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by

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Abstract: This paper introduces a utility formulation to the well–known gambler's ruin problem. An agent who maximizes lifetime expected utility has to tradeoff short–term utility against longer–term survival prospects. The optimal tradeoff is established by way of characterizing the agent's value and optimal policy functions. Further, the scope of expected utility maximization is examined by contrasting the bankruptcy probabilities of an agent employing such a criterion with those of an agent who is more directly interested in survival. Economic applications of the theory are also discussed.

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1. Introduction

It is widely acknowledged that the threat of bankruptcy[1] directly affects economic decision–making and that, in the presence of uncertainty, agents may be unable to guarantee survival. Yet the problem of bankruptcy is either not modelled or effectively side–stepped in a number of dynamic economic models. In a general equilibrium model with complete Arrow–Debreu markets (and hence a single lifetime budget constraint), borrowing in the event of negative current net worth is permissible. In portfolio allocation or consumption–savings models in which there are borrowing constraints, a standard assumption is the presence of at least one safe asset; bankruptcy can always be avoided and in an optimal solution, under usual conditions, is in fact avoided.[2]

The objective of this paper is to investigate a simple dynamic model in which not all risks can be hedged and agents are restricted in their ability to borrow. I do not propose any general theory but rather study a specific single–agent decision problem – the "gambler's ruin" – which has been widely used in probability theory to analyze bankruptcy, after adapting it appropriately for economic applications. In this model, I examine two questions: maintaining the hypothesis of expected utility maximization, ***how does an agent optimally trade off utility against survival prospects, at different levels of wealth?*** Since there is always residual uncertainty, survival cannot be guaranteed in such a model. So, ***how do the bankruptcy prospects of an expected utility maximizer differ from those of an agent who is more explicitly interested in survival?***

The model I study incorporates one important generalization of the gambler's ruin problem. In the classic formulation of that problem (see Dubins–Savage (1965)) the agent's action is identified with a bet (whose returns are uncertain). Consequently, different actions are distinguishable only from the long–term view of which ones are better suited to survival. In economic applications, actions will correspond to effort or consumption or choice of projects etc., and consequently will need to be distinguished in terms of short–term utility consequences as well. And that is exactly what I do in this paper; I introduce a utility formulation in the pure ruin or pure survival problem.[3]

In the formulation studied here, the agent's wealth follows a diffusion process and an action is the choice of the incremental mean and variance of the process (and each action additionally yields some instantaneous utility). I first investigate the maximization of expected discounted lifetime utility (subject, of course, to the bankruptcy constraint). I show that the agent's value function is a strictly increasing, strictly concave C^2 function which satisfies the Bellman differential equation. (The smoothness property and the optimality equation are surprisingly difficult to prove and the approach has to be completely different from that employed in proving the same results in the gambler's ruin problem; this point is explained in greater detail in the sequel). I further show that there is a (stationary) Markov optimal policy for the agent's problem. In this policy, as the agent's wealth increases, he ***either*** picks higher variance ***or*** lower mean and mean–variance ratios. Indeed when the instantaneous utility function is separable in mean and variance, at higher wealth the agent is more willing to assume risk and picks ***both*** a higher variance as well as a lower mean[4] (and consequently higher instantaneous utility).

When agents can, in fact, go bankrupt it is not immediate that expected utility maximization is the only rational decision criterion; indeed people seem to sometimes employ a "pessimistic" criterion like the maximization of survival probabilities instead. I characterize the optimal policy in that case and show that it has a very simple form; it involves a constant action at every instant. Moreover, the optimal action is easily computed and, as the discount rate approaches zero, involves picking the action with the highest mean–variance ratio. By way of comparison, I provide a bound on the ratio of the probability of failure under survival maximization to the same probability under expected utility maximization. I show that, although both probabilities approach zero as the agent's initial wealth becomes unbounded, the rate of approach is faster for survival maximizers and hence the ratio of the two probabilities also tends to zero.

Finally, I show that the generalized gambler's ruin structure covers a number of interesting economic models including the well–known consumption–savings problem (see Deaton (1991)). I present a brief discussion of that problem under the usual assumption of borrowing constraints as well as the additional assumption of bankruptcy. I demonstrate that this additional constraint can change the optimal consumption policy quite significantly. I also believe that the current framework is sufficiently flexible so that it can be adapted, in future work, to study many–agent problems. That will allow an examination of several questions related to bankruptcy which arise in a market context; the most interesting of these is possibly the Alchian (1950)– Friedman (1950) argument that firms that do not maximize profits eventually go bankrupt.

In Section 2, I formulate the problem precisely. Section 3 contains results on the pure survival maximization problem while Section 4 presents existence and characterization results for expected utility maximization. Some comparative results and implications are discussed in Section 5. Section 6 presents two computable examples as well as a discussion of the consumption–savings problem. All proofs are in the appendix.

2. Basic Definitions and Assumptions

I examine a continuous time[5] optimization model in which the agent's wealth follows a diffusion process; $[Y(t): t \geq 0]$ on $[0, \infty)$ given by a stochastic differential equation

$$dY(t) = m(t)dt + v^{1/2}(t)\ dB(t),\ Y(0) = y \qquad (2.1)^6$$

where $[B(t): t \geq 0]$ is a standard Brownian motion and $[m(t),v(t)]$ are the instantaneous mean and variance of the incremental (normal) distribution $dY(t)$ and these are chosen from a set A. The initial wealth level is denoted y. Suppose further that the process is absorbed at the origin and let T denote the random time of absorption (which could be infinite and whose distribution depends on the strategy followed). The choice $[m(t),v(t)]$ yields a flow payoff at instant t which is denoted $U(m(t),v(t))$. At the time of absorption the decision maker receives a terminal payment (and from this point on, I normalize this payment to zero). Future payoffs are discounted at a rate $\delta \in (0,\infty)$.

The following ***assumptions*** on U and A are maintained :

(A1) The utility function $U(m,v)$: $\mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is continuous and strictly concave.

(A2) Variances are bounded away from zero: there is $b > 0$, s.t. $(m,v) \in A \Rightarrow v \geq b$.

(A3) The set A of feasible controls is a convex and compact subset of $\mathbb{R} \times \mathbb{R}_+$.

I study two distinct control problems. The first, the ***expected utility maximization*** problem, is to pick an admissible strategy to maximize expected discounted lifetime payoffs:

$$\text{(P1)} \qquad \text{Maximize } E\ \delta \int_0^T e^{-\delta t}\ U(m(t),v(t))\ dt \qquad (2.2)$$

Let $\overline{U} = \sup U(m,v),\ (m,v) \in A$. For this problem to be interesting we must have $\overline{U} > 0$. Optimization involves a tradeoff between instantaneous payoffs and a movement away from zero. A second optimization problem, ***survival maximization***, is to maximize expected discounted time to bankruptcy:

$$\text{(P2)} \qquad \text{Maximize } 1 - E\ e^{-\delta T} \qquad (2.3)$$

Admissibility of Strategies: Let $[B(t):t \geq 0]$ be a standard Brownian motion on some complete probability space $(\Omega, \mathfrak{F}, P)$ with a filtration $[\mathfrak{F}(t):t \geq 0]$ such that for each t, $\mathfrak{F}(t)$ is complete with respect to the measure P. Let $C[0,\infty)$ denote the space of continuous functions on the non–negative real line and let $Y[0,\infty)$ denote a generic element; a continuous function on $[0,\infty)$. $\Xi(t)$ will be the notation for the smallest σ–algebra of subsets of $C[0,\infty)$ which contains all sets of the form $\{Y[0,\infty)\colon Y(s) \leq b, s \leq t, b \in \mathbb{R}\}$. A ***strategy***, $[\pi(t) = m(t),v(t)\colon t \geq 0]$, is a specification $\pi(t)\colon C[0,\infty) \to A$, for all $t \geq 0$, and it is admissible for initial state y if it is progressively measurable with respect to $\Xi(t)$ and there exists at least one solution to the stochastic differential equation (2.2) which is $\mathfrak{F}(t)$–measurable for each t.[7] A strategy is said to be a ***stationary Markov strategy*** if $\pi(t) = \beta(y(t))$ for some measurable function $\beta\colon [0,\infty) \to \mathbb{R}$.

(P1) and (P2) are clearly stationary dynamic programming problems, and I shall denote the value functions, respectively, by $V_1(y)$ and $V_2(y)$. Any solution of (P1)–(P2) will be called an optimal strategy or policy. If a stationary Markov policy is optimal in the class of all admissible strategies, it will be called a stationary Markov optimal policy.

3. The Survival Problem: Existence and Characterization

In this section I study the simpler optimization problem; survival maximization or (P2). Since such a decision–maker does not have to trade off immediate utility against future wealth, the analysis is considerably simplified; indeed, I demonstrate explicitly the optimal strategy. The agent picks a constant control regardless of his wealth level; the only determinant of this choice is the discount rate.

The pure survival problem has been studied extensively as a gambling problem, first by Dubins–Savage (1965) and subsequently by a number of others (for example, Heath et.al. (1987), Orey et.al. (1987), Pestien–Sudderth (1985), and Sudderth–Weerasinghe (1989) etc.). These authors have examined a number of different criteria but almost all of them are relevant for undiscounted problems.[8] For completeness therefore, I present a characterization for the discounted case.

Pick any feasible action, $(m,v) \in A$ and define $-\lambda(m,v)$ as the negative root of the quadratic, $1/2\ vx^2 + mx - \delta = 0$; $\lambda(m,v) = \dfrac{m + \sqrt{m^2 + 2v\delta}}{v}$. Let $(\hat{m},\hat{v})$

be defined by $\lambda(\hat{m},\hat{v}) \in \text{argmax}\ \lambda(m,v)$, $(m,v) \in A$.

Theorem 3.1 **An optimal policy for a survival maximizing decision–maker is to use the control $(\hat{m},\hat{v})$ at all wealth levels. The lifetime returns to this policy, i.e. the value function, is $V_2(y) = 1 - e^{-\hat{\lambda} y}$, where $\hat{\lambda} \equiv \lambda(\hat{m},\hat{v})$. The value function is therefore strictly increasing, strictly concave and twice continuously differentiable.**[9]

Remark: Note that as $\delta \downarrow 0$, $\lambda(m,v) \to 2(m/v)$ if $m > 0$. Further, the criterion itself tends to a maximization of the probability of survival. So this suggests that the optimal policy for the undiscounted problem of maximizing survival probability, is to pick the control with the highest mean–variance ratio. This is similar to the results of Pestien–Sudderth (1985) and Sudderth–Weerasinghe (1989) in somewhat different versions of this problem.

4. The Expected Utility Problem: Existence and Characterization

In the general problem (P1) there is a tradeoff between immediate payoffs and continuation values; consequently, constant controls are typically not optimal. Hence, it is impossible, except in very special cases, to directly "guess" an optimal policy (and use a verification theorem to establish its optimality); we have to establish the existence of a value function with appropriate differentiability properties and the existence of optimal policies by analytical methods. As I explain below, there is no general result in the optimal control literature, that I am aware of, which directly covers this existence problem. Theorem 4.1 is the **existence** result for (P1). I then present **characterization** results for the value and optimal policy function. In particular, I investigate how the optimal action choice changes with the agent's wealth level. Again, the presence of a utility function considerably complicates the analysis.

The following theorem is the basic existence result:

Theorem 4.1 **The value function is C^2, $V(0)=0$, $\lim_{y \to \infty} V(y)=\overline{U}$ and satisfies**

$$\max_{(m,v)\in A} \{\ 1/2\ v\ V''(y) + m\ V'(y) - \delta\ V(y) + \delta\ U(m,v)\} = 0 \quad (4.1)$$

Furthermore, V is the unique solution of (4.1) in the class of functions which are twice continuously differentiable on $[0,\infty)$ and equal to the endpoints 0 and $\overline{U}$.

Finally, there is a continuous function β^*, such that the stationary Markov

strategy formed by this function is an optimal policy.

Remark 1: Theorem 4.1 is similar to a result proved in Krylov (1981, Theorem 1.4.5). There are two main differences between the results. Firstly, Krylov's result is valid when the state space is a compact interval (and indeed it does not seem that the proof would extend to unbounded domains). Secondly the result proved there requires the immediate payoff function U to satisfy a Lipschitz condition (and this rules out standard boundary conditions like Inada conditions). On the other hand, in Krylov's formulation, state dependence in the payoffs is admissible and he does not impose the condition of strict concavity on the payoff function. Furthermore, he allows a more general discounting structure including the undiscounted case, $\delta = 0$. Note also that the results from the portfolio allocation literature (for example, Karatzas–Shreve (1987)) cannot be employed for two reasons: it is critical for those results that there be a risk–free asset and in that problem, given any consumption, the mean and variance necessarily have a linear relationship to each other.

Remark 2: The result is also valid when the agent stops if he **either** goes bankrupt **or** reaches a high level of wealth, i.e. Theorem 4.1 covers the two–sided absorption problem as well. The characterization results that follow can also be derived for this case by employing similar techniques as those used here. Hence, I do not discuss the two–sided absorption problem in this paper.

I turn now to the characterization results:

Theorem 4.2 **i) The value function is strictly increasing in y.**
ii) Suppose that π^* is an optimal strategy. Then,

$$V'(y) = V'(0)\ Ee^{-\delta T^*(y)} \tag{4.2}$$

where,

$$T^*(y) = \min\ \{t\colon Y(t) = 0 \mid Y(0) = y,\ \pi^*\}.$$

iii) The value function is strictly concave. Further V′′ is strictly increasing in y.

Remark: The characterization (4.2), which will be seen to be extremely useful, is true even when the immediate payoff function is not concave; in particular, the strict concavity and monotonicity of the value function require neither monotonicity nor concavity of the utility function.

Consider the stationary Markov optimal strategy β^*. The function defining this strategy is given by the (unique) maximizer of the Bellman equation (4.1). From that equation it immediately follows that as the agent becomes wealthier, the change in his optimal mean–variance choice is (partly) determined by the behavior of the local risk–aversion index, V'', and the marginal valuation, V', in y. But evidently knowing the behavior of these two indices alone, which knowledge can be inferred from Theorem 4.2, is not sufficient; from (A.20) the reader can see that what we need to know is how V''/V', the coefficient of relative risk aversion, changes in y. Although, in general, this information is impossible to deduce, I now present results which develop alternative and increasingly more detailed characterizations of β^*, under some further restrictions.

Proposition 4.3 **Suppose $y' > y$. Let $\beta^*(y) \equiv m,v$ and $\beta^*(y') \equiv m',v'$. Then, it must be the case that either or both of the following monotonicity relations are satisfied: i) $v' \geq v$, ii) $m' \leq m$ and $m'/v' \leq m/v$. In words, either the optimal variance increases with wealth or the mean and mean–variance ratios decrease. In particular, if all actions have the same variance, then the optimal mean decreases with wealth.**

Remark: Proposition 4.3 holds without **any** assumptions whatsoever on U. It is my conjecture that under monotonicity and strict concavity restrictions on U, **both** i) and ii) will hold for an optimal policy; I have however been able to prove this only under some further conditions. I do have counterexamples to show that, without concavity assumptions, only i) or ii) but not both need hold (see Section 6).

β^* is said to be an interior optimal policy if $\beta^*(y) \in \text{int } A$, for all $y \in \mathbb{R}_+$. Suppose that U is differentiable and define $H_W: A \to \mathbb{R}$ as $H(m,v) = U(m,v) - vU_2(m,v) - mU_1(m,v)$.

Proposition 4.4. **Suppose β^* is interior. Then, y' > y implies that**

$$H(m',v') > H(m,v) \tag{4.3}$$

Further, the optimal policy β^* is a one to one function.

(4.3) establishes that in general the order of usage of optimal controls is nothing as straightforward as the mean–variance ratio or some other such simple index (unlike the pure survival problem). The utility function is said to be separable if there exist functions $\xi(m)$ and $\phi(v)$ such that

$$U(m, v) = \xi(m) - \phi(v).$$

Proposition 4.5 **Suppose that β^* is interior and U is separable. Then, $y' > y$ implies that $m' < m$, $v' > v$. Consequently, if the utility function is decreasing in m and increasing in v, then $U(m',v') > U(m,v)$. In an optimal policy at low wealth levels, the decision–maker picks lower utility actions which have however larger means and smaller variances.**

If the feasible set A has more structure, one can even dispose of the interiority assumption. We say that A is a rectangle if $A = [\underline{m}, \overline{m}] \times [\underline{v}, \overline{v}]$, $0 < \underline{v} \leq \overline{v}$.

Proposition 4.6 **Suppose that A is a rectangle and U is separable. Then, $y' > y$ implies $m' \leq m$, $v' \geq v$, (and under monotonicity assumptions), $U(m',v') \geq U(m,v)$.**

5. A Comparative Analysis of Expected Utility and Survival Maximization

In this section I present some results on the comparative survival properties of expected utility and survival maximizing agents. I will contrast the probability of failure under the optimal policies implied by each of those criteria; from Theorem 3.1 we know that this probability is strictly positive for both decision–makers. Note that since I consider a discounted problem, the appropriate index is $Ee^{-\delta T}$; $(1 - Ee^{-\delta T}$ converges to the survival probability as $\delta \downarrow 0)$.

It will be instructive to begin with a discussion of the comparative survival properties of myopic utility maximization; a myopic utility maximizer picks the constant control $(\overline{m},\overline{v})$ where $U(\overline{m},\overline{v}) \equiv \overline{U} = \max_{m,v} U(m,v)$. Denote $\lambda(\overline{m},\overline{v})$ as $\overline{\lambda}$ (and recall that $\overline{\lambda} \leq \hat{\lambda}$). Define the ***relative survivability*** of a myopic agent as

$$R(y) = \frac{Ee^{-\delta \hat{T}(y)}}{Ee^{-\delta \overline{T}(y)}}$$

the ratio of expected discounted times to failure, from initial wealth y, if the decision rules are respectively those of a myopic maximizer and a survival maximizer.

Proposition 5.1 **The relative survivability of a myopic maximizer is given by**

$$R(y) = e^{(\overline{\lambda}-\hat{\lambda})y} \tag{5.1}$$

Relative survivability of a myopic maximizer is a decreasing convex function with $R(0) = 1$ and $\lim_{y\to\infty} R(y) = 0$.

I turn now to comparing expected utility maximization against survival maximization. Since expected utility maximization does not have a simple representation for its optimal policy, the estimate here is going to be less precise. Let ρ be the relative survivability of an expected utility maximizer

$$\rho(y) = \frac{E e^{-\delta \hat{T}(y)}}{E e^{-\delta T^*(y)}}$$

where $T^*(y)$ is the (random) time to failure under the optimal policy of an expected utility maximizer if initial wealth is y. Since $\bar{U}(1 - Ee^{-\delta T^*}) \geq V(y) \geq \bar{U}(1 - Ee^{-\delta \bar{T}})$, it follows that $Ee^{-\delta T^*} \leq Ee^{-\delta \bar{T}}$ and hence $\rho(y) \geq e^{(\bar{\lambda}-\hat{\lambda})y}$. Additionally, I can show:

Proposition 5.2 **a) The relative survivability of expected utility maximization is a ratio of two convex functions which satisfies**

$$1 \geq \rho(y) \geq [1 + \frac{\varphi}{E e^{-\delta \hat{T}(y)}}]^{-1} \tag{5.2}$$

where $\varphi \equiv \bar{U}/\hat{U} - 1$.
b) Survival and expected utility maximization are perfectly congruent if and only if $\bar{U} = \hat{U}$. Further, $\rho(0) = 1$ and, if survival and expected utility maximization are not perfectly congruent, $\lim_{y\to\infty} \rho(y) = 0$.

6. Two Examples and An Economic Application

In this section I first present two simple computable examples of expected utility maximization. The examples show that a variety of policies are consistent with optimality. Since computable examples are hard to come by when controls have direct payoffs, the examples will have an artificial flavor; strong resrictions will be placed on the payoff function U and feasible set A to facilitate computation. In particular, the convexity assumptions which were used to prove Theorem 4.1 will not be satisfied by these examples (and that does not create problems because the verification result, that a solution of (4.1) and an associated stationary Markov policy are in fact optimal, is true even without such convexity restrictions).

Example 6.1: $U(m,v) = m^{\gamma} v^{\theta}$, $2\theta + \gamma \in (0,1)$. $A = \{(m,v)\colon v = m^2, m \geq b > 0\}$.

The solution I will check for is $V(y) = k(y+b)^{\alpha}$ and $\beta(y) = (y+b)/c, (y+b/c)^2$, where $k > 0$ and $c > 0$ and $\alpha \in (0,1)$ are choice variables whereas $b > 0$ is an exogeneous constraint defined by the set of controls A. Moreover, the candidate policy is feasible if $c < 1$. There are two conditions to verify: $1/2\beta_2(y)V''(y) + \beta_1(y)V'(y) - \delta V(y) + \delta U(\beta(y)) = \max_{m,v,} \{1/2vV''(y) + mV'(y) - \delta V(y) + \delta U(m,v)\} = 0$. It is straightforward to check that the first equation implies that $\alpha = 2\theta + \gamma$ and further that $k = 2\delta c^{2-\alpha}[\alpha(1-\alpha) + 2\delta c^2 - 2\alpha c]$. The second equation implies that $k = \delta c^{2-\alpha}[1 - \alpha - c]^{-1}$. It is a somewhat tedious exercise in algebra to then show that these last two equations have positive solutions for all large δ (in fact for $\delta \geq \alpha[2(1-\alpha)]^{-1}$). Note, furthermore, that a positive solution for k implies that $c < 1 - \alpha < 1$.

In this example, the relative risk–aversion index, $-V''/V'$, decreases in y. Hence, in the optimal solution, the lower the wealth level the more conservative the agent's choice; the mean increases (and consequently the variance increases) as y increases– although the mean–variance ratio decreases at the same time.

Example 6.2: $U(m,v) = |m|^{\gamma} v^{\theta}$, $2\theta + \gamma \in (0,1)$. $A = \{(m,v)\colon v=m^2, m \leq -b < 0\}$

By a procedure identical to that of the previous example it is possible to show that $V(y) = k(y+b)^{\alpha}$ and $\beta(y) = -(y+b)/c, (y+b/c)^2$ for $k > 0$, $1 > c > 0$ and $\alpha \in (0,1)$. So in this example, the mean decreases while the variance and the mean–variance increase in y; the intuition is similar to that given above.[10]

Remark: In these examples, the feasible control set is not compact. However, it follows from a theorem in Karatzas and Shreve (1987, Proposition 2.13, p.291) that there are diffusions consistent with each of the exhibited strategies. A standard argument, but in two steps, then shows that V is an upper bound for the returns to arbitrary strategies, and that this bound is attained by β. The first step is an argument for the truncated problem on state space $[0,k]$ and then, in the second step, let $k \to \infty$.

I turn now to a brief discussion of two economic applications of the current analysis – the optimal consumption/savings problem of an agent subject to stochastic labor income and principal–agent models.

The consumption–savings problem (see, for example Yaari (1976), Schechtman (1976) and for a more recent treatment, Deaton (1991)) investigates the nature of the optimal consumption policy for an agent subject to uncertain income. The recent literature has emphasized limited borrowing opportunities and indeed has focussed on the case where the agent is completely unable to borrow (although he can lend). One insight of this literature is that in the presence of borrowing constraints, an agent will use his wealth as a buffer and smooth consumption by accumulating wealth; in the limit as the discount rate and interest rate go to zero, we get the permanent income hypothesis, that an agent will consume exactly his mean income every period, if he can (see Schechtman (1976) and also Deaton (1991)).

An important assumption in this literature is that wealth is always invested in a safe asset and labor income is strictly positive in all states of the world, i.e. there is no bankruptcy. The current analysis can be used to determine the consequences of incorporating bankruptcy in the above problem (in addition to the zero borrowing condition). In particular it can be shown that, unlike the standard model, as the discount rate goes to zero, the optimal consumption at all wealth levels goes to zero as well. This result is proved in the appendix and should be contrasted with the permanent income hypothesis result discussed above.

One other economic application may be briefly mentioned. In the principal–agent literature, incentive schemes which require an agent to maintain output above a given performance index or else face dismissal have been widely studied. The agent's best response problem when faced with such a dismissal scheme is an example of the control problem studied in this paper (see Dutta–Radner (1991) for details).

Appendix

Proofs of Section 3

Suppose we have a C^2 function W which satisfies the Bellman equation

$$\max_{m,v} \{1/2\ v\ W''(y) + m\ W'(y)\} - \delta W(y) + \delta = 0 \tag{A.0}$$

Standard arguments by way of Ito's lemma then shows that $W \geq V_2$. Take the constant control $(\hat{m},\hat{v})$ and consider the function $W \equiv [1-e^{-\hat{\lambda} y}]$. From the definition of $\hat{\lambda}$ it is easy to show that this function satisfies $1/2\ \hat{v}\ W'' + \hat{m}\ W' - \delta W + \delta = 0$. A second usage of Ito's lemma then shows that W is in fact the expected discounted time to failure for a decision-maker using the constant control $(\hat{m},\hat{v})$; $W(y) = 1 - Ee^{-\delta \hat{T}(y)}$. In particular, $W \leq V_2$. Further, using the property that $\hat{\lambda} = \max \lambda(m,v)$, it is possible to show that (A.0) is satisfied by this function. The theorem follows.■

Proofs of Section 4

It is simpler for a logical development of the proofs of Theorems 4.1 and 4.2 to actually prove them in approximately the reverse order. More precisely, I will first prove that the value function is increasing and, if C^1, satisfies (4.2). This will be used to prove Theorem 4.1 after which I will conclude by proving Theorem 4.2iii).

Proof of Theorem 4.2 i) Consider two initial states y and y′ with $y' > y$. A candidate policy from y′ is: use the constant control that generates $\overline{U}$ (say $\overline{a}$) till the first time the state hits y. Thereafter, use π where π is ϵ–optimal from y. Then,

$$V(y') \geq \overline{U}\ (1 - Ee^{-\delta \overline{T}}) + Ee^{-\delta \overline{T}}\ (V(y) - \epsilon) \tag{A.1}$$

where $\overline{T} = \min \{t: Y(t) = y \mid Y(0) = y', \pi \equiv \overline{a}\}$

Since ϵ is arbitrary and $\overline{U} \geq V(y)$, it follows from (A.1) that the value function is monotonically increasing (it will in fact be seen to be strictly monotonic from the strict concavity property that we will shortly prove). ■

Proof of (4.2) when the value function is C^1 and there is an optimal strategy: Let π be any strategy and fix $\Theta > 0$. Take any sample path $Y[0,\infty)$ and let the Θ–translate of $Y[0,\infty)$ be the sample path $Z[0,\infty)$ s.t. $Z(s) = Y(s)-\Theta$, $0 \leq s$. Define the Θ–translate of the strategy π, call it $\pi(\Theta)$, as follows: for every sample path take its Θ–translate.

If at time t, $Z[0,\infty)$ has not yet hit zero, i.e. $Y(s)-\Theta > 0$, for all $0 \leq s \leq t$, then take the same action as taken under π at $Z[0,\infty)$. If $Y(s)-\Theta = 0$, for some $0 \leq s \leq t$, then take any action; suppose $(\overline{m},\overline{v})$. It is easy to see that if π is $\Xi(t)$ measurable then so is $\pi(\Theta)$ and furthermore, there is a solution to (2.1) which is $\mathfrak{S}(t)$– measurable.

Fix an initial state $y > 0$ and let π^* be an optimal policy from y. Consider an alternative starting state $y + \Theta$ and take $\pi^*(\Theta)$ as the policy. Then, defining $T^*(y)$ as the first time to hit 0, with initial state y and π^* as policy, we have,

$$V(y + \Theta) - V(y) \geq Ee^{-\delta T^*(y)} V(\Theta) \qquad (A.2)$$

It immediately follows that

$$V'(y) \geq V'(0)\, Ee^{-\delta T^*(y)} \qquad (A.3)$$

Consider instead the starting state $y - \Theta$. Consider now the $(-\Theta)$–translate of π^*, as a policy for this initial state. By arguments as above we have

$$V(y) - V(y - \Theta) \leq Ee^{-\delta T^*(y,\Theta)} V(\Theta), \qquad (A.4)$$

where $$T^*(y, \Theta) = \min \{t : Y(t) = \Theta \mid Y(0) = y, \pi^*\}$$

From (A.4) it follows that

$$V'(y) \leq V'(0)\, Ee^{-\delta T^*(y)} \qquad (A.5)$$

(8.5) follows by the dominated convergence theorem, utilizing the fact that $T^*(y,\Theta) \uparrow T^*(y)$ a.e. From (A.4) and (A.5), (4.2) follows. Notice that the strict concavity of the value function is an immediate consequence of (4.2). □

Proof of Theorem 4.1: In proving Theorem 4.1 the following notation will be useful:

$$F_1(b,c) = \max_{(m,v)\in A} \left\{ \frac{2m}{v} b - \frac{2\delta}{v} c + \frac{2\delta}{v} U(m,v) \right\} \qquad (A.6)$$

For a stationary Markov policy $\beta:\mathbb{R}_+ \to A$ and a $C^2[0,\infty)$ function w, define

$$L_\beta w(y) = \tfrac{1}{2}\beta_2(y)w''(y) + \beta_1(y)w'(y) - \delta w(y) \qquad (A.7)$$

where $\beta(y) \equiv \beta_1(y),\beta_2(y) = m,v$

$$F[w](y) = \max_{(m,\ v)\in A} \left\{ \tfrac{1}{2}vw''(y) + mw'(y) - \delta w(y) + \delta U(m,v) \right\} \qquad (A.8)$$

The proof of Theorem 4.1 will be in two steps.

Step 1. Consider the k–step truncated optimization problem, in which the agent controls the diffusion process starting at some $y \in (0, k)$, where $k > 0$. The process is absorbed the first time it hits either 0 or k. Let V_k denote the value function for this problem. Continue to assume that returns after absorption are normalized to zero, irrespective of whether the absorption was at $y = 0$ or $y = k$.

Lemma A.1 i) V_k is C^2 and is the unique solution to the constrained Bellman equation

$$F[V_k](y) = \max_{m,\ v\in A}\left\{\frac{1}{2}vV''_k(y) + mV'_k(y) - \delta V_k(y) + \delta U(m,v)\right\} = 0,\ y\in[0,k) \qquad (A.9)$$

$$V_k(0) = V_k(k) = 0 \qquad (A.10)$$

ii) **There is $M < \infty$, independent of k, such that**

$$\|V_k\| + \|V'_k\| + \|V''_k\| < M \qquad (A.11)$$

where $\|\cdot\|$ denotes the sup–norm.

Step 2. I shall then let $k \uparrow \infty$, and argue that limits are well–defined and indeed define the value function of the $k = \infty$ problem, i.e. prove

Lemma A.2 **There is a C^2 function $\tilde{V}$, such that $(V_k, V'_k) \to (\tilde{V}, \tilde{V}')$. Moreover, $F[\tilde{V}](y) = 0$, for all $y\in [0, \infty)$. Further $\tilde{V} = V$.**

Proof of Lemma A.1: The proof proceeds by way of several auxiliary lemmas. The underlying idea is the Bellman–Howard improvement routine. Parts of the proof are adapted from Krylov (1980, Theorem 1.4.5).

Lemma A.3 **Suppose $\beta : [0,k] \to A$ is a continuous function. Then**

i) there is a unique C^2 function w_β such that

$$L_\beta w_\beta(y) + \delta U(\beta(y)) = 0,\quad y\in[0, k] \qquad (A.12)$$

$$w_\beta(0) = w_\beta(k) = 0$$

ii) there is $M_k < \infty$, such that

$$\|w_\beta\| + \|w'_\beta\| + \|w''_\beta\| < M_k \qquad (A.13)$$

Proof: This result is proved as Lemma 1.4.6 in Krylov (1980). □

The next result is standard and shows that the function w_β given by Lemma 4.3

is in fact the lifetime returns from using β as a stationary Markov policy.

Lemma A.4 **Consider the stationary Markovian policy in which $\pi_t(Y[0,\infty)) = \beta(Y(t))$ where β is continuous. Let**

$$I_\beta(y) = E_\beta\, \delta \int_0^T e^{-\delta s}\, U(\beta(Y(s)))\, ds$$

where $\quad T = \min\{t : Y(t) = 0 \text{ or } k \mid Y(0) = y, \beta\}$

Then, I_β is the unique C^2 function that solves $L_\beta I_\beta + \delta U(\beta) \equiv 0$, $y \in [0, k]$, and further satisfies $I_\beta(0) = I_\beta(k) = 0$.

Proof: By Lemma A.3, there is a unique C^2 solution of (A.12). Since β is continuous, from Theorem 5.2 of Skorohod (1982), the stochastic differential equation (2.1) has a solution $[Y(t): t \geq 0]$, unique in probability law. By an application of Ito's lemma to $e^{-\delta t} w_\beta(Y(t))$ we get

$$w_\beta(y) = E_\beta \left\{\delta \int_0^{T\Lambda t} U(\beta(Y(s)))\, e^{-\delta s}\, ds + e^{-\delta(T\Lambda t)}\, w_\beta(Y(T\Lambda t))\right\}$$

Letting $t \to \infty$, $T\Lambda t \to T$, and the dominated convergence theorem yields

$$w_\beta(y) = E_\beta\, \delta \int_0^T e^{-\delta s}\, U(\beta(Y(s)))\, ds \qquad \blacksquare$$

Let β_0 be an arbitrary continuous function, and denote its associated lifetime returns w_0. Consider,

$$F[w_0]\,(y) = \max_{m,\ v \in A} \left\{\tfrac{1}{2} v w''_0(y) + m w'_0(y) - \delta w_0(y) + \delta U(m,v)\right\} \quad (A.13)$$

By the strict concavity of the utility function and the convexity of A, the argmax in (A.13) is single–valued. By the Maximum Theorem (Berge (1963), p.116), the function of maximizers is in fact a continuous function. Denote this function β_1 and the associated returns w_1. In this manner, we can construct a sequence of C^2 functions w_n and stationary policy functions β_n, $n \geq 0$ such that

$$L_n w_n + \delta U_n = 0 \leq L_{n+1} w_n + \delta U_{n+1}$$

(where $L_n w_n \equiv L_{\beta_n}$, $U_n \equiv U(\beta_n)$ etc.)

Lemma A.5 **For all n, $y \in [0, k]$, $w_{n+1}(y) \geq w_n(y)$**

Proof: Let the continuous function $h_n \geq 0$ be defined by

$$L_{n+1}(w_{n+1} - w_n) + \delta h_n \equiv 0$$

Then, an application of Ito's lemma to the function $w_{n+1} - w_n$, yields

$$w_{n+1}(y) - w_n(y) = E\ \delta \int_0^{T_n} e^{-\delta s}\ h(Y(s))\ ds \geq 0 \qquad \blacksquare$$

To continue the proof of Lemma A.1, let us now define $\tilde{w} = \lim_{n\to\infty} w_n$. This limit is well–defined by Lemma A.5, for all $y \in [0,k]$. Further w_n' is uniformly bounded by (A.13). Since

$$w_n(y) - w_n(z) = \int_z^y w_n'(x)\ dx \tag{A.14}$$

it follows that w_n is in fact an equicontinuous family. Hence, $\tilde{w}$ is a continuous function and $w_n \to \tilde{w}$ uniformly on any compact subset of $\mathbb{R}_+$. Similarly,

$$w_n'(y) - w_n'(z) = \int_z^y w_n''(x)\ dx$$

and hence w_n' is also an equicontinuous (and uniformly bounded) family. By the Arzela–Ascoli theorem, there is a subsequence (retain notation) such that w_n' is convergent on it, to φ say. Taking limits, along this subsequence,

$$\tilde{w}(y) - \tilde{w}(z) = \int_z^y \varphi(x)\ dx \tag{A.15}$$

I further used the dominated convergence theorem in arriving at (A.15). Moreover, (A.15) establishes that $\tilde{w}'(y) = \varphi(y)$. Hence, along the full sequence, $w_n' \to \tilde{w}'$, and of course $\tilde{w}$ is C^1. I have hence proved

Lemma A.6 **$w_n \to \tilde{w}$, where $\tilde{w}$ is a C^1 function. Further, $w_n' \to \tilde{w}'$.**

It remains to show that in fact $\tilde{w}$ is C^2 and further that $F[\tilde{w}] = 0$. This last step of the proof of Lemma A.1 is identical to the proof of the analogous step in Krylov (1981) Theorem 1.4.5 and hence is not reproduced here. Clearly, $\tilde{w}(0)=\tilde{w}(k)=0$.

A standard application of Ito's lemma now shows that $\tilde{w} \geq V_k$. Further, let β be argmax $F[\tilde{w}]$ (and hence it is a continuous function). Then, a second application of

Ito's lemma to the diffusion process generated by the stationary policy β, establishes that $I_\beta \geq \tilde{w}$. The first half of Lemma A.1 is completely proved. I now turn to the proof of Lemma A.1 ii).

Lemma A.7 **Let β_k denote the stationary Markovian optimal policy for the kth truncated problem. Define**

$$T_k^0(y) = \min \{t > 0: Y(t) = 0 \mid Y(0) = y, \beta_k\}$$

$$T_k^k(y) = \min \{t \geq 0: Y(t) = k \mid Y(0) = y, \beta_k\}$$

$$T_k(y) = \min (T_k^0(y), T_k^k(y))$$

$$V_k'(y) = E_{\beta_k} [e^{-\delta T_k^0(y)} \cdot P(T_k^0 = T_k) V_k'(0) + e^{-\delta T_k^k(y)} \cdot P(T_k^k = T_k) V_k'(k)] \quad \text{(A.16)}$$

Proof: The proof of Lemma A.7 is very similar to the proof of Theorem 4.2 ii), and hence I omit it. ■

It is immediate from (A.16) that $\|V_k'(y)\| \leq \max (\|V_k'(0)\|, \|V_k'(k)\|)$. To establish an upper bound on $\|V_k'\|$ it clearly suffices to establish such a bound on $\|V_k'(0)\|$ and $\|V_k'(k)\|$. I now show that such a bound exists and is, in fact, independent of k. Note that $W \equiv \bar{U} [1-e^{-\hat{\lambda} y}] \geq V_k(y)$ for all y and k, and of course $W(0) = V_k(0) = 0$. Hence,

$$V_k'(0) \leq W'(0) = \hat{\lambda}$$

Of course, by definition $V_k'(0) \geq 0$. It is easy to see that a symmetric set of arguments could be repeated for $V_k'(k)$.

Further $\|V_k\| \leq \bar{U}$. Finally note that

$$V_k''(y) = - \frac{2m_k}{v_k} V_k'(y) + \frac{2\delta}{v_k} V_k(y) - \frac{2\delta}{v_k} U(m_k, v_k) \qquad \text{(A.17)}$$

Clearly, the uniform upper bounds on $\|V_k'\|$, $\|V_k\|$ and $\|U\|$, imply a uniform upper bound independent of k on $\|V_k''\|$. Hence, Lemma A.1 is fully proved. □

I now turn to the second step in the proof of Theorem 4.1. The objective here is to use the properties of the k–th truncation value function to establish analogous properties for the value function V. First of all, clearly V_k is a monotone sequence of functions, and hence there is $\tilde{V}$ such that $V_k \uparrow \tilde{V}$, as $k \to \infty$. By an argument identical to that in Lemma A.6, we get

Lemma A.8 $\tilde{V}$ **is a** C^1 **function. Moreover,** $V_k' \to \tilde{V}'$, **as** $k \to \infty$.

Note that $F[V_k](y)=0$ is equivalent to $F_1(V_k',V_k) + V_k''=0$. Hence,

$$V_k'(y) - V_k'(0) + \int_0^y F_1(V_k(x),\ V_k'(x))\ dx = 0 \tag{A.18}$$

But $V_k'(y)$ converges to $\tilde{V}'(y)$ for all y and by the maximum theorem $F_1(V_k',V_k)$ converges to $F_1(\tilde{V}',\tilde{V})$. From (A.18) it then follows that

$$\tilde{V}'(y) - \tilde{V}'(0) + \int_0^y F_1(\tilde{V}(x),\ \tilde{V}'(x))\ dx = 0 \tag{A.19}$$

From the fundamental theorem of calculus, the maximum theorem and (A.19) it follows that $\tilde{V}$ has a continuous second derivative $\tilde{V}'' = -F_1(\tilde{V},\tilde{V}')$. Further $F[\tilde{V}] = 0$. A standard application of Ito's lemma then establishes that $\tilde{V} = V$. Further given the strict concavity of the utility function, there is a unique element to argmax F[V] and this selection, by the maximum theorem, is a continuous function. A further use of Ito's lemma establishes that this selection in fact achieves the returns V, i.e. is optimal. The proof of Lemma A.2 and Theorem 4.1 is complete. □

From the strict concavity and monotonicity of V, $V'' + F_1(V',V) = 0$ implies that V'' increases with y. The proof of Theorem 4.2 is complete. ■

Proof of Proposition 4.3: Suppose y' > y. From the fact that (m,v) = argmax F[V](y) and (m',v') = argmax F[V](y') it follows that

$$(v - v')(V''(y) - V''(y')) + (m - m')(V'(y) - V'(y')) \geq 0 \tag{A.20}$$

Further $F[V] = 0$ is equivalent to $V'' + F_1(V',V) = 0$ and there is a common argmax for F and F_1. It then follows that

$$(m/v - m'/v')(V'(y) - V'(y')) - \delta(1/v - 1/v')(V(y) - V(y')) \geq 0 \tag{A.21}$$

From (A.20), (A.21) and the properties of V established in Theorem 4.2, the proposition follows.■

Proof of Proposition 4.4: First–order conditions yield

$$1/2\ V''(y) = -\ \delta U_2(m,v) \tag{A.21}$$

$$V'(y) = -\ \delta U_1(m,v) \tag{A.22}$$

Substituting (A.21) – (A.22) back into F[V] = 0, yields

$$H(m,v) = V(y)$$

From the strict monotonicity of the value function it follows that the stationary Markov optimal policy is in fact a one to one function. □

Proof of Proposition 4.5: Follows from (A.21), (A.22) and the fact the utility function is assumed to decrease in m and increase in v. ■

Proof of Proposition 4.6: Since $A = [m, \overline{m}] \times [v,\overline{v}]$ and U is separable, F[V](y) can be re–written as

$$\max_{(m,\ v)\epsilon A} \left\{\frac{1}{2}\ vV''(y) + mV'(y) + \delta U(m,v)\right\} = \max_m \{mV'(y) + \delta\xi(m)\} + \max_v \left\{\frac{1}{2}vV''(y) - \delta\phi(v)\right\} = \delta V(y)$$

Let $W_1(y) \equiv \max_m \{V'(y) + \delta\xi(m)\}$. From the strict concavity of V it immediately follows that $m' < m$. For similar reasons $v' > v$ and hence $U(m,v) < U(m',v')$. □

Proofs of Section 5:

Proof of Proposition 5.1: By the arguments used in proving Theorem 3.1 it can be shown that $1 - Ee^{-\delta T(y)}$ for the constant control $(\overline{m},\overline{v})$ is given by $1 - e^{-\overline{\lambda}y}$. The proposition follows.■

Proof of Proposition 5.2: That $Ee^{-\delta T^*(y)}$ is convex follows from (4.2) and the fact that V′′ is increasing in the initial weallth. From the definitions it follows that

$$\overline{U}\ [1 - Ee^{-\delta T^*(y)}]\ \geq\ \hat{U}\ [1 - Ee^{-\delta \hat{T}(y)}] \tag{A.23}$$

From (A.23) it follows that

$$\frac{\overline{U}}{\hat{U}} - 1 \geq Ee^{-\delta\hat{T}(y)}\,[1/\rho - 1] \tag{8.24}$$

From (A.24) the first part of the proposition follows. If $\hat{U} = \overline{U}$ it is clear that $(\hat{m},\hat{v})$ is an optimal policy, as can be verified from the optimality equation. From the bound (5.2), the necessity of $\hat{U} = \overline{U}$ follows whenever $(\hat{m},\hat{v})$ is an optimal solution for the expected utility maximization problem. Finally, if $\overline{U} \neq \hat{U}$, then from Theorem 4.1 and Proposition 4.3 it follows that controls other than $(\hat{m},\hat{v})$ are used for all sufficiently high wealth levels. By a straightforward amendment of the arguments used in proving Proposition 5.1 it can be shown that $Ee^{-\delta T^*}$ goes to zero more slowly than $Ee^{-\delta\hat{T}}$.■

Proof of the claim in Section 6:

To fix ideas, suppose that the agent receives uncertain income, with mean μ, every period. The agent's choice of a consumption rate c, $c \in [0,\overline{c}]$, determines the average increment of wealth, $m = \mu - c$. The agent goes bankrupt at wealth level zero; in this formulation, there is no choice over the variance of incremental wealth.

I show now that when $\delta\downarrow 0$ and the variance is fixed, the solution of (P1) tends to the constant use of the control with the highest mean. Note that (4.1) implies that the optimality equation tends to $\max_{(m,v)}\{1/2\ vV'' + mV'\} = 0$. It is easy to show (using the techniques used in proving Theorem 3.1) that the unique solution to this differential equation is $V(y) = 1 - e^{-(2\overline{m}/v)y}$ where $\overline{m} \equiv \max m,\ (m,v) \in A$.■

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Footnotes

[1]Bankruptcy will be defined by the condition that net assets are non–positive. In practice, an agent has some discretion on when to declare bankruptcy. I ignore this possibility in the discussion that follows.

[2]Indeed, Robinson (1962), in discussing theories of economic growth, notes critically that "an equilibrium position which contains consumption of exhaustible resources or starvation of some group is in course of upsetting itself from within, and chance events may upset it from without."

[3]Note that several recent papers in economics have studied the pure survival problem. These include Majumdar–Radner (1990, 1991) and Mitra–Roy (1990). A literature that follows the Dubins–Savage gambling problem includes Heath et.al. (1987), Orey et.al. (1987), Pestien–Sudderth (1985) and Sudderth–Weerasinghe (1989).

[4]This result requires either of two additional conditions (see Propositions 4.5 and 4.6).

[5]The reason for adopting a continuous time formulation rather than one in discrete time is that if time is discrete an agent can go bankrupt with any non–positive terminal wealth and there is no obvious way in which to assign termination values to different levels of terminal wealth; different assignments will clearly change the optimal choices an agent faced with possible bankruptcy will make. To avoid these "overshooting at the bankruptcy barrier" problems, I work with a continuous time model in which sample paths are continuous.

[6]An alternative formulation is one in which the incremental returns depend on the action as well as the level of wealth; say $dW(t) = m(t)W(t)dt + v^{1/2}(t)W(t)dB(t)$. As is well–known this can be reduced to (2.1) by dividing through by W(t) in the above equation and defining $\ln W \equiv Y$.

[7]My definition of a strategy is what Krylov (1981) calls a natural strategy. A strategy can be defined more generally as a stochastic process progressively measurable with respect to $\mathfrak{F}(t)$ (see, for example, Krylov (p.23) or Karatzas–Shreve (p.375)). All of the results that follow are valid under this broader definition.

[8]The undiscounted criteria that have been examined include: maximization of expected time to failure (Heath et. al.(1987)); in a problem with absorption either at zero or b > 0, maximization of the probability that absorption is in fact at b (Pestien–Sudderth (1985) and in a finite horizon model, the maximization of the probability that by the terminal date the process has reached b (Sudderth–Weerasinghe (1989)). Orey et. al. (1987) have looked at a discounted problem which is the converse of the pure survival problem. In their problem the decision–maker would like to get to zero rapidly.

[9]A referee points out that the from the proof of Theorem 3.1 it is immediate that the set of feasible controls A needs to be neither convex nor compact for the result to hold. Instead, it suffices to know that there exists $\hat{m},\hat{v}$ such that $\lambda(\hat{m},\hat{v}) = \max \lambda(m,v)$. A second referee points out that the reasoning behind this result is similar to that underlying Theorem 2.1 in Orey et.al.

[10]In this example, it is straightforward to show that there are positive solutions, for the constants k and c, of the Bellman equation (4.1) and its first–order condition. To

ensure that the solution for c is a fraction we need some restrictions; a sufficient condition is that $2\theta+\gamma < 1/2$ and $2\delta > (4-2\theta-\gamma)(1-2\theta-\gamma)$.

[11]This problem was first studied by Yaari (1976), subsequently by Schechtman (1976) and more recently by a number of authors who have used this framework to study macroeconomic consumption issues (for an authoratative treatment and a complete set of references, see Deaton (1991)).