The Core and Competitive Equilibria in Finite Economies

McKenzie, Lionel

Working Paper No. 369
December 1993

The Core and Competitive Equilibria in Finite Economies

Lionel W. McKenzie

Rochester Center for Economic Research
Working Paper No. 369

THE CORE AND COMPETITIVE EQUILIBRIA IN FINITE ECONOMIES*

Lionel W. McKenzie

University of Rochester

December 7, 1993

*This paper will appear in a festschrift for John Chipman in celebration of his 65*th* birthday.

THE CORE AND COMPETITIVE EQUILIBRIA IN FINITE ECONOMIES*

Lionel W. McKenzie

University of Rochester

December 7, 1993

The approach to competitive markets by way of bargaining among traders was developed by Edgeworth (1881). Walras (1874–77) considered markets led by market managers. In these markets prices are announced which lead to offers to buy and sell, which are aggregated over the market. Then price lists are revised in the light of supply and demand, upward for an excess demand, downward for an excess supply. Trades are carried out when demand and supply are in balance. On the other hand Edgeworth considered bargaining between individual traders in which no bargains are final until a point is reached where no group of traders can conclude a new bargain which they prefer to their existing bargains. Edgeworth showed in the simplest case of trading in two goods that such a situation, in which no new bargains are possible which are preferred by some participants, will approach a competitive equilibrium as the number of traders increases indefinitely. This result has been generalized to the case of many goods and to production economies in recent years. The path breaking paper was that of Debreu and Scarf (1963). Most of the results described in this paper may be found in McKenzie (1988 and 1990) and McKenzie and Shinotsuka (1991). There are also similar theorems in Hildenbrand and Kirman (1988). My purpose here is to show precisely where assumptions must be strengthened or may be weakened in establishing the major theorems of the subject for the case of economies with a finite number of agents and a finite number of goods.

---

* I am grateful to John Boyd and Tomoichi Shinotsuka for their assistance.

CORE ALLOCATIONS WHICH ARE COMPETITIVE ALLOCATIONS

Consider an economy where the consumption sets $C^h \subset \mathbb{R}^n$, $h = 1,\ldots,H$, represent the sets of net possible trades by consumers. $C = \Sigma_{h=1}^H C^h$ is the set of possible net trades by all consumers together. Positive quantities represent amounts received by the consumer and negative quantities represent quantities provided by the consumer. A *strictly preferred set* $P^h(x)$ contains all commodity bundles preferred to x. A *weakly preferred set* $R^h(x)$ contains all commodity bundles to which x is not preferred. P^h is *irreflexive* if $x \notin P^h(x)$. From the interpretation of P^h it is clear that it is irreflexive. We will also use the expression xP^hy for $x \in P^h(y)$ and correspondingly for R^h. That is, P^h or R^h may denote a relation or a correspondence determined by the relation. There is a production possibility set $Y \subset \mathbb{R}^n$. We will use the activities model of production, so Y will be a convex cone with vertex at the origin. However diminishing returns can be accommodated by introducing entrepreneurial factors (McKenzie (1959)). These activities are available to all consumers. An *allocation* of trades $\tilde{x}$ is a list $(x^1,\cdots,x^H)$ such that $x^h \in C^h$ for all h. We may also write $\{x^h\}$ for an allocation when no confusion will result. A *feasible allocation* is an allocation that satisfies $\Sigma_1^H x^h \in Y$. If $x \in C^h$ then the h*th* consumer is *locally not satiated* at x if an arbitrary neighborhood U of x contains $y \in U \cap C^h$ with $y \in P^h(x)$. The economy E_1 is given by the list $(C^1,\cdots,C^H, P^1,\cdots,P^H,Y)$.

In this section we make two assumptions on preferences.

1. C^h is convex and not empty for all h.

2. For all h, P^h is open valued relative to C^h. If $\tilde{x}$ is feasible, the h*th* consumer is locally not satiated at x^h.

We also make two assumptions on production possibilities.

3. The production set Y is a convex cone with vertex at the origin.

4. $Y \cap \mathbb{R}_+^n = \{0\}$.

Finally two assumptions relate consumption sets and production sets.

Let $x_S = \Sigma_{h \in S}\, x^h$ and $C_S = \Sigma_{h \in S}\, C_S^h$ where S is a subset of consumers. An economy is said to be *irreducible* if for every nontrivial partition of the set of consumers into two subsets S_1 and S_2 the following condition holds. If $(x^1, \ldots, x^H)$ is a feasible allocation then there is $w \in C_{S_2}$ and $y' \in Y$ such that $z_{S_1} = y' - \alpha w$ for some $\alpha > 0$ and $z^h \in P^h(x^h)$ for all $h \in S_1$.

5. $Y \cap C^h \neq \phi$ and relative interior $Y \cap$ relative interior $C \neq \phi$.

6. The economy E_1 is irreducible.

Assumption 3 implies that production processes are independent and divisible. Assumption 4 is only a convenience since goods produced out of nothing need not be economized.

Irreducibility means that given any feasible allocation of trades if the consumers are divided into two nonempty sets each set will have a possible trade some multiple of which, together with a possible output variation, improves the allocation to the members of the other set. This idea and assumptions like 5 and 6 will play important roles whenever the existence of a competitive equilibrium is to be proved. The first part of Assumption 5 is a survival assumption for isolated individuals. The second part of Assumption 5 guarantees that there is $z \in C^h$ with $pz < 0$ for some h. Then Assumption 6 provides that if one consumer satisfies this condition all do. Assumption 6 together with Assumption 5 will imply that prices which support preferred sets together with the production set will be competitive equilibrium prices. We will mean by the *relative interior* of a set $V \subset \mathbb{R}^n$ the interior of V relative to the smallest affine subspace containing V. We use this notion only with reference to convex sets. If a convex set is not empty, its relative interior is not empty.

The central idea leading to the concept of the core is that of an improving coalition. Given an allocation $(x^1,\cdots,x^H)$, an *improving coalition* is a subset of consumers B such that for each $h \in B$ there is $z^h \in C^h$ with $\Sigma_{h \in B}\, z^h \in Y$ and $z^h \in P^h(x^h)$ for all h. Then the *core* of the economy is defined as the set of all feasible allocations for which there is no improving coalition. This concept corresponds to Edgeworth's idea of equilibrium since if an allocation is in the core there will be no group of traders who can negotiate a new bargain among themselves, taking account of their production possibilities, which will improve the position of all the traders in the group. We will show that when the consumers are duplicated without limit any allocation that remains in the core indefinitely can be realized as a competitive equilibrium. Also the part of the core that assigns the same allocation to the replicas of each person will shrink to the set of competitive equilibria of the original economy.

Under these assumptions there is not a set of firms which exist independently of the choices of the agents. Rather there is a set of production possibilities, which are generated by activities available to any subset of agents that chooses to use them. For this reason the profit condition of competitive equilibrium will require that no profits are available and the demand condition will assume that households receive no income net of their sales of goods and services. However the conditions for competitive equilibrium will be equivalent under the present assumptions to the usual ones. We will say that $\{x^h\}$ is a *competitive allocation* if there is a price vector p and an output y such that

I. $px^h \leq 0$, and $z \in P^h(x^h)$ implies $pz > 0$, for all h.

II. $y \in Y$ and $py = 0$, while $z \in Y$ implies $pz \leq 0$.

III. $\Sigma_{h=1}^{H}\, x^h = y$.

Since an activity can operate at any positive level a positive profit is inconsistent with profit maximization. Since an activity can operate at zero level a negative profit is also inconsistent with with profit maximization. This leads to Condition

II. Thus in equilibrium profits are zero and household income is 0 for all h. Zero income, along with maximization of preference within budget constraints, leads to Condition I. Condition III is the balance of supply and demand.

Local nonsatiation alone implies that a competitive allocation lies in the core. In a Pareto improvement it is only required that some consumers benefit, not necessarily all consumers, while none suffers. The stronger criterion for improving coalitions may be defended as providing all members of an improving coalition with an incentive to act. Also the weaker definition is not sufficient for our proof that the equal treatment core is closed. This result is needed to prove that the equal treatment core converges to the set of competitive equilibria of the original economy as the economy is replicated and also to prove existence of equilibrium.

Theorem 1. Local nonsatiation implies that a competitive allocation for E_1 is in the core.

Proof. Let $\{x^h\}$ be a competitive allocation. It is implied by local nonsatiation that $px^h = 0$. Suppose B is an improving coalition by means of the allocation $\{z^h\}_{h \in B}$ where $\Sigma_{h \in B}\ z^h = z \in Y$. Then $z^h \in P^h(x^h)$ for all $h \in B$. However $z^h \in P^h(x^h)$ implies that $pz^h > px^h$ from the demand condition of competitive equilibrium. Therefore $\Sigma_{h \in B}\ pz^h > \Sigma_{h \in B}\ px^h = 0$. But $z \in Y$ implies $pz \leq 0$ by condition II of competitive equilibrium. Thus no such improving coalition B can exist and $\{x^h\}$ lies in the core. □

For each h let the number of consumers identical to the consumer with index h be increased to r by adding new consumers and index the expanded set by hs where $h = 1,\cdots,H$ and $s = 1,\cdots,r$. The economy that is replicated r times will be referred to as E_r. The economy E_r for $r > 1$ is given by the list $(C^{11},\cdots,C^{1r},\cdots,C^{H1},\cdots,C^{Hr},P^{11},\cdots,P^{Hr},Y)$. Allocations for E_r may be written $\{x^{hs}\}$. We will consider allocations $\{x^{hs}\}$ in the core in which $x^{h1} = x^{hs}$ for

$s = 1$ to r. These allocations form the *equal treatment core.* That is, the replicas of a given consumer receive the same allocation that he does. There will be no ambiguity if equal treatment allocations are indicated by the expression $\{x^h\}_r$.

Lemma 1. As r increases the allocations $\{x^h\}_r$ in the equal treatment core of E_r form a non-increasing sequence of nested sets.

Proof. If B is an improving coalition for the allocation $\{x^h\}_r$ when $r = s$, it is also an improving coalition when $r = s + 1$. Therefore as r increases no new allocations $\{x^h\}_r$ can appear in the equal treatment core. □

The basic result is

Theorem 2. Under Assumptions 1–6 if $\{x^h\}_r$ is in the equal treatment core of E_r for all r as $r \to \infty$ then $\{x^h\}$ is a competitive allocation for E_1.

Proof. Assume that $\{x^h\}_r$ is an allocation in the equal treatment core for all values of r. For any $x \geq 0$ let $P^h(x)$ be the set of trades preferred to the trade x by consumers who are duplicates of the original consumer with index h. We may refer to them as consumers of the h*th* type. By Assumption 2, $P^h(x)$ is not empty and $P^h(x)$ is open relative to C^h. Let $P(x^1,\cdots,x^H) = P(\tilde{x})$ be the convex hull of the $P^h(x^h)$.

Suppose $Y \cap P(\tilde{x}) \neq \phi$. Then there is a set of consumers B and weights α_i such that $\Sigma_{i\in B}\ \alpha_i z^i = y \in Y$, $\alpha_i > 0$, $\Sigma_{i\in B}\ \alpha_i = 1$, and $z^i \in P^{h(i)}(x^{h(i)})$ where the i*th* consumer of the set B is a replica of the h(i)*th* original consumer. The consumers may be chosen so that the number of consumers in B is less than or equal to n+1 (Fenchel (1953), p. 37). For any positive integer s let a_i^s be the smallest integer greater than or equal to $s\alpha_i$. By the first part of Assumption 5, for each $i \in B$ there is $y^i \in C^{h(i)} \cap Y$. Let $w_s^i = (s\alpha_i/a_i^s)(z^i - y^i) + y^i$. Since w_s^i is a convex combination of z^i and y^i it lies in $C^{h(i)}$ by Assumption 1. Moreover $w_s^i \to z^i$ as $n \to \infty$. It is this argument that requires the convexity of

C_h and the survival assumption for isolated individuals.

Since the preferred sets are open relative to the $C^{h(i)}$ by Assumption 2, we have $w_s^i \in P^{h(i)}(x^{h(i)})$ for all i for some number s which is large enough. Also

$$\Sigma_{i\in B}\ a_i^s w_s^i = \Sigma_{i\in B}\ (s\alpha_i z^i - s\alpha_i y^i + a_i^s y^i) = sy + \Sigma_{i\in B}(a_i^s - s\alpha_i)y^i.$$

The fact that $0 \leq a_i^s - s\alpha_i$ and $y^i \in Y$ implies that $\Sigma_{i\in B}\ a_i^s w_s^i \in Y$, since Y is a convex cone by Assumption 3. Let $T = \{\tau |\ h(i) = \tau$ for some $i \in B\}$. Let $B_\tau = \{i \in B |\ h(i) = \tau\}$. Put $r = \max_{\tau\in T}\ (\Sigma_{i\in B_\tau}\ a_i^s)$. If the economy has been replicated r times there are enough traders of each type τ in T to offer the net trade $\Sigma_{i\in B_\tau}\ a_i^s w_s^i$ needed from the τth type to achieve the improved allocation for all $i \in B$. Then the improving coalition can be formed if the original economy has been replicated r times. This contradicts the hypothesis. Therefore $Y \cap P(x^1,\cdots,x^H) = \phi$. In other words $\{x^h\}_r$ in the core for all r implies that the production set Y intersected with the convex hull of the $P^h(x^h)$, the sets of preferred trades of the original consumers, is empty.

Consider $P(\tilde{x})$ and Y in the smallest linear subspace L that contains both C and Y. By a separation theorem for convex sets (Berge (1963), p. 162) there is a vector $p \in L$, $p \neq 0$, such that $pz \geq 0$ for all $z \in P(x^1,\cdots,x^H)$ and $pz \leq 0$ for all $z \in Y$. Since $P^h(x^h) \subset P(x^1,\cdots,x^H)$, $pz \geq 0$ for all $z \in P(x^h)$. By Assumption 2 there is a point $z \in P^h(x^h)$ in every neighborhood of x^h. Thus $px^h \geq 0$ must hold for all h. Since $(x^1,\cdots,x^h)$ is an allocation it is also true that $\Sigma_{h=1}^H\ x^h = y \in Y$. Therefore $p\cdot\Sigma_{h=1}^H\ x^h \leq 0$. This implies that $px^h = 0$ must hold for all h.

To complete the proof it is necessary to show that in fact $pz > 0$ for any $z \in P_h(x^h)$ for all h. We first prove two lemmas.

<u>Lemma</u> <u>2</u>. Let sets A and B be convex sets in R^n. Suppose $0 \in A$. If relative interior A $\cap$ relative interior B $\neq \phi$ there is no hyperplane which

separates A and B in the smallest linear subspace containing both.

Proof. Let L be the smallest linear subspace containing A and B. If H is a hyperplane separating A and B in L then there is $q \in L$, $q \neq 0$, such that $H = \{z \in L |\ qz = \mu\}$ and H separates A and B in L. However, q may be chosen so that $qz \leq \mu$ for all $z \in A$ and $qz \geq \mu$ for all $z \in B$. Then $y \in A \cap B$ implies $qy = \mu$. But $y \in$ relative interior A implies $qz = \mu$ for all $z \in A$. Similarly $qz = \mu$ for all $z \in B$. Then $0 \in A$ implies $\mu = 0$. Since A and B span L this implies $qz = 0$ for all $z \in L$. Then $q \in L$ implies that $q = 0$. Since this is a contradiction of the choice of q no such separation is possible. □

<u>Lemma</u> <u>3</u>. If $pz \geq 0$ for all $z \in P^h(x)$ where $x \in C^h$, and there is $w \in C^h$ such that $pw < 0$, then $pz > 0$ holds for all $z \in P^h(x)$.

Proof. Assume there is $z \in P^h(x)$ where $pz = 0$. Since P^h is open in C^h and C^h is convex by Assumption 1, there is $z' \neq z$ on the line segment from w to z and close to z such that $z' \in P^h(x)$. But $pz' < 0$ in contradiction to the hypothesis. Thus $pz > 0$ holds for all $z \in P^h(x)$. □

The vector p supports Y. Thus $pz \leq 0$ for all $z \in Y$. Since relative interior Y ∩ relative interior C contains a point w by the second part of Assumption 5, and $0 \in Y$, it is implied by Lemma 2 that $pw < 0$. Therefore $pw^h < 0$ for some $w^h \in C^h$ for some h. Let $pw^h < 0$ hold for $h \in S_1$ and $pw^h \geq 0$ hold for $h \in S_2$ where S_2 and S_1 partition the set of consumers. Lemma 3 implies that $pz > 0$ holds for all $z \in P^h(x^h)$ for $h \in S_1$. However by irreducibility there is $w \in C_{S_2}$ and $y' \in Y$ such that $z_{S_1} = y' - \alpha w$ for $\alpha > 0$ and $z^h \in P^h(x^h)$ for all $h \in S_1$. Since $py' \leq 0$ and $pz_{S_1} > 0$, we have $pw < 0$ in contradiction to the definition of S_2. Therefore S_2 must be empty. Then $S_1 = (1,..,H)$ and $pz > 0$ for $z \in P(x^h)$ for all h. Thus the demand condition of competitive equilibrium is met. Since $px^h = 0$ and $y = \Sigma_{h=1}^{H} x^h$ we have

$py = 0$. But $pz \leq 0$ for all $z \in Y$ by the support property. Therefore the profit condition is met. The balance condition is implied by the definition of a feasible allocation. Thus $\{x^h\}_1$ is a competitive allocation for E_1. □

2. CONVERGENCE OF THE EQUAL TREATMENT CORE TO THE SET OF EQUILIBRIA

With somewhat strengthened assumptions it is possible to go further and prove convergence of the set of allocations in the equal treatment core to the set of competitive allocations. We introduce

1′. C^h is convex and not empty, and *closed and bounded from below*, for all h.

2′. Relative to C^h, P^h is open valued with *open lower sections* for all h. If $\tilde{x}$ is feasible, the h*th* consumer is locally not satiated at x^h.

3′. The production set Y is a *closed* convex cone with vertex at the origin.

The new assumptions are in italics.

Let K_r be the set of allocations $\{x^h\}_r$ in the equal treatment core of E_r where there are r members of each type and let W be the set of competitive allocations $\{x^h\}$ in E_1. Let the distance $d(K_r,W)$ of the equal treatment core to the set of competitive allocations in E_1 be given by

$$d(K_r,W) = \max_{x \in K_r} \min_{z \in W} |x - z| + \max_{z \in W} \min_{x \in K_r} |x - z|.$$

Convergence of the equal treatment core to the set of competitive allocations of E_1 is defined by $d(K_r,W) \to 0$ as $r \to \infty$.

Let T be the set of equal treatment allocations which are feasible, that is, $T = \{\{x^h\}_r | \Sigma_{h=1}^H x^h \in Y\}$. Because of equal treatment and Assumption 3, that Y is a cone, the index r is irrelevant to feasibility. We first prove

Lemma 4. With Assumptions 1′ and 3′ the set T is compact.

Proof. Since T lies in a metric space, to prove compactness it suffices to show that every infinite sequence of feasible allocations has a point of accumulation which is contained in T (Berge (1963), p. 90). Let $\tilde{x}^s = (x^{1s},\cdots,x^{Hs})$, $s = 1,2,\ldots$, be a sequence of allocations with $\Sigma_{h=1}^H x^{hs} = y^s \in Y$. I claim that $\tilde{x}^s$ is bounded. If not, since the C^h are bounded below it must be that $x_i^{hs} \to \infty$ for some h and i. But for a subsequence (retain notation) $y^s/|y^s| \to y \geq 0$ and $\neq 0$. Since Y closed implies that $y \in Y$ Assumption 4 is violated. This shows that x^{hs} is bounded for each h. Since C^h is closed there is a point of accumulation $(\bar{x}^1,\cdots,\bar{x}^H)$ for the sequence $(x^{1s},\cdots,x^{Hs})$, $s = 1,2,\cdots$, with $\bar{x}^h \in C^h$ for all h. The point $(\bar{x}^1,\cdots,\bar{x}^H)$ represents a feasible allocation and lies in T. □

The new closedness and boundedness assumptions were needed for Lemma 4.

We may now prove

Theorem 3. Given Assumptions 1′, 2′, 3′, 4, 5, 6, if the equal treatment core of E_r is not empty, as the number of replications increases without limit, the equal treatment core of E_r converges to the set of competitive allocations of the original economy E_1.

Proof. From the definition of a competitive allocation the competitive allocations of the original economy are the competitive allocations with equal treatment in the replicated economy. Since by Theorem 1 competitive allocations are always in the core, the second term of the distance formula is 0 for all r. Thus the theorem requires that the first term be shown to converge to 0. Suppose that $\{z^h\}_r$ is an allocation for which there is an improving coalition B. Recall T is the set of equal treatment allocations which are feasible. Let the allocation to the replicas of the hth consumer be perturbed by Δx^h where $\Sigma_{h=1}^H \Delta x^h = 0$, $|\Delta x^h| < \epsilon > 0$, and $x^h + \Delta x^h \in C^h$, for all h. By Assumption 2′ the preference correspondences P^h have open lower sections relative to C^h. Thus if ϵ is sufficiently small the new allocation $\{x^h + \Delta x^h\}_{h \in B}$ is still dominated by the same net trades achievable within B alone that dominated $\{x^h\}_{h\epsilon B}$. Note that this argument uses the strong definition of an improving coalition. Then the set of allocations $T\backslash K_r$ for which there is an improving coalition is open relative to the set of feasible allocations. Also the set T of feasible equal treatment allocations T is closed in $\mathbb{R}^n$ by Lemma 4. Therefore the set of allocations K_r is closed in $\mathbb{R}^n$. This is the argument that requires open lower sections for the preference correspondences P^h.

By Lemma 1 an allocation in the equal treatment core for the sth replica economy is in the core for all $r < s$. Suppose there are allocations $\{w^h\}_r$ in K_r and at a distance of at least $\epsilon > 0$ from any allocation in W for indefinitely large r. Since T is compact by Lemma 4, there would be an accumulation point $\{z^h\}$ of the sequence $\{w^h\}_r$ where $\{z^h\}$ lies at least ϵ from any allocation in W. But Lemma 1 implies that a subsequence $\{w^h\}_r$ (save notation) which converges to $\{z^h\}$ as $r \to \infty$ provides for any s a subsequence (save notation) along which $r > s$ holds which converges to $\{z^h\}$ as $r \to \infty$ and which is contained in K_s. Since K_s is closed $\{z^h\}$ lies in K_s for all s. Therefore $\{z^h\}$ is a competitive allocation of E_1 by Theorem 2, contradicting the inference from the definition of the sequence $\{w^h\}_r$ that $\{z^h\}$ lies at least ϵ from the set of competitive allocations of E_1. Thus no such sequence can exist and it must be that K_r converges to W as $r \to \infty$. □

The Pareto optimum is an allocation for which the coalition of the whole is not an improving coalition, even in a weak sense. No other improving coalitions are considered, in particular not the coalitions composed of single consumers. On the other hand the competitive equilibrium does not allow any improving coalition. Moreover competitive equilibrium requires that each consumer's allocations have zero value at some price vector. In a sense the core is an intermediate notion, especially if we take a weaker definition of an improving coalition and thus a stronger definition of the core. This definition would serve for the proof of Theorem 2 with no major change, but not for the proof of Theorem 3. Let a *weakly improving coalition* B for an allocation $\{x^h\}$ be a coalition for which there is an allocation $\{z^h\}$ with $\Sigma_{h\in B}\ z^h = \Sigma_{h\in B}\ x^h$ such that $x^h \notin P^h(z^h)$ for any $h \in B$ and $z^h \in P^h(x^h)$ for some $h \in B$. Then the *strong core* is the set of allocations for which there is no weakly improving coalition. If an allocation is in the strong core then for any coalition there is no allocation within its feasible set which is a Pareto improvement over its core allocation. The analogous relation holds between the ordinary core allocation and the *weak Pareto optimum*, defined as an allocation such that no reallocation can improve the position of every consumer.

The set of competitive equilibrium allocations is the subset of the strong core in which the core allocations have a zero value at a supporting price vector. Thus although a limit of equal treatment allocations in the strong core as the number of replications increases without limit is a competitive equilibrium, it is not clear that this limit lies in the strong core. However we will find that a slight strengthening of the irreducibility assumption, which we need for proving that the equal treatment core is not empty, will imply that the strong and weak cores coincide.

3. NONEMPTINESS OF THE CORE

It has been proved that every competitive equilibrium is in the core. We will show that the existence of an allocation in the core may be proved independently of the existence of equilibrium allocations. Then the existence of an allocation in the equal treatment core can be used together with Theorem 2 to prove the existence of a competitive equilibrium for E_1. The economy E_1 is defined, as described, by means of consumption sets $C^h \subset \mathbb{R}^n$, strict preference correspondences P^h, for $h = 1,\cdots,H$, and a production set Y. $C = \Sigma_{h=1}^H\ C^h$.

The assumptions are strengthened in some respects and weakened in others. Assumptions 1′, 3′, and 4 are retained. The *graph* of P^h is the set $\{(x,y) \in C^h \times C^h | \ y \in P^h(x)\}$. Assumption 2′ is replaced by the slightly stronger assumption

2′′. For all h, P^h is *convex valued. The graph of P^h is open relative to $C^h \times C^h$.*

Assumption 6 is not needed and Assumption 5 is replaced by the weaker assumption

5′. For all h, $C^h \cap Y$ is not empty.

It will be noted that P^h is now assumed to be convex valued although this assumption was not needed in Theorem 3 to prove convergence of the equal treatment core to the set of competitive equilibria for E_1. This assumption is also critical for the existence of competitive equilibrium. On the other hand irreducibility, interiority, and local nonsatiation are not needed here. The assumption that the graph of P^h is open is a slight strengthening of the assumption in 2′ that P^h is open valued and has open lower sections. On the other hand, irreducibility and interiority assumptions are important for the existence of a competitive equilibrium but not for the existence of a point in the core.

We first record some properties of the set of feasible allocations F_S where $S \subset (1,\cdots,H)$. Let $x_S = (x^h)_{h\in S}$. $F_S = \{(x^h)_{h\in S} | \ x^h \in C^h$ for all $h \in S$ and $\Sigma_{h\in S}\, x^h \in Y\}$.

<u>Lemma 5</u>. F_S is nonempty, compact, and convex.

Proof. Assumption 5′ and the fact that Y is a cone imply that F_S is not empty. F_S is compact by the argument of the proof of Lemma 4. Convexity of F_S follows from the convexity of the C^h and Y. □

Let $\mathcal{B}$ be a nonempty family of subsets of $\{1,\cdots,H\}$. Define

$\mathscr{B}_h = \{S \in \mathscr{B} \mid h \in S\}$. A family $\mathscr{B}$ is *balanced* if there exist nonnegative weights $\{\lambda_S\}$ with $\Sigma_{S\in\mathscr{B}_h} \lambda_S = 1$ for all h. Let $I = \{1,\cdots,H\}$. The economy is said to be *O–balanced*, if for any balanced family $\mathscr{B}$ with balancing weights $\{\lambda_S\}$ which satisfies $x_S \in F_S$ for all $S \in \mathscr{B}$, it follows that $x_I \in F_I$ where $x_I^h = \Sigma_{S\in\mathscr{B}_h} \lambda_S x_S^h$.

<u>Lemma</u> <u>6</u>. The economy E_1 is O–balanced.

Proof. To show that x_I is feasible it is necessary and sufficient to show that $x_I^h \in C^h$ for all h and $\Sigma_{h\in S} x_I^h \in Y$. Since $x_S^h \in C^h$ for each $S \in \mathscr{B}_h$ by the feasibility of x_S and x_I^h is a convex combination of the x_S^h for $S \in \mathscr{B}_h$, x_I^h lies in C^h by the convexity of C^h. On the other hand, since Y is a cone with vertex at the origin $\Sigma_{h\in S} x_S^h \in Y$ implies that $\Sigma_{h\in S} \lambda_S x_S^h \in Y$ and $\Sigma_{S\in\mathscr{B}} \Sigma_{h\in S} \lambda_S x_S^h \in Y$. But $\Sigma_{S\in\mathscr{B}} \Sigma_{h\in S} \lambda_S x_S^h = \Sigma_{h\in I} \Sigma_{S\in\mathscr{B}_h} \lambda_S x_S^h = \Sigma_h x_I^h$. This completes the proof. □

The following theorem is from Border (1985). It is rephrased to accord with our terminology.

<u>Theorem</u> (Border). Let E be an economy given by the list $(C^1,\cdots,C^H,P^1,\cdots,P^H,Y)$ satisfying

1. For each $h = 1,\cdots,H$, C^h is a nonempty convex subset of $\mathbb{R}^n$.

2. For any $S \subset I$, F_S is a nonempty compact subset of $\Pi_{h\in S} C^h$.

3. For each h, (a) P^h has an open graph in $C^h \times C^h$, (b) $x \notin P^h(x)$, (c) P^h is convex valued (but possibly empty).

4. E is O–balanced.

Then the core of E is not empty.

We may now assert

<u>Theorem</u> <u>4</u>. Under Assumptions 1′, 2′′, 3′, 4, and 5′ the economy E has a nonempty core.

Proof. Condition 1 in Border's theorem is contained in Assumption 1. Lemma 5 implies Condition 2. Condition 3 is implied by Assumption 2′′ and the definition of P^h. Condition 4 is provided by Lemma 6. Applying Border's theorem the conclusion follows. □

4. EXISTENCE OF COMPETITIVE EQUILIBRIUM

We have proved the core not to be empty under assumptions which are weaker in some respects and stronger in other respects than the assumptions used to show that allocations that remain in the equal treatment cores of replicated economies are competitive allocations. We also found that the equal treatment core converges to the set of competitive allocations. This suggests that if the nonemptiness of the core can be extended to the equal treatment core it will be possible to prove that a competitive equilibrium exists by using the strongest form of each assumption from earlier sections. However to prove that the equal treatment core is not empty it is necessary to strengthen the irreducibility assumption although no irreducibility assumption was needed to prove the core nonempty.

If C is a convex set, $x \in C$ is an *extreme point* of C if x is not a convex combination of two points of C distinct from x. We will say that the economy is *strongly irreducible* if it is irreducible and, whenever S_1, S_2 is a nontrivial partition of $\{1,\cdots,H\}$, and $x_{S_1} + x_{S_2} \in Y$ where x_{S_2} is not an extreme point of C_{S_2}, there are $z_{S_1} + z_{S_2} \in Y$ with $z^h P^h x^h$ for $h \in S_1$. Another version of this concept was introduced in Boyd and McKenzie (1993).

By Theorem 2 we have found that under weak conditions the equal treatment core K_r converges to the set of competitive equilibria W of E_1. Bounding and closing C^h and closing Y, and introducing convexity of preference with an open graph, it was possible to prove that the core of E_r is not empty.

This result did not require irreducibility. However in order to obtain the existence of competitive equilibrium we will require stronger forms of both Assumptions 2′′ and 6.

Assumption 2′′′. For all h, P^h is convex valued *and transitive. For all* $x \in C^h$, $R^h(x)$ *is the closure of* $P^h(x)$. The graph of P^h is open relative to $C^h \times C^h$.

Assumption 6′. The economy is *strongly* irreducible.

Recall that x is *indifferent* with y, xI^hy, if xR^hy and yR^hx where xR^hy means not yP^hx. We define $I^h(x) = \{y|\ yI^hx\}$. Thus $y \in I^h(x)$ means $x \notin P^h(y)$ and $y \notin P^h(x)$. The strengthening of the assumption on preferences in 2′′′ and the introduction of strong irreducibility in 6′ are needed to prove that equal treatment allocations in the core for replicates of a given consumer are indifferent. This result is needed in turn to show that K_r is not empty if E_r is not empty.

Lemma 8. For $x \in C^h$, x is locally nonsatiated for all h.

Proof. Since $x^h \in R^h(x^h)$ Assumption 2′′′ implies there is z in every neighborhood of x^h with $z \in P^h(x^h)$. □

Lemma 8 allows us to appeal to results which depend on local nonsatiation.

Lemma 9. If $x \in P^h(z)$ and $z \in R^h(y)$ then $x \in P^h(y)$.

Proof. Since $R^h(y)$ is the closure of $P^h(y)$ there is $z^\nu \to z$ where $z^\nu \in P^h(y)$. Since the graph of P^h is open, for ν large enough $x \in P^h(z^\nu)$. Thus by transitivity $x \in P^h(y)$. □

Lemma 9 allows us to prove

Lemma 10. Under Assumptions 1′, 2′′′, and 6′, suppose $\{x^{hk}\}_r$, $h = 1,\cdots,H$, and $k = 1,\cdots,r$, is an allocation in the core of E_r. Then for given h, $x^{hk}I^hx^{hk'}$ holds for all k and k′.

Proof. Let the allocation $\{x^{hk}\}_r$ where $h = (1,\ldots,H)$ and $k = (1,\ldots,r)$ lie in the core for the economy E_r. I claim that $x^{hk}I^h x^{hj}$ for all h, k, j. Suppose not. Consider a replica with index hj(h), for each original consumer with index h, where hj(h) satisfies $x^{hk}R^h x^{hj(h)}$ for all $k = 1,\cdots,r$. That is, hj(h) has an allocation which no better, and perhaps poorer, than the allocation of any other replica of h. The existence of hj(h) is guaranteed by the irreflexivity and transitivity of preference, and it is for this reason that transitivity is introduced. Consider the coalition $B = \{1j(1),\cdots,Hj(H)\}$ and the allocation to the member of B with index hj(h) of $(1/r)\ \Sigma_{k=1}^{r}\ x^{hk} = x^h$. Since R^h is the closure of P^h by Assumption 2''', R^h is convex valued. Therefore for each h we have $x^h R^h x^{hj(h)}$. Also if $x^{ik}P^i x^{ij(i)}$ for some i,k, then x^i is a convex combination which includes an interior point of $R^i(x^{ij(i)})$, so x^i cannot lie on the boundary of $R^i(x^{ij(i)})$ relative to C^h. Therefore $x^i P^i x^{ij(i)}$ holds.

Now $\Sigma_{h=1}^{H}\ x^h = (1/r)\ \Sigma_{k=1}^{r}\Sigma_{h=1}^{H}\ x^{hk}$ which lies in Y since $\{x^{hk}\}$ is feasible and Y is a cone. Thus $\{x^h\}_1$ is a feasible allocation. By strong irreducibility and convexity we will see that it is possible to spread the gain received by i from the allocation $\{x^h\}_1$ to all h. Let $S_1 = \{h|\ h \neq i\}$ and $S_2 = \{i\}$. Since x^i is not an extreme point of C^i, strong irreducibility implies that there is a feasible allocation $\{z^h\}$ with $z^h P^h x^h$ for $h \in S_1$. Take the convex combination $w^h = \{\lambda x^h + (1-\lambda)z^h\}$ for $0 \leq \lambda \leq 1$. $\{w^h\}_{h=1}^{H}$ is a feasible allocation. Moreover w^h is preferred by all $h \in S_1$ to x^h and, using Lemma 9, to $x^{hj(h)}$ as well. Since $x^i P^i x^{ij(i)}$ and the graph of P^i is open, for λ sufficiently close to 1, w^i is preferred by i to x^h. Thus B is an improving coalition and $\{x^{hk}\}_r$ is not an allocation in the core of E_r contrary to the assumption. Therefore $x^{hk}I^h x^{hk'}$ must hold for all k,k'. □

Note that Lemma 10 implies that the weak and strong cores coincide under strong irreducibility.

<u>Lemma</u> <u>11</u>. Under the assumptions of Lemma 10 the equal treatment core

K_r of E_r is nonempty if the core of E_r is nonempty.

Proof. According to Lemma 10 for any allocation in the core the allocations received by the replicas of a given h in the original economy are indifferent. By the convex valuedness of the correspondence R^h the equal treatment allocation in which each replica of h receives x^h, as defined in the proof of Lemma 10, satisfies $x^h R^h x^{hk}$ for all h, k. Then by Lemma 9 there is no improving coalition for the allocation $\{x^{hk}\}_r$ there is also no improving coalition for the allocation $\{y^{hk}\}_r$ in which $y^{hk} = x^h$ for all h, k. Therefore $\{y^{hk}\}_r$ is in the core of E_r. □

<u>Lemma</u> <u>12</u>. Under Assumptions 1′, 2′′′, 3′, 4, 5′, 6′, the equal treatment core K_r of E_r is not empty.

Proof. By Theorem 4 the core of E_r is not empty. By Lemma 11 this implies that the equal treatment core K_r of E_r is not empty. □

We may now prove

Theorem 5. Under Assumptions 1′, 2′′′, 3′, 4, 5, and 6′ the economy E_1 has a competitive equilibrium, and K = W.

Proof. The assumptions imply the assumptions of Section 3, so by Theorem 4 the core of E_r is not empty for any r. Since the assumptions also imply the assumptions of Lemma 12, the set of equal treatment allocations in the core is not empty for any E_r. From the proof of Theorem 3 the set K_r is closed. Also the K_r are nested by Lemma 1. Therefore $K = \cap_{r=1}^{\infty} K_r$ is not empty. But the assumptions imply the assumptions of Section 1, so, by Theorem 2, K is included in the set of competitive allocations for E_1. Indeed, by Theorem 3 the equal treatment core of E_r converges to the set of competitive allocations of E_1 as $r \to \infty$. Since both sets are closed K is precisely the set W of competitive allocations. Finally the proof of Theorem 2 provides a price vector p which supports the allocations of K in a competitive equilibrium given by $(p,y,\tilde{x})$ where $y = \Sigma_{h=1}^{\infty} x^h$. □

Once transitivity is introduced the Scarf theorem (1967) becomes available for proving that the core is not empty, so it would be enough to stay with open lower sections and open values, rather than requiring an open graph. On the other hand, use of the weak core requires strong irreducibility, which is needed in the proof that the equal treatment core is closed. However we know from the theorem of Gale and Mas–Colell (1975) that existence of competitive equilibrium can be proved without transitivity. Indeed we know from Moore (1975) and McKenzie (1981) that the survival of isolated individuals is also not needed for proving the existence of competitive equilibrium. Indeed individual survival, convexity of R^h, and transitivity may be dispensed with together (see McKenzie (1981)).

The line of proof that we have followed is that used by Boyd and McKenzie to prove a theorem for the case of an infinite number of goods, except that they assume transitivity already at the stage where it is proved that the core is not empty. This is to allow the use of the Scarf theorem for a nonempty core, which is proved in the utility space, rather than the Border theorem, which is proved in the goods space. The Border proof makes essential use of the finite dimensionality of the goods space (see Yannelis (1991)). Yannelis provides a proof of nonemptiness of the core for the case of a goods space with infinite dimension. However the proof of existence is still blocked by need for transitivity to establish that consumers of the same type have allocations in the core which are indifferent. Whether the line of proof we have used here for the finite case can be improved to match the results of Gale and Mas–Colell, Moore, and McKenzie, is an open question so far as I know.

A further strengthening of the assumptions can reduce the core of E_r to the equal treatment core. Assume that preferences are strictly convex in the sense that xI^hy and $z = \alpha x + (1-\alpha)y$, for $x,y \in C^h$ and $0 < \alpha < 1$ implies zP^hx.

Then we may prove a result used by Debreu and Scarf (1963).

Lemma 13. Assume P^h is strictly convex, $R^h(x^h)$ is the closure of $P^h(x^h)$ when $\tilde{x}$ is feasible, and E_1 is strongly irreducible. Then all allocations in the core of E_r are equal treatment allocations.

Proof. The conclusion of the lemma is equivalent to the statement that an allocation in the core of E_r has $x^{hk} = x^{hk'}$ for all k, $k' = 1,\cdots,r$. Suppose not. For any h let $x^{hj(h)}$ satisfy $x^{hk}R^h x^{hj(h)}$ for all k. Then the consumer with index hj(h) is a worse off consumer among the replicates of the h*th* original consumer. By strict convexity of preference $(1/r)\Sigma_{k=1}^{r}\ x^{hk}\ R^h\ x^{hj(h)}$ for $h = 1,\cdots,H$ with P^h in place of R^h at least once. Strong irreducibility implies that P^h can be realized for all $hj(h) \in B = (1j(1),\cdots,Hj(H))$. But $\bar{x}^h = (1/r)\Sigma_{k=1}^{r}\ x^{hk} \in C^h$ by convexity of C^h and $(1/r)\Sigma_{h=1}^{H}\ \Sigma_{k=1}^{r}\ x^{hk} \in Y$ since Y is a cone. Thus B is an improving coalition with the allocation $\{\bar{x}^h\}$, so $\{x^{hk}\}_r$ is not in the core of E_r, contrary to assumption. This implies that $x^{hk} = x^{hk'}$ for all k, $k' = 1,\cdots,r$, and all allocations in the core of E_r are equal treatment allocations. □

REFERENCES

Border, Kim C., 1985. Fixed Point Theorems with Applications to Economics and Game Theory. Cambridge University Press.

Boyd, John H., III., and Lionel W. McKenzie, 1993. The Existence of Competitive Equilibrium over an Infinite Horizon with Production and General Consumption Sets. International Economic Review, 34, 1–20.

Debreu, Gerard, and Herbert Scarf, 1963. A Limit Theorem on the Core of an Economy. International Economic Review, 4, 235–46.

Edgeworth, Francis Y., 1881. Mathematical Psychics. Kegan Paul, London.

Fenchel, W., 1951. Convex Cones, Sets, and Functions. Mathematics Department, Princeton University (mimeographed).

Gale, David, and Andreu Mas–Colell, 1975. An Equilibrium Existence Theorem for a General Model without Ordered Preferences. Journal of Mathematical Economics, 2, 9–16.

Hildenbrand, W., and Alan Kirman, 1988. Equilibrium Analysis, Variations on a Theme by Edgeworth and Walras. North–Holland, New York.

McKenzie, Lionel W., 1959. On the Existence of General Equilibrium for a Competitive Market. Econometrica 27, 54–71.

McKenzie, Lionel W., 1981. The Classical Theorem on Existence of Competitive Equilibrium. Econometrica 49, 819–41.

McKenzie, Lionel W., 1988. A Limit Theorem on the Core. Economics Letters, 27, 7–9.

McKenzie, Lionel W., 1990. A Limit Theorem on the Core, Addendum. Economics Letters, 109–110.

McKenzie, Lionel, and Tomoichi Shinotsuka, 1991. A Limit Theorem on the Core, Addendum II. Economics Letters, 37, 331.

Moore, James, 1975. The Existence of 'Compensated Equilibrium' and the Structure of the Pareto Efficient Frontier. International Economic Review, 15, 267–300.

Scarf, Herbert, 1967. The Core of an n–Person Game. Econometrica 35, 50–69.

Yannelis, Nicholas C., 1991. The Core of an Economy without Ordered Preferences. In Equilibrium Theory in Infinite Dimensional Spaces, edited by M. Ali Khan and N. C. Yannelis. Springer Verlag, Berlin.

Walras, Leon, 1874–77. Eléments d'Economie Politique Pure. Corbaz, Lausanne. Trans. by W. Jaffe from the 1926 definitive edition as Elements of Pure Economics, George Allen and Unwin, 1954, Orion, New York.