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by

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William Thomson

Abstract. The objective of this paper is the construction of "simple" games implementing in Nash equilibria several solutions to the problem of fair division. These solutions are the no–envy solution, which selects the allocations such that no agent would prefer someone else's bundle to his own, and variants of this solution. Components of strategies can be interpreted as allocations, consumptions, permutations, points in simplices of dimensionalities equal to the number of goods or to the number of agents, and integers. We also propose a simple game implementing the pareto solution and games implementing the intersections of the pareto solution with each of these solutions.

Key words. Nash implementation. No–envy. Divide and permute.

1. ***Introduction.*** We are concerned here with the implementation by "simple" games of solutions to the problem of fair division.

An informal description of this objective is as follows. A bundle of goods have to be divided among a group of agents with equal claims on them. Given a class of such ***problems of fair division***, consisting of a space of possible preferences for the agents involved, a ***solution*** is a mapping associating with each problem in the class a subset of the set of feasible allocations for that problem. Suppose that a solution φ has been chosen as being the most desirable. Given some admissible problem, computing the allocations selected by φ for that problem requires that preferences be known. Unfortunately, agents will typically benefit from not reporting their true preferences. It is indeed well–known that on sufficiently wide domains there are no well–behaved "strategy–proof" solutions, that is, solutions such that each agent finds it in his best interest to reveal his true preferences independently of the preferences announced by the others; then, truth–telling is a "dominant" strategy in the associated direct revelation game. In the two–person case, this was shown by Hurwicz (1972) for classical economies under the requirements of efficiency and individual rationality, and by Thomson (1987) under the requirements of efficiency and one of several alternative requirements of fairness; Zhou (1991) generalizes both of these contributions by showing that strategy–proofness and efficiency together imply dictatorship; Barbera and Jackson (1992) complete this program by identifying in the n–person case the implications of strategy–proofness alone, that is, even without imposing the requirement of efficiency.

Unless preferences are required to belong to some restricted domain[1], obtaining truthful information as a dominant strategy is therefore incompatible with efficiency and

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[1]For instance, in a one–commodity economy with single–peaked preferences, the solution known as the uniform rule is implementable in dominant strategies (Sprumont, 1991).

minimal distributional objectives. We will then weaken our incentive requirement as follows. Given a solution, we will only ask whether there is a game such that for each admissible economy, its set of Nash equilibrium allocations coincides with the set of allocations that the solution would have selected for this economy. If such a game exists, the solution is said to be "implementable". The properties that a solution should satisfy to be implementable are now well–understood, and general algorithms have been proposed to construct games achieving the implementation when it is possible. However, these algorithms, being designed to solve the problem in very general situations, produce games that have the disadvantage of involving complex strategy spaces. Indeed, strategies include either whole preference profiles or whole indifference sets for several agents. In economic applications, these are infinite dimensional objects. Fortunately, when it comes to implementing specific solutions, the particular features of the space of feasible outcomes and of the space of admissible preferences can be used to achieve considerable simplifications. For instance, a variety of games whose strategy spaces are subsets of finite dimensional euclidean spaces have been constructed to implement the Walrasian and Lindahl solutions. Hurwicz (1979b) and Walker (1981) are early examples. Our objective here is the construction of such simple games to implement several solutions to the problem of fair division.

We first consider (i) the solution that associates with each problem its set of "envy–free" allocations: an allocation is envy–free if no agent would prefer someone else's bundle to his own. Alternatively, we require that each agent prefers what he receives to: (ii) the average of what the other agents receive; (iii) the average received by any group to which he does not belong; (iv) any point in the convex hull of the consumptions of all the agents. (The relations between these various notions are explained below). In each case, we show that implementation is possible by a game whose strategy spaces are the cross–product of a subset of a finite dimensional euclidean space with some finite set. We also propose a simple game (v) implementing the pareto solution, the solution that associates with each economy its set of pareto–efficient allocations. Here, strategy spaces are slightly more complicated but they are still finite dimensional euclidean spaces. Finally, (vi) we show how the distributional objectives embodied in the various solutions listed above can be reconciled with efficiency by "combining" the games implementing them with the game implementing the pareto solution. These results are achieved by identifying a class of solutions that contains each of the above mentioned examples, and constructing a general game for the class. However, a separate game is also constructed for the no–envy solution, which we name "Divide and Permute".

For each of the games we construct, the components of strategies have a straightforward economic interpretation as allocations, consumptions, or prices, and (except for Divide and Permute), one component is an integer that is used as a device to select an agent who is granted the right to choose in a certain set. This set is determined as a function of the other components of the strategies. The use of such "integer" constructions has been criticized by Jackson (1992), who shows however that not having access to them severely limits the range of solutions that can be implemented. An appealing aspect of our games is that at equilibrium, agents receive the consumptions that have been announced. This feature essentially corresponds to what Dutta, Sen and Vohra (1993) calls "truth–telling". Without it, there would be little meaning to evaluating how complicated a game is by the dimensionality of its strategy spaces because of the possibility of information "smuggling" by means of such mathematical devices as Peano's space filling curve. Hurwicz (1977) was the first one to recognize this possibility in this context (see also Chakravorti, 1991) and avoided it by placing restrictions on the outcome function.

A contribution with a similar aim as ours, namely the investigation of the implications of requiring games to be "simple", is the paper just mentioned by Dutta, Sen and Vohra. These authors characterize the class of solutions that can be implemented by what they call "elementary" games, that is, games such that at equilibrium, the set of consumptions that each agent can attain by varying his own strategy, given the strategies chosen by the others, is a subset of the half–space whose boundary is a hyperplane of support to his upper contour set at his assigned consumption. The pareto solution is one of the solutions covered by their theorem, but none of the solutions to the problem of fair division we discuss here can be handled because they all fail to satisfy a condition that is critical to their approach, namely that the desirability of an allocation be verifiable only on the basis of local information (in the case of smooth preferences, the marginal rates of substitution)[2]. Therefore our results throw some light on the strength of the requirement that the game be elementary. Similarly, Sjostrom (1991a) considers implementation by "demand games", that is, games in which strategies are points in the commodity space that can be interpreted as agents' desired bundles. He shows that the no–envy solution can be so implemented but that neither the pareto solution nor its intersection with the no–envy solution can, indicating that limiting oneself to demand games may also be too restrictive. We finally comment on the recent results independently obtained by Saijo, Tatamitani and Yamato (1993). These authors identify the conditions that a solution has to satisfy in order to be implemented by a game where strategies are required to be interpretable as one of the following: consumptions, pairs of consumptions, allocations, and each of these together with a price. They apply their results to the no–envy solution and its intersection with the pareto solution. They show that the former can be implemented by a game where each agent announces two consumptions, and its intersection with the pareto solution by a game in which each agent announces a consumption and a price. In the games they consider, all agents have the same

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[2]Although one should note that if preferences are smooth, the smallest of the correspondences that we consider coincides then with the Walrasian solution, to which the condition does apply.

strategy space, whereas here, two agents are made to play a special role. The implications in terms of dimensionality of the requirement that strategy spaces be the same for all agents may be a question worthy of further study. For a more complete comparison of the results of Saijo, Tatamitani et Yamato with those of the current paper, one would also need to verify whether the other examples of solutions examined here satisfy the conditions they derive. We will also leave this question to future research.

Finally, we note that although our main objective was to construct games with strategy spaces of low dimensionality, we did not address the issue of identifying the dimensionality that is minimal for implementation. A recent contribution to this issue is Reichelstein and Reiter (1988). The above mentioned papers of Dutta, Sen and Vohra, Sjostrom and Saijo, Tatamitani and Yamato provide useful additional information on this matter.

2. ***The Model.*** There are ℓ private goods and n agents indexed by $i \in N = \{1,...,n\}$. Each agent $i \in N$ is equipped with a continuous, convex, and strictly monotone preference relation on $\mathbb{R}^{\ell}_{+}$, denoted by R_i. Let P_i be the associated strict preference relation and I_i the indifference relation. There is a bundle $\Omega \in \mathbb{R}^{\ell}_{++}$ of goods to be divided. This bundle is assumed to be fixed and known. Therefore, a problem of fair division, or an ***economy***, is simply a list $R = (R_1,...,R_n)$ of preference relations. Let $Z = \{z \in \mathbb{R}^{\ell n}_{+} | \Sigma z_i = \Omega\}$ be the set of feasible allocations and $Z_0 = \{z_0 \in \mathbb{R}^{\ell}_{+} \mid z_0 \leqq \Omega\}$ be the set of possible consumptions for any agent.

Given a class of economies $\mathscr{R}^n$, a ***solution*** is a correspondence $\varphi: \mathscr{R}^n \to Z$ associating with each economy in that class a non-empty subset of the set of feasible allocations, each point of which is interpreted as a recommendation.

A game form, or simply a ***game***, is a pair $\Gamma = (S,h)$, where $S = S_1 \times ... \times S_n$ is a cross-product of ***strategy spaces*** and $h{:}S \to \mathbb{R}^{\ell n}_{+}$ is an ***outcome function.*** Given an economy $R \in \mathscr{R}^n$, let ***$E(\Gamma,R) \subseteq S$*** be the set of (Nash)–***equilibria of Γ when played in R*** and ***$E_Z(\Gamma,R)$*** be the corresponding set of ***equilibrium allocations***: $z \in E_Z(\Gamma,R)$ if there exists $s \in E(\Gamma,R)$ such that $z = h(s)$. The game Γ ***implements the solution $\varphi: \mathscr{R}^n \to Z$*** if for all $R \in \mathscr{R}^n$, $E_Z(\Gamma,R) = \varphi(R)$. A solution is ***implementable*** if there exists a game that implements it.

Consider now an abstract set of alternatives A, a domain $\mathscr{R}$ of preference relations defined over A, and a correspondence $\varphi: \mathscr{R}^n \to A$. Given $i \in N$, $R_i \in \mathscr{R}$ and $a \in A$, let ***$L(R_i,a)$*** $= \{b \in A \mid aR_ib\}$ be the ***lower contour set of R_i at a.***

Maskin (1977) showed that if a correspondence is implementable, then it satisfies the following condition:

Definition. The correspondence $\varphi: \mathscr{R}^n \to A$ is ***monotonic*** if for all $R, R' \in \mathscr{R}^n$, and for all $a \in \varphi(R)$, if $L(R_i,a) \subseteq L(R_i',a)$ for all $i \in N$, then $a \in \varphi(R')$.

A correspondence satisfies "no veto power" if, when an alternative is at the top of the preferences of all but possibly one agent, then it is selected by the solution. Maskin showed that if there are at least three agents, and if a correspondence is *monotonic* and satisfies *no veto power*, then it is implementable. *No veto power* is trivially satisfied in private good economies as soon as there are at least three agents and at least one good with respect to which their preferences are strictly monotone, since then its hypothesis is never met. Having assumed that preferences are strictly monotone, *monotonicity* is the only relevant condition on correspondences to be implementable, and we will make no more mention of *no veto power.*

Maskin's proof is constructive: he exhibits an algorithm producing, for each implementable correspondence, a game implementing it. Although he restricts his attention to the case when the set of feasible alternatives is finite, his result can be extended to general domains, as shown by Repullo (1987) and Saijo (1988). Unfortunately, the games used by all of these authors have the drawback of involving complex strategy spaces, as explained above, and our objective is to construct games with simple strategy spaces. Implementation of the Walras and Lindahl solutions by simple games has been achieved by several authors, but few attempts have been made to implement by simple games solutions to the problem of fair division. The exceptions of which we are aware are the following: Crawford (1979) proposes a game implementing a selection from Pazner and Schmeidler (1978)'s "egalitarian–equivalent" solution (see also Demange, 1984). Since this selection is not *monotonic*, the implementation is not achieved in Nash equilibrium; instead, it involves stage games and the implementation is in perfect equilibrium. Here, following the bulk of the literature, we consider normal form games and implementation in Nash equilibrium. Another exception is Tadenuma and Thomson (1992), who offer an implementation of the no–envy solution for a class of economies with one indivisible object and one infinitely divisible good by means of a game in which each strategy space is the real line[3,4]. A final reference is Sjostrom (1991a), already discussed in the introduction.

3. ***Divide and Permute : an implementation of the no–envy solution.*** The requirement that plays the most important role in the literature on fair allocation is that no agent should prefer someone else's consumption to his own.

No–envy solution, F (Foley, 1967). The allocation $z \in Z$ is ***envy–free for*** $R \in \mathcal{R}^n$ if $z_i R_i z_j$ for all $i, j \in N$.

It is easy to see that the no–envy solution is *monotonic*. Therefore, it can be implemented by the Maskin–Repullo–Saijo games. The following is a simple game achieving the same result. Let Π^n be the class of permutations of order n and π_0 be the identity permutation.

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[3]Whether this game can be generalized to handle the multiple object case remains to be determined however.

[4]The implementation of solutions to the problem of fair division in economies with one good when preferences are single–peaked is considered by Thomson (1990), Sjostrom (1991b), and Yamato (1992,1993), but these papers do not address the issue of implementation by simple games.

Game Γ^F *(Divide and Permute)*[5]: $S_1 = S_2 = Z \times \Pi^n$ and $S_3 = \ldots = S_n = \Pi^n$.

Given $s = ((z^1,\pi_1), (z^2,\pi_2),\pi_3,\ldots,\pi_n) \in S$, let

$$h(s) = (0,0,\ldots,0) \quad \text{if } z^1 \neq z^2$$
$$= \pi_n \circ \pi_{n-1} \circ \ldots \circ \pi_1(z^1) \quad \text{if } z^1 = z^2$$

The game can be informally described as follows. The first two agents are dividers (each proposes an allocation), and everyone proposes a permutation; if the dividers disagree, they are penalized, and so is everybody else. If they agree on an allocation, each agent (including them) can reach any of its components by appropriately choosing his permutation, and this, independently of the permutations chosen by the others.

We will use the following additional piece of notation. Given a list $s \in S$, and $i \in N$, ***$att_i(s)$*** is ***the set of consumptions attainable by agent i at s***, namely $\{z_i \in \mathbb{R}^{\ell}_{+} | z_i = h_i(s_i',s_{-i})$ for some $s_i' \in S_i\}$[6].

Theorem **1**: The game Γ^F (Divide and Permute) implements the no–envy solution.

Proof ***Claim*** **1.** *If $z \in E_Z(\Gamma^F;R)$, then $z \in F(R)$.* Let $s = ((z^1,\pi_1),(z^2,\pi_2),\pi_3,\ldots,\pi_n)$ be an equilibrium supporting z. We have $att_1(s) = \{0,z^2_1,z^2_2,\ldots,z^2_n\}$; the first consumption results from choosing any $s_1' = (z',\pi) \in S_1$ such that $z' \neq z^2$; each of the

[5]The name "Divide and Permute" is meant to bring to mind the well–known two–person "Divide and Choose" procedure (one agent divides and the other chooses). However, a number of important distinctions should be noted. In Divide and Choose, only one agent proposes a division; here, two agents do so. Divide and Choose is a stage game; here, we consider normal form games. Divide and Choose provides a ***partial*** implementation of the no–envy solution (only the allocation in the envy–free set the most favorable to the divider is obtained at equilibrium; Kolm, 1972; Crawford, 1977); here, we obtain ***full*** implementation (each envy–free allocation is obtained at some equilibrium.) Finally, a nuber of complications arise in extending Divide and Choose to the n–person case (See Thomson, 1993, for a presentation of the literature devoted to such extensions); here, the n–person case poses no special problem. In fact, for $n \geq 3$, additional desirable properties can be imposed on the game (See Remark 3 below).

[6]It is for convenience of the notation that we write $att_i(s)$ instead of $att_i(s_{-i})$.

remaining ones is obtained by choosing a pair $s_1' = (z^2,\pi)$ for some appropriate π. Since at least one of the components of z^2 contains a positive amount of at least one good, and agent 1 has strictly monotone preferences, $s_1 = (z^1,\pi_1)$ is a best response to s_{-1} only if $z^1 = z^2$, and therefore, $z = \pi_n o \pi_{n-1} o \ldots \pi_2 o \pi_1(z^1)$. Similarly, $att_2(s) = \{0,z_1^1,z_2^1,\ldots,z_n^1\}$. Now, given that $z^1 = z^2$, if follows that for all $i \in N$, $i \neq 1, 2$, $att_i(s) = \{z_1,z_2,\ldots,z_n\}$. Therefore equilibrium occurs only because the π_i's are such that for each $i \in N$, $(\pi_n o \pi_{n-1} o \ldots \pi_2 o \pi_1(z^1))_i$ maximizes R_i on $\{z_1,z_2,\ldots,z_n\}$, which means that $z \in F(R)$.

Claim 2. If $z \in F(R)$, then $z \in E_Z(\Gamma^F;R)$. Indeed, let $s = ((z,\pi_0), (z,\pi_0), \pi_0,\ldots,\pi_0)$. Then, $h(s) = z$. Here, $att_1(s) = \{0,z_1,\ldots,z_n\}$, and since $z_1R_1z_i$ for all $i \in N$ and z_1R_10, (z,π_0) is a best response to s_{-1} for agent 1. Similarly, (z,π_0) is a best response to s_{-2} for agent 2. Finally, for each $i \in N$, $i \neq 1, 2$, $att_i(s) = \{z_1,\ldots,z_n\}$. Since $z_iR_iz_j$ for all $j \in N$, π_0 is a best response for agent i to s_{-i}.

Q.E.D.

Remark 1. It would suffice to have each agent announce a transposition (a permutation exchanging only two components), instead of an arbitrary permutation.

Remark 2. Implementation occurs even for $n = 2$. This is worth noting since, as is well–known (Moore, 1991), implementation for $n = 2$ is often more difficult to achieve than for $n > 2$.

Remark 3. If $n > 2$, the outcome function can be respecified so that no resource be ever thrown away, that is, so that $\Sigma h_i(s) = \Omega$ for all $s \in S$ (not just at equilibrium). In the case $z^1 \neq z^2$, set $h(s) = (0,0,\Omega,0\ldots,0)$ for instance (any distribution of Ω between agents 3 to n would do it).[7]

Remark 4. Divide and Permute can be used to implement the no–envy solution on domains of economies with heterogeneous goods such as land or time, the amount to

[7]I owe this remark to B. Dutta.

divide being modelled as a measurable subset of a finite–dimensional euclidean space with preferences defined over its measurable subsets. (The issue of existence of envy–free and efficient allocations in this context is addressed by Weller, 1985, and Berliant, Dunz and Thomson, 1992). For time (the one–dimensional case), it is often natural to assume that agents have preferences defined over intervals and that of two intervals ordered by inclusion, the larger one is preferred to the smaller one. In this case, an envy–free allocation is necessarily efficient, so that Divide and Permute achieves both efficiency and distributional objectives.

4. Implementation of other solutions. In this section, we consider a variety of other solutions to the problem of fair division, but instead of dealing with each of them separately, we offer a general procedure. The no–envy solution is covered by this procedure, However, Divide and Permute seems particularly natural for that solution and we felt that there would be some advantage to giving it also.

We begin by listing the examples of solutions to which the general procedure applies. First, instead of requiring that each agent prefer his own consumption to the consumption of any other agent, we require that each agent prefer his consumption to the average of the consumptions of the others. This definition can be found in Thomson (1979, 1982), Baumol (1986) and Kolpin (1991), to which we refer the reader for motivation.

Average no–envy solution, A. The allocation $z \in Z$ is an ***average envy–free allocation for*** $R \in \mathcal{R}^n$ if for all $i \in N$, $z_i R_i a_i(z)$ where $a_i(z) = (\sum_{j \neq i} z_j/(n-1))$.

Alternatively, we require that each agent prefer his consumption to the average consumption of any group not containing him. See Zhou (1992) for motivation. Given $i \in N$, let $\mathcal{G}_i = \{G \subsetneq N \mid i \notin G\}$ denote the set of groups not containing agent i.

Strict no–envy solution, ***C*** (Zhou, 1992). The allocation $z \in Z$ is ***strictly envy–free for*** $R \in \mathscr{R}^n$ if for all $i \in N$, and for all $G \in \mathscr{G}_i$, $z_i R_i \sum_{j \in G} z_j / |G|$.

Finally, we require that each agent prefer his consumption to any point in the convex hull of the n consumptions it comprises. Given $z_1,\ldots,z_n \in \mathbb{R}^{\ell}_{+}$, $H\{z_1,\ldots,z_n\}$ denotes their convex hull.

Super no–envy solution, K (Kolm, 1973): The allocation $z \in Z$ is ***super envy–free for R*** $\in \mathscr{R}^n$ if for all $i \in N$, and for all $z_i' \in H\{z_1,\ldots,z_n\}$, $z_i R_i z_i'$.

Note that if n=2, no–envy and average no–envy coincide. However, if $n > 2$, there is no logical relation between these concepts (Thomson, 1982). The strict no–envy solution is a subsolution of both the no–envy and the average no–envy solutions[8]. The super no–envy solution is the smallest of the distributional criteria discussed in this paper. It is a subsolution of the strict no–envy solution, and if preferences are smooth, it actually coincides with the Walrasian solution operated from equal division.

The three solutions listed above are *monotonic.* Next, we present a game intended to implement any solution in the following broad class, which includes them.

Definition. A solution φ belongs to the ***family*** Φ if there is a list $(\psi_i)_{i \in N}$ of correspondences $\psi_i : Z \to Z_0 = \{ z_0 \in \mathbb{R}^{\ell}_{+} \mid z_0 \leqq \Omega \}$ such that

(i) for all $i \in N$ and for all $z \in Z$, $z_i \in \psi_i(z)$ and there is $z_i' \in \psi_i(z)$ with $z_i' \neq 0$

(ii) for all $R \in \mathscr{R}^n$, and for all $z \in Z$, $z \in \varphi(R)$ if and only if for all $i \in N$, and for all $z_i' \in \psi_i(z)$, $z_i R_i z_i'$

(iii) there is a list $(V_i)_{i \in N}$ of sets and a list $(\tau_i)_{i \in N}$ of functions $\tau_i : V_i \times Z \to \mathbb{R}^{\ell}_{+}$ such that for all $i \in N$, for all $z \in Z$, $\mathrm{Im}\{\tau_i(.,z)\} = \psi_i(z)$.

[8]It can also be related by means of a consistency property to the average no–envy solution. It is the "largest" consistent solution contained in it (Thomson, 1992).

Although the definition of Φ may appear somewhat technical, it has the advantage of being quite general and in particular, the three examples given above belong to the family Φ, as now demonstrated[9]:

For the average no-envy solution, let $\psi_i(z) = \{a_i(z),z_i\}$, $V_i = \{0,1\}$ with generic element denoted by k_i and $\tau_i(k_i,z) = k_iz_i + (1-k_i)a_i(z_i)$;

For the strict no-envy solution, let $\psi_i(z) = \{z_i' \in \mathbb{R}_+^\ell \mid$ for some $G_i \in \mathcal{G}_i$, $z_i' = \sum_{j\in G_i} z_j/|G_i|\}\cup\{z_i\}$, $V_i = \mathcal{G}_i\cup\{i\}$ with generic element denoted by G_i, and $\tau_i(G_i,z) = \sum_{j\in G_i} z_j/|G_i|$;

For the super no-envy solution, let $\psi_i(z) = \{z_i' \in \mathbb{R}_+^\ell \mid$ for some $\lambda_i \in \Delta^{n-1}$, z_i' $\sum_{j\in N} \lambda_{ij}z_j\}$, $V_i = \Delta^{n-1}$ with generic element λ_i, and $\tau_i(\lambda_i,z) = \sum_{j\in N} \lambda_{ij}z_j$.

Note that all the members of Φ are *monotonic.* We now present a game implementing any one of them. In the specification of the outcome function, when reference has been made to agent i, the only non-zero component of an allocation such as $(0,..,z_i,..,0)$ should be understood to appear in the i^{th} place. To implement a given $\varphi \in \Phi$, we use its associated sets $(V_i)_{i\in N}$, so that the simplicity of the implementation will obviously be directly related to that of the $(V_i)_{i\in N}$. Our claim of having achieved simplicity for our examples rests on the fact that their associated $(V_i)_{i\in N}$ (see above) are indeed simple.

Game Γ^φ: $S_1 = Z\times N\times V_1$, $S_2 = Z\times N\times V_2$, and for each $i \in N\backslash\{1,2\}$, $S_i = N\times V_i$.

Given $s = ((z^1,t_1,v_1),(z^2,t_2,v_2),(t_3,v_3),\ldots,(t_n,v_n)) \in S$, let $i(s) = \Sigma t_i \pmod n$ and

$$h(s) = \begin{cases} (0,\ldots,0) & \text{if } z^1 \neq z^2 \\ (0,..,\tau_{i(s)}(v_{i(s)},z^1),\ldots,0) & \text{if } z^1 = z^2 \text{ and } \tau_{i(s)}(v_{i(s)},z^1) \neq z^1_{i(s)} \\ z^1 & \text{if } z^1 = z^2 \text{ and } \tau_{i(s)}(v_{i(s)},z^1) = z^1_{i(s)} \end{cases}$$

[9]For the examples considered in the paper, it would actually suffice to choose the sets V_i to be subsets of the simplex. At the price of a somewhat greater complexity, our formulation covers a wider family of solutions.

Just like in the previous game, this outcome function is designed so as to ensure that at equilibrium, the first two agents announce the same allocation. Once they agree on some $z \in Z$, then for every $i \in N$, agent i gets the opportunity to choose between any point of $\psi_i(z^1)$ (a set which includes z_i^1) by adjusting his integer so as to win the modulo game. Each point of $\psi_i(z^1)$ is obtained by an appropriate choice of v_i.

Theorem 2. Given any $\varphi \in \Phi$, the game Γ^{φ} implements the solution φ.

Proof. Claim 1. *If* $z \in E_Z(\Gamma^{\varphi};R)$, *then* $z \in \varphi(R)$. Let $s = ((z^1,t_1,v_1),(z^2,t_2,v_2),(t_3,v_3),\dots,(t_n,v_n))$ be an equilibrium supporting z. First, we must have $z^1=z^2$. Indeed, $att_1(s) = \{0\} \cup \psi_1(z^2)$, the consumption 0 being obtained for any $s_1' = (z'^1,t_1',v_1')$ such that $z'^1 \neq z^2$, and each consumption $z_1' \in \psi_1(z^2)$ being obtained for $s_1' = (z^2,t_1',v_1')$, where t_1' is such that $t_1' + \sum_{i \neq 1} t_i \pmod{n} = 1$, and v_1' such that $\tau_1(v_1',z^2) = z_1'$. By strict monotonicity of preferences, and the fact that at least one point of $\psi_1(z^2)$ is not equal to 0, then necessarily for some strategy available to agent 1, the consumption he receives is strictly preferred to 0. Therefore, since $s_1 = (z^1,t_1,v_1)$ is a best response to s_{-1} for agent 1, we have $z^1=z^2$. Now, we cannot have $\tau_{i(s)}(v_{i(s)},z^1) \neq z^1_{i(s)}$. Indeed, for any $j \neq i(s)$, $att_j(s) = \{0\} \cup \psi_j(z^1)$, where the consumption 0 is obtained by playing s_j, and each point $z_j' \in \psi_j(z^1)$ is obtained by switching to t_j' such that $t_j' + \sum_{k \neq j} t_k \pmod{n} = j$ and v_j' such that $\tau_j(v_j',z^1) = z_j'$. Since, again by strict monotonicity of preferences, necessarily one of these attainable consumptions is strictly preferred to 0, the claim is proved.

Claim 2. *If* $z \in \varphi(R)$, *then* $z \in E_Z(\Gamma^{\varphi};R)$. Indeed, let $s = ((z,1,v_1),(z,1,v_2),(1,v_3),\dots,(1,v_n))$, where v_i is such that for all $i \in N$, $\tau_i(v_i,z) = z_i$. Then, $h(s) = z$. We have $att_1(s) = \{0\} \cup \psi_1(z)$, where the consumption 0 is obtained for any $s_1' = (z',t',v_1')$ with $z' \neq z$, and each consumption $z_1' \in \psi_1(z)$ is obtained for $s_1' = (z,2,v_1')$ such that $\tau_1(v_1',z) = z_1'$. Since $z_1 R_1 z_1'$ for all $z_1' \in \psi_1(z)$ and $z_1 R_1 0$, s_1

is a best response to s_{-1} for agent 1. Similarly, s_2 is a best response to s_{-2} for agent 2. Finally, for any $i \in N$, $i \neq 1, 2$, we also have $att_i(s) = \psi_i(z)$ where each consumption $z_i' \in \psi_i(z)$ is obtained for $s_i' = (i+1, v_i')$, and v_i' is such that $\tau_i(v_i', z) = z_i'$. Again, since $z_i R_i z_i'$ for all $z_i' \in \psi_i(z)$, s_i is a best response to s_{-i} for agent i.

Q.E.D.

***Remark* 5.** As claimed earlier, the no–envy solution is covered by the theorem. Take $\psi_i(z) = \{z_1,\ldots,z_n\}$, $V_i = N$ with generic element k_i and $\tau_i(k_i,z) = z_{k_i}$.

5. ***Implementation of the Pareto solution.*** The pareto solution is the solution that associates with each economy its set of pareto efficient allocations.

Pareto solution, P^*. The allocation $z \in Z$ is ***pareto–efficient for $R \in \mathscr{R}^n$*** if there is no $z' \in Z$ such that $z_i' R_i z_i$ for all $i \in N$ and $z_i' P_i z_i$ for at least one $i \in N$.

Under strict monotonicity of preferences, the pareto solution is *monotonic.* (For an example showing that if preferences are only weakly monotone, the property may not hold, see Thomson, 1985). The solution that associates with each economy its set of efficient allocations such that at some supporting prices the value of every agent's consumption is positive is also *monotonic.* We will find it convenient to work with this variant of the pareto solution, which we will refer to as the "strong" pareto solution. Note that under strict monotonicity of preferences, the two differ only in that the allocations at which some agent receives nothing are excluded by the strong pareto solution. These allocations are also excluded by all of the distributional criteria considered above.

Strong pareto solution, P. The allocation $z \in Z$ is ***strongly pareto–efficient for $R \in \mathscr{R}^n$*** if it is pareto–efficient and at some supporting prices $p \in \Delta^{\ell-1}$, $pz_i > 0$ for all $i \in N$.

Consider the following game, where $D = \{(z,p) \in Z \times \Delta^{\ell-1} | pz_i > 0 \text{ for all } i \in N\}$ and given $(z_i,p) \in \mathbb{R}_+^{\ell} \times \Delta^{\ell-1}$, $B(z_i,p) = \{z_i' \in \mathbb{R}_+^{\ell} \mid z_i' \leqq \Omega,\ pz_i' \leqq pz_i\}$.

Game Γ^P: $S_1 = S_2 = D \times N \times Z_0$ and $S_3 = \ldots = S_n = N \times Z_0$.

Given $s = ((z^1,p_1,t_1,z_1),(z^2,p_2,t_2,z_2),(t_3,z_3),\ldots,(t_n,z_n)) \in S$, let $i(s) = \Sigma t_i \pmod n$ and

$h(s) = (0,\ldots,0)$	if $(z^1,p_1) \neq (z^2,p_2)$ or
	if $(z^1,p_1) = (z^2,p_2)$ and $z_{i(s)} \notin B(z^1_{i(s)},p_1)$
$= (0,\ldots,z_{i(s)},\ldots,0)$	if $(z^1,p_1) = (z^2,p_2)$, $z_{i(s)} \in B(z^1_{i(s)},p_1)$, and $z_{i(s)} \neq z^1_{i(s)}$
$= z^1$	if $(z^1,p_1) = (z^2,p_2)$ and $z_{i(s)} = z^1_{i(s)}$

The inspiration for this game is the second fundamental welfare theorem. Indeed, it works as follows: the first two agents, the dividers, announce an allocation–price pair; the allocations can be interpreted as recommendations for the entire economy and the prices as supporting prices; the outcome function is specified so as to guarantee that the dividers agree on some allocation–price pair; in addition, each agent announces (i) an integer, which is used to determine who is granted the right to object to the dividers' recommendation, and (ii) a consumption that can be interpreted as the consumption that he feels he should receive; to be accepted, an objection should be reasonable, in that its value at the common prices announced by the dividers should not exceed the value of the consumption the dividers intended for him. If the objection is not reasonable, every one ends up with nothing. If it is, the objector receives what he requested and the others receive nothing.

***Theorem* 3.** The game Γ^P implements the strong pareto solution.

Proof. *Claim 1. If $z \in E_Z(\Gamma^P;R)$, then $z \in P(R)$.* First, if $s = ((z^1,p_1,t_1,z_1),(z^2,p_2,t_2,z_2),(t_3,z_3),\ldots,(t_n,z_n)) \in E_Z(\Gamma^P;R)$, then $(z^1,p_1) = (z^2,p_2)$. Indeed, $att_1(s) = B(z^2_1,p_2)$: agent 1 receives 0 by announcing $(z'^1,p_1') \neq (z^2,p_2)$; he receives any point $z_1' \in B(z^2_1,p_2)$ by announcing $s_1' = (z^1,p_1,t_1,z_1)$ such that $(z^1,p_1) =$

(z^2,p_2), and $t_1 + \Sigma_{i \neq 1} t_i \pmod n = 1$ and $z_1 = z_1'$. Since preferences are strictly monotonic and $B(z_1^2,p_2)$ contains positive points, if follows that at equilibrium, $(z^1,p_1) = (z^2,p_2)$. Similarly, $att_2(s) = B(z_2^1,p_2)$. Next, if the equality $(z^1,p_1) = (z^2,p_2)$ holds, then for each $i \in N$, $att_i(s) = B(z_i^1,p_1)$: indeed, agent i can reach any point z_i' in this set by announcing $s_i' = (t_i',z_i')$ such that $t_i' + \Sigma_{j \neq i} t_j \pmod n = i$. Equilibrium requires that he obtains his preferred point in that set. This is possible only if for each $i \in N$, $z_i' = z_i$. And then, $z \in P(R)$.

Claim 2. If $z \in P(R)$, then $z \in E_Z(\Gamma^P;R)$. Indeed, if $z \in P(R)$, then by the second fundamental welfare theorem, there exists $p \in \Delta^{\ell-1}$ such that for all $i \in N$ and for all $z_i' \in \mathbb{R}_+^\ell$ such that $pz_i' \leqq pz_i$, $z_i R_i z_i'$. Now, let $s = ((z,p,1,z_1),(z,p,1,z_2),(1,z_3),\ldots(1,z_n))$. Then, $att_i(s) = B(z_i,p)$ for all $i \in N$. We omit the straightforward proof that s is indeed an equilibrium and z the corresponding equilibrium allocation.

Q.E.D.

6. ***Implementation of equitable and efficient solutions.*** In the final section, we show how to take care of both distributional and efficiency objectives. Essentially, we combine the game just proposed to implement the strong pareto solution, first with Divide and Permute – this is the game $\Gamma^{F \cap P}$ below – and then with the game designed for the implementation of any solution φ in the family Φ, whevener this intersection is well–defined – this is the game $\Gamma^{\varphi \cap P}$. Note that the intersection of two *monotonic* solutions is also *monotonic*, provided it is a well–defined solution. This proviso is met for each of the examples in which we are interested since they all contain the Walrasian solution operated from equal division.

Given $(z_i,p) \in \mathbb{R}_+^\ell \times \Delta^{n-1}$, the sets D and $B(z_i,p)$ are defined as for the game Γ^P.

Game $\Gamma^{F\cap P}$: For $i \in \{1,2\}$, let $S_i = D \times N \times Z_0 \times \Pi^n$ and for each $i \in N \backslash \{1,2\}$, let $S_i = N \times Z_0 \times \Pi^n$.

Given $s = ((z^1,p_1,t_1,z_1,\pi_1),(z^2,p_2,t_2,z_2,\pi_2),(t_3,z_3,\pi_3),\ldots,(t_n,z_n,\pi_n)) \in S$, let $i(s) = \Sigma t_i \pmod n$ and

$h(s) = (0,\ldots,0)$ if $(z^1,p_1) \neq (z^2,p_2)$ or
if $(z^1,p_1) = (z^2,p_2)$ and $z_{i(s)} \notin B(z^1_{i(s)},p_1)$

$= (0,\ldots,z_{i(s)},\ldots,0)$ if $(z^1,p_1) = (z^2,p_2)$, $z_{i(s)} \in B(z^1_{i(s)},p_1)$ and $z_{i(s)} \neq z^1_{i(s)}$

$= \pi_n \circ \pi_{n-1} \circ \ldots \pi_1(z^1)$ if $(z^1,p_1) = (z^2,p_2)$, and $z_{i(s)} = z^1_{i(s)}$

Game $\Gamma^{\varphi\cap P}$: For $i \in \{1,2\}$, let $S_i = D \times N \times Z_0 \times V_i$ and for each $i \in N \backslash \{1,2\}$, let $S_i = N \times Z_0 \times V_i$.

Given $s = ((z^1,p_1,t_1,z_1,v_1),(z^2,p_2,t_2,z_2,v_2),(t_3,z_3,v_3),\ldots,(t_n,z_n,v_n)) \in S$, let $i(s) = \Sigma t_i \pmod n$ and $h(s)$ is as in the above game for the first three cases. Otherwise

$h(s) = (0,\ldots,\tau_{i(s)}(v_{i(s)},z^1),\ldots,0)$ if $(z^1,p_1) = (z^2,p_2)$, $z_{i(s)} = z^1_{i(s)}$ and
$\tau_{i(s)}(v_{i(s)},z^1) \neq z^1_{i(s)}$

$= z^1$ if $(z^1,p_1) = (z^2,p_2)$ $z_{i(s)} = z^1_{i(s)}$ and
$\tau_{i(s)}(v_{i(s)},z^1) = z_{i(s)}$.

***Theorem* 4.** The game $\Gamma^{F\cap P}$ implement the solution $F\cap P$.

***Theorem* 5.** For all $\varphi \in \Phi$ such that $\varphi\cap P$ is well-defined, the game $\Gamma^{\varphi\cap P}$ implement the solution $\varphi\cap P$.

We omit the proofs of Theorems 4 and 5, which are similar to the proofs of Theorems 2 and 3, limiting ourselves to noting that as before, they involve showing that at equilibrium, agents 1 and 2 announce the same allocation–price pair, and that at those prices, the value of the consumption that the winner of the modulo game announces for himself is equal to the value of his component of that common allocation. Given agents 1 and 2's common announcement z, each agent's attainable set contains all the feasible consumptions whose value is less than the value of his component of z, and it may contain additional points. These additional points are the components of z for $\Gamma^{F\cap P}$ and the elements of $\psi_{i(s)}(z)$ for $\Gamma^{\varphi\cap P}$. By feasibility, no one can end up above his budget set. This implies the existence of parallel lines of support to the indifference curves at each of the equilibrium consumptions, and therefore efficiency. The attainability of the additional consumptions guarantees that the relevant equity criterion is met.

***Remark* 6.** Implementation occurs even if $n = 2$.

***Remark* 7.** If $n > 2$, the outcome function can be modified so that no resource be ever thrown away.

***Remark* 8.** The spaces of feasible allocations in models with heterogeneous goods such as land or time do not have a convex structure and apart from the no–envy solution, the examples of solutions examined above cannot be applied to them.

7. ***Conclusion.*** The purpose of this paper has been to show that very simple games can be devised to implement solutions to the problem of fair division. We hope that the availability of such games will help in bringing the theory of implementation closer to applications.

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