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Abstract

We consider the problem of fair allocation in economies with indivisible goods. Our primary concept is that of an envy-free allocation, that is, an allocation such that no agent would prefer anyone else's bundle to his own. Since there typically is a large set (a continuum) of such allocations, the need arises to identify well-behaved selections from the no-envy solution. First, we establish the non-existence of "population monotonic" selections. Then we propose a variety of selections motivated by intuitive considerations of fairness.

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1 Introduction

We consider the problem of allocating a finite number of indivisible "objects" and a single infinitely divisible good, thought of as money, among a group of agents with equal rights on these resources. An example is the allocation of jobs among workers together with the determination of their salaries. Another example is the distribution of the various indivisible parts of an estate (paintings, silverware), and of the divisible part (cash, securities) among a group of heirs. A ***solution*** is a systematic method of solving all such problems: a solution associates with each problem of fair allocation a set of feasible allocations, seen as recommendations for that problem. A fundamental example is the ***no-envy solution***, which selects the set of allocations at each of which no agent prefers the bundle of any other agent to his own.

The no-envy notion is quite appealing intuitively[1]. Under our assumptions, it is even fully compatible with efficiency since an envy-free allocation is necessarily efficient. However, the set of envy-free allocations is often large, and in such situations the no-envy requirement is insufficient. Our objective is to identify desirable, and preferably "small", subsolutions of the no-envy solution.

In previous work (Tadenuma and Thomson, 1991), we followed the axiomatic approach, focusing on a certain property of consistency[2]. We showed that, although the no-envy solution is consistent, essentially no proper selection is. Here, we will look for solutions satisfying the following monotonicity property with respect to the number of agents: when new agents come in, while resources remain the same, all agents initially present are affected in the same direction; they all lose or they all gain. Our first main result is that this approach results in an impossibility too.

We then propose selections, directly based on intuitive considerations of fairness. We introduce two ways of measuring how well each agent is treated in relation to each other agent at an envy-free allocation. First, given the total resources received by the pair $\{i, j\}$, we identify the extreme points of

[1]The fast-growing literature on the no-envy concept and related concepts is reviewed in Thomson (1994b).

[2]A solution is ***consistent*** if the recommendation it makes for an economy is never contradicted by the recommendation it makes for "reduced" economies obtained from it by imagining the departure of some of the agents with their alloted bundles.

the set of envy-free allocations in the two-person economy consisting of these two agents and these resources. The distance between the two extreme points can be seen as a measure of the "equity surplus" available to the two agents. We then compute the percentage of the equity surplus that is received by each agent.

Alternatively, given a pair of agents $\{i, j\}$, we determine the amount of money necessary to add to agent j's bundle so that agent i be indifferent between his bundle and agent j's revised bundle. We propose to use this "compensation" as a measure of how well agent i is treated in relation to agent j.

Equipped with these two measures, we then construct solutions that treat agents as equally as possible.

2 The model

Let $\mathcal{Q}$ be an infinite set of ***agents***, with members denoted by i, j,..., and $\mathcal{A}$ an infinite set of ***objects***, with members denoted by α, β, ... Each agent can consume at most one of these objects. There is also a single infinitely divisible good called ***money***. An ***economy*** is a list $e = (Q, A, M, R_Q)$ where Q is a finite set of agents drawn from $\mathcal{Q}$, A is a finite set of objects drawn from $\mathcal{A}$, and $M \in \mathbb{R}$ is an amount of money. We assume that the number of agents, $|Q|$, is greater than or equal to the number of objects, $|A|$. If there are more agents than objects, some of the agents receive no object. In this case we say that these agents receive "the null object," denoted by $\emptyset$.

Each agent $i \in Q$ is equipped with a preference relation on $\{A \cup \emptyset\} \times \mathbb{R}$, denoted by R_i, P_i denoting the strict preference relation associated with R_i and I_i the indifference relation. The symbol R_Q denotes the list $(R_i)_{i \in Q}$. A typical consumption for an agent is a pair (α, m_0) composed of object $\alpha \in \{A \cup \emptyset\}$ and m_0 units of money. As in most of the recent literature on the subject, we let m_0 be positive or negative; we might want m_0 to be negative when, for example, the cost of providing the objects has to be covered by the agents[3]. Each preference relation R_i is assumed to be reflexive,

[3]Moreover we allow m_0 to be unbounded below. Under this assmption on the consumption set together with the assumptions on preferences, we can guarantee the existence of envy-free allocations, as established by Alkan, Demange and Gale (1991). Note, however, that our selection methods can be defined on other classes of economies in which envy-free

transitive and complete, and such that:

(a.1) for all $\alpha \in \{A \cup \emptyset\}$, and for all m_i, $m'_i \in \mathbb{R}$, if $m_i > m'_i$, then $(\alpha, m_i)P_i(\alpha, m'_i)$

(a.2) for all α, $\beta \in \{A \cup \emptyset\}$, and for all $m_i \in \mathbb{R}$, there is $m'_i \in \mathbb{R}$ such that $(\alpha, m_i)I_i(\beta, m'_i)$.

Let $\mathcal{E}$ be the class of all economies[4].

Starting from an economy $(Q, A, M, R_Q) \in \mathcal{E}$, we will have occasions to consider economies involving some subgroup Q' of Q, some subset A' of A, and some amount M' of money. Then the notation $(Q', A', M', R_{Q'})$ designates this economy in which the preference relations of the members of Q' are restricted to the domain $\{A' \cup \emptyset\} \times \mathbb{R}$.

Let $e = (Q, A, M, R_Q) \in \mathcal{E}$ be given. A ***feasible allocation for e*** is a pair $z = (\sigma, m)$ where $\sigma : Q \to \{A \cup \emptyset\}$ is a mapping such that $|\sigma^{-1}(\alpha)| = 1$ for all $\alpha \in A$, and m is a vector in $\mathbb{R}^Q$ satisfying $\sum_{i \in Q} m_i = M$. Mapping σ assigns objects to agents so that $|Q| - |A|$ agents receive the "null object", and each of the other agents receives exactly one object in A. Agent i's bundle is denoted by $z_i = (\sigma(i), m_i)$ for all $i \in Q$, and we also write a feasible allocation z as $z = (z_i)_{i \in Q} = ((\sigma(i), m_i))_{i \in Q}$ when no confusion may arise. Let $Z(e)$ be the set of feasible allocations for e.

We would like to be able to make recommendations for all problems in the class just described. A ***solution*** φ is a correspondence that associates with each economy $e \in \mathcal{E}$ a nonempty subset $\varphi(e)$ of $Z(e)$. A solution provides for each economy a set of feasible allocations regarded as desirable for the economy. A familiar example is the following:

The Pareto solution, P: For all $e = (Q, A, M, R_Q) \in \mathcal{E}$, $P(e) = \{z \in Z(e) \mid \nexists z' \in Z(e)$ such that $\forall i \in Q, z'_i R_i z_i$, and $\exists i \in Q, z'_i P_i z_i\}$.

allocations exist, e.g., a class of economies in which the consumption of money is bounded below and there is "sufficiently large" amount of money (Maskin, 1987, Alkan, Demange and Gale, 1991, and Aragones, 1992).

[4]This model was examined by Svensson (1983), Maskin (1987), Alkan, Demange and Gale (1991), Tadenuma and Thomson (1991), Alkan (1989), Moulin (1992), and Aragones (1992). Luce and Raiffa (1957), Kolm (1972), Crawford and Heller (1979), van Damme (1987), Tadenuma and Thomson (1993a, b), and Moulin (1990) studied versions of this model.

We are interested in solutions satisfying the following fundamental notion of equity: simply, no agent prefers the bundle of any other agent to his own (Foley, 1967).

The no-envy solution, F: For all $e = (Q, A, M, R_Q) \in \mathcal{E}, F(e) = \{z \in Z(e) \mid \forall i, j \in Q, z_i R_i z_j\}$.

In ***classical*** economies, that is, economies where all goods are infinitely divisible, the set of envy-free allocations and the set of Pareto-efficient allocations are not usually related by inclusion. In the class of economies considered here, any envy-free allocation is efficient, as shown by Svensson (1983).

Although the no-envy concept is very attractive intuitively, the set of envy-free allocations may be quite large, and in these situations the no-envy solution does not make a precise enough recommendation. This is what motivates the search for well-behaved selections from the no-envy solution. We will attack this problem on two fronts, by first examining the existence of selections satisfying a property of population-monotonicity; this approach resulting in an impossibility result, we will then define selections based on simple considerations of fairness.

3 Population-monotonicity

When the number of agents with equally valid claims on some fixed resources increases, it seems quite natural to require that all of the agents initially present should lose. This property, which was examined by Thomson (1983) in the context of bargaining theory, was shown by Alkan (1989) to be violated by all selections from the no-envy solution in the present context (see also Moulin, 1990).

However, consider the following variant of this property, first studied by Chun (1986) in the context of quasi-linear choice problems: if new agents come in, all of the old agents gain, or they all lose. This version is particularly useful when there are public goods or externalities, and in such contexts it can sometimes be met when the stronger version cannot. Should we expect a possibility result here too? Our first result is that unfortunately there are no selections from the no-envy solution satisfying this property. In fact, an impossibility holds even if we allow solution correspondences and only require that there be two allocations, one among the allocations the solution would select in the initial economy, and one among the allocations it would select in the enlarged economy, such that all agents in the initial economy prefer one to the other[5].

Weak population-monotonicity: For all $e = (Q, A, M, R_Q) \in \mathcal{E}$ and $e' = (Q', A', M', R'_{Q'})$, if $Q \subset Q'$ and $(A, M, R_Q) = (A', M', R'_Q)$, then there are $z \in \varphi(e)$ and $z' \in \varphi(e')$ such that either $z_i R_i z'_i$ for all $i \in Q$ or $z'_i R_i z_i$ for all $i \in Q$.

Theorem 1 *There is no* weakly population-monotonic *subsolution of the no-envy solution*[6].

Proof: Let $e = (Q, A, M, R_Q) \in \mathcal{E}$ be such that $Q = \{1, 2\}$, $A = \{\alpha, \beta\}$, $M = 0$, and for all $m_0 \in \mathbb{R}$, $(\alpha, m_0) I_1 (\beta, m_0) I_1 (\emptyset, m_0 + 1)$ and $(\alpha, m_0) I_2 (\beta, m_0) I_2 (\emptyset, m_0 + 10)$. Then $F(e) = \{((\alpha, 0), (\beta, 0)), ((\beta, 0), (\alpha, 0))\}$. Let $e' = (Q', A, M, R_{Q'})$ be such that $Q' = \{1, 2, 3, 4\}$ and for all $m_0 \in \mathbb{R}$, $(\alpha, m_0) I_3 (\beta, m_0) I_3 (\emptyset, m_0 + 4)$, and $R_3 = R_4$.

We claim that if $z \in F(e)$ and $z' \in F(e')$, then $z'_1 P_1 z_1$ and $z_2 P_2 z'_2$. Indeed, at z',

(i) Since $z' \in P(e')$, agent 2 receives one of the two objects.

(ii) Since $z' \in P(e')$, either agent 3 or agent 4 receives the remaining object. Because they have identical preferences, in order for envy not to exist between them, the one who does not receive an object should receive 4 more units of money than the other.

(iii) For envy not to exist between the two agents receiving the objects, they receive the same amount of money. The same holds for the two agents who receive the null object.

Thus, $z'_1 = (\emptyset, 2)$ and $z'_2 = (\alpha, -2)$ or $(\beta, -2)$, and therefore, $z'_1 P_1 z_1$ and $z_2 P_2 z'_2$. Q.E.D.

[5]Alkan (1989) examines other monotonicity properties. For a survey of the various notions of population-monotonicity and of their applications, see Thomson (1994a).

[6]Theorem 1 was conjectured by H. Moulin.

4 Refinements based on intuitive considerations of fairness

In order to achieve some refinement of the set of envy-free allocations, we now introduce two alternative measures of how well each agent is treated in relation to each other agent. One is based on the notion of an "equity surplus" (Section 4.1), the other is based on the notion of a "compensation" (Section 4.2). These measures are then used to define selections from the no-envy solution (Section 4.3).

One of our objectives in formulating these definitions was to remain within the ordinal framework to which the no-envy concept owes much of its success. Just like the no-envy concept, our selections require only information about agents' preferences. This is not because we feel that cardinal information about agents' "intensities" of preferences are irrelevant to the problem of fair allocation, but rather because we do believe that it is important to see how far an ordinal approach can take us.

4.1 Equity surplus

Our first proposal is based on the observation that with each two-person economy $(\{i, j\}, \{\alpha, \beta\}, M_0, \{R_i, R_j\})$ can be associated a number measuring the "size" of its set of envy-free allocations. Indeed, for any such economy, the assignment of objects is the same at all envy-free allocations except for a degenerate case (Proposition 1). Then, we determine, for either one of the two agents, the difference in the amounts of money he holds at his least preferred and most preferred allocations in the set. This difference, which we refer to as the "equity surplus" for the two-person economy, indicates how much freedom we have in solving the distribution problem without generating envy. Finally, we compute the share of that surplus received by each of the two agents.

Proposition 1 *For all $e = (Q, A, M, R_Q) \in \mathcal{E}$ with $Q = \{i, j\}$*
(i) *if there is $\tilde{z} = \{\tilde{z}_i, \tilde{z}_j\} \in Z(e)$ such that $\tilde{z}_i I_i \tilde{z}_j$ and $\tilde{z}_j I_j \tilde{z}_i$, then $F(e) = \{\tilde{z}, \tilde{z}'\}$ where $\tilde{z}' = (\tilde{z}_j, \tilde{z}_i)$;*
(ii) *otherwise, for all $z = (\sigma, m), z' = (\sigma', m') \in F(e)$, $\sigma = \sigma'$.*

Proof: Part (i). Clearly, $\tilde{z}, \tilde{z}' \in F(e)$. Let $\tilde{z} = (\tilde{\sigma}, \tilde{m})$ and $\tilde{z}' = (\tilde{\sigma}', \tilde{m}')$. Let $z = (\sigma, m) \in Z(e)$ be such that $z \neq \tilde{z}, \tilde{z}'$. Then, either (1) $\sigma = \tilde{\sigma}$ and $m \neq \tilde{m}$, or (2) $\sigma = \tilde{\sigma}'$ and $m \neq \tilde{m}'$. If (1) holds, then for one agent in Q, say agent i, $m_i < \tilde{m}_i$. By feasibility, $m_j > \tilde{m}_j$. Then, $z_j = (\sigma(j), m_j) = (\tilde{\sigma}(j), m_j) P_i (\tilde{\sigma}(j), \tilde{m}_j) I_i (\tilde{\sigma}(i), \tilde{m}_i) P_i (\tilde{\sigma}(i), m_i) = (\sigma(i), m_i) = z_i$. Hence, $z \notin F(e)$. By the same argument, if (2) holds, then $z \notin F(e)$. Thus, $F(e) = \{\tilde{z}, \tilde{z}'\}$.

Part (ii). Assume that there is no $z \in Z(e)$ such that $z_i I_i z_j$ and $z_j I_j z_i$. Let $z = (\sigma, m), z' = (\sigma', m') \in F(e)$. Then, at least one of the agents strictly prefers his own bundle to the other at z. Let us assume $z_i P_i z_j$. Suppose $\sigma \neq \sigma'$. Then, $\sigma(i) = \sigma'(j)$, and $\sigma(j) = \sigma'(i)$. If $m'_j \geqq m_i$, then $m'_i \leqq m_j$, and $z'_j = (\sigma'(j), m'_j) = (\sigma(i), m'_j) R_i (\sigma(i), m_i) P_i (\sigma(j), m_j) R_i (\sigma(j), m'_i) = (\sigma'(i), m'_i) = z'_i$. Hence, $z' \notin F(e)$. If $m'_j < m_i$, then $m'_i > m_j$, and $z'_i = (\sigma'(i), m'_i) = (\sigma(j), m'_i) P_j (\sigma(j), m_j) R_j (\sigma(i), m_i) P_j (\sigma(i), m'_j) = (\sigma'(j), m'_j) = z'_j$. Thus, we have $z' \notin F(e)$, which is a contradiction. Q.E.D.

In order better to understand the significance of Proposition 1, it is worth noting that if $|Q| > 2$ the assignment of objects need not be the same at all envy-free allocations, as shown next.

Proposition 2 *There exists* $e = (Q, A, M, R_Q) \in \mathcal{E}$ *with* $|Q| > 2$ *such that*
(i) *there is no* $\tilde{z} \in Z(e)$ *with* $\tilde{z}_i I_i \tilde{z}_j$ *for all* $i, j \in Q$, *and*
(ii) *there are* $z = (\sigma, m), z' = (\sigma', m') \in F(e)$ *with* $\sigma \neq \sigma'$.

Proof: Let $e = (Q, A, M, R_Q) \in \mathcal{E}$ be such that $Q = \{1, 2, 3\}$, $A = \{\alpha, \beta, \gamma\}$, $M = 6$, and $(\alpha, 1) I_1 (\beta, 2) I_1 (\gamma, 4)$, $(\alpha, 4) I_1 (\beta, 3) I_1 (\gamma, 5)$, $(\alpha, 2) I_2 (\beta, 1) I_2 (\gamma, 4)$, $(\alpha, 3) I_2 (\beta, 4) I_2 (\gamma, 5)$, and $(\alpha, 3) I_3 (\beta, 3) I_3 (\gamma, 0)$. Let $z = ((\alpha, 1), (\beta, 1), (\gamma, 4))$ and $z' = ((\beta, 3), (\alpha, 3), (\gamma, 0))$. Then $z, z' \in F(e)$. Since $z'_1 P_1 z_1$, there is no $\tilde{z} \in Z(e)$ such that $\tilde{z}_i I_i \tilde{z}_j$ for all $i, j \in Q$. (By the same argument as in the proof of part (i) in Proposition 1, if there is such $\tilde{z}$, then for all $z \in F(e)$ and all $i \in Q$, $z_i I_i \tilde{z}_i$ must hold.) Yet we have $\sigma \neq \sigma'$. Q.E.D.

Similarly, the assignment of objects need not be the same at all efficient allocations, even in two-person economies[7].

[7]However, in an economy with "quasi-linear" preferences (as defined in section 4.2), the assignment of objects is the same at all efficient allocations except for the degenerate case (i) in Proposition 1.

Proposition 3 *There exists $e = (Q, A, M, R_Q) \in \mathcal{E}$ with $Q = \{i, j\}$ such that*
(i) *there is no $\tilde{z} \in Z(e)$ with $\tilde{z}_i I_i \tilde{z}_j$ and $\tilde{z}_j I_j \tilde{z}_i$, and*
(ii) *there are $z = (\sigma, m), z' = (\sigma, m') \in P(e)$ with*
$\sigma \neq \sigma'$.

Proof: Let $e = (Q, A, M, R_Q) \in \mathcal{E}$ be such that $Q = \{1, 2\}$, $A = \{\alpha, \beta\}$, $M = 6$. Let $(\alpha, 3)I_1(\beta, 4)$, $(\alpha, 6)I_1(\beta, 5)$, $(\alpha, 1)I_2(\beta, 2)$, and $(\alpha, 4)I_2(\beta, 3)$. Let $z = ((\alpha, 3), (\beta, 3))$ and $z' = ((\beta, 5), (\alpha, 1))$. We claim that $z, z' \in P(e)$. Indeed, $z \in P(e)$ since $z \in F(e)$ and $F(e) \subseteq P(e)$. Now if z' is Pareto-dominated by some allocation z'', then $z'' = ((\alpha, m''_1), (\beta, m''_2))$ for some $m''_1 > 6$ and $m''_2 > 2$. But since $M = 6$, we have $z'' \notin Z(e)$. Q.E.D.

Given $e = (Q, A, M, R_Q) \in \mathcal{E}$, $z = (\sigma, m) \in F(e)$, and $i, j \in Q$ with $i \neq j$, let $\underline{m}_{ij}(z) \in \mathbb{R}$ and $\bar{m}_{ji}(z) \in \mathbb{R}$ be the amounts of money such that
(i) $\underline{m}_{ij}(z) + \bar{m}_{ji}(z) = m_i + m_j$ and
(ii) $(\sigma(i), \underline{m}_{ij}(z))I_i(\sigma(j), \bar{m}_{ji}(z))$.
By assumptions (a.1) and (a.2) on preferences, $\underline{m}_{ij}(z)$ and $\bar{m}_{ji}(z)$ exist and are unique. Symmetrically $\underline{m}_{ji}(z) \in \mathbb{R}$ and $\bar{m}_{ij}(z) \in \mathbb{R}$ are defined.

By Proposition 1, the two allocations $((\sigma(i), \underline{m}_{ij}(z)), (\sigma(j), \bar{m}_{ji}(z)))$ and $((\sigma(i), \bar{m}_{ij}(z)), (\sigma(j), \underline{m}_{ji}(z)))$ are indeed the "end-points" of the segment of envy-free allocations for the economy $(\{i, j\}, \{\sigma(i), \sigma(j)\}, m_i + m_j, \{R_i, R_j\})$[8]. The former allocation is the worst for agent i and the best for agent j in this set, and the latter the opposite. Obviously, $\bar{m}_{ij}(z) - \underline{m}_{ij}(z) = \bar{m}_{ji}(z) - \underline{m}_{ji}(z)$[9]. Let $s_{\{i,j\}}(z)$ denote this difference. The number $s_{\{i,j\}}(z)$ is the ***equity surplus for $\{i,j\}$ at z.***

Now, we quantify how fairly agent i is treated at z relative to agent j by the proportion of the equity surplus that agent i receives in the economy $(\{i, j\}, \{\sigma(i), \sigma(j)\}, m_i + m_j, \{R_i, R_j\})$. Let $p_{ij}(z) = (m_i - \underline{m}_{ij}(z))/s_{\{i,j\}}(z)$ if $s_{\{i,j\}}(z) \neq 0$, and $p_{ij}(z) = \frac{1}{2}$ if $s_{\{i,j\}}(z) = 0$. Note that $0 \leqq p_{ij}(z) \leqq 1$, $0 \leqq p_{ji}(z) \leqq 1$ and $p_{ij}(z) + p_{ji}(z) = 1$. The number $p_{ij}(z)$ is ***agent i's share of the equity surplus for $\{i,j\}$ at z.***

[8]They are also the two egalitarian-equivalent and efficient allocations (Pazner and Schmeidler, 1978) for this economy.

[9]In the degenerate case (i) in Proposition 1, $\bar{m}_{ij}(z) - \underline{m}_{ij}(z) = \bar{m}_{ji}(z) - \underline{m}_{ji}(z) = 0$ for all $z \in F(e)$.

When $s_{\{i,j\}}(z) = 0$, each of the two agents is indifferent between his bundle and the other's at z, and in that sense, both agents are treated equally. This is the reason why we define $p_{ij}(z) = \frac{1}{2}$ if $s_{\{i,j\}}(z) = 0$. However, no matter what value we assign to $p_{ij}(z)$ when $s_{\{i,j\}}(z) = 0$, there arises a problem: discontinuity of the share function p_{ij} (with respect to the second argument) at z where $s_{\{i,j\}}(z) = 0$. We describe in Appendix A an example in which p_{ij} is not continuous[10]. Continuity is a natural requirement for a good measure: Small perturbations of a money distribution should not lead to drastically different values of the measure. In order to obtain this property, we replace p_{ij} with the following continuous approximation.

Given $\delta > 0$, define a function $t^{\delta} : \mathbb{R}_+ \to [0,1]$ by $t^{\delta}(x) = 0$ if $x \geqq \delta$, and $t^{\delta}(x) = 1 - \frac{x}{\delta}$ if $0 \leqq x < \delta$. Then define $\tilde{p}_{ij}(z) = \frac{1}{2}t^{\delta}(s_{\{i,j\}}(z)) + [1 - t^{\delta}(s_{\{i,j\}}(z))]p_{ij}(z)$. Note that we can set δ arbitrarily close to 0, and that for all $z \in F(e)$ such that $s_{\{i,j\}}(z) \geqq \delta$, $\tilde{p}_{ij}(z) = p_{ij}(z)$. That is, $\tilde{p}_{ij}(z)$ is the same as $p_{ij}(z)$ except where the equity surplus $s_{\{i,j\}}(z)$ is "very small".

When the equity surplus is sufficiently small, and as it approaches zero, the importance of how the surplus be divided is decreasing. Thus we put less and less weight on the exact share of the surplus when we evaluate the allocation. And at the limit we judge that the two agents are treated equally.

4.2 Compensations

The second measure we propose is the ***distance that agent i is from envying agent j at z***, that is, the maximal amount of money that can be added to the bundle of agent j without causing agent i to envy him. We quantify how well agent i is treated in relation to agent j by this number. Let $d_{ij}(z) = \max\{m_0 \in \mathbb{R} | z_i R_i (\sigma(j), m_j + m_0)\}$.

Alternatively, we could take the maximal amount of money that can be ***subtracted*** from the bundle of agent i without causing him to envy agent j. From a conceptual viewpoint, we do not feel that there is much of a difference between the two approaches. However, it should be noted that the two selection methods based on them will in general give different answers. A special case where they would not is when preferences are "quasi-linear": agent i's preferences are quasi-linear if for all $(\alpha, m_i), (\beta, m_i') \in \{A \cup \emptyset\} \times \mathbb{R}$,

[10]The discontinuity problem does not arise for the second measure, d_{ij}, proposed in the next subsection.

for all $t \in \mathbb{R}$, if $(\alpha, m_i) I_i (\beta, m_i')$, then $(\alpha, m_i + t) I_i (\beta, m_i' + t)$.

4.3 Selections

Now, given an economy $e = (Q, A, M, R_Q)$ and an allocation $z \in F(e)$, we keep the complete record of how well each agent is treated ***in relation to each other agent***. This record contains $|Q| \times (|Q| - 1)$ such numbers. Let $p(z) = (p_{ij}(z))_{i,j \in Q, i \neq j} \in \mathbb{R}^{|Q| \times (|Q|-1)}$, and $d(z) = (d_{ij}(z))_{i,j \in Q, i \neq j} \in \mathbb{R}^{|Q| \times (|Q|-1)}$. By taking the average of the numbers pertaining to a given agent, we obtain a measure of how well this agent is treated ***on average*** in relation to all the other agents. Let $\tilde{p}_i^a(z) = \frac{1}{|Q|-1} \sum_{j \in Q, j \neq i} \tilde{p}_{ij}(z)$, and $d_i^a(z) = \frac{1}{|Q|-1} \sum_{j \in Q, j \neq i} d_{ij}(z)$. Let $\tilde{p}^a(z) = (\tilde{p}_i^a(z))_{i \in Q} \in \mathbb{R}^Q$, and $d^a(z) = (d_i^a(z))_{i \in Q} \in \mathbb{R}^Q$.

We have four distinct records on the basis of which we can evaluate the allocation. Using each of these records, we propose to treat agents as equally as possible. A most natural way to do this is perhaps to choose allocations whose associated record has maximal minimum coordinate [11].

The equal-share solution, S: For all $e = (Q, A, M, R_Q) \in \mathcal{E}$, $S(e) = \{z \in Z(e) | \min_{i,j \in Q, i \neq j} \tilde{p}_{ij}(z) \geqq \min_{i,j \in Q, i \neq j} \tilde{p}_{ij}(z') \text{ for all } z' \in Z(e)\}$.

The equal-average-share solution, $\mathbf{S}^a$: For all $e = (Q, A, M, R_Q) \in \mathcal{E}$, $S^a(e) = \{z \in Z(e) | \min_{i \in Q} \tilde{p}_i^a(z) \geqq \min_{i \in Q} \tilde{p}_i^a(z') \text{ for all } z' \in Z(e)\}$.

The equal-compensation solution, C: For all $e = (Q, A, M, R_Q) \in \mathcal{E}$, $C(e) = \{z \in Z(e) | \min_{i,j \in Q, i \neq j} \ d_{ij}(z) \geqq \min_{i,j \in Q, i \neq j} \ d_{ij}(z') \text{ for all } z' \in Z(e)\}$.

The equal-average-compensation solution, $\mathbf{C}^a$: For all $e = (Q, A, M, R_Q) \in \mathcal{E}$, $C^a(e) = \{z \in Z(e) | \min_{i \in Q} \ d_i^a(z) \geqq \min_{i \in Q} \ d_i^a(z') \text{ for all } z' \in Z(e)\}$.

[11] Diamantaras and Thomson (1990), following Chaudhuri (1986), measured the distance that an agent is from envying another agent in classical economies by the maximal rate of proportional expansion of that agent's bundle compatible with no-envy, and proposed the subsolution of the no-envy solution defined by maximizing the minimum coordinate of the list of such distances for all pairs of agents.

A further refinement can be obtained by using the ***lexicographic order***: For any two vectors x, $y \in \mathbb{R}^n$, say that ***x is lexicographically greater than y*** if $x_1 > y_1$, or $[x_1 = y_1$ and $x_2 > y_2]$, ..., or $[x_1 = y_1, ..., x_{n-1} = y_{n-1}$, and $x_n > y_n]$. First, rearrange the coordinates of the record associated with each allocation in increasing order. Then, select the allocations whose reordered records are lexicographically maximal. In social choice and game theory, maximization (or minimization) in the lexicographic ordering is a standard procedure to perform selections[12].

Note that none of the solutions proposed above depends on utility representations, and for two-person economies, each solution chooses the allocation at which the two agents are treated equally according to the corresponding measure[13].

Other proposals of refinements were made by Alkan, Demange, and Gale (1991), who consider the general case, and Aragones (1992), who considers the quasi-linear case. These refinements are mainly based on the maximin criteria.

The next theorem establishes the existence of allocations satisfying our definitions.

Theorem 2 *The equal-share solution, the equal-average-share solution, the equal-compensation solution, and the equal-average-compensation solution are all well-defined: For all $e \in \mathcal{E}, S(e), S^a(e), C(e)$ and $C^a(e)$ are non-empty.*

The proof of Theorem 2 is standard. It relies on the following lemmata, the proofs of which are relegated to Appendix B.

Let $e = (Q, A, M, R_Q) \in \mathcal{E}$ be given. Define $\Sigma^F(e) = \{\sigma : Q \to \{A \cup \emptyset\} \mid (\sigma, m) \in F(e)$ for some $m \in \mathbb{R}^Q\}$. Let $\sigma \in \Sigma^F(e)$ be given. Then define $F^\sigma(e) = \{m \in \mathbb{R}^Q \mid (\sigma, m) \in F(e)\}$.

Lemma 1 *For all $e \in \mathcal{E}$ and all $\sigma \in \Sigma^F(e)$, $F^\sigma(e)$ is compact.*

Given $e = (Q, A, M, R_Q) \in \mathcal{E}$, $\sigma \in \Sigma^F(e)$, and $i, j \in Q$ with $i \neq j$, define the function $\tilde{p}^\sigma_{ij} : F^\sigma(e) \to \mathbb{R}$ by $\tilde{p}^\sigma_{ij}(m) = \tilde{p}_{ij}(\sigma, m)$.

[12]See the lexicographic maximin social choice rule (Sen, 1970) and the nucleolus (Schmeidler, 1969).

[13]Tadenuma (1989) establishes the single-valuedness of the equal-average-compensation solution.

Lemma 2 *For all $e = (Q, A, M, R_Q) \in \mathcal{E}$, all $\sigma \in \Sigma^F(e)$, and all $i, j \in Q$ with $i \neq j$, the function $\tilde{p}_{ij}^{\sigma}$ is continuous on $F^{\sigma}(e)$.*

Proof of Theorem 2: Let $e = (Q, A, M, R_Q) \in \mathcal{E}$ be given. By Lemma 2, for all $\sigma \in \Sigma^F(e)$, $\min_{i,j \in Q, i \neq j} \tilde{p}_{ij}^{\sigma}$ is a continuous function on $F^{\sigma}(e)$. It follows from Lemma 1 that for all $\sigma \in \Sigma^F(e)$, there exists $m(\sigma) \in F^{\sigma}(e)$ such that for all $m' \in F^{\sigma}(e)$, $\min_{i,j \in Q, i \neq j} \tilde{p}_{ij}^{\sigma}(m(\sigma)) \geqq \min_{i,j \in Q, i \neq j} \tilde{p}_{ij}^{\sigma}(m')$. Since $\Sigma^F(e)$ is a finite set, there exists $\sigma^* \in \Sigma^F(e)$ such that for all $\sigma \in \Sigma^F(e)$, $\min_{i,j \in Q, i \neq j} \tilde{p}_{ij}^{\sigma^*}(m(\sigma^*)) \geqq \min_{i,j \in Q, i \neq j} \tilde{p}_{ij}^{\sigma}(m(\sigma))$. Clearly, for all $(\sigma, m) \in F(e)$, $\min_{i,j \in Q, i \neq j} \tilde{p}_{ij}(\sigma^*, m(\sigma^*)) \geqq \min_{i,j \in Q, i \neq j} \tilde{p}_{ij}(\sigma, m)$. Hence $(\sigma^*, m(\sigma^*)) \in S(e)$.

The proofs of the non-emptiness of $S^a(e), C(e)$ and $C^a(e)$ are analogous.
Q.E.D.

5 Conclusion

We have considered the problem of defining selections from the no-envy solution in economies with indivisible goods. The need for such selections arises because economies often admit large sets of envy-free allocations. We have proposed solutions based on intuitive considerations of fairness. The reason for such a direct approach is that the axiomatic approach followed in an earlier paper (Tadenuma and Thomson, 1991), and in section 2 has led to a dead end. Indeed, we had shown there that there is no proper subsolution of the no-envy solution satisfying a certain consistency property, stating that no recommendation made by a solution is ever contradicted by any recommendation it would make for the problem of distributing among any subgroup the resources they have collectively received under that initial recommendation, and we show in section 2 of the present paper that there is no *weakly population-monotonic* selection from the no-envy solution. These results indicate that it will be impossible to define subsolutions of the no-envy solution that are well-behaved in all respects. Nevertheless, intuitive considerations of fairness do allow for the definition of appealing solutions.

We have considered economies in which the number of agents is at least as great as the number of objects. If there are more objects than agents, then the set of envy-free allocations is not in general included in the set of efficient allocations. If we enlarge the class of economies to allow this case, then we should also look for selections from the *intersection of the no-envy solution and the Pareto solution.* Since the no-envy solution coincides with the intersection of the no-envy and the Pareto solutions on the class of economies considered in the paper, and Theorem 1 is an impossibility result, it still holds true on this enlarged class of economies. On this enlarged class, we can still define selections from the intersection of the no-envy and the Pareto solutions in the same manner. We can also establish the non-emptiness of each of these solutions[14].

[14]Alkan, Demange and Gale (1991) showed that the intersection of the set of envy-free and efficient allocations is non-empty. Thus, all we need to show is that given $e = (Q, A, M, R_Q)$ and $\sigma : Q \to \{A \cup \emptyset\}$, the set $P^{\sigma}(e) = \{m \in \mathbb{R}^Q | (\sigma, m) \in P(e)\}$ is closed in $\mathbb{R}^Q$.

Appendices

A An example in which the share function is not continuous

Let $e = (Q, A, M, R_Q) \in \mathcal{E}$ be such that $Q = \{1, 2, 3\}$, $A = \{\alpha, \beta, \gamma\}$, $M = 15$; and

(i) for all $i \in \{1, 2\}$, $(\alpha, 5) I_i (\beta, 5) I_i (\gamma, 10)$,

(ii) for all $\epsilon > 0$, $(\alpha, 5 - \epsilon) I_1 (\beta, 5 - \epsilon) I_1 (\gamma, 10 - \epsilon)$, and $(\alpha, 5 - \frac{1}{2}\epsilon) I_2 (\beta, 5 - \epsilon) I_2 (\gamma, 10 - \epsilon)$,

(iii) $(\alpha, 10) I_3 (\beta, 10) I_3 (\gamma, 5)$.

Let $\sigma : Q \to A$ be such that $\sigma(1) = \alpha, \sigma(2) = \beta$, and $\sigma(3) = \gamma$. Let $m = (5, 5, 5)$. Then, $(\sigma, m) \in F(e)$ and $s_{\{1,2\}}(\sigma, m) = 0$,. Let $\{\epsilon^n\}$ be a sequence such that $0 < \epsilon^n < 1$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \epsilon^n = 0$. For each $n \in \mathbb{N}$, let $m^n = (5 - \epsilon^n, 5 - \epsilon^n, 5 + 2\epsilon^n)$. Then, for all $n \in \mathbb{N}$, $(\sigma, m^n) \in F(e)$, and $\lim_{n\to\infty} m^n = m$. Now for all $n \in \mathbb{N}$, $s_{\{1,2\}}(\sigma, m^n) = \frac{1}{3}\epsilon^n$, $m^n_{\sigma(1)} - \underline{m}_{12}(\sigma, m^n) = 0$, and $p_{12}(\sigma, m^n) = 0$. Hence, $\lim_{n\to\infty} p_{12}(\sigma, m^n) = 0$. Thus, if $p_{12}(\sigma, m) \neq 0$, the function p_{12} is not continuous with respect to the second argument at (σ, m).

Let $p_{12}(\sigma, m) = 0$. Consider the sequence $\{m'^n\}$ such that for each $n \in \mathbb{N}$, $m'^n = (5 - \frac{1}{2}\epsilon^n, 5 - \epsilon^n, 5 + \frac{3}{2}\epsilon^n)$. Then, for all $n \in \mathbb{N}$, $(\sigma, m'^n) \in F(e)$, $\lim_{n\to\infty} m'^n = m$, and $\lim_{n\to\infty} p_{12}(\sigma, m'^n) = 1 \neq p_{12}(\sigma, m)$.

Therefore, no matter what value we assign to $p_{12}(\sigma, m)$, the function p_{12} is not continuous with respect to the second argument at (σ, m).

B Proofs of Lemmata

Lemma 1 *For all $e \in \mathcal{E}$ and all $\sigma \in \Sigma^F(e)$, $F^\sigma(e)$ is compact.*

Proof: Let $e = (Q, A, M, R_Q) \in \mathcal{E}$ and $\sigma \in \Sigma^F(e)$ be given. First, we show that the complement of $F^\sigma(e)$, $(F^\sigma(e))^c$ is an open set in $\mathbb{R}^Q$. Let $m \in (F^\sigma(e))^c$. Then, there are $i, j \in Q$ such that $(\sigma(j), m_j) \, P_i \, (\sigma(i), m_i)$. Let

$\hat{m}_j \in \mathbb{R}$ be such that $(\sigma(j), \hat{m}_j) I_i (\sigma(i), m_i)$. By assumptions (a.1) and (a.2), $\hat{m}_j$ is well-defined and $\hat{m}_j < m_j$. Let $\tilde{m}_j = (m_j + \hat{m}_j)/2$ and let $\tilde{m}_i \in \mathbb{R}$ be such that $(\sigma(i), \tilde{m}_i) I_i (\sigma(j), \tilde{m}_j)$. Again by (a.1) and (a.2), $\tilde{m}_i$ exists and $\tilde{m}_i > m_i$. Observe that $(\sigma(j), m_j) P_i (\sigma(j), \tilde{m}_j) I_i (\sigma(i), \tilde{m}_i) P_i (\sigma(i), m_i)$.

Let $\epsilon = \min\{(m_j - \tilde{m}_j), (\tilde{m}_i - m_i)\} > 0$. For all $m' \in \mathbb{R}^Q$, if $d(m', m) < \epsilon$ where $d(m', m)$ is the Euclidean distance between m' and m, then $m'_j > m_j - \epsilon \geqq \tilde{m}_j$ and $m'_i < m_i + \epsilon \leqq \tilde{m}_i$, and hence $(\sigma(j), m'_j) P_i (\sigma(j), \tilde{m}_j) I_i (\sigma(i), \tilde{m}_i) P_i (\sigma(i), m'_i)$. Thus $m' \in (F^\sigma(e))^c$. Therefore $(F^\sigma(e))^c$ is open in $\mathbb{R}^Q$, and $F^\sigma(e)$ is closed in $\mathbb{R}^Q$.

Next we show that $F^\sigma(e)$ is bounded in $\mathbb{R}^Q$. Let $i \in Q$ be given. For each $j \in Q$, if $(\sigma(i), M) R_j (\sigma(j), 0)$, then let $m_i(R_j) = 0$, and if $(\sigma(j), 0) P_j (\sigma(i), M)$, then let $m_i(R_j) \in \mathbb{R}$ be such that $(\sigma(i), M + m_i(R_j)) I_j (\sigma(j), -m_i(R_j))$. By assumptions (a.1) and (a.2), $m_i(R_j)$ is well-defined and nonnegative. Note that $(\sigma(i), M + m_i(R_j))\ R_j\ (\sigma(j), -m_i(R_j))$ always holds. Let $\bar{M}_i = M + \sum_{j \neq i} m_i(R_j)$.

We claim that for all $m \in F^\sigma(e)$, and for all $i \in Q$, $m_i \leqq \bar{M}_i$. To establish the claim, suppose, on the contrary, that for some $i \in Q$, $m_i > \bar{M}_i$. By feasibility, $\sum_{j \in Q} m_j = M$. Hence, $\sum_{j \neq i} m_j = M - m_i < M - \bar{M}_i = -\sum_{j \neq i} m_i(R_j)$. Thus, there is $j \neq i$ such that $m_j < -m_i(R_j)$. It follows that $(\sigma(i), m_i) P_j (\sigma(i), \bar{M}_i) R_j (\sigma(i), M + m_i(R_j)) R_j (\sigma(j), -m_i(R_j)) P_j (\sigma(j), m_j)$, and we have $m \notin F^\sigma(e)$, a contradiction. Hence we also have that for all $i \in Q$, $m_i = M - \sum_{j \neq i} m_j \geqq M - \sum_{j \neq i} \bar{M}_j$.

We have shown that $F^\sigma(e)$ is closed and bounded in $\mathbb{R}^Q$, and hence it is compact. Q.E.D.

Lemma 2 *For all $e = (Q, A, M, R_Q) \in \mathcal{E}$, all $\sigma \in \Sigma^F(e)$, and all $i, j \in Q$ with $i \neq j$, the function $\tilde{p}^\sigma_{ij}$ is continuous on $F^\sigma(e)$.*

Proof: Let $e = (Q, A, M, R_Q) \in \mathcal{E}$, $\sigma \in \Sigma^F(e)$, and $i, j \in Q$ with $i \neq j$ be given. Let $m \in F^\sigma(e)$. We distinguish two cases.

Case 1: $s_{\{i,j\}}(\sigma, m) > 0$.

Let $\{m^n\}$ be a sequence such that $m^n \in F^\sigma(e)$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} m^n = m$.

Let $n \in \mathbb{N}$ be given. Recall that $(\sigma(i), \underline{m}_{ij}(\sigma, m^n)) I_i (\sigma(j), \bar{m}_{ji}(\sigma, m^n))$ and $(\sigma(i), \underline{m}_{ij}(\sigma, m)) I_i (\sigma(j), \bar{m}_{ji}(\sigma, m))$. It follows from assumption (a.1) that the sign of $(\underline{m}_{ij}(\sigma, m^n) - \underline{m}_{ij}(\sigma, m))$ is also the sign of $(\bar{m}_{ji}(\sigma, m^n) - \bar{m}_{ji}(\sigma, m))$. Hence,

$$\begin{aligned}
&|\underline{m}_{ij}(\sigma, m^n) - \underline{m}_{ij}(\sigma, m)| \\
&\leqq |(\underline{m}_{ij}(\sigma, m^n) - \underline{m}_{ij}(\sigma, m)) + (\bar{m}_{ji}(\sigma, m^n) - \bar{m}_{ji}(\sigma, m))| \\
&= |(\underline{m}_{ij}(\sigma, m^n) + \bar{m}_{ji}(\sigma, m^n)) - (\underline{m}_{ij}(\sigma, m) + \bar{m}_{ji}(\sigma, m))| \\
&= |(m_i^n + m_j^n) - (m_i + m_j)|
\end{aligned}$$

From the above inequality, we have $\lim_{n\to\infty} |\underline{m}_{ij}(\sigma, m^n) - \underline{m}_{ij}(\sigma, m)| \leqq \lim_{n\to\infty} |(m_i^n + m_j^n) - (m_i + m_j)| = 0$. Hence, $\lim_{n\to\infty} \underline{m}_{ij}(\sigma, m^n) = \underline{m}_{ij}(\sigma, m)$. By the same argument, $\lim_{n\to\infty} \bar{m}_{ij}(\sigma, m^n) = \bar{m}_{ij}(\sigma, m)$. Therefore, $\lim_{n\to\infty} s_{\{i,j\}}(\sigma, m^n) = s_{\{i,j\}}(\sigma, m)$ and $\lim_{n\to\infty}(m_i^n - \underline{m}_{ij}(\sigma, m^n)) = m_i - \underline{m}_{ij}(\sigma, m)$. Since $s_{\{i,j\}}(z) \neq 0$, we have

$$\lim_{n\to\infty} p_{ij}(\sigma, m^n) = \frac{m_i - \underline{m}_{ij}(\sigma, m)}{s_{\{i,j\}}(\sigma, m)} = p_{ij}(\sigma, m)$$

Because the function $t^\delta : \mathbb{R}_+ \longrightarrow [0, 1]$ is continuous,

$$\begin{aligned}
&\lim_{n\to\infty} \tilde{p}_{ij}^\sigma(m^n) = \lim_{n\to\infty} \tilde{p}_{ij}(\sigma, m^n) \\
&= \frac{1}{2} \lim_{n\to\infty} t^\delta(s_{\{i,j\}}(\sigma, m^n)) + [1 - \lim_{n\to\infty} t^\delta(s_{\{i,j\}}(\sigma, m^n))] \lim_{n\to\infty} p_{ij}(\sigma, m^n) \\
&= \frac{1}{2} t^\delta(s_{\{i,j\}}(\sigma, m)) + [1 - t^\delta(s_{\{i,j\}}(\sigma, m))] p_{ij}(\sigma, m) = \tilde{p}_{ij}^\sigma(m)
\end{aligned}$$

Thus the function $\tilde{p}_{ij}^\sigma$ is continuous at m.

Case 2: $s_{\{i,j\}}(\sigma, m) = 0$.

Let $\{m^n\}$ be a sequence such that $m^n \in F^\sigma(e)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} m^n = m$. As in Case 1, $\lim_{n\to\infty} s_{\{i,j\}}(\sigma, m^n) = s_{\{i,j\}}(\sigma, m) = 0$. Because the function $t^\delta : \mathbb{R}_+ \to [0, 1]$ is continuous, $\lim_{n\to\infty} t^\delta(s_{\{i,j\}}(\sigma, m^n)) = t^\delta(s_{\{i,j\}}(\sigma, m)) = 1$. Since $0 \leqq p_{ij}(\sigma, m^n) \leqq 1$ for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty}[1 - t^\delta(s_{\{i,j\}}(\sigma, m^n))] p_{ij}(\sigma, m^n) = 0$. Hence,

$$\begin{aligned}
\lim_{n\to\infty} \tilde{p}_{ij}^\sigma(m^n) &= \frac{1}{2} \lim_{n\to\infty} t^\delta(s_{\{i,j\}}(\sigma, m^n)) + \lim_{n\to\infty}[1 - t^\delta(s_{\{i,j\}}(\sigma, m^n))] p_{ij}(\sigma, m^n) \\
&= \frac{1}{2} = \tilde{p}_{ij}^\sigma(m)
\end{aligned}$$

Thus the function $\tilde{p}_{ij}^\sigma$ is continuous at m. Q.E.D.

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