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1. Introduction

The canonical collective choice problem can be thought of as attempting to select an outcome from a certain set of feasible alternatives, where this selection is based in some manner on the preferences of the relevant individuals. The classical approach to this problem is to aggregate the individual preferences into a social preference relation, and then determine whether there exists any maximal elements with respect to this relation; such elements would then be the "best" outcomes to select and are referred to as the *core outcomes* associated with the individuals' preferences and the particular method of preference aggregation.

There are essentially two classes of results concerning the existence of core outcomes in the "spatial model", that is, when the feasible set of alternatives (or policy space) X constitutes a convex subset of some finite–dimensional Euclidean space. The first class, associated with Greenberg (1979), Schofield (1984) and Strnad (1985), assumes continuous convex preferences and a compact policy space X, and states that if the dimension of X is less than some critical value (with this value depending on the specifics of the preference aggregation) then core outcomes will always exist. Conversely, if this dimension is greater than or equal to the value a core outcome does not necessarily exist, in that there are simple examples of continuous (in fact, continuously differentiable), convex preferences for which no core outcome exists. We will have no quarrel with this class of results here.

The second class, as exemplified by Schofield (1980) and McKelvey and Schofield (1986), assumes preferences are continuously differentiable (but not necessarily convex), and investigates the *generic* existence of the core. That is, while for dimensions of X higher than the above critical value examples from the first class of results show that core outcomes might not exist, the results in this second class show how above a second critical value core outcomes *almost always* do not exist (in a manner to be made precise below). It is this second class of results with which we take issue. Specifically, the purpose of this paper is to demonstrate, not that these genericity results with respect to the core are necessarily wrong, but rather that the *proofs* of the results are wrong. These proofs, found in both Schofield (1980) and McKelvey and Schofield (1986), make use of known results pertaining to singularities of smooth mappings. The claim here is that the correct application of this proof technique generates critical values for the dimension of the policy space which are strictly above those posited by Schofield and McKelvey, in some instances, actually *doubling* this critical value.

2. The Model

There exists a set $N = \{1,...,n\}$ of voters, with $\infty > n > 2$, and a convex policy space $X \subseteq \mathbb{R}^p$; since in what follows attention is restricted to interior points and local optimum conditions, we will assume X is equal to all of $\mathbb{R}^p$. Voter i has preferences over X represented by a smooth utility function $u_i:X\to\mathbb{R}$, that is, where the partial derivatives of u_i of all orders exist and are continuous. The differential $du_i(x):\mathbb{R}^p\to\mathbb{R}$ defined by

$$du_i(x)(v) = \lim_{t\to 0} [u_i(x+tv) - u_i(x)]/t$$

is a linear map, and therefore can be represented by the gradient vector

$$\nabla_i(x) \equiv [\partial u_i(x)/\partial x_1,...,\partial u_i(x)/\partial x_p];$$

thus for any $v \in \mathbb{R}^p$ $du_i(x)(v) = v.\nabla_i(x)$ (i.e. scalar product). Let $u = (u_1,...,u_n):X\to\mathbb{R}^n$ denote a *utility profile*, and define $U(X)^n$ to be the space of all smooth utility profiles on X.

A (strict) collective preference relation P over the set X is characterized by a collection $\mathcal{D}$ of *decisive coalitions* of voters: for all $x,y \in X$, xPy if and only if there exists $C \in \mathcal{D}$ such that $u_i(x) > u_i(y)$ for all $i \in C$. The collection $\mathcal{D}$ is required to be *monotonic*: $C \in \mathcal{D}$ & $C' \supset C \Rightarrow C' \in \mathcal{D}$, and *proper*: $C \in \mathcal{D} \Rightarrow N \backslash C \notin \mathcal{D}$. A particularly important class of such collections for our purposes are *q–rules*, that is, collections of the form

$$\mathcal{D} = \{ C \subseteq N : |C| \geq q\} \equiv \mathcal{D}_q,$$

where q is some integer. Such a collection is obviously monotonic, and as long as q is greater than or equal to $(n+1)/2$ it will be proper as well.

The *core* with respect to the pair $(\mathcal{D},u)$ consists of those points $x \in X$ such that yPx for no $y \in X$:

$$\mathcal{C}(\mathcal{D},u) = \{ x \in X : \sim[\exists\ y \in X \ \&\ C \in \mathcal{D} \text{ s.t. } u_i(y) > u_i(x)\ \forall\ i \in C]\}.$$

It is well–known that if the collection $\mathcal{D}$ is *collegial*, that is, if

$$k(\mathcal{D}) \equiv \bigcap_{C \in \mathcal{D}} C \neq \phi\ ,$$

then the core is typically non–empty (simply choose one of the best points for any voter i from $k(\mathcal{D})$), so in what follows we will assume the collection is non–collegial. With respect to q–rules, it is apparent that the only collegial rule is where $q = n$ (i.e. unanimity is required), and so this restriction for q–rules is that the integer q be less than n.

The set of core points can be characterized as follows: for any coalition $C \subseteq N$ define the *Pareto set* for C at the utility profile u by

$$\mathcal{P}(C,u) = \{ x \in X : \sim[\exists\ y \in X \text{ s.t. } u_i(y) > u_i(x)\ \forall\ i \in C]\}.$$

The following is then immediate:

Lemma 1: For all $(\mathcal{D},u)$, $\mathcal{C}(\mathcal{D},u) = \bigcap_{C \in \mathcal{D}} \mathcal{P}(C,u)$.

Thus, the set of core points are precisely those elements in the Pareto sets of all decisive coalitions.

For smooth preferences such as those assumed here we have the following necessary condition describing elements of $\mathcal{P}(C,u)$:

Lemma 2 (Smale, 1973): If $x \in \mathcal{P}(C,u)$, then there exists $\lambda = (\lambda_1,\dots,\lambda_{|C|}) \in \mathbb{R}_+^{|C|}\backslash\{0\}$ such that $\sum_{i\in C} \lambda_i\nabla_i(x) = 0$.

Proof: Suppose $x \in \mathcal{P}(C,u)$ but for all such vectors λ, $\sum_{i\in C} \lambda_i\nabla_i(x) \neq 0$. Define

$$W = \{\lambda\in\mathbb{R}_+^{|C|}: \lambda_i \geq 0 \ \forall \ i \ \& \ \sum_{i\in C} \lambda_i = 1\}$$

$$Y = \{y\in\mathbb{R}^p: y = \sum_{i\in C} \lambda_i\nabla_i(x) \text{ for some } \lambda\in W\} .$$

Then Y is compact and convex, and $0 \notin Y$; hence by the separating hyperplane theorem there exist $v \in \mathbb{R}^p$ such that $v.y > 0 \ \forall \ y\in Y$.

Clearly $\nabla_i(x) \in Y$ for all $i\in C$, and hence $v.\nabla_i(x) > 0 \ \forall \ i\in C$ as well. And since

$$v.\nabla_i(x) = \lim_{t\to 0} [u_i(x+tv) - u_i(x)]/t > 0,$$

we have that $u_i(x+tv) > u_i(x) \ \forall \ i\in C$ for small t, contradicting $x \in \mathcal{P}(C,u)$. □

In words, a necessary condition for x to be Pareto optimal with respect to the coalition C is that the gradient vectors of the members of C are *semi-positively dependent* at x, for this guarantees there is no direction v away from x that all members of the coalition C prefer to move.

Define

$$\mathcal{I}(C,u) = \{x\in X: \{\nabla_i(x)\}_{i\in C} \text{ are semi-positively dependent}\}$$

and let

$$\mathcal{I}(\mathcal{D},u) = \bigcap_{C\in\mathcal{D}} \mathcal{I}(C,u).$$

By Lemmas' 1 and 2, then, we have that for all pairs $(\mathcal{D},u)$,

$$\mathcal{C}(\mathcal{D},u) \subseteq \mathcal{I}(\mathcal{D},u) , \qquad (1)$$

where we label $\mathcal{I}(\mathcal{D},u)$ the *infinitesimal core.* Therefore if one can show that the infinitesimal core is empty, then necessarily the core will be empty as well.

The "singularity approach" to the question of core existence backs away from this requirement of semi-positive dependence and considers points where a coalition's gradient vectors are only *linearly* dependent, the rationale being the latter is a much more well-understood concept in mathematics. So for all coalitions $C \subseteq N$ and utility profiles u, define

$$\Lambda(C,u) = \{ x \in X : \{\nabla_i(x)\}_{i \in C} \text{ are linearly dependent}\};$$

thus for all $C \subseteq N$, $\mathcal{I}(C,u) \subseteq \Lambda(C,u)$. Finally, define

$$\Lambda(\mathcal{D},u) = \bigcap_{C \in \mathcal{D}} \Lambda(C,u),$$

so that for all pairs $(\mathcal{D},u)$ we have

$$\mathcal{I}(\mathcal{D},u) \subseteq \Lambda(\mathcal{D},u). \tag{2}$$

Therefore if one can show that $\Lambda(\mathcal{D},u)$ is empty, then by (1) and (2) there will not exist any core points. For values of p sufficiently large this will "almost always" be the case, in a manner to be made precise; the issue in this paper concerns exactly how large "sufficiently large" must be.

Endow the space $U(X)^n$ of smooth profiles with the Whitney C^∞ topology (cf. Golubitsky and Guillemin, 1973); under this topology two utility profiles are close if the values of the functions as well as partial derivatives of all orders are close. A subset V of $U(X)^n$ is *dense* if for any $u \in U(X)^n$ and any neighborhood W of u, it is the case that $W \cap V \neq \phi$. We say that a property K of utility profiles is *generic* if the set {u satisfies K} constitutes an open and dense subset of $U(X)^n$.

Schofield (1980) and McKelvey and Schofield (1986) prove one or both of the following:

Claim: (i) For an arbitrary collection $\mathcal{D}$, if $p \geq n-1$ then $\Lambda(\mathcal{D},u) = \phi$ generically;

(ii) For an arbitrary q-rule, if $p \geq q$ then $\Lambda(\mathcal{D}_q,u) = \phi$ generically.

Specifically, Theorems' 1 and 2 in Schofield (1980) state these bounds with respect to the infinitesimal core $\mathcal{I}(\mathcal{D},u)$, but then the proofs focus on showing (i) and (ii) to be true, and then invoke (2) above. McKelvey and Schofield (1986) deal only with q–rules, and in fact state a stricter bound than (ii) for the infinitesimal core of such rules (Theorem 1). However, they state this claim as a result (Corollary 1) and employ it in their proof of their Theorem 1. More importantly, this proof relies critically on a result (Theorem 4) which logically *implies* (ii); indeed, McKelvey and Schofield use Theorem 4 to prove (ii), a result which is trivial assuming Theorem 4 to be true.

Our argument is that both parts of this claim are in fact not true, in that there exists open sets of utility profiles for which this claim fails to hold. Therefore the proofs of Theorems' 1 and 2 in Schofield (1980) as well as Theorem 4 in McKelvey and Schofield (1986) (and hence the proof of their Theorem 1) are incorrect. We will demonstrate this argument with simple examples which take care of both parts of the claim simultaneously: the examples will be of q–rules where q is equal to n–1 (n=3 and q=2, or n=4 and q=3) and p is greater than or equal to q. From now on, therefore, we will restrict attention to q–rules; for ease of notation, let $\Lambda(q,u) = \Lambda(\mathcal{D}_q,u)$.

3. Singularities

The method of proof found in Schofield (1980) and McKelvey and Schofield (1986) involves first identifying the dimension of the set $\Lambda(C,u)$, and then arguing that as one intersects, say, $\Lambda(C,u)$ with $\Lambda(C',u)$ the dimension of this intersection typically falls by a certain amount. For example, when we think of two planes intersecting in $\mathbb{R}^3$, we envision this intersection as generally constituting a one–dimensional object. Therefore, if upon taking this intersection over *all* of the decisive coalitions this dimension turns out to be less than zero, the set $\Lambda(q,u)$, which equals this intersection, must be empty.

The key to this method is that these intersections be *transversal*, so that existing dimension–counting arguments can be applied (more on transversality below). However a problem arises when considering the intersection of $\Lambda(C,u)$ with $\Lambda(C',u)$ when $C \cap C'$ is non–empty, since here the sets $\Lambda(C,u)$ and $\Lambda(C',u)$ are essentially "glued together" by the members of $C \cap C'$. That is, $\Lambda(C \cap C',u)$ will necessarily be a subset of both $\Lambda(C,u)$ and $\Lambda(C',u)$, and hence will be in their intersection. Therefore the intersection of $\Lambda(C,u)$ with $\Lambda(C',u)$ will not be as clean as transversality would require, calling into question this method of determining the emptiness or non–emptiness of the set $\Lambda(q,u)$, and hence of the core.

On the other hand, it turns out that determining whether a point x is in $\Lambda(q,u)$ is equivalent to verifying a relatively simple property of the differential of the utility profile $u:X \to \mathbb{R}^n$. As with the individual utilities, the map $du(x):\mathbb{R}^p \to \mathbb{R}^n$ is linear, and therefore can be represented by the $(n \times p)$ *Jacobian matrix*

$$J_u(x) = \begin{bmatrix} \nabla_1(x) \\ \nabla_2(x) \\ \vdots \\ \nabla_n(x) \end{bmatrix}.$$

The *rank* of the mapping $du(x)$ is defined to be the dimension of its image; equivalently the rank of $du(x)$ is given by the rank of the matrix $J_u(x)$, i.e. the maximum number of linearly independent rows or columns. Let $z = \min\{p,n\}$, and for all $k = 0,\dots,z$, define

$$S_k(u) = \{\ x \in X : \text{rank } J_u(x) = k\ \},$$

and let

$$S(u) = \bigcup_{k=0}^{z-1} S_k(u)\ .$$

The set $S(u)$ then consists of the *singularities* of the mapping $u:X \to \mathbb{R}^n$, that is, points where the rank of the Jacobian $J_u(.)$ is less than maximal. The set $\Lambda(q,u)$ will be a subset of $S(u)$:

***Lemma* 3**: For all (q,u), $\Lambda(q,u) = \bigcup_{k=0}^{q-1} S_k(u)$.

***Proof*.**
$x \in \Lambda(q,u) <=>$

$x \in \Lambda(C,u)\ \forall\ C \in \mathcal{D}_q <=>$

$\{\nabla_i(x)\}_{i \in C}$ are linearly dependent $\forall\ C \in \mathcal{D}_q <=>$

any set of $m \geq q$ gradient vectors are linearly dependent $<=>$

any set of $m \geq q$ rows in $J_u(x)$ are linearly dependent $<=>$

rank $J_u(x) \leq q-1 <=>$

$x \in S_k(u)$ for some $k = 0,\ldots,q-1 <=>$

$x \in \bigcup_{k=0}^{q-1} S_k(u)$. □

Therefore elements of $\Lambda(q,u)$ will be precisely those points in X where the rank of the Jacobian matrix $J_u(.)$ is at most q–1.

Example 1: $n = 3$, $p = q = 2$, and

$$u_1(x,y) = x$$
$$u_2(x,y) = xy$$
$$u_3(x,y) = y^2/2$$

The Jacobian matrix then is

$$J_u(.) = \begin{bmatrix} 1 & 0 \\ y & x \\ 0 & y \end{bmatrix},$$

and hence the origin (0,0) is a point where the rank of the Jacobian is 1, $S_1(u) = \{(0,0)\}$; further, there are no points where the rank of the Jacobian is 0. Thus $\Lambda(2,u) = \{(0,0)\}$.

Example 2: $n = 4$, $p = q = 3$, and

$$u_1(x,y,z) = x$$
$$u_2(x,y,z) = y$$
$$u_3(x,y,z) = yz$$
$$u_4(x,y,z) = z^2/2$$

The Jacobian matrix is then

$$J_u(\cdot) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & z & y \\ 0 & 0 & z \end{bmatrix}$$

and hence the Jacobian is of rank 2 for all points along the x–axis, $S_2(u) = \{(x,y,z)\colon y = z = 0\}$, while there are no points for which the Jacobian is of rank 1 or 0. Thus $\Lambda(3,u) = \{(x,y,z)\colon y=z=0\}$.

Example 3: $n = p = 4$, $q = 3$, and

$$u_1(x,y,z,w) = x$$
$$u_2(x,y,z,w) = y$$
$$u_3(x,y,z,w) = xz + yw$$
$$u_4(x,y,z,w) = z^2/2 + w^2/2$$

The Jacobian is

$$J_u(\cdot) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ z & w & x & y \\ 0 & 0 & z & w \end{bmatrix},$$

and hence the Jacobian is of rank 2 when x=y=z=w=0, while there are no points for which the Jacobian is of rank 1 or 0. Therefore $\Lambda(3,u) = \{(0,0,0,0)\}$.

Let $\mathcal{L}(p,n)$ denote the space of linear maps from $\mathbb{R}^p$ to $\mathbb{R}^n$, so in particular for all $x \in X$ we have that $du(x) \in \mathcal{L}(p,n)$. As before, we can identify $\mathcal{L}(p,n)$ with the set of all real $(n\times p)$ matrices, which is evidently equivalent to $\mathbb{R}^{n\times p}$ (with each coordinate in the latter giving one of the $n\times p$ entries in the matrix). For any $k = 0,\ldots,z$ let $\mathcal{L}_k(p,n)$ be the elements in $\mathcal{L}(p,n)$ of rank k.

Next let $du: X \to \mathcal{L}(p,n)$ be the mapping which assigns to every point x in X the linear map du(x). Then we have that the point $x \in X$ is an element of the set $S_k(u)$ if and only if $du(x) \in \mathcal{L}_k(p,n)$, or equivalently that $S_k(u)$ is the preimage under du of the set $\mathcal{L}_k(p,n)$:

$$S_k(u) = du^{-1}(\mathcal{L}_k(p,n)).$$

That is, a singularity point of the utility profile u is an element of X where the mapping du intersects the set $\mathcal{L}_k(p,n)$ for some integer $k < z$. Thus in example 1 the mapping $du: X \to \mathcal{L}(2,3)$ intersects the set $\mathcal{L}_1(2,3)$ when $(x,y) = (0,0)$ and at the point in $\mathcal{L}_1(2,3)$ associated with the rank 1 matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

As mentioned above, our goal is to construct an open set of utility profiles $V \subset U(X)^n$ such that for all $u \in V$ we have $\Lambda(q,u) \neq \phi$. With this in mind, suppose we "tremble" the utilities in example 1 in the following manner:

$$\tilde{u}_1(x,y) = x + \epsilon y$$

$$\tilde{u}_2(x,y) = xy$$
$$\tilde{u}_3(x,y) = \delta x + y^2/2,$$

where ϵ and δ are real numbers close to zero. Then the Jacobian is

$$J_{\tilde{u}}(.) = \begin{bmatrix} 1 & \epsilon \\ y & x \\ \delta & y \end{bmatrix},$$

and so at $x = \delta\epsilon^2$ and $y = \delta\epsilon$ the rank of the Jacobian is 1; therefore $\Lambda(2,\tilde{u}) \neq \phi$. Alternatively, consider a different tremble:

$$\hat{u}_1(x,y) = x + \epsilon y^2/2$$
$$\hat{u}_2(x,y) = xy + \sigma x$$
$$\hat{u}_3(x,y) = \delta x^2/2 + y$$

The Jacobian is then

$$J_{\hat{u}}(.) = \begin{bmatrix} 1 & \epsilon y \\ y & x+\sigma \\ \delta x & y \end{bmatrix},$$

and so at $x = -\sigma$, $y = 0$ the rank of the Jacobian is again 1, implying $\Lambda(2,\hat{u}) \neq \phi$. In fact, it turns out that *any* tremble for which the Jacobian is sufficiently close to the original will possess a point where the Jacobian is rank 1, and indeed such a point will be close to the original singularity at (0,0). The reason for this is that not only does the differential of the original utility profile intersect the set of rank 1 linear maps $\mathcal{L}_1(2,3)$, but it does so *transversally.*

4. Transversal intersections

Consider by way of example the function $f:\mathbb{R}\to\mathbb{R}^2$ defined by $f(x) = (x,x^2)$, and a one-dimensional manifold M in $\mathbb{R}^2$, i.e. a subset of $\mathbb{R}^2$ which locally "looks like" $\mathbb{R}$, where $M = \{(y,z)\in\mathbb{R}^2: z = y^3\}$; see Figure 1. The function f(.) then intersects the manifold M at two points in $\mathbb{R}^2$, namely at (0,0) and at (1,1). Suppose we now "tremble" the function f in a particular manner, say $\tilde{f}(x) = (x,x^2+\epsilon)$, where $\epsilon > 0$ is small. As is apparent upon inspection, the first intersection disappears whereas the second remains (albeit at a slightly different point). Further, it is clear upon inspection that no matter how one trembles the function f, the second intersection will remain. Thus the latter "crosswise" intersection of the function f(.) with the manifold M will be stable under perturbations, while the non-crosswise intersections will not. The concept of transversality characterizes precisely this idea of crosswise (and hence stable) intersections, and so we begin this section with its formal definition.

Let $f:\mathbb{R}^s\to\mathbb{R}^t$ be an arbitrary smooth function, and M an m-dimensional manifold in $\mathbb{R}^t$; define the *codimension* of M as $c \equiv t-m$. For each point $y\in M$ there exists a neighborhood Y of y and a mapping $g:Y\to\mathbb{R}^c$, where the differential dg is of rank c, such that $Y \cap M = g^{-1}(0)$. That is, in a neighborhood of y we can characterize the set M as the zero set of a function g, where this function is referred to as the *defining mapping* of M at y. Then f(.) is said to *intersect the manifold* M *transversally at* $x\in\mathbb{R}^s$ if $f(x) \in M$ and the composite map $h \equiv g\circ f:\mathbb{R}^s\to\mathbb{R}^c$ is such that the differential dh is of rank c at x, where g(.) is the defining mapping of M at $y = f(x)$; that is, 0 is a regular value of $g\circ f$. This definition of transversality is equivalent to the more common one, that the image of df(x) plus the tangent space to M at f(x) span all of $\mathbb{R}^t$ (cf. Guillemin and Pollack, 1974); for our purposes the former definition is more useful, since below we will be able to identify the relevant defining mappings quite easily.

To see this definition of transversality in action, consider the above example with $f(x) = (x,x^2)$. In this instance we are in fact given the defining mapping for the manifold M globally, namely $g(y,z) = z - y^3$. The composite map $h \equiv g\circ f:\mathbb{R}\to\mathbb{R}$

is then $h(x) = x^2 - x^3$, implying $dh(x) = [2x - 3x^2]$. Therefore $dh(0) = [0]$ while $dh(1) = [-1]$, and hence the differential of the composite map h is of rank 0 at $x = 0$ and is of rank 1 at $x = 1$. Thus the function f(.) intersects the manifold M transversally at $x = 1$, but non–transversally at $x = 0$.

Returning to our original problem, we have a mapping $du:\mathbb{R}^p \to \mathcal{L}(p,n)$, and a subset $\mathcal{L}_k(p,n)$ of $\mathcal{L}(p,n)$ describing the rank k linear maps. As previously discussed, we can identify $\mathcal{L}(p,n)$ with $\mathbb{R}^{n\times p}$, and it can be shown that $\mathcal{L}_k(p,n)$ constitutes a manifold in $\mathcal{L}(p,n)$ of codimension equal to $(p-k)(n-k)$ (Levine, 1971 p. 11). The importance of transversal intersections, and the formalization of the "trembling" arguments above, are captured in the following:

Theorem (Levine, 1971 p. 44): Suppose $u \in U(X)^n$ is such that du intersects the set $\mathcal{L}_k(p,n)$ transversally at $x \in X$. Then for any neighborhood Y of x there exists a neighborhood V of u in $U(X)^n$ such that if $u' \in V$ then du' intersects $\mathcal{L}_k(p,n)$ transversally at some $x' \in Y$.

Thus if the utility profile u is such that du(x) intersects $\mathcal{L}_k(p,n)$ transversally, then not only do all nearby profiles intersect $\mathcal{L}_k(p,n)$ but they do so at points in X near to x. In this sense, the stability observed in the transversal intersection of $f(.) = (x,x^2)$ with M in the above example will exist whenever we have a transversal intersection of du(.) with $\mathcal{L}_k(p,n)$.

Given the above theorem, then, all that needs to be verified is that the intersections exhibited in any of the examples from section 3 are transversal; in fact we will show that they *all* are. Note first that in each example the points of intersection in $\mathcal{L}_k(p,n)$ are all of the form

$$J = \begin{array}{r} k\left\{\vphantom{I_k}\right. \\ \vphantom{0} \end{array}\!\left[\begin{array}{c|c} \overbrace{I_k}^{k} & 0 \\ \hline 0 & \underbrace{0}_{p-k} \end{array}\right]\!\begin{array}{l} \vphantom{I_k} \\ \left.\vphantom{0}\right\} n-k \end{array}$$

where I_k is the k×k identity matrix and the remaining submatrices are all zeros. Let

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

be an n×p matrix "close" to J, where A is k×k, B is k×(p–k), C is (n–k)×k, and D is (n–k)×(p–k); for such matrices the square submatrix A will be non–singular and hence invertible (since A will be close to I_k). Post–multiply M by the non–singular matrix

$$N = \left[\begin{array}{c|c} I_k & -A^{-1}B \\ \hline 0 & I_{p-k} \end{array}\right]$$

to get the matrix

$$M' = \left[\begin{array}{c|c} A & 0 \\ \hline C & D-CA^{-1}B \end{array}\right] .$$

Since the product of a non–singular matrix and a rank r matrix generates a rank r matrix, we have rank(M) = rank(M′). Finally, since the submatrix A is non–singular and hence of rank k, the matrix M′ (and therefore the matrix M) is of rank k if and only if

$$D - CA^{-1}B = 0,$$

where the RHS is an (n–k)×(p–k) matrix of zeros. Thus let $g:\mathcal{L}(p,n)\to\mathbb{R}^{(n-k)\times(p-k)}$ be defined by $g(M) = D - CA^{-1}B$; the function g then is the defining mapping of the manifold $\mathcal{L}_k(p,n)$ at $J \in \mathcal{L}_k(p,n)$.

To see that the differential du intersects the manifold $\mathcal{L}_k(p,n)$ transversally at J, consider the composite mapping $h = g\circ du:\mathbb{R}^p\to\mathbb{R}^{(n-k)\times(p-k)}$. In example 1, we have that h(x,y) = (x,y) and hence dh(0) is trivially of rank 2, as required; in example 2 h(x,y,z) = (y,z) and again dh(0) is of rank 2, as required; and in example 3 h(x,y,z,w) = (x,y,z,w) and hence dh(0) is of rank 4, again as required. Therefore, any one of these examples, together with an appeal to the above theorem, demonstrates that the claim described above is false; we have "robust" examples of utility profiles where $p \geq n-1$ and yet $\Lambda(q,u) \neq \phi$ (in fact, example 3 shows that the inequality in the first part of the claim cannot be replaced with "$p \geq n$" either).

5. Counting dimensions

The singularity approach *does* provide an upper bound on the generic existence of core points, it's just that the correct bounds are much weaker than those identified in the above claim. For an arbitrary smooth function $f:\mathbb{R}^s\to\mathbb{R}^t$ and a manifold M in $\mathbb{R}^t$ of codimension c, say that f is *transversal to* M if at every point x in $\mathbb{R}^s$ either i) $f(x) \notin M$, or ii) f intersects M transversally at x; equivalently, f is transversal to M if all intersections are transversal. For example, in Figure 1 the function f is not transversal to M whereas the function $\tilde{f}$ is.

Given this definition the function f would be transversal to M if it *never* intersected M, transversally or otherwise. In this case, of course, the preimage of M, $f^{-1}(M)$, would necessarily be empty; otherwise, $f^{-1}(M)$ itself will be a manifold in $\mathbb{R}^s$

of codimension c (Levine, 1971 p.23). If on the other hand $c = (t-m) > s$, then $\dim f^{-1}(M) = s - c < 0$ and again $f^{-1}(M)$ will be empty, since the only manifold with a negative dimension is the empty set. That is, if $t-m > s$ then the only way f can be transversal to M is to have f never intersect M at all.

Now from above we have that the manifold $\mathcal{L}_k(p,n)$ in $\mathcal{L}(p,n)$ is of codimension $(n-k)(p-k)$, and hence of dimension $pn - (n-k)(p-k)$. For instance, in example 1 we have that $\mathcal{L}(2,3)$ is six dimensional, with $\mathcal{L}_1(2,3)$ being a four dimensional subset of $\mathcal{L}(2,3)$. Further, $S_k(u)$, the set of points $x \in X$ for which the differential $du(x)$ is of rank k, is given by the preimage $du^{-1}(\mathcal{L}_k(p,n))$ of the set $\mathcal{L}_k(p,n)$ under the mapping du. Therefore, if du is transversal to $\mathcal{L}_k(p,n)$ then $S_k(u)$ will either be empty or else a manifold in $\mathbb{R}^p$ of codimension $(p-k)(n-k)$, and hence

$$\dim S_k(u) = p - (n-k)(p-k) \ . \qquad (3)$$

In example 1, for instance, we have a zero–dimensional set $S_1(u)$, as this dimension–counting argument would predict; in example 2, we have a one–dimensional set $S_2(u)$; and in example 3, we again have a zero–dimensional set $S_2(u)$.

In general, whenever the inequality

$$p < (n-k)(p-k) \qquad (4)$$

holds we know that if du is transversal to $\mathcal{L}_k(p,n)$ it will be the case that $S_k(u)$ will be empty. Further, it turns out that transversal intersections of du and $\mathcal{L}_k(p,n)$ happen to be the generic state of affairs: by the Transversality Theorem of Thom (cf. Golubitsky and Guillemin, 1973), for all $k < z$ the set

$$\{ \ u \in U(X)^n \ : \ du \text{ is transversal to } \mathcal{L}_k(p,n)\}$$

is *residual*, where a residual subset of $U(X)^n$ is one which contains a countable intersection of open dense sets.. And since the space $U(X)^n$ equipped with the Whitney topology is a Baire space, i.e. one in which all residual subsets are dense, du will generically be transversal to $\mathcal{L}_k(p,n)$. In particular, we have that $S_k(u)$ will generically be empty whenever the parameters (p,n,k) are such that (4) holds.

Next, note that the RHS of (4) is decreasing in k; therefore if (p,n,k) satisfies (4) then (p,n,k′), with $k' < k$, will satisfy (4) as well. Thus by Lemma 3, in order to insure that $\Lambda(q,u)$ is generically empty we simply need to check that $k = q-1$ satisfies (4), which gives us the following inequality guaranteeing the generic emptiness of $\Lambda(q,u)$:

$$p < (n-q+1)(p-q+1) \ . \tag{5}$$

Thus, by (1) and (2) from section 2 we have that whenever (p,n,q) are such that (5) holds the core $\mathcal{C}(\mathcal{D}_q,.)$ will generically be empty.

We can in addition generate an upper bound for the generic existence of the core for general rules (i.e. not necessarily q–rules) using exactly the same arguments. Recall that throughout we have been assuming the collection of decisive coalitions $\mathcal{D}$ is non–collegial, so in particular for every $i \in N$ there exists some coalition $C \in \mathcal{D}$ such that $i \notin C$; by monotonicity it must be that the coalition $N\backslash\{i\} \in \mathcal{D}$ as well. Hence if $x \in X$ is a core point with respect to the pair $(\mathcal{D},u)$ then necessarily $x \in \mathcal{P}(C,u)$ for all $C \in \mathcal{D}_{n-1}$, and x would be a core point for the q–rule with $q = n-1$ as well. Therefore a sufficient condition for $\mathcal{C}(\mathcal{D},u)$ to be empty is that $\mathcal{C}(\mathcal{D}_{n-1},u)$ be empty, and from (5) we know that a sufficient condition for the latter is

$$p < 2(p-n+2), \text{ or}$$

$$p > 2(n-2) = 2n - 4. \tag{6}$$

That is, if (p,n) satisfy (6) then for *any* non–collegial rule the core will generically be empty (recall the original claim had $p > n-2$). In addition, we know that this is the lowest possible general bound using this approach, since this is precisely the bound for one particular rule, namely the q–rule with $q = n-1$.

Finally, examples' 1 – 3 in section 3 merely demonstrate that when the above inequalities are not satisfied for those particular parameters (p,n,q) one can find open sets of utility profiles such that on this set $\Lambda(q,u)$ is non–empty. Is this a general result? That is, suppose that for some q–rule the parameters (p,n,q) are such that

(5) does *not* hold; is it necessarily the case that one can find an open set of profiles such that on this set $\Lambda(q,u)$ is non–empty? Whitney (1958) answers this question in the affirmative: whenever (p,n,k) are such that (4) fails to hold, there exist open sets of mappings u for which $S_k(u)$ is non–empty. In this sense, then, there is nothing inherently special about examples' 1 – 3.

6. Caveats and Extensions

Three comments are in order. The first is that the examples in section 3 all had one of the voters possessing non–convex (in particular, saddle–point) preferences; hence one reasonable conjecture would be that by restricting attention to convex preferences such examples might not exist. This however turns out not to be true, as the following example (due to John Nachbar) aptly demonstrates:

Example 4: $n = 3$, $p = q = 2$, and

$$u_1(x,y) = x + y$$
$$u_2(x,y) = -x^2/2 - k.y^2/2$$
$$u_3(x,y) = -k.x^2/2 - y^2/2 \text{ , where } k > 1.$$

The Jacobian matrix is

$$J_u(.) = \begin{bmatrix} 1 & 1 \\ -x & -ky \\ -kx & -y \end{bmatrix} ,$$

so that at $(x,y) = (0,0)$ the rank of J_u is 1. To see that this intersection of du with $\mathcal{L}_1(2,3)$ is transversal, use the defining mapping $g(M) = D - CA^{-1}B$ as in section 4 to get the composite map $h = g\circ du = (x-ky,kx-y)$. Then dh is represented by the matrix

$$\begin{bmatrix} 1 & -k \\ k & -1 \end{bmatrix},$$

which (since $k \neq 1$) is clearly of rank 2. Thus the non–convexity of the preferences in the earlier examples was not crucial.

The second comment again has to do with the examples in section 3, and that is that all of these actually dealt with *majority rule* core points. However, these do not address Schofield's (1983) work on the generic non–existence of such points, for in the latter use is made of the *pivotal gradient conditions* (McKelvey and Schofield, 1987), which place additional requirements on core points over and above those found in Lemma 3 above.

Finally, what has been shown here is that certain *proofs* of the generic non–existence of the core are wrong; however this does not necessarily imply that core points exist when, for example, $p \geq n-1$. Rather, the proof technique of relaxing the requirement of semi–positive dependence to linear dependence, and subsequently invoking results from singularity theory, does not generate the dimensional bounds on core existence that were claimed. It would appear, given the message of the previous comment, that by requiring semi–positive dependence (which is essentially equivalent to the pivotal gradient conditions) and not simply linear dependence one may be able to generate tighter bounds on the existence of core points.

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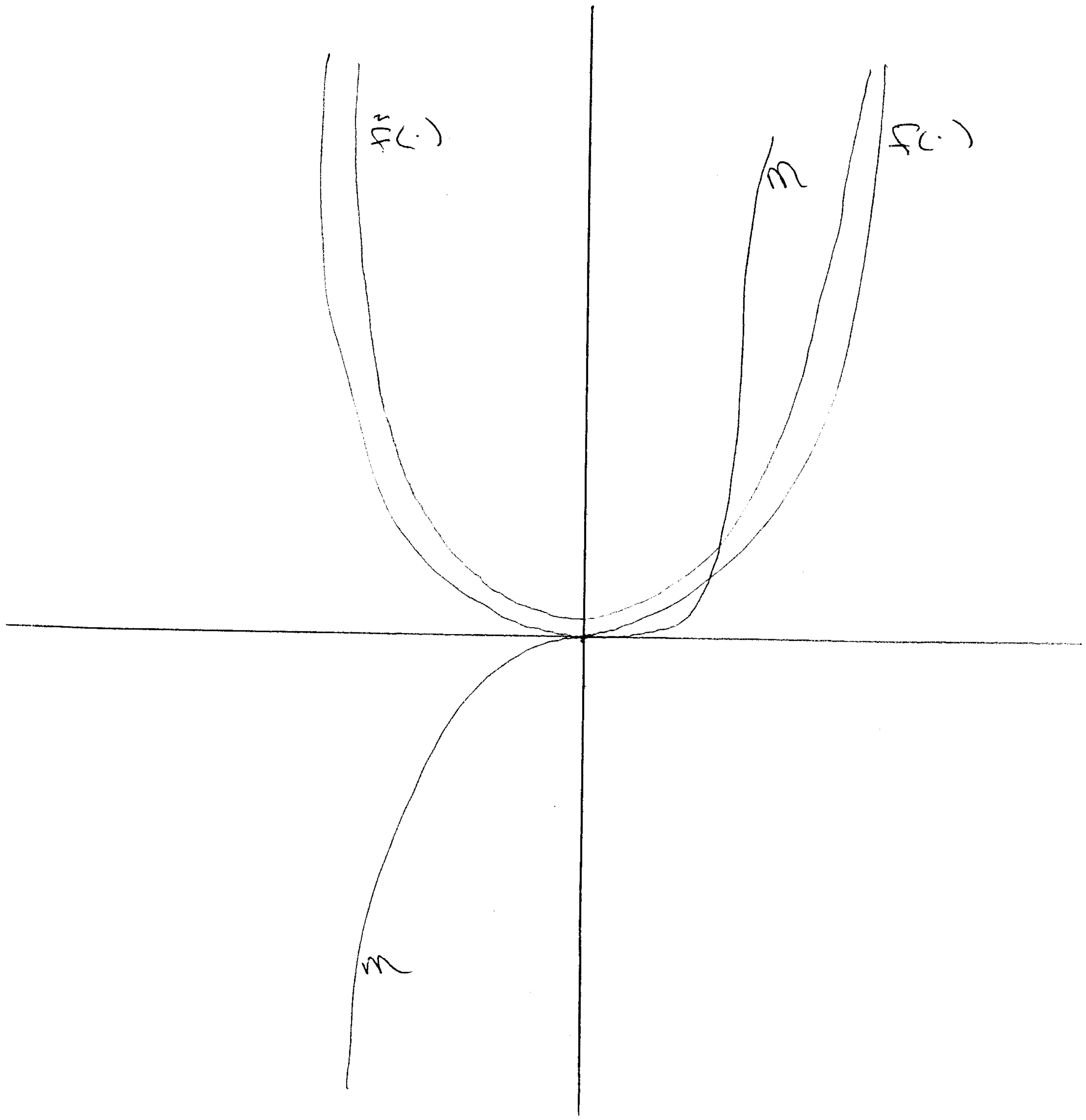

Figure 1