanatory. Assumption 3 ensures that the principal cannot, with certainty, rule out certain actions on the basis of observed rewards. Such an assumption is standard in the literature on imperfect monitoring (e.g. Abreu, et.al., 1990). The MLRP condition has been extensively employed in a number of papers both in the principal–agent literature (e.g. Grossman and Hart, 1983) and elsewhere (Milgrom, 1981). An important implication of this assumption is that $F_i(.|a)$ is ordered according to first–order stochastic dominance.

With regard to the agents' preferences, an important concept is the following: let $g(x,y)$ be a function from (subsets of) $\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}$. Then $g(.)$ is *supermodular* in x and y if whenever $(x,y) >> (\hat{x},\hat{y})$, we have that $g(x,y) - g(x,\hat{y}) > g(\hat{x},y) - g(\hat{x},\hat{y})$; if the above inequality holds weakly $g(.)$ will be weakly supermodular.[15] Supermodularity requires that as (say) y increases, the impact on g of increasing x increases as well; note that if $g(.)$ is additively separable in x and y then it is not supermodular, but it is weakly so. This condition has been widely utilized in studying adverse selection models (e.g. Spence, 1979), games with strategic complementarities (e.g. Milgrom and Roberts, 1990) and a variety of other situations.[16]

Now to motivate the assumptions we place on the agents' utility functions, consider the quasi–linear case described in section 2 above, where the per–period utility of a type ω agent who takes an action a, and receives an income compensation of y, is $y - c(a;\omega)$. Suppose, as is standard in principal–agent models, that $c(a;.)$ is continuous in a, decreasing in ω, and submodular in (a,ω), i.e. $(a,\omega) >> (\hat{a},\hat{\omega})$ implies $c(a;\omega) - c(a;\hat{\omega}) < c(\hat{a};\omega) - c(\hat{a};\hat{\omega})$. Let α be a given pair of sharing rules, and $E_i(\alpha)(a,\omega)$ the expected payoffs to the agent in period i. Now note the following: regardless of the form of the sharing rules embodied in α, is it true that

[15]If the function $g(.)$ is twice–continuously differentiable, then this condition becomes $\partial g^2/\partial x \partial y > 0$.

[16]See Milgrom and Shannon (1992) for a detailed analysis of supermodularity (and its variants), as well as further references of applications.

(i) $E_i(\alpha)(.,.)$ is supermodular in (a,ω) and is continuous in a; and

(ii) $E_i(\alpha)(a,.)$ is increasing on Ω.

Further, since the first period reward does not enter into any of the second period functions, we trivially have that

(iii) $E_2(\alpha)(.)$ is weakly supermodular in (r,ω) for each fixed $a \in A$, and weakly supermodular in (r,a) for each fixed $\omega \in \Omega$; and

(iv) $E_2(\alpha)(.)$ is nondecreasing on S for each (a,ω).

All we require of the agents' reduced form utilities u_1,u_2 is that conditions' (i) – (iv) hold, as well as the condition that E_2 is strictly quasi–concave on A, with the latter guaranteeing a unique best action by the agent of any type in the second period; alternatively, we can dispense with this additional restriction if conditions' (iii) and (iv) hold in their *strict*, rather than weak, incarnations, with the latter approach being adopted below. Hence as long as the manner in which first period rewards interact with second period preferences obey these restrictions, we are fine. For example, if first period reward enters into the second period cost function as (say) a substitute for a or ω, then conditions (iii) and (iv) will follow. Alternatively, if the second period sharing rule α_2 is a function of the first period reward $\hat{r}$, then as long as α_2 is non–decreasing in $\hat{r}$ and supermodular in $(r,\hat{r})$ then (iii) and (iv) follow as well.

In terms of the per–period utility functions u_1,u_2, for the agents, then, we have:

<u>Assumption 5</u>: $u_1(a,\omega)$ is supermodular in (a,ω), and is continuous in a.

<u>Assumption 6</u>: $u_2(a,.)$ is increasing on Ω.

<u>Assumption 7</u>: u_2 is supermodular in (r,a),(a,ω), and (r,ω), and is continuous in a.

<u>Assumption 8</u>: u_2 is increasing on Ω for each (r,a), and is non–decreasing on S for each (a,ω).

Finally, with respect to the principal's preferences we have:

Assumption 9: $v_1(.)$ and $v_2(.;\hat{r})$ are increasing on S; and $v_2(r;.)$ is non–decreasing on S.

We will prove the following result:

Theorem 1: *There exists an equilibrium in anonymous strategies* $(\sigma^*,\mu^*,\gamma^*)$ *where* σ^* *is a cut–off strategy,* (μ^*,γ^*) *is a type–monotone strategy, and where* γ_k^* *is non–decreasing on S for each k.*

Proof. Theorem 1 is proved via several lemmata. We begin with an examination of the agents' best–response problem when the principal adopts a cut–off strategy.

The agents' best response problem

Let σ be any cut–off strategy for the principal, with cut–off point $\bar{r} \in S$. Henceforth, we will refer to this strategy simply by the point $\bar{r}$.

A) *the agents' second period problem*

In the second period, an agent of type ω who has been retained after generating a reward of r in the first period solves the following problem:

$$\max \{u_2(r,a,\omega) \mid a \in A\}.$$

Since v is continuous and A is compact, a maximum exists. Let

$$V(r,\omega) = \max \{u_2(r,a,w) \mid a \in A\}$$

$$A(r,\omega) = \operatorname{argmax} \{u_2(r,a,\omega) \mid a \in A\}.$$

Lemma 1. Let $a \in A(r,\omega)$, $\hat{a} \in A(\hat{r},\hat{\omega})$, and suppose $(r,\omega) \geq (\hat{r},\hat{\omega})$, $(r,\omega) \neq (\hat{r},\hat{\omega})$. Then $a \geq \hat{a}$.

Proof. By definition of a and $\hat{a}$,

$$u_2(r,a,\omega) \geq u_2(r,\hat{a},\omega)$$
$$u_2(\hat{r},\hat{a},\hat{\omega}) \geq u_2(\hat{r},a,\hat{\omega}).$$

Thus,

$$u_2(r,a,\omega) - u_2(\hat{r},a,\hat{\omega}) \geq u_2(r,\hat{a},\omega) - u_2(\hat{r},\hat{a},\hat{\omega}),$$

or

$$[u_2(r,a,\omega) - u_2(\hat{r},a,\omega)] + [u_2(\hat{r},a,\omega) - u_2(\hat{r},a,\hat{\omega})]$$
$$\geq [u_2(r,\hat{a},\omega) - u_2(\hat{r},\hat{a},\omega)] + [u_2(\hat{r},\hat{a},\omega) - u_2(\hat{r},\hat{a},\hat{\omega})].$$

Now if $a < \hat{a}$, this last inequality violates the supermodularity in (r,a) and (a,ω). (If u_2 is only weakly supermodular in (r,a) but is strictly quasi–concave on A, then if $a \neq \hat{a}$ all of the above inequalities would be strict and the result would again follow.)□

Lemma 2. $V(r,\omega)$ is continuous and non–decreasing in r, and increasing in ω.

Proof. Immediate from Assumption 8. □

Lemma 3. $V(r,\omega)$ is weakly supermodular in (r,ω), i.e. $(r,\omega) >> (\hat{r},\hat{\omega})$ implies

$$V(r,\omega) - V(\hat{r},\omega) \geq V(r,\hat{\omega}) - V(\hat{r},\hat{\omega}).$$

Proof. Pick $a \in A(r,\omega)$ and $\hat{a} \in A(\hat{r},\hat{\omega})$. Let a_1,a_2 be any points from $A(\hat{r},\omega)$ and $A(r,\hat{\omega})$, respectively. By Lemma 1, $a \geq a_1$ and $a_2 \geq \hat{a}$. We consider two cases:

Case 1: $a_1 \geq a_2$.

Here, we have

$$V(r,\omega) - V(r,\hat{\omega}) = u_2(r,a,\omega) - u_2(r,a_2,\hat{\omega})$$
$$= [u_2(r,a,\omega) - u_2(r,a_2,\omega)] + [u_2(r,a_2,\omega) - u_2(r,a_2,\hat{\omega}]$$
$$\geq [u_2(r,a,\omega) - u_2(r,a_2,\omega)] + [(u_2(\hat{r},a_2,\omega) - u_2(\hat{r},a_2,\hat{\omega})]$$

(by the supermodularity of u_2 in (r,ω))

$$\geq [u_2(r,a,\omega) - u_2(r,a_2,\omega)] + [(u_2(\hat{r},a_2,\omega) - u_2(\hat{r},\hat{a},\hat{\omega})] \qquad (*)$$

(since $\hat{a}$ is optimal at $(\hat{r},\hat{\omega})$).

Now note that

$u_2(r,a_1,\omega) - u_2(\hat{r},a_1,\omega) \geq u_2(r,a_2,\omega) - u_2(\hat{r},a_2,\omega)$ (since $a_1 \geq a_2$ and u_2 supermodular in (r,a)), and so

$$u_2(\hat{r},a_2,\omega) - u_2(r,a_2,\omega) \geq u_2(\hat{r},a_1,\omega) - u_2(r,a_1,\omega) \geq u_2(\hat{r},a_1,\omega) - u_2(r,a,\omega),$$

where this last inequality follows since $a \in A(r,\omega)$. Rearranging terms, then, we have

$$u_2(r,a,\omega) - u_2(r,a_2,\omega) \geq u_2(\hat{r},a_1,\omega) - u_2(\hat{r},a_2,\omega).$$

Substituting in (*), we finally obtain

$$\begin{aligned} V(r,\omega) - V(r,\hat{\omega}) &\geq u_2(\hat{r},a_1,\omega) - u_2(\hat{r},\hat{a},\hat{\omega}) \\ &= V(\hat{r},\omega) - V(\hat{r},\hat{\omega}), \end{aligned}$$

as desired.

Case 2: $a_1 < a_2$.

Similar to case 1, with the obvious changes (use $(r,\hat{r})$ in place of $(\omega,\hat{\omega})$ to get the result). □

To sum up, the set of optimal second period actions $A(r,\omega)$ for an agent of type ω who generated a reward of r in the first period, is ordered in (r,ω), i.e.

$(r,\omega) > (\hat{r},\hat{\omega})$, $a \in A(r,\omega)$ and $\hat{a} \in A(\hat{r},\hat{\omega})$ imply $a \geq \hat{a}$.

Moreover, the corresponding value function $V(r,\omega)$ is non–decreasing in r, increasing in ω, and supermodular in (r,ω).

B) *the agents' first period problem*

In his first period, an agent of type ω solves

$$\max \{u_1(a,\omega) + \delta \int_{r \geq \bar{r}} V(r,\omega) dF_1(r|a) \mid a \in A\},$$

where $\bar{r}$ is the cut–off point. For notational simplicity let

$$H(\bar{r},a,\omega) = \delta \int_{r \geq \bar{r}} V(r,\omega) dF_1(r|a).$$

Then the problem is

$$\max \{u_1(a,\omega) + H(\bar{r},a,\omega) \mid a \in A\}.$$

Lemma 4. H is weakly supermodular in (a,ω) for each $\bar{r}$.

Proof. For $\omega > \hat{\omega}$,

$$H(\bar{r},a,\omega) - H(\bar{r},a,\hat{\omega}) = \delta\{ \int_{r \geq \bar{r}} [V(r,\omega) - V(r,\hat{\omega})] dF_1(r \mid a)\}.$$

By Lemma 3, the difference $[V(r,\omega) - V(r,\hat{\omega})]$ is non–decreasing in r. Since F_1 is ordered according to stochastic dominance in a, the result follows. □

Lemma 5. $H(.,.,.)$ is continuous in its arguments.

Proof. This is an immediate consequence of the atomlessness of F_1, and the joint continuity of f_1 on $S_1 \times A$. □

Now define

$$A^*(\bar{r},\omega) = \operatorname{argmax} \{u_1(a,\omega) + H(\bar{r},a,\omega) \mid a \in A\}.$$

By Lemma 5 and A compact $A^*(\bar{r},\omega) \neq \phi$; by Lemma 4 and u_1 supermodular in (a,ω) the elements of $A^*(\bar{r},\omega)$ are ordered in ω for each fixed $\bar{r}$; and by the Maximum Theorem, $A^*(\bar{r},\omega)$ is an upper–semicontinuous correspondence in $(\bar{r},\omega)$. Thus, letting $\Sigma^*(\bar{r},\omega) = P(A^*(\bar{r},\omega))$ be the set of mixed best actions for an agent of type ω facing a cut–off of $\bar{r}$, we obtain the following:

(i) Σ^* is a non–empty valued, convex valued, upper–semicontinuous correspondence on $S \times \Omega$;

(ii) For each fixed $\bar{r}$, the elements of $\Sigma^*(\bar{r},\omega)$ are ordered in ω in the sense that if $\mu \in \Sigma^*(\bar{r},\omega)$ and $\hat{\mu} \in \Sigma^*(\bar{r},\hat{\omega})$, and $\omega > \hat{\omega}$, then there exist intervals $[a,b] \subset A$ and $[\hat{a},\hat{b}] \subset A$ such that $a \geq \hat{b}$ and $\mu([a,b]) = 1 = \hat{\mu}([\hat{a},\hat{b}])$.

The principal's best response problem

Let anonymous strategies $(\mu_k, \gamma_k)_{k=1}^{n}$ for the agents be given, where (i) $\mu_k \in P(A)$ is a mixed action that is the first period action of a type ω_k agent; and (ii) $\gamma_k:S \to A$ is a function that specifies for each realized first period reward $r \in S$, a second period action $\gamma_k(r) \in A$ for an agent of type ω_k. Suppose also that

– For $k = 1,\ldots,n-1$, $\gamma_{k+1}(r) \geq \gamma_k(r)$ $\forall$ $r \in S$

– $\gamma_k(.)$ is non–decreasing on S for each $k \in \{1,\ldots,n\}$

– The μ_k's are ordered in the sense that there exist (possibly degenerate) intervals $[a_k, b_k] \subset A$ such that

$$a_{k+1} \geq b_k,\ k = 1,\ldots,n-1$$
$$\mu_k([a_k, b_k]) = 1,\ k = 1,\ldots,n$$

We will show that the principal's best response to such behavior by the agents is to use a cut–off strategy; and that the set of optimal cut–offs is an upper–semicontinuous convex valued correspondence in the agents' strategies.

The expected second period reward to the principal from retaining the incumbent, denoted $R(r,\mu,\gamma)$, is given by

$$R(r,\mu,\gamma) = \Sigma_k\{\beta_k(r)\int v_2(.;r)f_2(.\,|\,\gamma_k(r))d.\},$$

where $\beta_k(.)$ is as defined in section 2 above. The following lemma implies almost immediately that $R(r,\mu,\gamma)$ is increasing in r.

Lemma 6. $\varphi(r\,|\,\mu_k)/\varphi(r\,|\,\mu_j)$ is non–decreasing in r if $k > j$.

Proof. $$\frac{\varphi(r\,|\,\mu_k)}{\varphi(r\,|\,\mu_j)} = \int_A \left\{\frac{f_1(r\,|\,\hat{a})}{\int_A f_1(r\,|\,\bar{a})\mu_j(d\bar{a})}\right\} \mu_k(d\hat{a})\ .$$

Now

$$\left[\frac{f_1(r|\hat{a})}{\int f_1(r|a)\mu_j(da)}\right]^{-1} = \int_{a_j}^{b_j} \left[\frac{f_1(r|a)}{f_1(r|\hat{a})}\right]\mu_j(da)\,.$$

Since the support of μ_k is contained in $[a_k,b_k]$, we may take $\hat{a} \in [a_k,b_k]$ without loss of generality. Since $k > j$, we have $a_k \geq b_j$ by hypothesis, so $\hat{a} \geq b_j$. Thus the bracketed term on the RHS of the last expression is non–increasing in r by Assumption 4, and so then is the integral of this term. It follows then easily that $\varphi(r|\mu_k)/\varphi(r|\mu_j)$ is non–decreasing in r. □

Lemma 7. $R(r,\mu,\gamma)$ is non–decreasing in r.

Proof. As r increases, lemma 6 shows that beliefs (weakly) shift towards higher types. Further, $v_2(\hat{r};.)$ is nondecreasing in r, and since $\gamma_k(.)$ is non–decreasing in r and $\gamma_{k+1}(r) \geq \gamma_k(r)$ for all k,r, the then result follows from the ordering of the reward distributions according to stochastic dominance. □

It is intuitively obvious from lemma 7 that a cut–off strategy for the principal is optimal. For, suppose $V^*(\mu,\gamma)$ represents the value function for the principal each time she hires a new agent. (By the stationarity of the problem, $V^*(\mu,\gamma)$ is independent of when this occurs.) Then, since any agent must be replaced after two periods, the choice facing the principal after a first period reward of r from the incumbent is of the form

$$\max \{ R(r,\mu,\gamma) + \delta V^*(\mu,\gamma),\ V^*(\mu,\gamma) \}.$$

Since $R(r,\mu,\gamma)$ is non–decreasing in r, a cut–off strategy must be optimal.

Unfortunately, formalizing these ideas (in particular, showing that $V^*(\mu,\gamma)$ is well–defined) is notationally tedious. On the other hand, this formalization is necessary to show that this optimal cut–off varies continuously with (μ,γ).

Given (μ,γ), the best response problem facing the principal is a stationary dynamic programming problem (SDP) whose components are given by the following:

(1) The *state space* is the Cartesian product of $\{0,1\}$,S, and Δ^{n-1}. A typical state is denoted (s,r,ν), where s is the number of periods the current incumbent has left, r is the reward produced by the incumbent in the period immediately preceding, and ν is the current belief regarding the incumbent's type.

(2) The *action space* A is the two point set {"keep" (k), "dismiss" (d)}, whose components are self–explanatory. The set of feasible actions $\Psi(s,r,\nu)$ at the state (s,r,ν) is given by

$$\Psi(1,.,.) = A$$

$$\Psi(0,.,.) = \{d\}.$$

(3) The *reward function* λ is defined as follows: when the action d is taken at any state (s,r,ν), the immediate reward received is

$$\lambda((s,r,\nu),d) = \Sigma_k \{\pi_k \int r\varphi(r\,|\,\mu_k)dr\}$$

while the action k at the state (s,r,ν) leads to a reward of

$$\lambda((s,r,\nu),k) = \Sigma_k \{\nu_k \int \hat{r} f_2(\hat{r}\,|\,\gamma_k(r))d\hat{r}\} .$$

(4) Finally, the *transition probabilities* $q(.\,|\,(s,r,\nu),d)$ and $q(.\,|\,(s,r,\nu),k)$ are defined as: when the action d is taken at any state (s,r,ν), the new state is

$$(1,\hat{r},\beta_k(\hat{r})) \text{ with probability } \Sigma_k(\pi_k \varphi(\hat{r}\,|\,\mu_k))d\hat{r}.$$

If the action k is taken at state $(1,r,\nu)$, then the new state is

$$(0,r,\nu) \text{ with probability } 1.$$

This completes the description of the principal's best response problem. Since the state and action spaces are compact, the feasible action Ψ correspondence is continuous, the reward function λ is continuous, and the transition probabilities q are weakly continuous, standard results in stationary dynamic programming show that this problem is well–defined; an optimal strategy exists; and the value function $W(s,r,\nu)$ is continuous in its arguments.

It is trivial that the Bellman equation for the problem at the state $(1,r,\nu)$ can be written as

$$W(1,r,\nu) = \max \{R(r,\mu,\gamma) + \delta W(0,r,\pi), W(0,r,\pi)\},$$

where $R(r,\mu,\gamma)$ was defined earlier.

Now let

$$C(\mu,\gamma;\pi) = \begin{cases} \sup . S & \text{if } R(r,\mu,\gamma) < (1-\delta)W(0,r,\pi)\ \forall\, r \\ \inf . S & \text{if } R(r,\mu,\gamma) > (1-\delta)W(0,r,\pi)\ \forall\, r\,, \\ \{r \mid R(r,\mu,\gamma) = (1-\delta)W(0,r,\pi)\} & \text{o.w.} \end{cases}$$

where recall $\pi \in \Delta^{n-1}$ denotes the common prior on agent types. Since $R(r,\mu,\gamma)$ is non–decreasing in r by lemma 3.7, it is clear that the principal's set of optimal actions given (μ,γ) is $C(\mu,\gamma;\pi)$, i.e., given any $\bar{r} \in C(\mu,\gamma;\pi)$, an optimal strategy for the principal when facing (μ,γ) is to retain the agent if and only if the first period reward exceeds $\bar{r}$.

Finally, note that in the SDP defining the principal's best response problem, the reward function λ is jointly continuous in states, actions, and the triple $(\mu,\gamma;\pi)$; while the transition probabilities are jointly weakly continuous in states, actions, and $(\mu,\gamma;\pi)$. It follows from Dutta, Majumdar, and Sundaram (1991, Theorem 1) that treating $(\mu,\gamma;\pi)$ as the *parameters* of this SDP, the solution (i.e. the optimal action correspondence) is jointly upper–semicontinuous in states and parameters. In particular, $C(\mu,\gamma;\pi)$ is upper–semicontinuous in $(\mu,\gamma;\pi)$.

Summing up, the principal's best response to the anonymous agent strategy (μ,γ) is to use a cut–off rule. The set of optimal cut–off rules $C(\mu,\gamma;\pi)$ define an upper–semicontinuous, convex valued correspondence in (μ,γ) and in π.

The fixed–point theorem

The principal's space of strategies may evidently taken to be S. The agent's second period action does not depend on the principal's cut–off strategy; thus we may take the agent's second period action vector $(\gamma_1,...,\gamma_n)$ to be any selection from $A(r,\omega)$ (recall that $A(r,\omega)$ is the set of optimal actions for an agent of type ω who produced the first period reward r), i.e. such that

$\gamma_k(r) \in A(r,\omega_k)\ \forall\ r \in S,\ k \in \{1,\ldots,n\}.$

Pick any selection $\gamma = (\gamma_1,\ldots,\gamma_n)$ and hold it fixed through the remainder of the argument. Note that by the double monotonicity of A(.,.), we must have

$\gamma_k(r) \geq \gamma_j(r)\ \forall\ r \in S,\ k>j$, and

$\gamma_k(.)$ non–decreasing on S for each k.

Now define $Z^n \subset \underbrace{P(A)\times\ldots\times P(A)}_{n\text{-times}}$ by

$$Z^n = \{\ (\mu_1,\ldots,\mu_n)\ |\ \mu_k \in P(A)\ \forall k,\ \exists\ [a_k,b_k] \subset A \text{ s.t. } a_{k+1} \geq b_k\ \forall k\ \&\ \text{supp.}\mu_k \subset [a_k,b_k]\ \forall k\}.$$

From the agent's best response problem, we know that when the principal adopts a cut–off strategy $\bar{r}$, the set of optimal first period actions for an agent of type ω_k is given by $\Sigma^*(\bar{r},\omega_k)$, where this is an upper–semicontinuous, convex valued correspondence. Define

$B^*(\bar{r}) = (\Sigma^*(\bar{r},\omega_1),\ldots,\Sigma^*(\bar{r},\omega_n))$.

Then B^* is an upper–semicontinuous convex valued correspondence from S into Z^n (that B^* maps into Z^n follows from lemmata 4–5).

From the principal's best response problem we know that for any $\mu \in Z^n$, the principal has an optimal response which is a cut–off strategy. The set of optimal cut–offs is given by $C(\mu,\gamma)$, which is an upper–semicontinuous and convex valued correspondence from Z^n into S.

Let $D{:}S\times Z^n \to S\times Z^n$ be defined as

$D(\bar{r},\mu) = (C(\mu,\gamma),B^*(\bar{r}))$.

Then D is an upper–semicontinuous convex valued map from $S\times Z^n$ into itself. Further, S is compact and convex by hypothesis, and it is easy to check that (under the weak topology) Z^n is also compact, and evidently convex. Therefore the existence of a fixed–point follows from Glicksberg's Fixed–Point Theorem.

Finally, such a fixed–point is an equilibrium by construction. Also by construction, it satisfies the remaining properties outlined in Theorem 1. QED

We close this section with a result establishing upper–hemicontinuity of the equilibrium correspondence in π, the common prior on all untried agents. Since the optimal second period actions by the agents are evidently not a function of the prior π, we fix second period actions at some γ^* and then characterize equilibria by the pair (σ^*,μ^*); thus given $\pi \in \Delta^{n-1}$, $E(\pi) = \{(\sigma^*,\mu^*)\}$ denotes the set of all anonymous equilibria of Theorem 1.

Theorem 2: *E is an upper–hemicontinuous correspondence on* Δ^{n-1}.

Proof: Let $\pi_s \to \pi$. Suppose $(\sigma^*_s,\mu^*_s) \in E(\pi_s)$ for each s. Since Z^n and S are compact, we may assume, without loss of generality, that $r^*_s \to r^* \in S$ (where r^*_s is the cut–off point associated with the strategy σ^*_s), and $\mu^*_s \to \mu^* \in Z^n$. From the proof of Theorem 1 we have that $C(.,.;.)$ is upper–hemicontinuous on $Z^n \times A^n \times \Delta^{n-1}$. By definition, $r^*_s \in C(\mu^*_s,\gamma^*,\pi_s)$ for each s, establishing $r^* \in C(\mu^*,\gamma^*,\pi)$. Also, $\mu^*_s \in B^*(r^*_s)$; and from the proof of Theorem 1 $B^*(.)$ is an upper–semicontinuous correspondence on Z^n. Therefore $\mu^* \in B^*(r^*)$, and thus $(\sigma^*,\mu^*) \in E(\pi)$, completing the proof. □

In particular, as the prior π converges to e_k for some k, where e_k is the k–th unit vector in $\mathbb{R}^n$ (i.e. the vector with 1 in the k–th place and zeros elsewhere), the equilibria converge to equilibria of the "pure moral hazard" case wherein the principal knows the agent to be of type ω_k with probability one. Such limiting behavior will be explored in detail below.

4. Equilibria in a Flat Environment

In this section we consider the equilibria characterized by Theorem 1 in an environment wherein the preferences of the agent as well as the reward distributions are the same in an agent's first period and second period of employment. Thus, let $F(.|a)$, $a \in A$, be the common reward distribution, while $u(a,\omega)$ $a \in A, \omega \in \Omega$, describe the agent's per period payoffs. Further, since in this environment the agent's first period reward does not effect her second period payoffs, we add the assumption that u(.) is strictly quasi–concave on A: let $a_k^m \equiv \text{argmax}\ \{u(a,\omega) \mid a \in A\}$.

Theorem 3: *Let $(\sigma^*,\mu^*,\gamma^*)$ be an equilibrium as in Theorem 1; then for* $k = 1,\ldots,n$, $a_k \geq \gamma_k^*$, *where* a_k *is the lower bound on the support of* μ_k^*; *i.e., agent's actions are (weakly) decreasing over time.*

Proof. Let $V(\omega) = \max\{u(a,\omega) \mid a \in A\}$ denote the agent's second period payoff, where $V(\omega) > 0$; and for $t \in \{0,1\}$ define

$$U(t,a,\omega) = u(a,\omega) + \delta t[1-F(\bar{r}|a)]V(\omega),$$

where $\bar{r}$ is the cut–off employed by the principal. Thus $U(0,a,\omega)$ gives the maximand for the agent's second period problem, while $U(1,a,\omega)$ gives the maximand for the first period. Now if $\bar{r}$ is equal to either inf.S or sup.S, or if $\delta = 0$, the agent will choose the same action (namely a_k^m) in either period and hence the conclusion follows; therefore assume $\bar{r} \notin \{\text{inf.S},\text{sup.S}\}$ and $\delta > 0$. Let

$$A(t,\omega) = \text{argmax}\ \{U(t,a,\omega) \mid a \in A\},$$

and select $a \in A(1,\omega)$, $\hat{a} \in A(0,\omega)$ (where these exist by the continuity of U(.) and the compactness of A). Then by definition

$$U(1,a,\omega) \geq U(1,\hat{a},\omega)$$

$$U(0,\hat{a},\omega) \geq U(0,a,\omega).$$

rearranging terms,

$$U(1,a,\omega) - U(1,\hat{a},\omega) \geq U(0,a,\omega) - U(0,\hat{a},\omega)$$

or

$$u(a,\omega) - u(\hat{a},\omega) + \delta V(\omega)[F(\bar{r}|\hat{a}) - F(\bar{r}|a)] \geq u(a,\omega) - u(\hat{a},\omega),$$

implying (since $\delta V(\omega) > 0$) $a \geq \hat{a}$. Finally, since in equilibrium $\mu_k^* \in P(A(1,\omega_k))$, the result follows. □

By adding some smoothness and interiority assumptions, we can generate "strict" rather than "weak" predictions:

Corollary 1: *Let* $(\sigma^*,\mu^*,\gamma^*)$ *be an equilibrium as in Theorem 1, and in addition assume* (i) $u(.)$ *is continuously differentiable,* (ii) $\delta > 0$, (iii) $A = [\underline{a},\bar{a}]$ *with* $\gamma_k^* \in$ int.A *for all* k, *and* (iv) $u(\bar{a},\omega) < 0$. *Further, assume* $\pi_k > 0$ *for at least two values of k. Then*

(a) $\gamma_{k+1}^* > \gamma_k^*$, $k = 1,\ldots,n-1$;

(b) $a_{k+1} > b_k$, $k = 1,\ldots,n-1$;

(c) $a_k > \gamma_k^*$, $k = 1,\ldots,n$; and

(d) $r^* \in$ int.S, *where* r^* *is the cut-off associated with the strategy* σ^*.

Proof: Since $\gamma_k^* \in$ int.A for all k, we have $\partial u(\gamma_k^*,\omega_k)/\partial a = 0$ for all k; that (a) holds is now immediate by Assumption 7 (i.e. the supermodularity of u(.) in (a,ω)). Next, suppose that $r^* \in$ {inf.S, sup.S}; then agents take their myopic action γ_k^* in both periods. However, since these actions are strictly monotone in type, we get that the principal's one–period reward from retaining the incumbent agent is strictly increasing in the first period reward (cf. Lemmata 3.6–7 in the proof of Theorem 1 above). And for a sufficiently high first period reward, this second period reward from retaining the agent will dominate that obtained from replacing him; consequently the equilibrium breaks down, and therefore in any equilibrium $r^* \notin$ {inf.S, sup.S}, proving (d).

Next, let

$$U(t,a,\omega) = u(a,\omega) + \delta t[1-F(r^*|a)]V(\omega)$$

as in the proof of Theorem 3. Then since $\delta > 0$ and $r^* \in$ int.S, it is easily seen that $U(..,\omega)$

is supermodular in (t,a), thereby implying (c); and $U(1,.,.)$ is supermodular in (a,ω), implying (b). □

The equilibria described in Corollary 1 possess a number of interesting features with regard to the interaction between the principal and the agents. The first is that there exists a "performance" effect from the principal's retention decision, in that all types of agent work harder in the first period than they would have otherwise (i.e. myopically) in response to the incentives to be employed a second period. Further, there is a "selection" effect as well, in that not all agent types are equally likely to be employed for this second period, but rather higher types are more likely than lower types. And while an agent employed for a second period will necessarily take a lower action than in the first period, this selection effect will outweigh the performance effect if the principal's posterior is sufficiently optimistic, which occurs (by MLRP) for sufficiently high observed rewards.

Corollary 1 requires there to exist at least two types who might actually exist, since otherwise no "learning" on the part of the principal takes place. It turns out that it is a simple matter to identify the anonymous equilibria of Theorem 3 when this condition fails to hold i.e. in the "pure moral hazard" version of our model in which $\pi = e_k$ for some k, where e_k is the k–th unit vector in $\mathbb{R}^n$.[17]

Corollary 2: *Suppose* $\pi = e_k$ *for some* k; *then in any anonymous equilibrium agents take the myopic action* a_k^m *in both periods.*

Proof: By Theorem 3 type k's (mixed) actions are weakly decreasing over time. Now if μ_k^* places positive probability on *any* action greater than a_k^m, the principal's unique best response is to always replace an incumbent agent, i.e. set r^* equal to sup.S; but then the

[17]This is the case that has been studied by, for instance, Radner (1986).

agent's unique best first period action is a_k^m, contradicting the hypothesis. □

Therefore, in the absence of uncertainty concerning the agents' types, and hence no possibility of selection effects, there will not exist performance effects either; that is, the principal cannot provide performance incentives in any anonymous equilibrium.

Note that there are two "cut–off" equilibria consistent with Corollary 3, namely where $r^* = \sup.S$ and where $r^* = \inf.S$. Further, by Theorem 2 we know that as the uncertainty about agent types vanishes, the equilibria of Theorem 3 converge to the equilibria of Corollary 2: in particular, the equilibrium first period actions of the agents converge to their myopic actions as π converges to some e_k.

Corollary 1, demonstrating that agents take higher actions in the first period of employment than in the second in the presence of adverse selection, and Corollary 2, showing how this fails to hold when there is no adverse selection, raise the following possibility: might it be the case that the principal is actually better off *with* adverse selection than without? In particular, suppose we compare the principal's equilibrium payoff under two scenarios, the first where only the "best" type of agents exist (i.e. $\pi_n = 1$) and the second where there is some uncertainty about agents' types ($\pi_n < 1$). From Corollary 2 we know that under the former the principal receives the reward generated by a_n^m, the best type's myopic action, in every period, whereas in the latter performance effects might be present, implying all types take higher actions in their first period of employment. Certainly, it appears plausible that under some appropriate set of circumstances – say, the probability of type ω_n is arbitrarily close to unity, and the "worse" types are only marginally worse than ω_n – the principal could be made better off.

Perhaps unfortunately, this intuition is misguided: the following result shows that the principal can never become worse off if only "better" types are added, and can never become better off if only worse types are added. Recall that e_k denotes the k–th unit vector in $\mathbb{R}^n$.

Theorem 4: *For any* $\pi \in \Delta^{n-1}$, *let* $V(\pi)$ *denote the payoff the principal obtains in some anonymous equilibrium; then* $V(e_n) \geq V(\pi) \geq V(e_1)$.

Proof: Pick any $\pi \in \Delta^{n-1}$, $\pi \neq e_n$. If $\pi = e_k$ for some $k \neq n$, the result is trivial, so suppose $\pi \neq e_k$ for any k. Let $\mathfrak{K}$ be the set of k for which $\pi_k > 0$, let $(\sigma^*,\mu^*,\gamma^*)$ be any equilibrium of the game given π, and for any first period reward r let $\beta(r) = (\beta_k(r))_{k\in\mathfrak{K}}$ represent the principal's beliefs regarding the incumbent agent's type in this equilibrium after observing r. In their last period, all agents necessarily take their myopic action a_k^m; thus, given that the current incumbent has generated a reward of r, the principal solves

$$\text{Maximize}\{V(\pi), \Sigma_{k\in\mathfrak{K}}\, \beta_k(r)E(a_k^m) + \alpha V(\pi)\},$$

where $E(a_k^m) \equiv \int v(r)dF(r|a_k^m)$. We consider two cases: first, suppose the set $\{r \mid \Sigma_{k\in\mathfrak{K}}\beta_k(r)E(a_k^m) \geq (1-\alpha)V(\pi)\}$ is non–empty. Then, since $E(a_n^m) \geq E(a_k^m)$ for all $k \neq n$, we also have $E(a_n^m) \geq (1-\alpha)V(\pi)$, and since $E(a_n^m) = (1-\alpha)V(e_n)$ by Corollary 2, we are done. So suppose the second case holds, i.e., for all r, we have $(1-\alpha)V(\pi) > \Sigma_{k\in\mathfrak{K}}\beta_k(r)E(a_k^m)$. Then, necessarily, all agents are dismissed after one period. But this in turn means that in their first period, all agents will take myopic actions, implying $V(\pi) = \Sigma_{k\in\mathfrak{K}}\pi_k E(a_k^m)/(1-\alpha)$ and hence that again $V(\pi) \leq V(e_n)$.

The second part of Theorem 4, that $V(\pi) \geq V(e_1)$ is proved analogously. □

Therefore the performance effects present in any adverse selection scenario are never enough to overcome the possibility of "adverse selection, namely, the possibility of retaining an incumbent who is less than the best type.

An intuitively rather appealing case we could consider – and one regarding which Theorem 4 has nothing to offer – is one where we introduce adverse selection by making the agent–type of the moral hazard model, an "average" type of the adverse selection model. Unfortunately, "average" has no single obvious meaning in our context. One option, for instance, is to suppose only that the myopic actions of the agents in the adverse selection model imply the same expected payoffs for the principal as that from the single agent–type in the moral hazard model; that is, we have $\Sigma_{k \in \mathcal{K}} \pi_k E(a^m_k) = E(a^m_\ell)$, where ω_ℓ is the single type of the moral hazard model, $\mathcal{K}$ is the set of agents in the adverse selection model, and π_k is the probability in the adverse selection model of type $k \in \mathcal{K}$.

In this case, it is not too difficult to see that there are conditions under which the principal can improve from the presence of adverse selection. Pick any equilibrium of the type in Theorem 3, and suppose $r^* \in S$ is the cut–off point in this equilibrium. Then, since all agents choose (weakly) higher actions than their myopic actions in the first period, the principal's first period reward, denoted say $\overline{R}$, is at least $E(a^m_\ell)$. Let $R(r)$ represent the principal's second–period reward from retaining an incumbent who produced a reward of r in the first period. We claim that $R(r^*) \geq \overline{R}$. For, suppose the reverse (strict) inequality obtained. Then, letting V^* denote the principal's value of this equilibrium, we have $\overline{R} + \alpha V^* > R(\overline{r}) + \alpha V^* \geq V^*$, so $V^* < \overline{R}/(1-\alpha)$. But this is impossible, since the principal can always set the cut–off equal to sup.S, a strategy which has a value of $\overline{R}/(1-\alpha)$. Thus, $R(r^*) \geq \overline{R} \geq E(a^m_\ell)$. Since R(.) is a non–decreasing function, we are done. Note in particular, that if the conditions of Corollary 1 are met, the principal does strictly better in any cut–off equilibrium under adverse selection.

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