On the Existence of Nonoptimal Equilibria

Greenwood, Jeremy and Gregory W. Huffman

Working Paper No. 370
December 1993

On the Existence of Nonoptimal Equilibria

Jeremy Greenwood and Gregory W. Huffman

Rochester Center for Economic Research
Working Paper No. 370

On the Existence of Nonoptimal Equilibria

in Dynamic Stochastic Economies*

by

Jeremy Greenwood

Department of Economics
University of Rochester
Rochester, New York, 14627

and

Gregory W. Huffman

Department of Economics
Southern Methodist University
Dallas, Texas, 75275
and
Research Department
The Federal Reserve Bank of Dallas
2200 North Pearl Street
Dallas, Texas, 75222

Forthcoming, Journal of Economic Theory

* We thank a referee and Nobuhiro Kiyotaki for helpful comments, and the Social Sciences and Humanities Research Council of Canada for financial support. The views expressed in this article are solely those of the authors and should not be attributed to the Federal Reserve Bank of Dallas, or to the Federal Reserve System.

Proposed Running Head: Existence of Nonoptimal Equilibria

Please send correspondence to: Gregory W. Huffman
Department of Economics
Umphrey Lee Building
Southern Methodist University
Dallas, Texas, 75275

ABSTRACT

The question of the existence of a stationary equilibrium for distorted versions of the standard neoclassical growth model is addressed in this paper. The conditions presented guaranteeing the existence of nontrivial equilibrium for the class of economies under study are simple and intuitively appealing, while the existence proof developed is elementary. Examples are given illustrating that economies with distortional taxation, endogenous growth with externalities, and monopolistic competition can all fit into the framework developed. (JEL Classification No. C62, E30, E60)

I. INTRODUCTION

A great deal of work has recently been devoted to the study of economic environments which give rise to nonoptimal equilibria (see, for example, Coleman [3], Greenwood and Huffman [10]). The analysis of these environments is not a straightforward application of results known from dynamic competitive analysis, as in Stokey and Lucas with Prescott [17], since the resulting equilibria are not optimal. The usual problem in these environments is that individual agents fail to take into account how their actions influence the behavior of other agents. In order to solve an individual agent's dynamic program, one needs to know the equilibrium law of motion governing the aggregate state of the world, but to know this in turn requires knowledge of the outcome of decision-making at the individual level. Proving existence of equilibria can then be problematic, since in these representative agent models it can be difficult to establish that the individual and aggregate laws of motion for decision variables coincide along an equilibrium path. Various assumptions have been suggested to guarantee existence of an equilibrium. It is shown below that a relatively minor restriction on the nature of the technology, with no unusually restrictive assumptions placed on preferences, is sufficient to guarantee the existence of an equilibrium.

The environment studied below is very similar to the standard neo-classical growth model, with the notable exception that behavior at the aggregate level can affect the decision-making of individual agents. In particular, there is an externality in the production technology whereby the productivity of each agent's investment in capital is influenced by the investment decisions of other agents. As is illustrated, economies with distortional taxation, endogenous growth with externalities, and monopolistic competition can be cast into this general framework.

The analysis conducted below borrows ingredients from Coleman [3] and Greenwood and Huffman [10]. In particular, following the innovative work of Coleman [3], the existence of equilibrium is established by constructing a monotone operator which maps the aggregate law of motion into itself. Coleman's [3] strategy for proving existence is generalized and significantly simplified in two ways: first, by adopting a less restrictive and more intuitive set of assumptions on tastes and technology outlined in Greenwood and Huffman [10]; and second, by employing a simpler line of argument to prove that the fixed point to the monotone operator is an equilibrium for the economy under study. The end result of combining components from the above two analyses is an existence proof that is much simpler than that contained in either work, despite the greater generality of the argument being pursued. Another benefit of the present approach is that the assumptions employed below would appear to be readily verifiable for a variety of environments.

II. THE ECONOMIC ENVIRONMENT

The economy under consideration is one which is very similar to the discrete-time standard neo-classical growth model. There is a continuum (of measure one) of identical agents who have preferences described as

$$E[\sum_{t=0}^{\infty}\beta^t U(c_t)], \qquad \beta \in (0,1),$$

where $E[\cdot]$ is the expectation operator, β is the discount factor, and c_t represents consumption in period t. The momentary utility function $U(\cdot):\Re_+ \rightarrow \Re_+$ is strictly increasing, strictly concave, and twice differentiable and $U'(0) = \infty$.

Each agent is endowed with the same production technology which is written in the following form:

$$y_t = F(k_t, K_t, \eta_t).$$

Here, y_t represents output of the consumption good in period t, and k_t is the agent's private input of capital into the production process which was chosen in period t-1. The variable η_t is a random technology shock which is known at the beginning of period t. It is drawn from the bounded set Λ and has a Markov distribution function which will be denoted as $G(\eta_{t+1}|\eta_t)$. The variable K_t represents the average or per capita quantity of capital supplied by all agents in the economy. Each agent, being of measure zero, behaves as though his choice of capital stock k_t has no influence on the average capital stock K_t. As is conventional, it is assumed that the production technology is strictly increasing and strictly concave in its first argument, and twice differentiable in its first two arguments. Other than the restrictions described below, no further constraints are placed on the technology with respect to the influence of the last two arguments.

Assumptions: $\forall\ \eta \in \Lambda$:

(i) $F(k,K,\eta) > 0,\ \forall\ k, K > 0,$ and $\lim_{K\to 0} F_1(K,K,\eta) = \infty$.

(ii) $\exists\ \bar{K} \ni F(\bar{K},\bar{K},\eta) \leq \bar{K}$.

(iii) For all $K \in (0, \bar{K}]$,

$F_1(K,K,\eta) + F_2(K,K,\eta) \geq 0,$

and

$F_{11}(K,K,\eta) + F_{21}(K,K,\eta) < 0.$

These assumptions seem relatively innocuous. The first assumption places relatively standard restrictions on the production technology. The second merely places a convenient upper bound on the level of output, or capital stock in the economy, which is satisfied for almost all parameterized versions of the concave technology. The third assumption places benign restrictions on how aggregate capital can affect the individual technology. It requires that the social marginal product of capital (or the sum of the marginal products of the individual and aggregate stocks) be positive. Additionally, it is necessary that the social returns to capital are diminishing in the individual capital stock. Observe these properties need only hold locally "along an equilibrium path" where k=K. It would seem easy to check that these simple properties hold for a wide range of economic problems. Additionally, it should be recognized that these appear to be much less restrictive assumptions than those imposed on the production and utility functions in Coleman [3].[1]

Finally, output in each period must be used for either consumption or investment purposes. Thus,

$$c_t + k_{t+1} \leq y_t,$$

where k_{t+1} is the amount of the consumption good used for investment at date t which becomes productive capital in the following period.

III. EXISTENCE OF EQUILIBRIA

The representative agent's dynamic programming problem for the environment under study can be cast as

$$V(k,K,\eta) = \max_{k'} \{U(F(k,K,\eta) - k') + \beta \int V(k',K',\eta')dG(\eta'|\eta)\}, \tag{P1}$$

where the aggregate K evolves exogenously according to the nontrivial law of motion $K' = Q(K,\eta)$. Assume that this aggregate law of motion is nontrivial in the sense that $Q(K,\eta) > 0$ for $K > 0$. Now let the optimal decision-rule associated with the above problem be denoted by $k' = q(k,K,\eta)$. An important property of this decision-rule is that $q(k,K,\eta) < F(k,K,\eta)$ if $k,K > 0$.[2] Standard arguments can be used to show that the any interior solution for this decision-rule must satisfy the Euler equation (1) shown below.

$$U'(F(k,K,\eta) - k') = \beta \int U'(F(k',K',\eta') - k'')F_1(k',K',\eta')dG(\eta'|\eta). \tag{1}$$

Now, the economy under study is one where all agents are identical. In equilibrium, therefore, the decision-rule regulating the individual's capital accumulation must coincide with the law of motion governing the evolution of the aggregate capital stock. The following definition makes this notion more precise:

Definition: A stationary equilibrium for the environment described above is a pair of functions $k' = q(k,K,\eta)$ and $K' = Q(k,\eta)$ such that

(i) the decision-rule for each agent, $k' = q(k,K,\eta)$, solves the optimization problem (P1), and

(ii) the individual's decision-rule is consistent with the law of motion for the aggregate per capita capital stock, or $q(K,K,\eta) = Q(K,\eta)$.

From the above definition of an equilibrium, it is clear that *if* an aggregate law of motion for capital of the form $0 < K' = Q(K,\eta) < F(K,\eta)$, for $K > 0$, exists, then it must satisfy equation (2)

$$U'(F(K,K,\eta) - K') = \beta \int U'(F(K',K',\eta') - K'')F_1(K',K',\eta')dG(\eta'|\eta), \tag{2}$$

where $K'' = Q(Q(K,\eta),\eta')$.

Proposition: There exists a nontrivial stationary equilibrium for the economy described above.

Proof:

A. Construction of the Aggregate Law of Motion

Consider the sequence of aggregate laws of motion $\{H^j(K,\eta)\}^{\infty}_{j=0}$ that are generated recursively as follows: First define $H^0(K,\eta) \equiv 0$. Second, define $H^{j+1}(K,\eta)$ to be the solution for x in the Euler equation shown below

$$U'(F(K,K,\eta) - x) = \beta \int U'(F(x,x,\eta') - H^j(x,\eta'))F_1(x,x,\eta')dG(\eta'|\eta). \quad (3)$$

Thus, equation (3) implicitly defines an operator mapping the function H^j into H^{j+1}.[3] Now assume that

$$0 \leq \frac{\partial H^j(K,\eta)}{\partial K} \leq [F_1(K,K,\eta) + F_2(K,K,\eta)], \quad (4)$$

which certainly holds for $j=0$. The left-hand side of equation (3) is strictly increasing in x, and the right-hand side is decreasing in x by Assumption (iii) and equation (4). Hence, there exists a unique solution to equation (3), and consequently $0 < H^j(K,\eta) < F(K,K,\eta)$ for $K>0$, $j\geq 1$, by Assumption (i). Note also that if the capital stock K increases by a unit, then output increases by the amount $[F_1(K,K,\eta) + F_2(K,K,\eta)]$. Then it is easily shown that

$$0 \leq \frac{\partial H^{j+1}(K,\eta)}{\partial K} \leq [F_1(K,K,\eta) + F_2(K,K,\eta)]. \quad (5)$$

It is also seen from equation (3) that replacing $H^j(K,\eta)$ by a function that is everywhere greater results in a larger solution for x. In other words, this procedure generates a monotonically increasing sequence of laws of motion for the aggregate capital stock $\{H^j(K,\eta)\}^{\infty}_{j=0}$ defined on $[0,\bar{K}]\times\Lambda$. Furthermore, this sequence is bounded from above by $\bar{K}$ [see Assumption (ii)]. Let

$$Q(K,\eta) = \lim_{j\to\infty} H^j(K,\eta),$$

so that $Q(\cdot,\cdot)$ is the pointwise limit of the functions $H^j(\cdot,\cdot)$ which exists because $\{H^j(K,\eta)\}^{\infty}_{j=0}$ is a monotonically increasing bounded sequence. Since $H^j(0,\eta) = 0$ for all j, then $Q(0,\eta) = 0$ as well. Note from (4) that $\partial H^{j+1}(K,\eta)/\partial K$ is bounded from above by $F_1(K,K,\eta) + F_2(K,K,\eta)$, for all j, and hence the sequence $\{H^{j+1}(K,\eta)\}^{\infty}_{j=0}$ is *equicontinuous* on $[\zeta, \bar{K}]$ for all $\zeta \in (0,\bar{K}]$. Therefore, the sequence $\{H^{j+1}(K,\eta)\}^{\infty}_{j=0}$ converges uniformly on each closed interval to $Q(\cdot,\cdot)$, which must also be a function continuous in its first argument (by the Arzela-Ascoli theorem and Theorem 7.13, Rudin [16]).[4] Furthermore, $Q(\cdot,\cdot)$ must also satisfy the restriction given by equation (4), with the derivative interpreted as a finite difference.

Is the function $Q(K,\eta) = \lim_{j\to\infty} H^j(K,\eta)$ constructed above a good candidate for an equilibrium aggregate law of motion? To answer this question, suppose that $Q(K,\eta) < F(K,K,\eta)$ for $K \in (0, \bar{K}]$ and $\eta \in \Lambda$. By construction, the function $K' = Q(K,\eta)$ satisfies equation (2). Furthermore, given this aggregate law of motion, there exists a unique decision-rule $k' = q(k,K,\eta)$ which solves the Euler equation (1). Clearly, when $k = K$, it transpires that $q(K,K,\eta) = Q(K,\eta)$ solves (1), since $Q(K,\eta)$ satisfied (2). Thus, for the case when $Q(K,\eta) < F(K,K,\eta)$ for all $K \neq 0$ an equilibrium has been found. To complete the existence proof it will be shown next that $Q(K,\eta) = \lim_{j\to\infty} H^j(K,\eta) \neq F(K,K,\eta)$ for $K \in (0, \bar{K}]$ and $\eta \in \Lambda$.

Remark: Observe that for any finite T, the sequence $\{H^j(K,\eta)\}^{T}_{j=0}$ gives the *unique* set of T+1 laws of motion that obtain from a T+1-period version of the economy under study.

B. *Nondegenerateness of the Aggregate Law of Motion*

It is important to ensure that the aggregate laws of motion do not converge to a degenerate case where investment equals total output. For instance, consider the possibility that $Q(K,\eta) = F(K,K,\eta)$ for all K and η. While this solution for the aggregate law of motion will trivially satisfy (2), it cannot constitute an equilibrium for the economy under study since (P1) demands a solution for the representative agent's decision-rule such that $q(k,K,\eta) < F(k,K,\eta)$ whenever $k, K > 0$.

To rule out such degenerate cases, consider the sequence of value functions $\{V^n\}^{\infty}_{n=0}$ generated from the mapping

$$V^n(k,K,\eta) = \max_{k'} \{U(F(k,K,\eta) - k') + \beta \int V^{n-1}(k',K',\eta')dG(\eta'|\eta)\}, \qquad \text{(P2)}$$

where $K' = Q(K,\eta)$. Without loss of generality, assume that $V^0 \equiv 0$. Well-known arguments can be employed to show that this mapping defines an operator T, such that $V^n = TV^{n-1}$, which has as its unique fixed point the function V characterized by (P1).

Now let $\{Q^j\}^n_{j=0}$ be a sequence of continuous functions converging uniformly to the function Q on $[0, \bar{K}]\times\Lambda$. Next, consider the sequence of value functions $\{\hat{V}^n\}^\infty_{n=0}$ generated from the mapping

$$\hat{V}^n(k,K,\eta) = \max_{k'} \{U(F(k,K,\eta) - k') + \beta\int \hat{V}^{n-1}(k',K',\eta')dG(\eta'|\eta)\}, \qquad \text{(P3)}$$

with $K' = Q^{n-1}(K,\eta)$. Set $\hat{V}^0 \equiv 0$. Let the optimal decision-rule associated with (P3) be denoted by $k' = q^{n-1}(k,K,\eta)$. A key step in the subsequent analysis is to show that $\lim_{n\to\infty} \hat{V}^n = V$, where V is the fixed point to equation (P1).[5]

Assume momentarily that indeed $\lim_{n\to\infty} \hat{V}^n = V$. Given this supposition, it is easy to demonstrate that $Q(K,\eta) < F(K,K,\eta)$ for $K \in (0, \bar{K}]$ and $\eta \in \Lambda$. Note that the sequence of functions $\{H^n\}^\infty_{n=0}$ constructed in the previous subsection satisfies the properties assumed for the sequence $\{Q^n\}^\infty_{n=0}$ that underlies the mapping (P3). Therefore let $Q^n = H^n$ for all n. Then by the construction of the mapping (P3), $q^n(K,K,\eta) = H^n(K,\eta)$ for $K \in [0, \bar{K}]$ and $\eta \in \Lambda$. Now, consider the sequence of functions $\{W^{n-1}\}_{n=1}^{\infty}$ where $W^{n-1}(k',K,\eta,\eta') \equiv \hat{V}^{n-1}(k',Q^{n-1}(K,\eta),\eta')$. Clearly, this sequence converges to the function $W(k',K,\eta,\eta') \equiv V(k',Q(K,\eta),\eta')$ by the assumption made above. Since the functions W and W^{n-1}, for all $n\geq2$, are each strictly concave in their first argument it follows that $q^{n-1}(k,K,\eta) \to q(k,K,\eta)$, pointwise.[6] Consequently, $Q(K,\eta) < F(K,\eta)$ for $K \neq 0$, since the solution to problem (P1) dictates that $q(k,K,\eta) < F(k,K,\eta)$ for $k,K > 0$.

All that remains to be established is that $\lim_{n\to\infty} \hat{V}^n \to V$. Toward this end, note the following fact for future use.

<u>Fact 1:</u> Since $V^n \to V$ uniformly, for all $\varepsilon > 0$, there exists an N such that for $n \geq N$

$$|V^n(k,K,\eta) - V(k,K,\eta)| \leq \varepsilon/3,$$

for all $k,K \in [0,\bar{K}]$, $\eta \in \Lambda$.

Next, define the continuous function π^q: $[0,\bar{K}]\times\Lambda^q \to \Re_+$ *recursively* by

$$\pi^q(K_0,\eta_0,\ldots,\eta_{q-1}) = Q(\pi^{q-1}(K_0,\eta_0,\ldots,\eta_{q-2}),\eta_{q-1}),$$

where $\pi^0 \equiv K_0$. The function π^q gives the period-q aggregate capital stock assuming that the initial period-zero capital stock is K_0, the history of shocks $\{\eta_t\}^{q-1}_{t=0}$ transpires, and the law of motion Q is followed.

Likewise, define the continuous function $\hat{\pi}^{m,q}:[0,\bar{K}]\times\Lambda^q \rightarrow \Re_+$, for $m \geq q \geq 1$, recursively by

$$\hat{\pi}^{m,q}(K_0,\eta_0,\ldots,\eta_{q-1}) = Q^{m-q}(\hat{\pi}^{m,q-1}(K_0,\eta_0,\ldots,\eta_{q-2}),\eta_{q-1}),$$

with $\hat{\pi}^{m,0} \equiv K_0$. The function $\hat{\pi}^{m,q}$ gives the period-q aggregate capital stock for the economy with an m-period horizon, assuming that the law of motion Q^{m-t-1} is followed in period t (for $t \leq q-1$) and contingent upon the initial period-zero capital having the value K_0, and the history of shocks $\{\eta_s\}^{q-1}_{s=0}$ being realized. Note that since $Q^j \rightarrow Q$ uniformly, $\lim_{m\rightarrow\infty} \hat{\pi}^{m,q} = \pi^q$ for all $q > 0$. This is easily seen to imply Fact 2, which will be used in the subsequent analysis.

Fact 2: Since $U(\cdot)$ and $F(\cdot)$ are continuous functions, and the sequence $\{Q^j\}$ converges uniformly to Q, for all q and $\varepsilon > 0$, there exists an M such that for $m \geq M \geq \max(q,1)$,

$$|U(F(k,\hat{\pi}^{m,t}(\cdot),\eta_t) - k') - U(F(k,\pi^t(\cdot),\eta_t) - k')| \leq \frac{(1-\beta)\varepsilon}{3},$$

for $k, k', K_0 \in [0, \bar{K}]$, $\{\eta_0, \eta_1, \ldots, \eta_t\} \in \Lambda^{t+1}$, and $0 \leq t \leq q$.

Finally, Fact 3 (which is trivial) is noted.

Fact 3: Let $B \equiv \max_{\eta\in\Lambda} [U(F(\bar{K}, \bar{K}, \eta))]$. For each $\varepsilon > 0$, there exists a P such that for all $p \geq P$,

$$\left(\frac{\beta^p}{1-\beta}\right)B < (\varepsilon/3).$$

Lemma: $\text{Lim}_{n\to\infty} \hat{V}^n = V$.

Proof: Pick $\varepsilon > 0$ and choose N and P such that Facts 1 and 3 hold. Next, let q = max[N,P] and choose M ≥ max(q,1) large enough so that Fact 2 holds as well. Now note that

$$\hat{V}^m(k_0,K_0,\eta_0) = \max_{\{k_{t+1}\}_{t=0}^{m-1}} E\left(\sum_{t=0}^{m-1} \beta^t U(F(k_t,\hat{\pi}^{m,t},\eta_t) - k_{t+1})\right),$$

where the expectation operator E(·) reflects the integration with respect to the finite probability distribution of $\{\eta_j\}_{j=1}^{m-1}$. First, set m ≥ M. By Fact 3 it follows that

$$\left| V(k_0,K_0,\eta_0) - \hat{V}^m(k_0,K_0,\eta_0) \right| \le \left| V(k_0,K_0,\eta_0) - \max_{\{k_{t+1}\}_{t=0}^{q-1}} E\left(\sum_{t=0}^{q-1} \beta^t U(F(k_t,\hat{\pi}^{m,t},\eta_t) - k_{t+1})\right) \right| + \frac{\varepsilon}{3},$$

since q ≥ P. Second, Fact 2 then implies

$$\left| V(k_0,K_0,\eta_0) - \hat{V}^m(k_0,K_0,\eta_0) \right| \le \left| V(k_0,K_0,\eta_0) - \max_{\{k_{t+1}\}_{t=0}^{q-1}} E\left(\sum_{t=0}^{q-1} \beta^t U(F(k_t,\pi^t,\eta_t) - k_{t+1})\right) \right| + \left(\frac{2\varepsilon}{3}\right)$$

$$= \left| V(k_0,K_0,\eta_0) - V^q(k_0,K_0,\eta_0) \right| + \left(\frac{2\varepsilon}{3}\right).$$

Third, since q > N, it immediately transpires from Fact 1 that

$$|V(k_0,K_0,\eta_0) - \hat{V}^m(k_0,K_0,\eta_0)| \le \varepsilon. \qquad \square$$

IV. DISCUSSION

The form for the production technology, y = F(k,K,η), is a general formulation that could embody many environments. The following three examples may help to illustrate this point.

Example 1: Distortional Taxation. Let the representative agent's production function be described by y = Y(k,η) + (1-δ)k, where Y has the standard production function properties. Note that under this formulation capital depreciates at the rate δ. Now suppose that there is a government in the economy that taxes income at the rate θ and provides a capital consumption allowance of μ. The resulting tax revenue is rebated back to the agent via a lump-sum transfer payment. This environment can be handled in the above framework by adopting the following specification for $F(k,K,\eta)$:

$$F(k,K,\eta) = (1 - \theta)Y(k,\eta) + [1-\delta(1 - \mu)]k + \theta Y(K,\eta) - \mu\delta K.$$

It is straightforward to check that Assumptions (i), (ii) and (iii) are satisfied. Additionally, this framework can also accommodate a progressive income tax structure.[7] To do this let θ be given by $\theta = \Theta(k,\eta)$. It is easily seen that, so long as the individual's after-tax income $[1-\Theta(k,\eta)][Y(k,\eta) + (1 - \delta(1 - \mu))k]$ is both strictly increasing and strictly concave in k, then the required assumptions are satisfied.

Furthermore, it is easy to incorporate a variable supply of labor into the analysis. Now, let the representative agent's preferences be described by $U(c - G(\ell))$, where $G(\cdot)$ is an increasing convex function.[8] The technology is specified as $y = Y(k,\ell,\eta)$, where ℓ represents his labor input. Also, suppose that the government taxes labor income at the rate λ. Define the function $L(k,\eta)$ by

$$L(k,\eta) = \underset{x}{argmax}\{(1-\theta)Y(k,x,\eta) - [(1-\theta)/(1-\lambda)]G(x)\}.$$

To recast the environment into the framework developed, let $F(k,K,\eta)$ be given by

$$F(k,K,\eta) = (1 - \theta)Y(k,L(k,\eta),\eta) + [1 - \delta(1 - \mu)]k - [(1 - \theta)/(1 - \lambda)]G(L(k,\eta))$$

$$+ \theta Y(K,L(K,\eta),\eta) - \mu\delta K + [(\lambda - \theta)/(1 - \lambda)]G(L(K,\eta)).$$

While now a little more difficult to verify, $F(k,K,\eta)$ satisfies Assumptions (i), (ii) and (iii). The representative agent's dynamic programming problem can again be characterized by (P1), given this formulation for $F(k,K,\eta)$.

Example 2: Endogenous Growth with Externalities. Consider a simple version of Romer's [15] well-known economic growth model. First, let the representative agent's preferences be described by $\sum_{t=0}^{\infty} \beta^t \ell n(c_t)$. Second, suppose the production technology is given by y =

$\eta Y(k,K)$, where Y is linearly homogeneous in its first two arguments with the standard properties holding otherwise. With this combination of tastes and technology it is possible for the economy to grow without bound. The model, however, has a stationary representation that can be handled in the framework developed above. To see this, define the variables $\hat{c}$, $\hat{k}'$, and $\hat{K}'$ by $\hat{c} = c/K$, $\hat{k}' = k'/K$, and $\hat{K}' = K'/K$.[9] Next, let $F(\hat{k},\hat{K},\eta)$ be represented by $F(\hat{k},\hat{K},\eta) = \eta Y(\hat{k}/\hat{K},1)$. Observe that F satisfies Assumptions (i), (ii) and (iii), with $F_1 + F_2 = 0$ when $\hat{k} = \hat{K}$.

Now, it is easy to deduce that the representative agent's dynamic programming problem (P1) can be rewritten for the current example in transformed form as

$$V(\hat{k},\hat{K},\eta) = \max_{k'} \{\ell n[F(\hat{k},\hat{K},\eta)-\hat{k}'] + \beta \int V(\hat{k}',\hat{K}',\eta')dG(\eta'|\eta)\}, \tag{P4}$$

where $\hat{K}' = Q(\hat{K},\eta)$. Since $F_1 + F_2 = 0$, when $\hat{k} = \hat{K}$ it is immediate from (4) that the equilibrium law of motion can be expressed more simply as $\hat{K}' = Q(\eta)$. Clearly then, the solution for the original model is homogeneous of degree one in the aggregate capital stock. Also, it is easy to check by solving (P4) that $Q(\eta) = \eta\beta F_1(1,1,1)$.[10] The economy can experience growth if $E(\eta)F_1(1,1,1) > 1/\beta$.

<u>**Example 3: Monopolistic Competition.**</u> A simple stochastic infinite horizon version of Kiyotaki's [12] model of monopolistic competition also fits into the framework developed. Imagine an economy inhabited by a continuum of agents distributed uniformly over the unit interval. Agents are identical except that each owns a firm producing a differentiated product. The generic individual's preferences are described by $\sum_{t=0}^{\infty} \beta^t \ell n[\int_0^1 c_t(\theta)^\rho d\theta]^{\frac{1}{\rho}}$, for $0 < \rho < 1$, where $c_t(\theta)$ represents his period-t consumption of the good produced by agent θ. The representative agent's firm produces output according to the linear production technology $y = \eta k$, where η is an aggregate technology shock. An individual's capital accumulation is governed by the technology $k' = [\int_0^1 i(\theta)^\rho d\theta]^{1/\rho}$, where $i(\theta)$ is the amount of goods purchased from agent θ for investment purposes. Finally, the agent's budget constraint for any given period reads $\int_0^1 p(\theta)[c(\theta) + i(\theta)]d\theta = p\eta k$, where $p(\theta)$ is the price for good θ.

Now, by carrying out the implied within-period maximization, it is easy to deduce that

$c(\theta) = [p(\theta)/P]^{1/(\rho-1)}[\int^1_0 p(\xi)c(\xi)d\xi/P]$ and $i(\theta) = [p(\theta)/P]^{1/(\rho-1)}[\int^1_0 p(\xi)i(\xi)d\xi/P]$, with the aggregate price level P being given by $[\int^1_0 p(\xi)^{\rho/(\rho-1)}d\xi]^{(\rho-1)/\rho}$. Next, define c and k′ by $c = [\int^1_0 c(\theta)^\rho d\theta]^{1/\rho}$ and $k' = [\int^1_0 i(\theta)^\rho d\theta]^{1/\rho}$. The agent's preferences can now be represented more simply by $\sum_{t=0}^{\infty} \beta^t \ell n c_t$. Also, using the above formulae it is apparent that $c = \int^1_0 p(\theta)c(\theta)d\theta/P$ and $k' = \int^1_0 p(\theta)i(\theta)d\theta/P$. This allows the individual's budget constraint to be given the more compact representation $c + k' = (p/P)\eta k$. Observe that the demand, y, for the agent's firm's product is described by $y = (p/P)^{1/(\rho-1)}Y$, where aggregate demand Y is defined by $Y = \int p(\theta)[C(\theta) + I(\theta)]d\theta/P$ and $C(\theta)$ and $I(\theta)$ represent the aggregate amounts of good θ that are demanded for consumption and investment purposes. Thus, $p/P = (\eta k/Y)^{\rho-1}$, as $y = \eta k$. Since in equilibrium each industry behaves the same, it must transpire that $Y = \eta K$, where K is the average or aggregate level of the capital stock. Therefore the representative agent's budget constraint can be reformulated as $c + k' = \eta k^\rho K^{1-\rho}$. The agent's intertemporal optimization problem amounts to choosing c and k′ in each period so as to maximize expected lifetime utility. The problem now has the form of Example 2. Observe that the economy can grow without bound.

FOOTNOTES

1. In particular, Assumption 5 in Coleman [3] assumes that there is not "too much" uncertainty about the production function disturbances. No such assumption is needed in the present analysis, as the degree of uncertainty is unrestricted.

2. Suppose to the contrary that $q(k,K,\eta) = F(K,\eta)$ for some k, $K > 0$. To show that this strategy of consuming nothing cannot be optimal, consider increasing current consumption, and correspondingly reducing current investment, by some amount $\varepsilon > 0$. On the one hand, the resulting per unit gain in current utility will be $[U(\varepsilon) - U(0)]/\varepsilon$. On the other hand, the per unit loss in expected future utility is $(\beta\int[V(k',K',\eta') - V(k'-\varepsilon,K',\eta')]dG(\eta' \mid \eta))/\varepsilon$. Now since $U'(0)=\infty$, this per unit gain can be made arbitrarily large by picking ϵ small enough. The per unit loss is bounded in size, however, since the solution for $V(\bullet)$ in (P1) is a continuous, bounded function that is both strictly increasing and strictly concave in its first argument.

3. Greenwood and Huffman [9], and Lucas and Stokey [13] use similar mappings to establish the existence, and characterize the properties, of equilibria for cash-in-advance models. Also, this mapping forms the basis for numerical algorithms used to solve nonlinear dynamic stochastic models that have been proposed by Baxter [1], Coleman [2], and Danthine and Donaldson [6]. See Danthine and Donaldson [5] for a discussion of these types of algorithms.

4. Specifically, the Arzela-Ascoli theorem implies that the limiting function is continuous in its first argument, while the theorem in Rudin [16] establishes that the convergence is uniform since the sequence is monotone.

5. Here the norm employed is $\| V \| = \sup_x |V(x_1, x_2, x_3)|$, where $x = (x_1, x_2, x_3) \in [0,\bar{K}]\times[0,\bar{K}]\times\Lambda$. Similarly, the norm for the convergence of the functions $\{Q^j\}^{\infty}_{j=0}$ is also the sup norm.

6. See Stokey, Lucas, with Prescott [17], Theorem 9.9.

7. Coleman's [3] analysis also allows for a progressive income tax.

8. This form of utility function has been used in Greenwood, Hercowitz and Huffman [8], and Hercowitz and Sampson [11], and Mendoza [14]. These analyses suggest that for the study of business cycles the restrictions imposed by this sort of utility may not be that severe.

9. Coleman [4] and Gomme [7] use this transformation in computational work to obtain stationary solutions for endogenous growth models.

10. $F(\hat{k},\hat{K},\eta) = \eta F(\hat{k},\hat{K},1)$, which implies that $F_1(\hat{k},\hat{K},\eta) = \eta F_1(\hat{k},\hat{K},1)$.

REFERENCES

1. Baxter, Marianne, Approximating Suboptimal Dynamic Equilibria: An Euler Equation Approach, *Journal of Monetary Economics* **28** (1991), 173-200.

2. Coleman, Wilbur John II, Solving the Stochastic Growth Model by Policy-Function Iteration, *Journal of Business and Economic Statistics* **8** (1990), 27-29.

3. Coleman, Wilbur John II, Equilibrium in a Production Economy with an Income Tax, *Econometrica* **59** (1991), 1091-1104.

4. Coleman, Wilbur John II, Money Interest, and Capital in a Cash-in-Advance Economy, Unpublished manuscript, Department of Economics, Duke University, 1993.

5. Danthine, Jean-Pierre and Donaldson, John B., Computing Equilibria of Non-optimal Economies, Forthcoming in "Frontiers of Business Cycle Theory" (Thomas F. Cooley, Ed.), Princeton University Press, N.J., 1993.

6. Danthine, Jean-Pierre and Donaldson, John B., Efficiency Wages and the Business Cycle Puzzle, *European Economic Review* **34** (1990), 1275-1301.

7. Gomme, Paul, Money and Growth Revisited: Measuring the Costs of Inflation in an Endogenous Growth Model, *Journal of Monetary Economics* **32** (1993), 51-77.

8. Greenwood, Jeremy, Zvi Hercowitz and Gregory W. Huffman, Investment, Capacity Utilization, and the Real Business Cycle, *American Economic Review* **78** (1988), 402-417.

9. Greenwood, Jeremy and Gregory W. Huffman, A Dynamic Equilibrium Model of Unemployment and Inflation, *Journal of Monetary Economics* **19** (1987), 203-228.

10. Greenwood, Jeremy and Gregory W. Huffman, Tax Analysis in a Dynamic Stochastic Model: On Measuring Harberger Triangles and Okun Gaps, Working Paper No. 164, Rochester Center for Economic Research, University of Rochester, (1988).

11. Hercowitz, Zvi and Michael Sampson, Output Growth, the Real Wage, and Employment Fluctuations, *American Economic Review* **81** (1991), 1215-1237.

12. Kiyotaki, Nobuhiro, Multiple Expectational Equilibria Under Monopolistic Competition, *Quarterly Journal of Economics* **CIII** (1988), 695-713.

13. Lucas, Robert E., Jr. and Nancy L. Stokey, Money and Interest in a Cash-Advance Economy, *Econometrica* **55** (1987), 491-513.

14. Mendoza, Enrique G., Real Business cycles in a Small Open Economy, *American Economic Review* **81** (1991), 797-818.

15. Romer, Paul M., Increasing Returns and Long-Run Growth, *Journal of Political Economy* **94** (1986), 1002-1037.

16. Rudin, Walter, "Principles of Mathematical Analysis," McGraw Hill, Inc., New York, 1976.

17. Stokey, Nancy L., and Robert E. Lucas Jr., with Edward C. Prescott, "Recursive Methods in Economic Dynamics," Harvard University Press, Cambridge, 1989.