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*University of Rochester, Rochester, New York 14627. Apart from some stylistic changes and an updated bibliography, this paper is essentially the text that will appear in the **Handbook of Game Theory**, (Robert Aumann and Sergiu Hart, eds), North-Holland. I am grateful to the National Science Foundation for its support under grant No. SES 8809822, to Hervé Moulin, Tayfun Sönmez, Sang Young Sonn, a referee, and the editors for their advice, and particularly to Youngsub Chun for his numerous and detailed comments. All remaining errors are mine.

Abstract

The purpose of this paper is to survey the literature devoted to the axiomatic analysis of the bargaining problem as formulated by Nash (1950). We cover the "classical" theory, starting from Nash's own work, and modern developments, including studies of the behavior of solutions under changes in population and changes in disagreement point. We briefly indicate applications of the theory to economics and strategic interpretations of the model.

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COOPERATIVE MODELS OF BARGAINING

1 Introduction

The axiomatic theory of bargaining originated in a fundamental paper by J.F. Nash (1950). There, Nash introduced an idealized representation of the **bargaining problem** and developed a methodology that gave the hope that the undeterminateness of the terms of bargaining that had been noted by Edgeworth (1881) could be resolved.

The canonical bargaining problem is that faced by management and labor in the division of a firm's income. Another example concerns the specification of the terms of trade among trading partners.

The formal and abstract model is as follows: Two agents have access to any of the alternatives in some set, called the feasible set. Their preferences over these alternatives differ. If they agree on a particular alternative, that is what they get. Otherwise, they end up at a prespecified alternative in the feasible set, called the disagreement point. Both the feasible set and the disagreement point are given in utility space. Let they be denoted by S and d respectively. Nash's objective was to develop a theory that would help predict the compromise the agents would reach. He specified a natural class of bargaining problems to which he confined his analysis, and defined a **solution** to be a rule that associates with each problem (S, d) in the class a point of S, to be interpreted as this compromise. He formulated a list of properties, or **axioms**, that he thought solutions should satisfy, and established the existence of a unique solution satisfying all the axioms. It is after this first **axiomatic characterization of a solution** that much of the subsequent work has been modelled.

Alternatively, solutions are meant to produce the recommendation that an impartial arbitrator would make. There, the axioms may embody normative objectives of fairness.

Although criticisms were raised from the very beginning against some of the properties Nash used, the solution he identified, now called the Nash solution, was often regarded as **the** solution to the bargaining problem until the mid-seventies. Then, other solutions were introduced and given appealing characterizations, and the theory expanded in several directions. Systematic investigations of the way in which solutions could, or should, depend on the various features of the problems to be solved, were undertaken. For instance, the crucial axiom on which Nash had based his characterization requires that the solution outcome be unaffected by certain contractions of the feasible set, corresponding to the elimination of some of the options initially available. But, is this independence fully justified? Often not. A more detailed analysis of the kinds of transformations to which a problem can be subjected led to the formulation of other conditions. In some cases, it seems quite natural that the compromise be allowed to move in a certain direction, and perhaps be required to move, in response to particular changes in the configuration of the options available.

The other parameters entering in the description of the problem may change too. An improvement in the fallback position of an agent, reflected in an increase in his coordinate of the disagreement point, should probably help him. Is it actually the case for the solutions usually discussed? This improvement, if it does occur, will be at a cost to the other agents. How should this cost be distributed among them?

The feasible set may be subject to uncertainty. Then, how should the agents be affected by it? And, what should the consequences of uncertainty in the disagreement point be? How should solutions respond to changes in the risk attitude of agents? Is it preferable to face an agent who is more, or less, risk-averse?

The set of agents involved in the bargaining may itself vary. Imagine some agents to withdraw their claims. If this affects the set of options available to the remaining agents favorably, it is natural to require that each of them be affected positively. Conversely, when the arrival of additional agents implies a reduction in the options available to the agents initially present, shouldn't they all be negatively affected? And, if some of the agents leave the scene, not empti-handed but with their payoffs, or promise of payoffs, shouldn't the situation, when reevaluated from the viewpoint of the agents left behind, be thought equivalent to the initial situation? If yes, each of them should still be attributed the very same payoffs as before. If not, renegotiations will be necessary that will greatly undermine the significance of any agreement.

What is the connection between the abstract models with which the theory of bargaining deals and more concretely specified economic models on the one hand, and strategic models on the other? How helpful is the theory of bargaining to the understanding of these two classes of problems?

These are a sample of the issues that we will discuss in this review. It would of course be surprising if the various angles from which we will attack the problem all led to the same solution. However, and in spite of the large number of intuitively appealing solutions that have been defined in the literature, only three solutions (and variants) will pass more than a few of the tests that we will formulate. They are Nash's original solution, which selects the point of S at which the product of utility gains from d is maximal, a solution due to Kalai and Smorodinsky (1975), which selects the maximal point of S at which utility gains from d are proportional to the maximal utilities within the set of feasible points dominating d, and the solution that simply equates utility gains from d, the Egalitarian solution. In contexts where interpersonal comparisons of utility would be inappropriate or impossible, the first two would remain the only reasonable candidates. We find this conclusion to be quite remarkable.

Bibliographic note. An earlier survey is Roth (1979c). Partial surveys are Schmitz (1977), Kalai (1985), Thomson (1985a), and Peters (1987a). Thomson and Lensberg (1989) analyze the case of a variable number of agents. Peters (1992) and Thomson (1994) are detailed accounts.

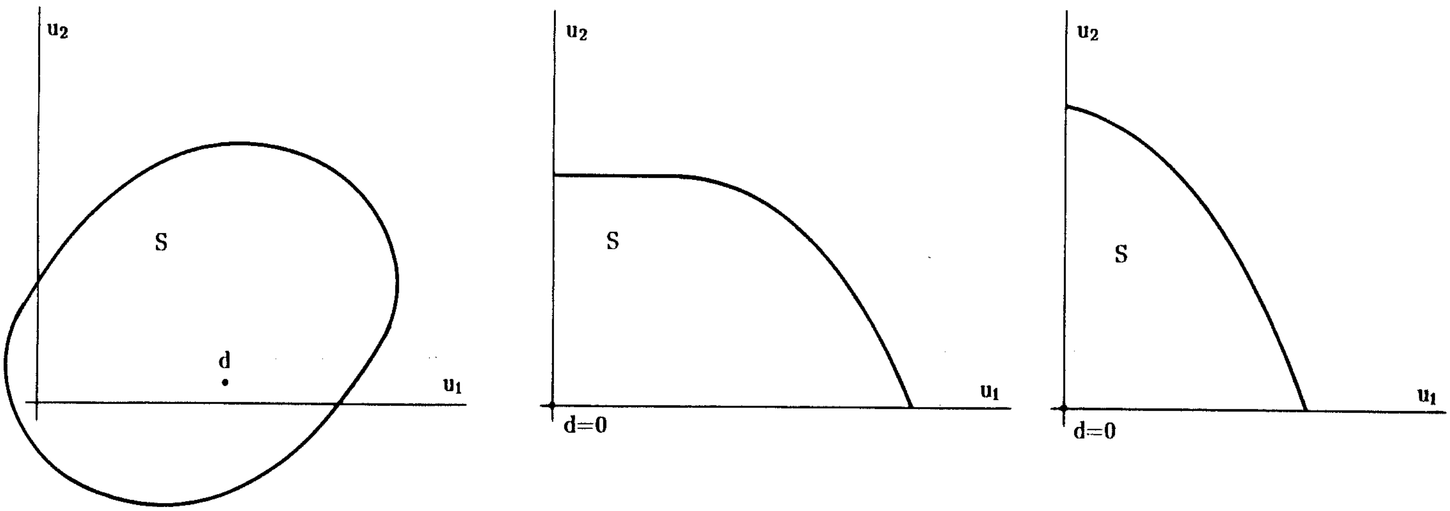
2 Domains. Solutions

A union contract is up for renewal; management and labor have to agree on a division of the firm's income, failure to agree resulting in a strike. This is an example of the sort of conflicts that we will analyze. We will consider the following abstract formulation: An n-person bargaining problem, or simply a **problem**, is a pair (S, d) where S is a subset of the n-dimensional euclidean space, and d is a point of S. Let Σ_d^n be the class of problems such that (Figure 1a):

(i) S is convex, bounded, and closed (it contains its boundary).

(ii) There is at least one point of S strictly dominating d.

Each point of S gives the utility levels, measured in some von Neumann-Morgenstern scales, reached by the agents through the choice of one of the



(a) (b) (c)

Figure 1: Various classes of bargaining problems: (a) An element of Σ^2_d. (b) An element of Σ^2_0. (c) A strictly comprehensive element of Σ^2_0.

alternatives, or randomization among those alternatives, available to them. Convexity of S is due to the possibility of randomization; boundedness holds if utilities are bounded; closedness is assumed for mathematical convenience. The existence of at least one $x \in S$ with $x > d$ is postulated to avoid the somewhat degenerate case when only some of the agents stand to gain from the agreement[1]. In addition, we will usually assume that

(iii) (S, d) is d-**comprehensive**: If $x \in S$ and $x \geq y \geq d$, then $y \in S$.

This property of (S, d) follows from the natural assumption that utility is freely disposable (above d). It is sometimes useful to consider problems satisfying the slightly stronger condition that the part of their boundary that dominates d does not contain a segment parallel to an axis. Along that part of the boundary of such a **strictly** d-**comprehensive** problem, "utility transfers" from one agent to another are always possible. Let $\partial S = \{x \in S | \nexists x' \in S \text{ with } x' > x\}$ be the undominated boundary of S.

In most of the existing theory the choice of the zero of the utility scales is assumed not to matter, and for convenience, we choose scales so that $d = 0$ and ignore d in the notation altogether. However, in some sections, the disagreement point plays a central role; it is then explicitly reintroduced.

[1]Vector inequalities: given x, $x' \in \mathbb{R}^n$, $x \geqq x'$ means $x_i \geqq x'_i$ for all i; $x \geq x'$ means $x \geqq x'$ and $x \neq x'$; $x > x'$ means $x_i > x'_i$ for all i.

When $d = 0$, we simply say that a problem is comprehensive instead of d-comprehensive. Finally, and in addition, we often require that

(iv) $S \subset \mathbb{R}^n_+$.

Indeed, an argument can be made that alternatives at which any agent receives less than what he is guaranteed at $d = 0$ should play no role in the determination of the compromise. (This requirement is formally stated later on.)

In summary, we usually deal with the class Σ^n_0 of problems S as represented in Figure 1b -1c. (the problem of Figure 1c is strictly comprehensive, whereas that of Figure 1b is only comprehensive; since its boundary contains a non-degenerate horizontal segment.) We occasionally consider **degenerate** problems, that is, problems whose feasible set contains no point strictly dominating the disagreement point.

Sometimes, we assume that utility can be disposed of in **any** amount: if $x \in S$, then any $y \in \mathbb{R}^n$ with $y \leqq x$ is also in S. We denote by $\Sigma^n_{d,-}$ and $\Sigma^n_{0,-}$, the classes of such **fully comprehensive problems** corresponding to Σ^n_d and Σ^n_0.

The class of games analyzed here can usefully be distinguished from the class of "games in coalitional form" (in which a feasible set is specified for each group of agents), and from various classes of economic and strategic models (in the former, some economic data are preserved, such as endowments and technology; in the latter, a set of actions available to each agent is specified, each agent being assumed to choose his action so as to bring about the outcome he prefers). In sections 7 and 8, we briefly show how the abstract model relates to economic and strategic models.

A **solution** defined on some domain of problems associates with each member (S, d) of the domain a unique point of S interpreted as a prediction, or a recommendation, for that problem.

Given $A \subset \mathbb{R}^n_+$, $cch\{A\}$ denotes the "convex and comprehensive hull" of A: it is the smallest convex and comprehensive subset of $\mathbb{R}^n_+$ containing A. If $x, y \in \mathbb{R}^n_+$, we write $cch\{x, y\}$ instead of $cch\{\{x, y\}\}$. Finally, $\Delta^{n-1} = \{x \in \mathbb{R}^n_+ \mid \sum x_i = 1\}$ is the $(n-1)$-dimensional unit simplex.

Bibliographic note. Other classes of problems have been discussed in the literature, in particular non-convex problems and problems that are unbounded below. In some studies, no disagreement point is given (Harsanyi 1955, Myerson 1977, Thomson 1981c). In others, an additional reference point is specified; if it is in S, it can be interpreted as a status quo (Brito, Buoncristiani and Intriligator 1977 choose it on the boundary of S), or as a first step towards the final compromise (Gupta and Livne 1988); if it is outside of S, it represents a vector of claims (Chun and Thomson 1988, Bossert 1992a,b, 1993, Herrero 1993, Herrero and Marco Gil 1993, Marco Gil 1994a,b). See also Conley, McLean, and Wilkie (1994), who apply the techniques of bargaining theory to multi-objective programming. Another extension of the model is proposed by Klemisch-Ahlert (1993). Some authors have considered multivalued solutions, (Thomson 1981a, Peters, Tijs and de Koster 1983), and others, probabilistic solutions (Peters and Tijs 1984b, Howe 1994).

Three solutions play a central role in the theory as it appears today. We introduce them first, but we also present several others so as to show how rich and varied the class of available solutions is. Their definitions, as well as the axioms to follow shortly, are stated for an arbitrary $S \in \Sigma_0^n$, [and $(S, d) \in \Sigma_d^n$].

For the best-known solution, introduced by Nash (1950), the compromise is obtained by maximizing the product of utility gains from the disagreement point.

Nash solution, N (Figure 2a) : $N(S)$ is the maximizer of the product $\prod x_i$ over S. [$N(S, d)$ is the maximizer of $\prod(x_i - d_i)$ for $x \in S$ with $x \geqq d$.]

The Kalai-Smorodinsky solution sets utility gains from the disagreement point proportional to the agents' "most optimistic expectations". For each agent, this is defined as the highest utility he can attain in the feasible set subject to the constraint that no agent should receive less than his coordinate of the disagreement point.

Kalai-Smorodinsky solution, K (Figure 2b) : $K(S)$ is the maximal point of S on the segment connecting the origin to $a(S)$, **the ideal point of** S, defined by $a_i(S) \equiv \max\{x_i | x \in S\}$ for all i. [$K(S, d)$ is the maximal point of S on the segment connecting d to $a(S, d)$, where $a_i(S, d) \equiv \max\{x_i | x \in S, x \geqq d\}$ for all i.]

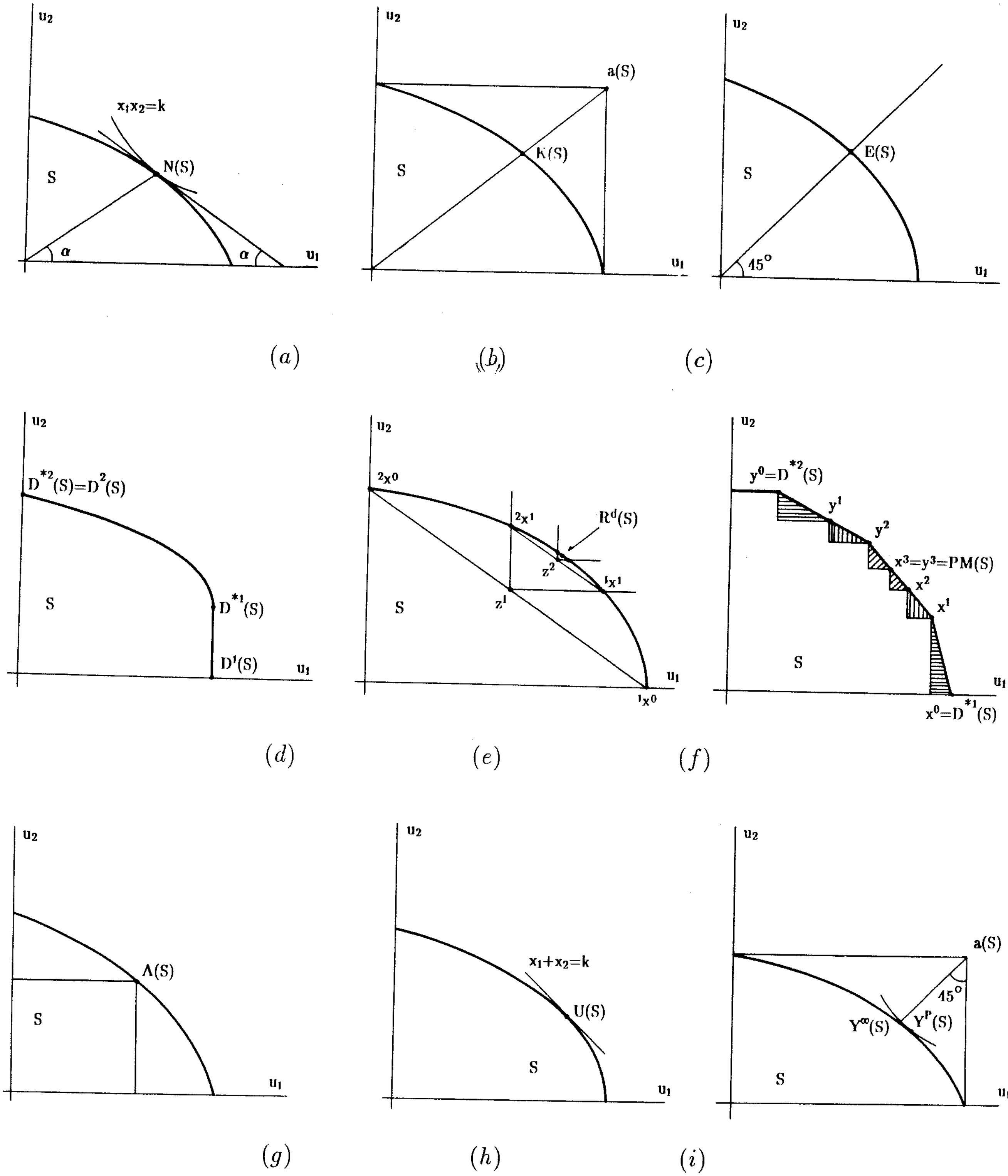


Figure 2: Examples of solutions. (a) The Nash solution. (b) The Kalai-Smorodinsky solution. (c) The Egalitarian solution. (d) The Dictatorial solutions. (e) The discrete Raiffa solution. (f) The Perles-Maschler solution. (g) The Equal Area solution. (h) The Utilitarian solution. (i) The Yu solutions.

The idea of equal gains is central to many theories of social choice and welfare economics. Here, it leads to the following solution:

Egalitarian solution, E (Figure 2c) : $E(S)$ is the maximal point of S of equal coordinates. [$E_i(S,d) - d_i = E_j(S,d) - d_j$ for all i, j.]

The next solutions are extreme cases of solutions favoring one agent at the expense of the others. They occur naturally in the construction of other solutions and sometimes serve as useful indicators of the strength of some proposed list of axioms (just as they do in Arrow-type social choice):

Dictatorial solutions, D^i and D^{*i} (Figure 2d) : $D^i(S)$ is the maximal point x of S with $x_j = 0$ for all $j \neq i$. [Similarly, $D^i_j(S,d) = d_j$ for all $j \neq i$.] If $n = 2$, $D^{*i}(S)$ is the point of $PO(S) \equiv \{x \in S | \nexists x' \in S \text{ with } x' \geq x\}$ whose i^{th}'s coordinate is maximal.

If S is strictly comprehensive, $D^i(S) = D^{*i}(S)$. If $n > 2$, the maximizer of x_i in $PO(S)$ may not be unique, and some rule has to be formulated to break possible ties. A lexicographic rule is often suggested.

The next two solutions are representatives of an interesting family of solutions based on processes of balanced concessions: agents work their way from their preferred alternatives (the dictatorial solution outcomes) to a final position by moving from compromise to compromise:

The (discrete) Raiffa solution, R^d (Figure 2e) : $R^d(S)$ is the limit point of the sequence $\{z^t\}$ defined by: $x^{i0} = D^i(S)$ for all i; for all $t \in \mathbb{N}$, $z^t = (\sum x^{i(t-1)})/n$, and $x^{it} \in WPO(S) \equiv \{x \in S | \nexists x' \in S \text{ with } x' > x\}$ is such that $x^{it}_j = z^t_j$ for all $j \neq i$. [On Σ^n_d, start from the $D^i(S,d)$ instead of the $D^i(S)$.]

A continuous version of the solution is obtained by having $z(t)$ move at time t in the direction of $(\sum x^i(t))/n$ where $x^i(t) \in WPO(S)$ is such that $x^i_j(t) = z_j(t)$ for all $j \neq i$.

The Perles-Maschler solution, PM (Figure 2f) : For $n = 2$. If ∂S is polygonal, $PM(S)$ is the common limit point of the sequences $\{x^t\}$, $\{y^t\}$, defined by: $x^0 = D^{*1}(S)$, $y^0 = D^{*2}(S)$; for each $t \in \mathbb{N}$, x^t, $y^t \in PO(S)$ are such that $x^t_1 \geq y^t_1$, the segments $[x^{t-1}, x^t], [y^{t-1}, y^t]$ are contained in $PO(S)$ and the products $|(x^{t-1}_1 - x^t_1)(x^{t-1}_2 - x^t_2)|$ and $|(y^{t-1}_1 - y^t_1)(y^{t-1}_2 - y^t_2)|$ are equal and maximal[2]. If ∂S is not polygonal, $PM(S)$ is defined by approximating S by a sequence of polygonal problems and taking the limit of the associated solution outcomes. [On Σ_d^n, start from the $D^i(S,d)$ instead of the $D^i(S)$.]

When ∂S is smooth, the solution can be given the following equivalent definition. Consider two points moving along ∂S from $D^{*1}(S)$ and $D^{*2}(S)$ so that the product of the components of their velocity vectors in the u_1 and u_2 directions remain constant: the two points will meet at $PM(S)$. The differential system describing this movement can be generalized to arbitrary n; it generates n paths on the boundary of ∂S that meet in one point that can be taken as the desired compromise.

The next solution exemplifies a family of solutions for which compromises are evaluated globally. Some notion of the sacrifice made by each agent at each proposed alternative is formulated and the compromise is chosen for which these sacrifices are equal. In a finite model, a natural way to measure the sacrifice made by an agent at an alternative would be simply to count the alternatives that the player would have preferred to it. Given the structure of the set of alternatives in the model under investigation, evaluating sacrifices by areas is appealing.

The Equal Area solution, *A* (Figure 2g) : For $n = 2$. $A(S)$ is the point $x \in PO(S)$ such that the area of S to the right of the vertical line through x is equal to the area of S above the horizontal line through x. [On Σ_d^2, ignore points that do not dominate d.][3]

The next solution has played a major role in other contexts. It needs no introduction.

Utilitarian solution, *U* (Figure 2h) : $U(S)$ is a maximizer in $x \in S$ of $\sum x_i$. [$U(S,d)$ is defined as the solution to the same maximization exercise.]

This solution presents some difficulties here. First, the maximizer may not be unique, and to circumvent this difficulty a tie-breaking rule has to be specified; for $n = 2$ it is perhaps most natural to select the midpoint of

[2]Equality of the products implies that the triangles of Figure 2f are matched in pairs of equal areas.

[3]There are several possible generalizations for $n \geqq 3$.

the segment of maximizers (if $n > 2$, this rule can be generalized in several different ways). A second difficulty is that as defined here for Σ_d^n, the solution does not depend on d. A partial remedy is to search for a maximizer of $\sum x_i$ among the points of S that do dominate d. In spite of these limitations, the utilitarian solution is often advocated. In some situations, it has the merit of being a useful limit case of sequences of solutions that are free of the limitations.

Agents cannot simultaneously obtain their preferred outcomes. An intuitively appealing idea is to try to get as close as possible to satisfying everyone, that is, to come as close as possible of what we have called the ideal point. The following one-parameter family of solutions reflects the flexibility that exists in measuring how close two points are from each other.

Yu solutions Y^p (Figure 2i) : Given $p \in]1, \infty[$, $Y^p(S)$ is the point of S for which the p-distance to the ideal point of S, $(\sum |a_i(S) - x_i|^p)^{1/p}$, is minimal. [On Σ_d^n, use $a(S,d)$ instead of $a(S)$.]

Bibliographic note: Versions of the Kalai-Smorodinsky solution appear in Raiffa (1953), Crott (1971), Butrim (1976), and the first axiomatization in Kalai and Smorodinsky (1975). A number of variants have been discussed, in particular by Rosenthal (1976,1978), Kalai and Rosenthal (1978) and Salonen (1985, 1987). The Egalitarian solution cannot be traced to a particular source but egalitarian notions are certainly very old. The Equal Area solution is analyzed in Dekel (1982), Ritz (1985), Anbarcı(1988), Anbarcıand Bigelow (1988), and Calvo (1989); the Yu solutions in Yu (1973) and Freimer and Yu (1976); the Raiffa solution in Raiffa (1953) and Luce and Raiffa (1957). The member of the Yu family obtained for $p = 2$ is advocated by Salukvadze (1971a,b). The extension of the Yu solutions to $p = \infty$ is to maximize $\min\{|a_i(S) - x_i|\}$ in $x \in S$ but this may not yield a unique outcome except for $n = 2$. For the general case, Chun (1988a) proposes, and axiomatizes, the Equal Loss solution, the selection from $Y^\infty(S)$ that picks the maximal point of S such that $a_i(S) - x_i = a_j(S) - x_j$ for all i, j. The solution is further studied by Herrero and Marco (1993). The Utilitarian solution dates back to the mid 19th century. The 2-person Perles-Maschler solution

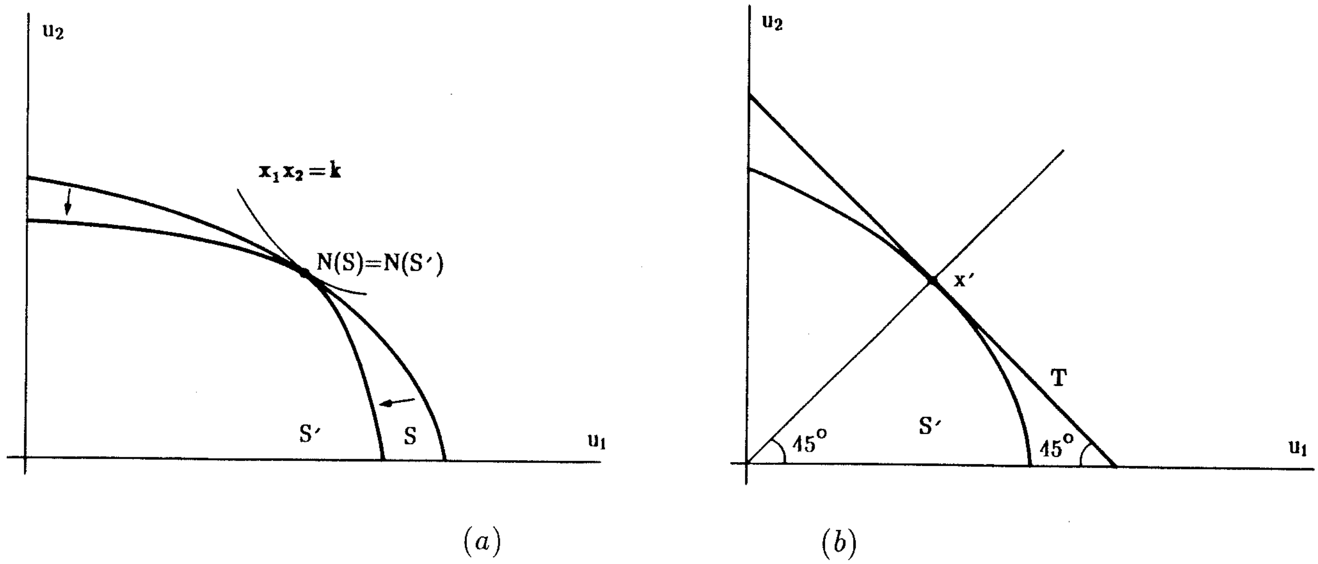


Figure 3: The Nash solution. (*a*) The Nash solution satisfies *contraction independence*. (*b*) Characterization of the Nash solution on the basis of *contraction independence* (Theorem 1).

appears in Perles-Maschler (1981) and its n-person extension in Kohlberg, Maschler and Perles (1983) and Calvo and Gutiérrez (1993).

3 The Main Characterizations

Here we present the classic characterizations of the three solutions that occupy center stage in the theory as it stands today.

3.1 The Nash solution

We start with Nash's fundamental contribution. Nash considered the following axioms, the first one of which is a standard condition: all gains from cooperation should be exhausted.

Pareto-optimality: $F(S) \in PO(S) \equiv \{x \in S | \nexists x' \in S \text{ with } x' \geq x\}$.

The second axiom says that if the agents cannot be differentiated on the basis of the information contained in the mathematical description of S, then the solution should treat them the same.

Symmetry: If S is invariant under all exchanges of agents, $F_i(S) = F_j(S)$ for all i, j.

This axiom applies to problems that are "fully symmetric", that is, are invariant under all permutations of agents. But some problems may only exhibit partial symmetries, which one may want solutions to respect. A more general requirement, which we will also use, is that solutions be independent of the name of agents.

Anonymity: Let $\pi : \{1, ..., n\} \to \{1, ..., n\}$ be a bijection. Given $x \in \mathbb{R}^n$, let $\tilde{\pi}(x) \equiv (x_{\pi(1)}, ..., x_{\pi(n)})$ and $\tilde{\pi}(S) \equiv \{x' \in \mathbb{R}^n | \exists x \in S \text{ with } x' = \tilde{\pi}(x)\}$. Then, $F(\tilde{\pi}(S)) = \tilde{\pi}(F(S))$.

Next, remembering that von-Neumann Morgenstern utilities are unique only up to positive affine transformations, we require that the solution should be independent of which particular members in the families of utility functions representing the agents' preferences are chosen to describe the problem.

Let $\Lambda_0^n : \mathbb{R}^n \to \mathbb{R}^n$ be the class of independent person by person, positive, linear transformations ("scale transformations"): $\lambda \in \Lambda_0^n$ if there is $a \in \mathbb{R}^n_{++}$ such that for all $x \in \mathbb{R}^n$, $\lambda(x) = (a_1x_1, ..., a_nx_n)$. Given $\lambda \in \Lambda_0^n$ and $S \subset \mathbb{R}^n$, $\lambda(S) \equiv \{x' \in \mathbb{R}^n | \exists x \in S \text{ with } x' = \lambda(x)\}$.

Scale invariance: $\lambda(F(S)) = F(\lambda(S))$.

Finally, we require that if an alternative is judged to be the best compromise for some problem, then it should still be judged best for any subproblem that contains it. It can also be thought as a requirement of informational simplicity: A proposed compromise is evaluated only on the basis of information about the shape of the feasible set in a neighborhood of itself.

Contraction independence: [4] If $S' \subseteq S$ and $F(S) \in S'$, then $F(S') = F(S)$.

In the proof of our first result we use the fact that for $n = 2$, if $x = N(S)$, then S has at x a line of support whose slope is the negative of the slope of the line connecting x to the origin (Figure 2a).

Theorem 1 (Nash 1950) *The Nash solution is the only solution on* Σ_0^n *satisfying Pareto-optimality, symmetry, scale invariance, and contraction independence.*

[4] This condition is commonly called "independence of irrelevant alternatives".

Proof: (for $n = 2$) It is easy to verify that N satisfies the four axioms (that N satisfies *contraction independence* is illustrated in Figure 3a). Conversely, let F be a solution on Σ_0^2 satisfying the four axioms. To show that $F = N$, let $S \in \Sigma_0^2$ be given and let $x \equiv N(S)$. Let $\lambda \in \Lambda_0^2$ be such that $x' \equiv \lambda(x)$ be on the 45° line. Such a λ exists since $x > 0$, as is easily checked. Also, the problem $S' \equiv \lambda(S)$ is supported at x' by a line of slope -1 (Figure 3b). Let $T \equiv \{y \in \mathbb{R}_+^2 \mid \sum y_i \leqq \sum x_i'\}$. The problem T is symmetric and $x' \in PO(T)$. By *Pareto-optimality* and *symmetry*, $F(T) = x'$. Clearly, $S' \subseteq T$ and $x' \in S'$, so that by *contraction independence*, $F(S') = x'$. The desired conclusion follows by *scale invariance*. Q.E.D.

No axiom is universally applicable. This is certainly the case of Nash's axioms and each of them has been the object of some criticism. For instance, to the extent that the theory is intended to predict how real-world conflicts are resolved, *Pareto-optimality* is certainly not appropriate, since such conflicts often result in dominated compromises. Likewise, we might want to take into account differences between agents pertaining to aspects of the environment that are not explicitly modelled, and differentiate among them even though they enter symmetrically in the problem at hand; then, we violate *symmetry*. *Scale invariance* prevents basing compromises on interpersonal comparisons of utility, but such comparisons are made in a wide variety of situations. Finally, if the contraction described in the hypotheses of *contraction independence* is "skewed" against a particular agent, why should the compromise be prevented from moving against him? In fact, is seems that solutions should in general be allowed to be responsive to changes in the geometry of S, at least to its main features. It is precisely considerations of this kind that underlie the characterizations of the Kalai-Smorodinsky and Egalitarian solutions reviewed later.

Bibliographic note: Nash's theorem has been considerably refined by subsequent writers. Without *Pareto-optimality*, only one other solution becomes admissible: it is the trivial **Disagreement solution**, which associates with every problem its disagreement point, here the origin (Roth 1977a, 1980). Dropping *symmetry*, we obtain the following family: given $\alpha \in \Delta^{n-1}$, the **Weighted Nash solution with weights** α is defined by maximizing over S the product $\prod x_i^{\alpha_i}$ (Harsanyi and Selten 1972); the Dictatorial solutions and some generalizations (Peters 1986b) also become admissible. Without *scale invariance*, many other solutions, such as the Egalitarian solution, are permitted.

The same is true if *contraction independence* is dropped; however, let us assume that a function is available that summarizes the main features of each problem into a **reference point** to which agents find it natural to compare the proposed compromise in order to evaluate it. By replacing in *contraction independence* the hypothesis of identical disagreement points (implicit in our choice of domains) by the hypothesis of identical reference points, variants of the Nash solution, defined by maximizing the product of utility gains from that reference point, can be obtained under weak assumptions on the reference function (Roth 1977b, Thomson 1981a).

Contraction independence bears a close relation to the axioms of revealed preference of demand theory (Lensberg 1987, Peters and Wakker 1987, Bossert, 1992b). An extension of the Nash solution to the domain of non-convex problems and a characterization appear in Conley and Wilkie (1991b). Non-convex problems are also discussed in Herrero (1989). A characterization without the expected utility hypothesis is due to Rubinstein, Safra and Thomson (1992).

We close this section with the statement of a few interesting properties satisfied by the Nash solution (and by many others as well). The first one is a consequence of our choice of domains: the Nash solution outcome always weakly dominates the disagreement point, here the origin. On Σ_d^n, the property would of course not necessarily be satisfied, so we write it for that domain.

Individual rationality: $F(S,d) \in I(S,d) \equiv \{x \in S | x \geqq d\}$.

In fact, the Nash solution (and again many others) satisfies the following stronger condition: all agents should strictly gain from the compromise. The Dictatorial and Utilitarian solutions do not satisfy the property.

Strong individual rationality: $F(S,d) > d$.

The requirement that the compromise depend only on $I(S, d)$ is implicitly made in much of the literature. If it is strongly believed that no alternative at which one or more of the agents receives less than his disagreement utility should be selected, it seems natural to go further and require of the solution that it be unaffected by the elimination of these alternatives.

Independence of non-individually rational alternatives: $F(S, d) = F(I(S, d), d)$.

Most solutions satisfy this requirement. A solution that does not, although it satisfies *strong individual rationality*, is the Kalai-Rosenthal solution, which picks the maximal point of S on the segment connecting d to the point $b(S)$ defined by $b_i(S) = \max\{x_i | x \in S\}$ for all i.

Another property of interest is that small changes in problems do not lead to wildly different solution outcomes. Small changes in the feasible set, small errors in the way it is described, small errors in the calculation of the utilities achieved by the agents at the feasible alternatives; or conversely, improvements in the description of the alternatives available, or in the measurements of utilities, should not have dramatic effects on payoffs.

Continuity: If $S^\nu \to S$ in the Hausdorff topology, and $d^\nu \to d$, then $F(S^\nu, d^\nu) \to F(S, d)$.

All of the solutions of Section 2 satisfy *continuity*, except for the Dictatorial solutions D^{*i} and the Perles-Maschler and Utilitarian solutions (the tie-breaking rules necessary to obtain single- valuedness of the Utilitarian solutions are responsible for the violations).

> **Bibliographic note:** Other continuity notions are formulated and studied by Jansen and Tijs (1983). A property related to *continuity*, which takes into account closeness of Pareto-optimal boundaries, is used by Peters (1986a) and Livne (1987a). Salonen (1992,1993) studies alternative definitions of *continuity* for unbounded problems.

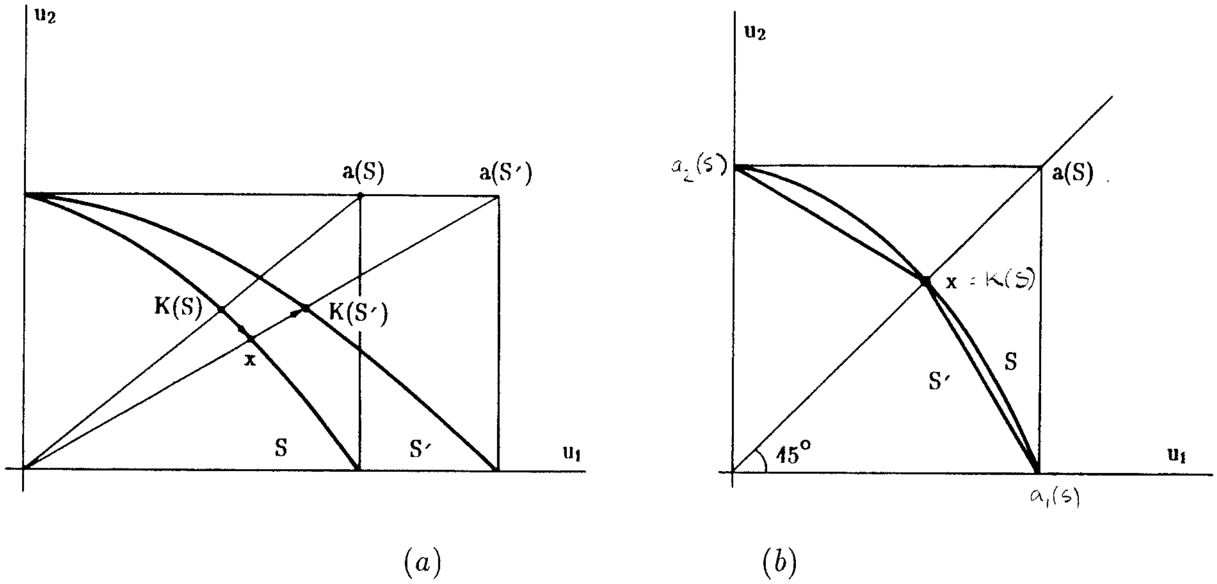


Figure 4: The Kalai-Smorodinsky solution. (*a*) The solution satisfies *individual monotonicity*: the feasible set expands in a direction favorable to agent 1 and he gains as a result. (*b*) Characterization of the solution on the basis of *individual monotonicity* (Theorem 2).

3.2 The Kalai-Smorodinsky solution

We now turn to the second one of our three central solutions, the Kalai-Smorodinsky solution. Just like the Egalitarian solution, examined last, the appeal of this solution lies mainly in its monotonicity properties. Here, we will require that an expansion of the feasible set "in a direction favorable to a particular agent" always benefits him: one way to formalize the notion of an expansion favorable to agent i is to say that the range of utility levels attainable by agent $j (j \neq i)$ remains the same as S expands to S', while for each such level, the maximal utility level attainable by agent i increases. Recall that $a_i(S) \equiv \max\{x_i | x \in S\}$.

Individual monotonicity (for $n = 2$): If $S' \supseteq S$, and $a_j(S') = a_j(S)$ for $j \neq i$, then $F_i(S') \geqq F_i(S)$.

By simply replacing *contraction independence* by *individual monotonicity* in the list of axioms shown earlier to characterize the Nash solution, we obtain a characterization of the Kalai-Smorodinsky solution.

Theorem 2 (Kalai-Smorodinsky 1975) The Kalai-Smorodinsky solution is the only solution on Σ_0^2 satisfying *Pareto-optimality*, *symmetry*, *scale invariance*, and *individual monotonicity*.

Proof: It is clear that K satisfies the four axioms (that K satisfies *individual monotonicity* is illustrated in Figure 4). Conversely, let F be a solution on Σ_0^2 satisfying the four axioms. To see that $F = K$, let $S \in \Sigma_0^2$ be given. By *scale invariance*, we can assume that $a(S)$ has equal coordinates (Figure 4b). This implies that $x \equiv K(S)$ itself has equal coordinates. Then let $S' \equiv cch\{(a_1(S), 0), x, (0, a_2(S))\}$. The problem S' is symmetric and $x \in PO(S')$, so that by *Pareto-optimality* and *symmetry*, $F(S') = x$. By *individual monotonicity* applied twice, we conclude that $F(S) \geqq x$, and since $x \in PO(S)$, that $F(S) = x = K(S)$. Q.E.D.

Before presenting variants of this theorem, we first note several difficulties concerning the possible generalization of the Kalai-Smorodinsky solution itself to classes of not necessarily comprehensive n-person problems for $n > 2$. On such domains the solution often fails to yield Pareto-optimal points, as shown by the example $S = convexhull\{(0,0,0), (1,1,0), (0,1,1)\}$ of Figure 5a: there $K(S)(= (0,0,0))$ is in fact dominated by all points of S (Roth, 1979d). However, by requiring comprehensiveness of the admissible problems, the solution satisfies the following natural weakening of *Pareto-optimality*:

Weak Pareto-optimality: $F(S) \in WPO(S) \equiv \{x \in S | \nexists x' \in S,\ x' > x\}$.

The other difficulty in extending Theorem 2 to $n > 2$ is that there are several ways of generalizing *individual monotonicity* to that case, not all of which permit the result to go through. One possibility is simply to write "for all $j \neq i$" in the earlier statement. Another is to consider expansions that leaves the ideal point unchanged (Figure 5b, Roth 1979d, Thomson 1980). This prevents the skewed expansions that were permitted by *individual monotonicity*. Under such "balanced" expansions, it becomes natural that **all** agents benefit: **restricted monotonicity** says that if $S' \supseteq S$ and $a(S') = a(S)$, then $F(S') \geqq F(S)$.

To emphasize the importance of comprehensiveness, we note that *weak Pareto-optimality*, *symmetry*, and *restricted monotonicity* are incompatible if that assumption is not imposed (Roth 1979d).

Bibliographic note: A lexicographic (see Section 3.3) extension of K that satisfies *Pareto-optimality* has been characterized by Imai (1983). Deleting *Pareto-optimality* from Theorem 2, a large family of solutions becomes admissible and without *symmetry*, the following generalizations are permitted: given

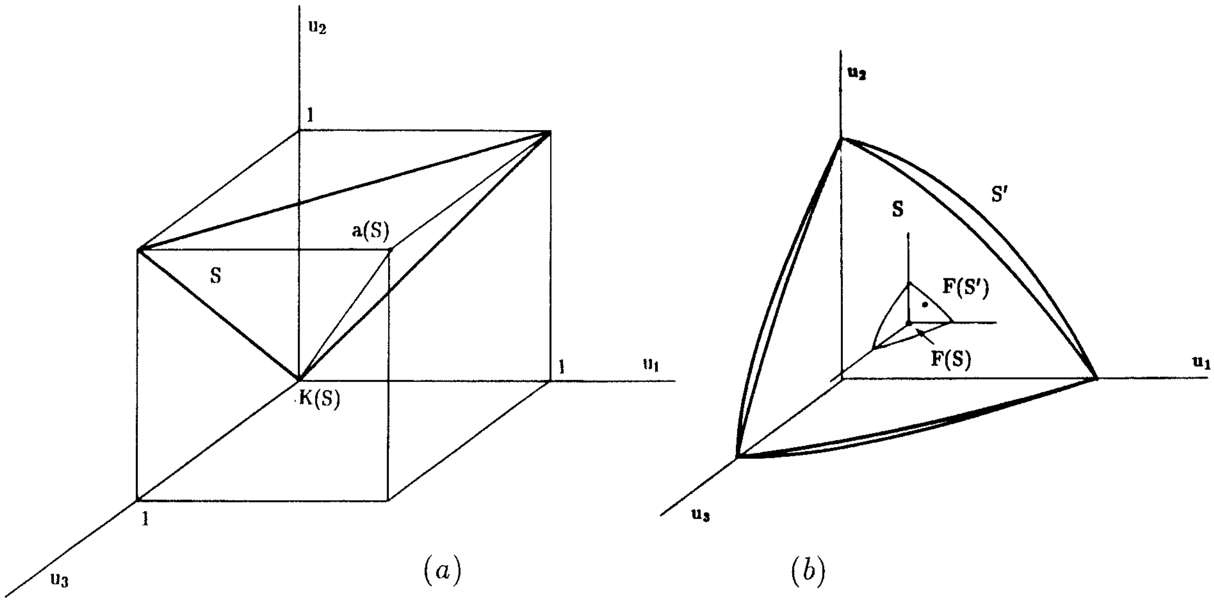


Figure 5: A difficulty with the Kalai-Smorodinsky solution for $n \geqq 2$. (a) If S is not comprehensive, $K(S)$ may be strictly dominated by all points of S. (b) The axiom of *restricted monotonicity*: An expansion of the feasible set leaving unaffected the ideal point benefits all agents.

$\alpha \in \Delta^{n-1}$, the **Weighted Kalai-Smorodinsky solution with weights** α, K^α, selects the point $K^\alpha(S)$ which is the maximal point of S in the direction of the α-weighted ideal point $a^\alpha(S) \equiv (\alpha_1 a_1(S), ..., \alpha_n a_n(S))$. These solutions satisfy *weak Pareto-optimality* (but not *Pareto-optimality* even if $n = 2$). There are other solutions satisfying *weak Pareto-optimality*, *scale invariance*, and *individual monotonicity*; they are normalized versions of the "Monotone Path solutions", discussed below in connection with the Egalitarian solution (Peters and Tijs 1984a; 1985b). Salonen (1985, 1987) characterizes two variants of the Kalai-Smorodinsky solution. These results, as well as the characterization by Kalai and Rosenthal (1987) of their variant of the solution, and the characterization by Chun (1988a) of the Equal Loss solution, are also close in spirit to Theorem 2. Anant, Basu and Mukherji (1990) and Conley and Wilkie (1991) discuss the Kalai-Smorodinsky solution in the context of non-convex problems.

3.3 The Egalitarian solution

The Egalitarian solution performs the best from the viewpoint of monotonicity and the characterization that we offer is based on this fact. The monotonicity condition that we use is that all agents should benefit from

any expansion of opportunities; this is irrespective of whether the expansion may be biased in favor of one of them, (for instance, as described in the hypotheses of *individual monotonicity*). Of course, if that is the case, nothing prevents the solution outcome from "moving more" in favor of that agent. The price paid by requiring this strong monotonicity is that the resulting solution involves interpersonal comparisons of utility (it violates *scale invariance*). Note also that it satisfies *weak Pareto-optimality* only, although $E(S) \in PO(S)$ for all strictly comprehensive S.

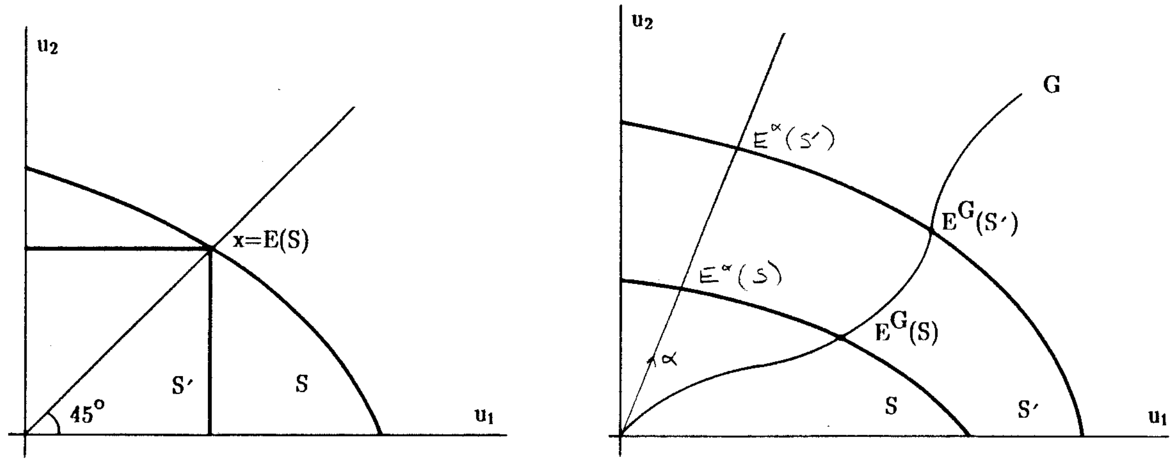
Strong monotonicity: If $S' \supseteq S$, then $F(S') \geqq F(S)$.

Theorem 3 (Kalai 1977) The Egalitarian solution is the only solution on Σ_0^n satisfying *weak Pareto-optimality*, *symmetry*, and *strong monotonicity.*

Proof: (for $n = 2$) Clearly, E satisfies the three axioms. Conversely, to see that if a solution F on Σ_0^2 satisfies the three axioms, then $F = E$, let $S \in \Sigma_0^2$ be given, $x \equiv E(S)$, and $S' \equiv cch\{x\}$ (Figure 6a). By *weak Pareto-optimality* and *symmetry*, $F(S') = x$. Since $S \supseteq S'$, *strong monotonicity* implies $F(S) \geqq x$. Note that $x \in WPO(S)$. If, in fact, $x \in PO(S)$, we are done. Otherwise, we suppose by contradiction that $F(S) \neq E(S)$ and we construct a strictly comprehensive problem S' that includes S, and such that the common value of the coordinates of $E(S')$ is smaller than $\max F_i(S)$. The proof concludes by applying *strong monotonicity* to the pair S, S'. Q.E.D.

It is obvious that comprehensiveness of S is needed to obtain *weak Pareto-optimality* of $E(S)$, even if $n = 2$. Moreover, without comprehensiveness, *weak Pareto-optimality* and *strong monotonicity* are incompatible (Luce and Raiffa 1957).

Bibliographic note: Deleting *weak Pareto-optimality* from Theorem 3, we obtain solutions defined as follows: given $k \in [0, 1]$, $E^k(S) \equiv kE(S)$. However, there are other solutions satisfying *symmetry* and *strong monotonicity* (Roth 1979a, 1979b). Without **symmetry**, the following solutions become admissible. Given $\alpha \in \Delta^{n-1}$, the **Weighted Egalitarian solution with weights** α, E^α, selects the maximal point of S in the direction α (Kalai 1977). *Weak Pareto-optimality* and *strong monotonicity* essentially characterize the following more general class: given a



(a) (b)

Figure 6: Egalitarian and Monotone Path solutions. (*a*) Characterization of the Egalitarian solution on the basis of *strong monotonicity* (Theorem 3). (*b*) The Weighted Egalitarian solution with weights α and the Monotone Path solution relative to the path G.

strictly monotone path G in $\mathbb{R}^n_+$, the **Monotone Path solution relative to** G, E^G, chooses the maximal point of S along G (Figure 6b, Thomson and Myerson 1980). For a derivation of the solution without expected utility, see Valenciano and Zarzuelo (1993).

It is clear that *strong monotonicity* is crucial in Theorem 3 and that without it, a very large class of solutions would become admissible. However, this axiom can be replaced by another interesting axiom (Kalai 1977b) pertaining to situations when opportunities expand over time, say from S to S': the axiom states that $F(S')$ can be indifferently computed in one step, ignoring the initial problem S altogether, or in two steps, by first solving S and then taking $F(S)$ as starting point for the distribution of the gains made possible by the new opportunities.

Decomposability: If $S' \supseteq S$ and $S'' \equiv \{x'' \in \mathbb{R}^n_+ | \exists x' \in S'$ such that $x' = x'' + F(S)\} \in \Sigma^n_0$, then $F(S') = F(S) + F(S'')$.

Bibliographic note: The weakening of *decomposability* obtained by restricting its application to cases where $F(S)$ is proportional to $F(S'')$, when used together with *Pareto-optimality*, *symmetry*, *independence of non-individually rational alternatives*, *scale invariance*, and *continuity* to characterize the Nash solution

(Chun 1988b). For a characterization of the Nash solution based on yet another decomposability axiom, see Ponsati and Watson (1994).

As already noted, the Egalitarian solution does not satisfy *Pareto-optimality*, but there is a natural extension of the solution that does. It is obtained by a lexicographic operation of a sort that is familiar in social choice and game theory. Given $z \in \mathbb{R}^n$, let $\tilde{z} \in \mathbb{R}^n$ denote the vector obtained from z by writing its coordinates in increasing order. Given x, $y \in \mathbb{R}^n$, x **is lexicographically greater than** y if $\tilde{x}_1 > \tilde{y}_1$ or $[\tilde{x}_1 = \tilde{y}_1$ and $\tilde{x}_2 > \tilde{y}_2]$, or, more generally, for some $k \in \{1, \ldots, n-1\}$, $[\tilde{x}_1 = \tilde{y}_1, ..., \tilde{x}_k = \tilde{y}_k$, and $\tilde{x}_{k+1} > \tilde{y}_{k+1}]$. Now, given $S \in \Sigma_0^n$, its **Lexicographic Egalitarian solution** outcome, $E^L(S)$, is the point of S that is lexicographically maximal. It can be reached by the following simple operation (Figure 7): let x^1 be the maximal point of S with equal coordinates (this is $E(S)$); if $x^1 \in PO(S)$, then $x^1 = E^L(S)$; if not, identify the greatest subset of the agents whose utilities can be simultaneously increased from x^1 without hurting the remaining agents. Let x^2 be the maximal point of S at which these agents experience equal gains. Repeat this operation from x^2 to obtain x^3, etc.,..., until a point of $PO(S)$ is obtained.

This algorithm produces a well-defined solution satisfying *Pareto-optimality*, even on the class of problems that are not necessarily comprehensive. Indeed, given a problem in that class, apply it to its comprehensive hull and note that taking the comprehensive hull of a problem does not affect its set of Pareto-optimal points. Problems in Σ_d^n can of course be easily accommodated. A version of the Kalai-Smorodinsky solution that satisfies *Pareto-optimality* on Σ_d^n for all n can be defined in a similar way.

Bibliographic note: For characterizations of E^L based on monotonicity considerations, see Imai (1983) and Chun and Peters (1988). Lexicographic extensions of the Monotone Path solutions are defined, and characterized by similar techniques for $n = 2$, by Chun and Peters (1989a). For parallel extensions and characterizations thereof, of the Equal Loss solution, see Chun and Peters (1991).

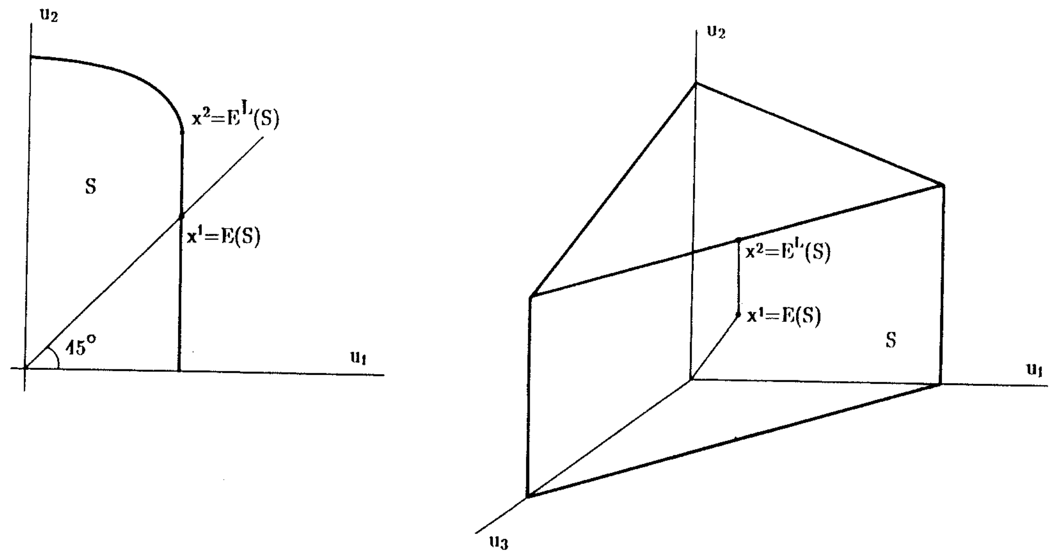


Figure 7: The Lexicographic Egalitarian solution for two examples. In each case, the solution outcome is reached in two steps. (*a*) A two-person example. (*b*) A three-person example.

4 Other Properties. The Role of the Feasible Set

Here, we change our focus, concentrating on properties of solutions. For many of them, we are far from fully understanding their implications, but taken together they constitute an extensive battery of tests to which solutions can be usefully subjected when they have to be evaluated.

4.1 Midpoint domination

A minimal amount of cooperation among the agents should allow them to do at least as well as the average of their preferred positions. This average corresponds to the often observed tossing of the coin to determine which one of two agents will be given the choice of his preferred alternative when no easy agreement on a deterministic outcome is obtained. Accordingly, consider the following two requirements (Sobel 1981, Salonen 1985, respectively), which correspond to two natural definitions of "preferred positions".

Midpoint domination: $F(S) \geqq [\sum D^i(S)]/n$.

Strong midpoint domination: $F(S) \geqq [\sum D^{*i}(S)]/n$.

Many solutions satisfy *midpoint domination*. Notable exceptions are the Egalitarian and Utilitarian solutions (Of course, this should not be a surprise since the point that is to be dominated is defined in a scale independent way); yet we have (compare with Theorem 1):

Theorem 4 (Moulin 1983) The Nash solution is the only solution on Σ_0^n satisfying *midpoint domination* and *contraction independence.*

Few solutions satisfy *strong midpoint domination* (the Perles-Maschler solution does however; Salonen 1985 defines a version of the Kalai-Smorodinsky solution that does too).

4.2 Invariance

The theory presented so far is a cardinal theory, in that it depends on utility functions, but the extent of this dependence varies, as we have seen. Are there solutions that are invariant under **all** monotone increasing, and independent agent by agent, transformations of utilities, i.e. solutions that depend only on ordinal preferences? The answer depends on the number of agents. Perhaps surprisingly, it is negative for $n = 2$ but not for all $n > 2$.

Let $\tilde{\Lambda}_0^n$ be the class of these transformations: $\lambda \in \tilde{\Lambda}_0^n$ if for each i, there is a continuous and monotone increasing function $\lambda_i : \mathbb{R} \to \mathbb{R}$ such that given $x \in \mathbb{R}^n$, $\lambda(x) = (\lambda_1(x_1), ..., \lambda_n(x_n))$. Since convexity of S is not preserved under transformations in $\tilde{\Lambda}_0^n$, it is natural to turn our attention to the domain $\tilde{\Sigma}_0^n$ obtained from Σ_0^n by dropping this requirement.

Ordinal invariance: For all $\lambda \in \tilde{\Lambda}_0^n$, $F(\lambda(S)) = \lambda(F(S))$.

Theorem 5 (Shapley 1969; Roth 1979) There is no solution on $\tilde{\Sigma}_0^2$ satisfying *strong individual rationality* and *ordinal invariance.*

Proof: Let F be a solution on Σ_0^2 satisfying *ordinal invariance* and let S and S' be as in Figure 8a. Let λ_1 and λ_2 be the two transformations from [0,1] to [0,1] defined by following the horizontal and vertical arrows of Figure 8a respectively. (The graph of λ_2 is given in Figure 8b; for instance, $\lambda_2(x_2)$, the image of x_2 under λ_2, is obtained by following the arrows from Figure 8a to Figure 8b). Note that the problem S is **globally** invariant under the

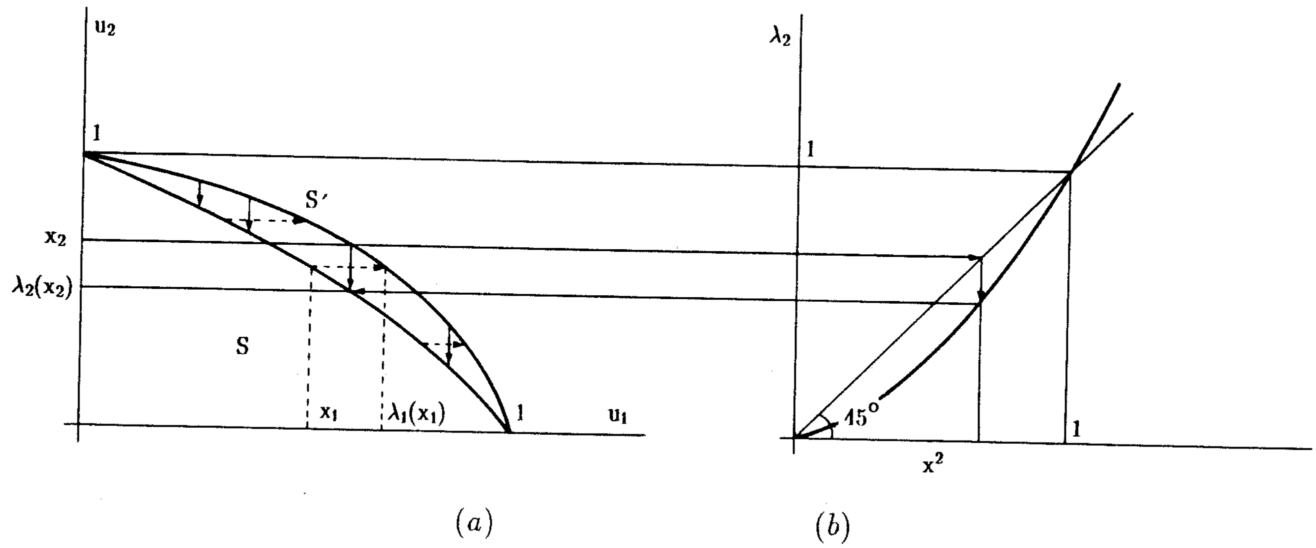


Figure 8: ***Strong individual rationality*** **and** ***ordinal invariance*** **are incompatible on** Σ_0^2 (Theorem 5) (*a*) S is globally invariant under the composition of the two transformations defined by the horizontal and the vertical arrows respectively. (*b*) An explicit construction of the transformation to which agent 2's utility is subjected.

transformation $\lambda \equiv (\lambda_1, \lambda_2) \in \tilde{\Sigma}_0^2$, with only three fixed points, the origin and the endpoints of $PO(S)$. Since none of these points is positive, F does not satisfy *strong individual rationality*. Q.E.D.

Theorem 6 (Shapley 1984; Shubik 1982) There are solutions on the subclass of $\tilde{\Sigma}_0^3$ of strictly comprehensive problems satisfying *Pareto-optimality* and *ordinal invariance*.

Proof: Given $S \in \tilde{\Sigma}_0^3$, let $F(S)$ be the limit point of the sequence $\{x^t\}$ where x^1 is the point of intersection of $PO(S)$ with $\mathbb{R}^{\{1,2\}}$ such that the arrows of Figure 9a lead back to x^1; x^2 is the point of $PO(S)$ such that $x_2^2 = x_2^1$ and a similarly defined sequence of arrows leads back to x^2; this operation being repeated forever (Figure 9b). The solution F satisfies *ordinal invariance* since at each step, only operations that are invariant under ordinal transformations are performed. Q.E.D.

Bibliographic note: There are other solutions satisfying these properties and yet other such solutions on the class of smooth problems (Shapley 1984).

In light of the negative result of Theorem 5, it is natural to look for a weaker invariance condition. Instead of allowing the utility transformations

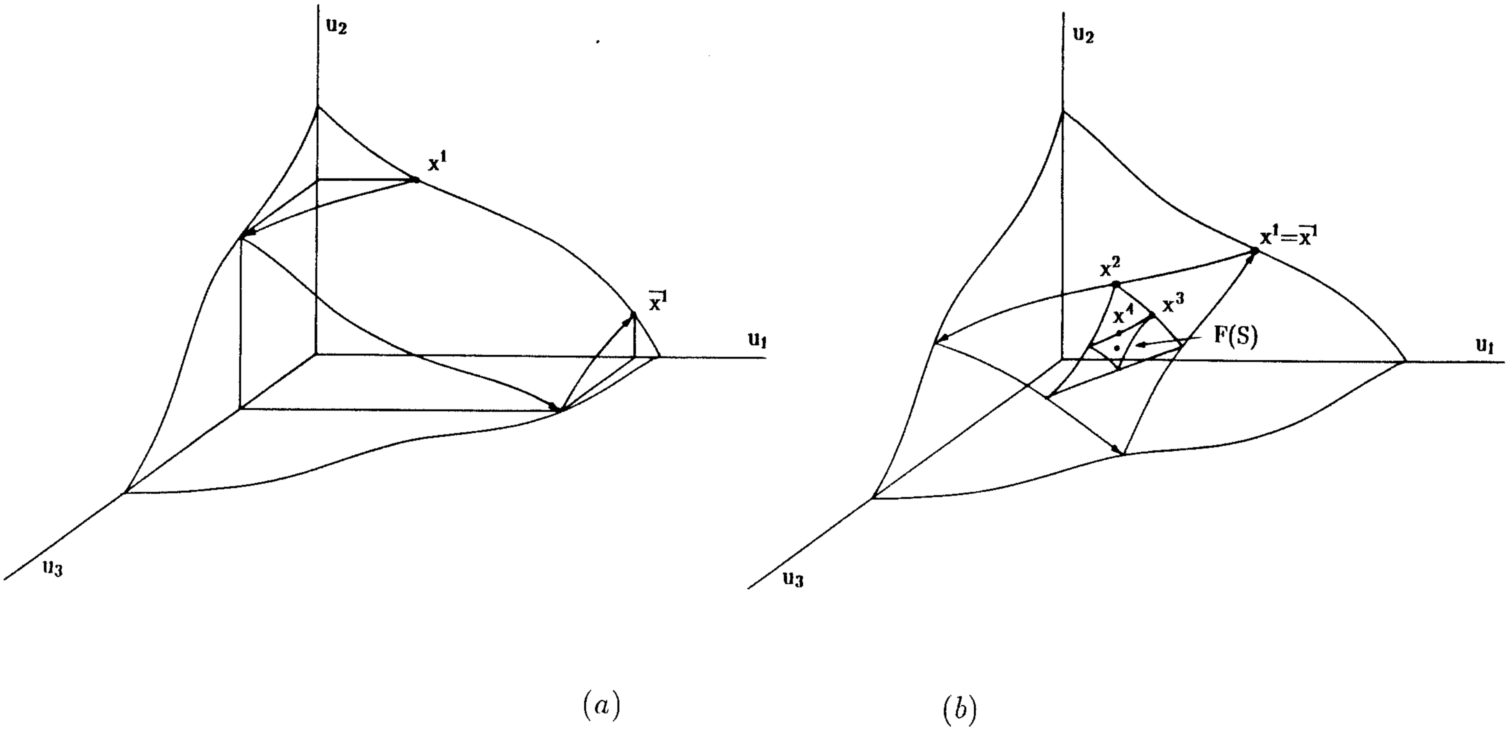


Figure 9: A solution on Σ_0^3 satisfying *Pareto-optimality, strong individual rationality,* and *ordinal invariance.* (a) The fixed point argument defining x^1; when x^1 moves down the intersection of ∂S with the coordinate subspace pertaining to agents 1 and 2, $\bar{x}^1$ moves up. (b) The solution outcome of S is the limit point of the sequence $\{x^t\}$.

to be independent across agents, require now that they be the same for all agents:

Weak ordinal invariance: For all $\lambda \in \tilde{\Lambda}_0^n$ such that $\lambda_i = \lambda_j$ for all i, j, $F(\lambda(S)) = \lambda(F(S))$.

This is a significantly weaker requirement than *ordinal invariance*. Indeed, it can be met for $n = 2$ and we have:

Theorem 7 (Roth 1979c; Nielsen 1983) The Lexicographic Egalitarian solution is the only solution on the subclass of $\tilde{\Sigma}_0^2$ of problems whose Pareto-optimal boundary is a connected set, to satisfy *weak Pareto-optimality*, *symmetry*, *contraction independence*, and *weak ordinal invariance*.

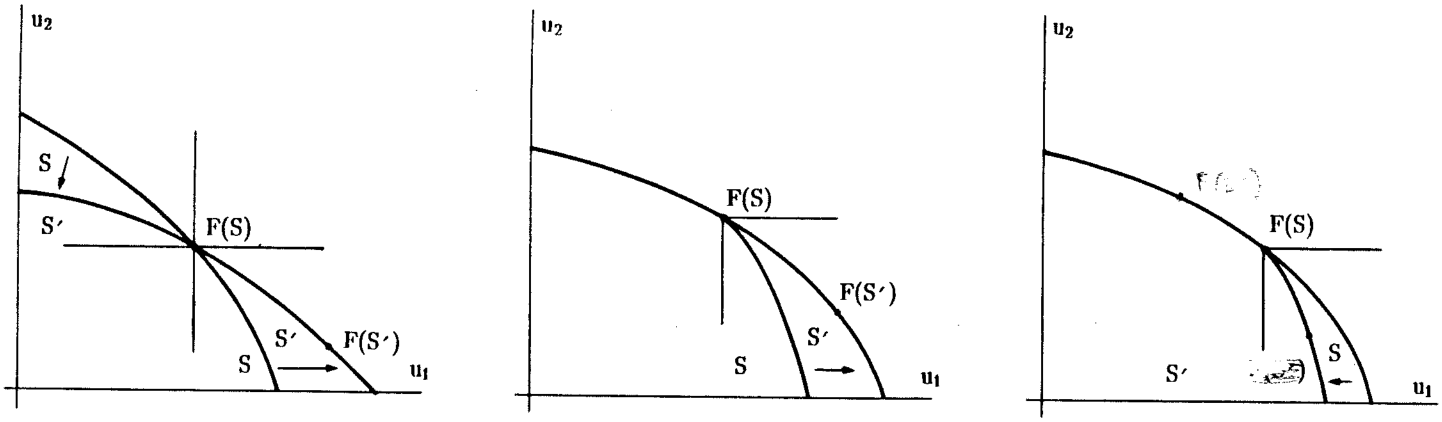
4.3 Independence and monotonicity

Here we formulate a variety of conditions describing how solutions should respond to changes in the geometry of S. An important motivation for the search for alternatives to the monotonicity conditions used in the previous pages is that these conditions pertain to transformations that are not defined with respect to the compromise initially chosen.

One of the most important conditions we have seen is *contraction independence*. A significantly weaker condition which applies only when the solution outcome of the initial problem is the only Pareto-optimal point of the final problem is:

Weak contraction independence: If $S' = cch\{F(S)\}$, then $F(S) = F(S')$.

Dual conditions to *contraction independence* and *weak contraction independence*, requiring invariance of the solution outcome under expansions of S, provided it remains on the boundary, have also been considered. Useful variants of these conditions are obtained by restricting their application to smooth problems. The Nash and utilitarian solutions can be characterized with the help of these conditions (Thomson 1981b,c). The smoothness restriction means that utility transfers are possible at the same rate in both directions along the boundary of S. Suppose S is not smooth at $F(S)$. Then, one could not eliminate the possibility that an agent who had been willing to concede along ∂S up to $F(S)$ might have been willing to concede further



(a) (b) (c)

Figure 10: Three monotonicity conditions. (*a*) Twisting. (*b*) Adding. (*c*) Cutting.

if the same rate at which utility could be transferred from him to the other agents had been available. It is then natural to think of the compromise $F(S)$ as somewhat artificial, and to exclude such situations from the range of applicability of the axiom.

> **Bibliographic note:** A number of other conditions that explicitly exclude kinks or corners have been formulated (Chun and Peters 1988, 1989a; Peters 1986a; Chun and Thomson 1990c). For a characterization of the Nash solution based on yet another expansion axiom, see Anbarcı(1991).

A difficulty with the two monotonicity properties used earlier, *individual monotonicity* and *strong monotonicity*, as well as with the independence conditions, is that they preclude the solution from being sensitive to certain changes in S that intuitively seem quite relevant. What would be desirable are conditions pertaining to changes in S that are defined **relative to the compromise initially established**. Consider the next conditions (Thomson and Myerson 1980), written for $n = 2$, which involve "twisting" the boundary of a problem around its solution outcome, only "adding", or only "subtracting", alternatives on one side of the solution outcome (Figure 10)

Twisting: If $x \in S' \backslash S$ implies $[x_i \geqq F_i(S)$ and $x_j \leqq F_j(S)]$ and $x \in S \backslash S'$ implies $[x_i \leqq F_i(S)$ and $x_j \geqq F_j(S)]$, then $F_i(S') \geqq F_i(S)$.

Adding: If $S' \supset S$, and $x \in S' \backslash S$ implies $[x_i \geqq F_i(S)$ and $x_j \leqq F_j(S)]$, then $F_i(S') \geqq F_i(S)$.

Cutting: If $S' \subset S$, and $x \in S \backslash S'$ implies $[x_i \geqq F_i(S)$ and $x_j \leqq F_j(S)]$, then $F_i(S') \leqq F_i(S)$.

The main solutions satisfy *twisting*, which is easily seen to imply *adding* and *cutting*. However, the Perles-Maschler solution does not even satisfy *adding*. *Twisting* is crucial to understanding the responsiveness of solutions to changes in agents' risk aversion (Section 5.3).

Finally, we have the following strong axiom of solidarity. Independently of how the feasible set changes, all agents gain together or all agents lose together. No assumptions are made on the way S relates to S'.

Domination: Either $F(S') \geqq F(S)$ or $F(S) \geqq F(S')$.

A number of interesting relations exist between all of these conditions. In light of *weak Pareto-optimality* and *continuity*, *domination* and *strong monotonicity* are equivalent (Thomson and Myerson 1980) and so are *adding* and *cutting* (Livne 1986a). *Contraction independence* implies *twisting* and so do *Pareto-optimality* and *individual monotonicity* together (Thomson and Myerson 1980). Many solutions (Nash, Equal Area) satisfy *Pareto-optimality* and *twisting* but not *individual monotonicity*. Finally, *weak Pareto-optimality*, *symmetry*, *scale invariance* and *twisting* together imply *midpoint domination* (Livne 1985a).

The axioms *twisting*, *individual monotonicity*, *adding* and *cutting* can be extended to the n-person case in a number of different ways.

4.4 Uncertain feasible set

Suppose that bargaining takes place today but that the feasible set will be known only tomorrow: It may be S^1 or S^2 with equal probabilities. Let F be a candidate solution. Then the vector of expected utilities today from waiting until the uncertainty is resolved is $x^1 \equiv [F(S^1) + F(S^2)]/2$ whereas solving the "expected problem" $(S^1 + S^2)/2$ produces $F[(S^1 + S^2)/2]$. Since x^1 is in general not Pareto-optimal in $(S^1 + S^2)/2$, it would be preferable for the agents to reach a compromise today. A necessary condition for this is that both benefit from early agreement. Let us then require of F that it gives all agents the incentive to solve the problem today: x^1 should dominate $F[(S^1+S^2)/2]$. Slightly more generally, and to accommodate situations when S^1 and S^2 occur with unequal probabilities, we formulate:

Concavity: For all $\lambda \in [0, 1]$, $F(\lambda S^1 + (1-\lambda)S^2) \geqq \lambda F(S^1) + (1-\lambda)F(S^2)$.

Alternatively, we could imagine that the feasible set is the result of the addition of two component problems and require that both agents benefit from looking at the situation globally, instead of solving each of the two problems separately and adding up the resulting payoffs.

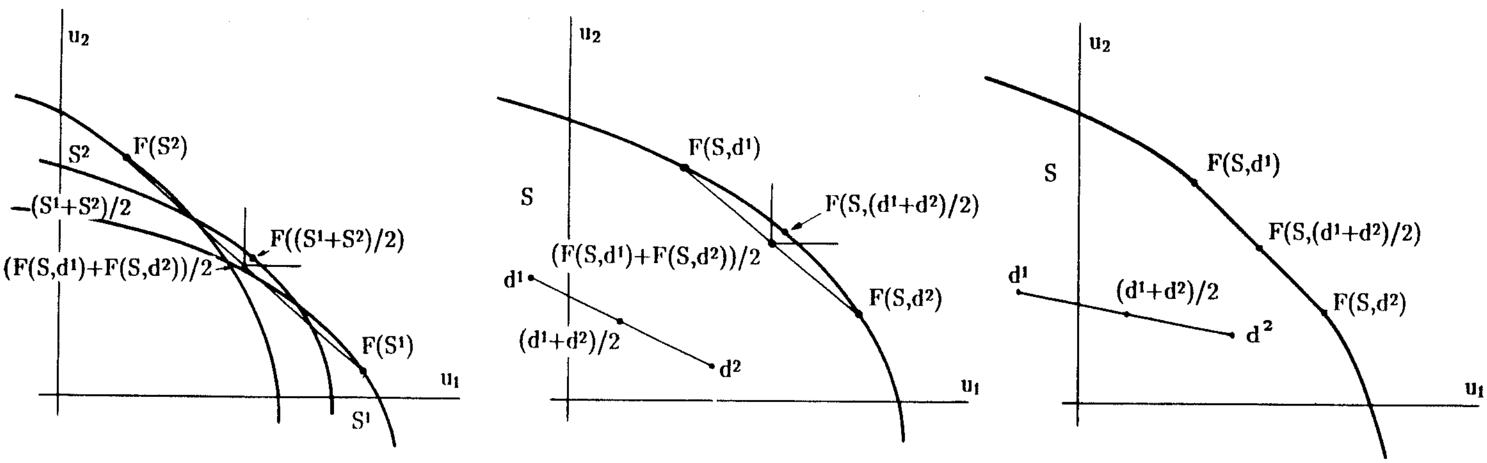
Super-additivity: $F(S^1 + S^2) \geqq F(S^1) + F(S^2)$.

Neither the Nash nor Kalai-Smorodinsky solution satisfies these conditions, but the Egalitarian solution does. Are the conditions compatible with *scale invariance*? Yes. However, only one *scale invariant* solution satisfies them together with a few other standard requirements. Let Γ_0^2 designate the class of problems satisfying all the properties required of the elements of Σ_0^2, but violating the requirement that there exists $x \in S$ with $x > 0$.

Theorem 8 (Perles and Maschler 1981) The Perles-Maschler solution is the only solution on $\Sigma_0^2 \cup \Gamma_0^2$ satisfying *Pareto-optimality*, *symmetry*, *scale invariance*, *super-additivity*, and to be *continuous* on the subclass of Σ_0^2 of strictly comprehensive problems.

> **Bibliographic note:** Deleting *Pareto-optimality* from Theorem 8, the solutions PM^λ defined by $PM^\lambda(S) \equiv \lambda PM(S)$ for $\lambda \in [0, 1]$ become admissible. Without *symmetry*, we obtain a two-parameter family (Maschler and Perles 1981). *Continuity* is indispensable (Maschler and Perles 1981) and so are *scale invariance* (consider E) and obviously *super-additivity*. Theorem 8 does not extend to $n > 2$: In fact, *Pareto-optimality*, *symmetry*, *scale invariance*, and *super-additivity* are incompatible on Σ_0^3 (Perles 1982).

Deleting *scale invariance* from Theorem 8 is particular interesting: then, a joint characterization of the Egalitarian and Utilitarian solutions on $\Sigma_{0,-}^n$ can be obtained (note the change of domains). In fact, *super-additivity* can then be replaced by the following strong condition, which says that agents are **indifferent** between solving problems separately or consolidating them into a single problem and solving that problem.



(a) *(b)* *(c)*

Figure 11: Concavity conditions. *(a)* ***(Feasible set) concavity***: the solution outcome of the average problem $\frac{S^1+S^2}{2}$ dominates the average $\frac{F(S^1)+F(S^2)}{2}$ of the solution outcomes of the two component problems S^1 and S^2. *(b)* ***Disagreement point concavity***: the solution outcome of the average problem $(S, \frac{d^1+d^2}{2})$ dominates the average $\frac{F(S,d^1)+F(S,d^2)}{2}$ of the solution outcomes of the two component problems (S, d^1) and (S, d^2). *(b)* *(c)* ***Weak disagreement point concavity***: this is the weakening of *disagreement point concavity* obtained by limiting its application to situations where the boundary of S is linear between the solution outcomes of the two components problems, and smooth at these two points.

Linearity: $F(S^1+S^2)=F(S^1)+F(S^2)$.

Theorem 9 (Myerson 1981) The Egalitarian and Utilitarian solutions are the only solutions on $\Sigma^n_{0,-}$ satisfying *weak Pareto-optimality*, *symmetry*, *contraction independence*, and *concavity*. The Utilitarian solutions are the only solutions on $\Sigma^n_{0,-}$ satisfying *Pareto-optimality*, *symmetry*, and *linearity* (In each of these statements the Utilitarian solutions are covered if appropriate tie-breaking rules are applied.)

On the domain $\Sigma^2_{0,-}$, the following weakening of *linearity* (and *superadditivity*) is compatible with *scale invariance*. It involves a smoothness restriction whose significance has been discussed earlier (Section 4.3).

Weak linearity: If $F(S^1)+F(S^2)\in PO(S^1+S^2)$ and ∂S^1 and ∂S^2 are smooth at $F(S^1)$ and $F(S^2)$ respectively, then $F(S^1+S^2)=F(S^1)+F(S^2)$.

Theorem 10 (Peters 1986a) The Weighted Nash solutions are the only solutions on $\Sigma^2_{d,-}$ satisfying *Pareto-optimality*, *strong individual rationality*, *scale invariance*, *continuity*, and *weak linearity*.

> **Bibliographic note:** The Nash solution can be characterized by an alternative weakening of *linearity* (Chun 1988b). Randomization between all the points of S and its ideal point, and all the points of S and its solution outcome have been considered by Livne (1988, 1989a,b) and used by him to formulate invariance conditions that can be used to characterize the Kalai-Smorodinsky and continuous Raiffa solutions.

To complete this section, we note that instead of considering the "addition" of two problems we could consider their "multiplication," and require the invariance of solutions under this operation. The resulting requirement leads to a characterization of the Nash solution.

Given $x,\ y\in\mathbb{R}^2_+$, let $x*y\equiv(x_1y_1,x_2y_2)$; given $S,\ T\in\Sigma^2_0$, let $S*T\equiv\{z\in\mathbb{R}^2_+|z=x*y$ for some $x\in S$ and $y\in T\}$. The domain Σ^2_0 is not closed under the $*$-operation, which explains the form of the condition stated next.

Separability: If $S*T\in\Sigma^2_0$, then $F(S*T)=F(S)*F(T)$.

Theorem 11 (Binmore 1984) The Nash solution is the only solution on Σ^2_0 satisfying *Pareto-optimality*, *symmetry*, and *separability*.

5 Other Properties. The Role of the Disagreement Point

In our exposition so far, we have ignored the disagreement point altogether. Here, we study its role in detail, and, for that purpose, we reintroduce it in the notation: a bargaining problem is now a pair (S, d) as originally specified in Section 2. We consider first increases in one of the coordinates of the disagreement point; then, situations when it is uncertain. In each case, we study how responsive solutions are to these changes. The solutions that will play the main role here are the Nash and Egalitarian solutions, and generalizations of the Egalitarian solution. We also study how solutions respond to changes in the agents' attitude toward risk.

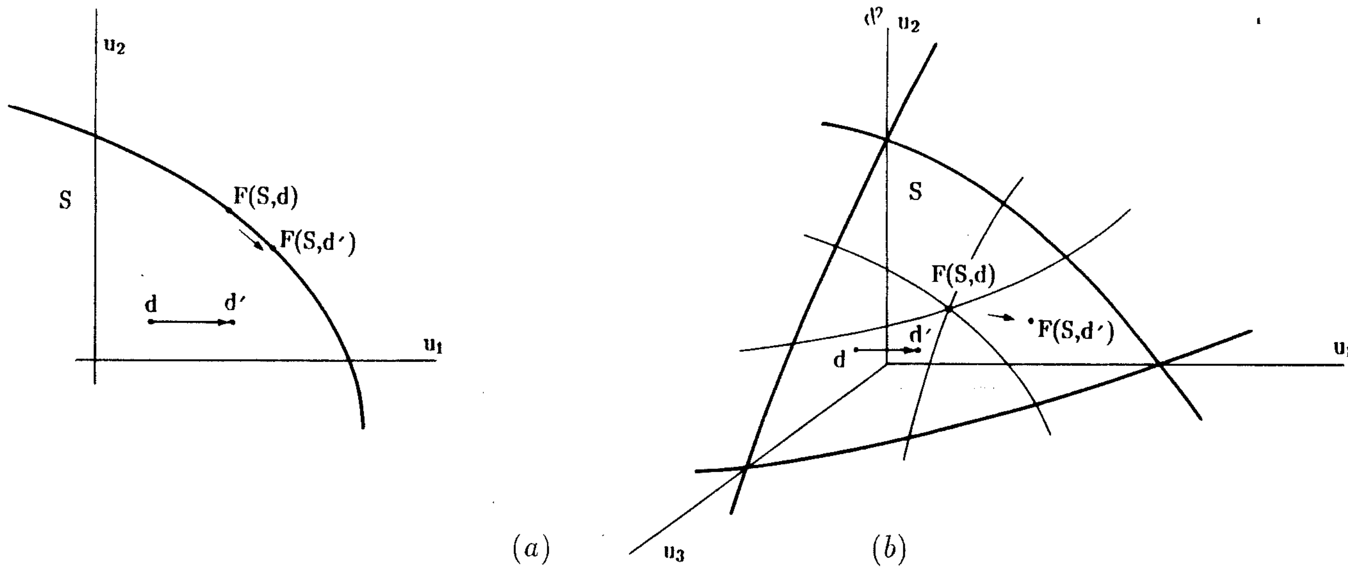
5.1 Disagreement point monotonicity

We first formulate monotonicity properties of solutions with respect to changes in d (Thomson 1987a). To that end, fix S. If agent i's fallback position improves while the fallback position of the others do not change, it is natural to expect that he will (weakly) gain. An agent who has less to lose from failure to reach an agreement should be in a better position to make demands.

Disagreement point monotonicity: If $d'_i \geqq d_i$ and for all $j \neq i$, $d'_j = d_j$, then $F_i(S, d') \geqq F_i(S, d)$.

This property is satisfied by all of the solutions that we have encountered. Even the Perles-Maschler solution, which is very poorly behaved with respect to changes in the feasible set, as we saw earlier, satisfies this requirement.

A stronger condition, which is of greatest relevance for solutions that are intended as normative prescriptions, is that under the same hypotheses as *disagreement point monotonicity*, not only $F_i(S, d') \geqq F_i(S, d)$ but in addition for all $j \neq i$, $F_j(S, d') \leqq F_j(S, d)$. The gain achieved by agent i should be at the expense (in the weak sense) of all the other agents (Figure 12b). For a solution that selects Pareto-optimal compromises, the gain to agent i has to be accompanied by a loss to at least one agent $j \neq i$. One could argue that an improvement in some agent k's payoff would unjustifiably further increase the negative impact of the change in d_i on all agents j, $j \notin \{i, k\}$.



(*a*) (*b*)

Figure 12: Conditions of monotonicity with respect to the disagreement point. (*a*) *Weak disagreement point monotonicity* for $n = 2$: an increase in the first coordinate of the disagreement point benefits agent 1. (*b*) *Strong disagreement point monotonicity* for $n = 3$: an increase in the first coordinate of the disagreement point benefits agent 1 and at the expense of both other agents.

(Of course, this is a property that is interesting only if $n \geqq 3$.) Most solutions, in particular the Nash and Kalai-Smorodinsky solutions and their variants, violate it. However, the Egalitarian solution does satisfy the property and so do the Monotone Path solutions.

5.2 Uncertain disagreement point

Next, we imagine that there is uncertainty about the disagreement point. Recall that earlier we considered uncertainty in the feasible set but in practice, the disagreement point may be subject to uncertainty just as well. Suppose, to illustrate, that the disagreement point will take one of two positions d^1 and d^2 with equal probabilities, and that this uncertainty will be resolved tomorrow. Waiting until tomorrow and solving then whatever problem has come up results in the expected payoff vector today $x^1 \equiv [F(S, d^1) + F(S, d^2)]/2$, which is typically Pareto dominated in S. Taking as new disagreement point the expected cost of conflict and solving the problem $(S, (d^1 + d^2)/2)$ results in the payoffs $x^2 \equiv F(S, (d^1+d^2)/2)$. If $x^1 \leqq x^2$, the agents will agree to solve the problem today. If neither x_1 dominates x_2 nor x_2 dominates x_1, their incentives to wait will be conflicting. The following requirement prevents this:

Disagreement point concavity: For all $\lambda \in [0,1]$, $F(S, \lambda d^1 + (1-\lambda)d^2) \geqq \lambda F(S, d^1) + (1-\lambda)F(S, d^2)$.

Of all the solutions seen so far, only the Weighted Egalitarian solutions satisfy this requirement. It is indeed a very strong requirement as indicated by the next result which is a characterization of a family of solutions that further generalize the Egalitarian solution: given δ, a continuous function from the class of n-person fully comprehensive feasible sets into Δ^{n-1}, and a problem $(S,d) \in \Sigma^n_{d,-}$, the **Directional solution relative to** δ, E^δ selects the point $E^\delta(S,d)$ which is the maximal point of S of the form $d + t\delta(S)$, for $t \in \mathbb{R}_+$.

Theorem 12 (Chun and Thomson 1990a) The Directional solutions are the only solutions on $\Sigma^n_{d,-}$ satisfying *weak Pareto-optimality*, *individual rationality, continuity*, and *disagreement point concavity.*

This result is somewhat of a disappointment since it says that *disagreement point concavity* is incompatible with full optimality, and permits *scale invariance* only when $\delta(S)$ is a unit vector (then, the resulting Directional solution is a Dictatorial solution). The following weakening of *disagreement point concavity* allows recovering full optimality and *scale invariance.*

Weak disagreement point concavity: If $[F(S,d^1), F(S,d^2)] \subset PO(S)$ and $PO(S)$ is smooth at $F(S,d^1)$ and $F(S,d^2)$, then for all $\lambda \in [0,1]$, $F(S, \lambda d^1 + (1-\lambda)d^2) = \lambda F(S,d^1) + (1-\lambda)F(S,d^2)$.

The boundary of S is linear between $F(S,d^1)$ and $F(S,d^2)$ and it seems natural to require that the solution should respond linearly to linear movements of d between d^1 and d^2. This "partial" linearity of the solution is required however only when the compromise is not at a kink of ∂S. Indeed, an agent who had been willing to trade off his utility against some other agent's utility up to such a point might perhaps have been willing to concede further: No further move has taken place because of the sudden change in the rates at which utility could be transferred. One can therefore argue that the initial compromise is little artifical and the smoothness requirement is intended to exclude these situations from the domain of applicability of the axiom (this is an agreement already encountered). The point is made earlier in Section 4.3.

Theorem 13 (Chun and Thomson 1990c) The Nash solution is the only solution on $\Sigma^n_{d,-}$ satisfying *Pareto-optimality*, *independence of non-individually rational alternatives*, *symmetry*, *scale invariance*, *continuity*, and *weak disagreement point concavity.*

A condition related to *weak disagreement point concavity* says that a move of the disagreement point in the direction of the desired compromise does not call for a revision of this compromise.

Star-shaped inverse: $F(S, \lambda d + (1-\lambda)F(S,d)) = F(S,d)$ for all $\lambda \in]0,1]$.

Theorem 14 (Peters and van Damme 1988) The Weighted Nash solutions are the only solutions on $\Sigma^n_{d,-}$ satisfying *strong individual rationality*, *independence of non-individually rational alternatives*, *scale invariance*, *disagreement point continuity*, and *star-shaped inverse.*

> **Bibliographic note:** Several conditions related to the above three have been explored. Chun (1987b) shows that a requirement of *disagreement point quasi-concavity* can be used to characterize a family of solutions that further generalize the directional solutions. He also establishes a characterization of the Lexicographic Egalitarian solution (1989). Characterizations of the Kalai-Rosenthal solution are given in Peters (1986c) and Chun (1990). Finally, the continuous Raiffa solution for $n = 2$ is characterized by Livne (1987b), and Peters and van Damme (1991). They use the fact that for this solution the set of disagreement points leading to the same compromise for each fixed S is a curve with differentiability, and certain monotonicity, properties. Livne (1988) considers situations where the disagreement point is also subject to uncertainty, but information can be obtained about it, and he characterizes a version of the Nash solution.

5.3 Risk-sensitivity

Here, we investigate how solutions respond to changes in the agents' risk aversion. Other things being equal, is it preferable to face a more risk-averse opponent? Should we expect this to be the case? To study this issue we need explicitly to introduce the set of underlying physical alternatives. Let

C be a set of **certain options** and L the set of **lotteries** over C. Given two von Neumann-Morgenstern utility functions u_i and $u'_i : L \to \mathbb{R}$, u'_i **is more risk-averse than** u_i if they represent the same ordering on C and for all $c \in C$, the set of lotteries that are u'_i-preferred to c is contained in the set of lotteries that are u_i-preferred to c. If $u_i(C)$ is an interval, this implies that there is an increasing concave function $k : u_i(C) \to \mathbb{R}$ such that $u'_i = k \circ u_i$. An n-**person concrete problem** is a list (C, e, u), where C is as above, $e \in C$, and $u = (u_1, ..., u_n)$ is a list of von Neumann-Morgenstern utility functions defined over C. The **abstract problem associated with** (C, e, u) is the pair $(S, d) \equiv (\{u(\ell) | \ell \in L\}, u(e))$.

The first property we formulate focuses on the agent whose risk-aversion changes. According to his old preferences, does he necessarily lose when his risk-aversion increases?

Risk-sensitivity: Given (C, e, u) and (C', e', u'), which differ only in that u'_i is more risk-averse than u_i, and such that the associated problems (S, d) and (S', d') belong to Σ^n_d, $F_i(S, d) \geqq u_i(\ell')$, where $u'(\ell') = F(S', d')$.

In the formulation of the next property, the focus is on the agents whose preferences are kept fixed. It says that all of them should benefit from the increase in some agent's risk-aversion.

Strong risk-sensitivity: Under the same hypotheses as **risk sensitivity**, $F_i(S, d) \geqq u_i(\ell')$ and in addition, $F_j(S, d) \leqq u_j(\ell')$ for all $j \neq i$.

The concrete problem (C, e, u) **is basic** if the associated (S, d) satisfies $PO(S) \subset u(C)$. Let $B(\mathcal{C}^n_d)$ be the class of basic problems. If (C, e, u) is basic and u'_i is more risk-averse than u_i, then (C, e, u'_i, u_{-i}) also is basic.

Theorem 15 (Kihlstrom, Roth and Schmeidler 1981, Nielsen 1984)
The Nash solution satisfies *risk-sensitivity* on $B(\mathcal{C}^n_d)$ but it does not satisfy *strong risk-sensitivity*. The Kalai-Smorodinsky solution satisfies *strong risk sensitivity* on $B(\mathcal{C}^n_0)$.

There is an important logical relation between *risk-sensitivity* and *scale invariance*.

Theorem 16 (Kihlstrom, Roth and Schmeidler 1981) If a solution on $B(\mathcal{C}_d^2)$ satisfies *Pareto-optimality* and *risk sensitivity*, then it satisfies *scale invariance*. If a solution on $B(\mathcal{C}_0^n)$ satisfies *Pareto-optimality* and *strong risk sensitivity*, then it satisfies *scale invariance*.

For $n = 2$, interesting relations exist between *risk sensitivity* and *twisting* (Tijs and Peters 1985) and between *risk sensitivity* and *midpoint domination* (Sobel 1981).

> **Bibliographic note:** Further results appear in de Koster, Peters, Tijs and Wakker (1983), Peters (1987a), Peters and Tijs (1981, 1983, 1985a), Tijs and Peters (1985), and Klemisch-Ahlert (1992a).

For the class of non-basic problems, two cases should be distinguished. If the disagreement point is the image of one of the basic alternatives, what matters is whether the solution is appropriately responsive to changes in the disagreement point.

Theorem 17 (based on Roth and Rothblum 1982 and Thomson 1987a) Suppose $C = \{c_1, c_2, e\}$. Suppose F is a solution on Σ_d^2 satisfying *Pareto-optimality*, *scale invariance*, and *disagreement point monotonicity*. Then, if u_i is replaced by a more risk-averse utility u_i', agent j gains if $u_i(\ell) \geqq \min\{u_i(c_1), u_i(c_2)\}$ and not otherwise.

> **Bibliographic note:** The n-person case is studied by Roth (1988). Situations when the disagreement point is obtained as a lottery are considered by Safra, Zhou and Zilcha (1990). An application to insurance contracts is discussed in Kihlstrom and Roth (1982).

6 Variable Number of Agents

Most of the axiomatic theory of bargaining has been written under the assumption of a fixed number of agents. Recently, however, the model has been enriched by allowing the number of agents to vary. Axioms specifying how solutions could, or should, respond to such changes have been formulated and new characterizations of the main solutions as well as of new solutions generalizing them have been developed. A detailed account of these developments can be found in Thomson and Lensberg (1989).

In order to accommodate a variable population, the model itself has to be generalized. There is now an infinite set of "potential agents", indexed by the positive integers. Any finite group may be involved in a problem. Let $\mathcal{P}$ be the set of all such groups. Given $Q \in \mathcal{P}$, $\mathbb{R}^Q$ is the utility space pertaining to that group, and Σ_0^Q the class of subsets of $\mathbb{R}_+^Q$ satisfying all of the assumptions imposed earlier on the elements of Σ_0^n. Let $\Sigma_0 \equiv \cup \Sigma_0^Q$. A **solution** is a function F defined on Σ_0 which associates with every $Q \in \mathcal{P}$ and every $S \in \Sigma_0^Q$ a point of S. All of the axioms stated earlier for solutions defined on Σ_0^n can be reformulated so as to apply to this more general notion by simply writing that they hold for every $Q \in \mathcal{P}$. As an illustration, the optimality axiom is written as:

Pareto-Optimality: For all $Q \in \mathcal{P}$ and for all $S \in \Sigma_0^Q$, $F(S) \in PO(S)$.

This is simply a restatement of our earlier axiom of *Pareto-optimality* for each group separately. To distinguish the axiom from its fixed population counterpart, we will capitalize it. We will similarly capitalize all axioms in this section.

Our next axiom, *Anonymity*, is also worth stating explicitly: it says that the solution should be invariant not only under exchanges of the names of the agents in each given group, but also under replacement of some of its members by other agents.

Anonymity: Given P, $P' \in \mathcal{P}$ with $|P| = |P'|$, $S \in \Sigma_0^P$ and $S' \in \Sigma_0^{P'}$, if there exists a bijection $\gamma : P \to P'$ such that $S' = \{x' \in \mathbb{R}^{P'} | \exists x \in S \text{ with } x'_i = x_{\gamma(i)} \forall i \in P\}$, then $F_i(S') = F_{\gamma(i)}(S)$ for all $i \in P$.

Two conditions specifically concerned with the way solutions respond to changes in the number of agents have been central to the developments reported in this section. One is an independence axiom, and the other a monotonicity axiom. They have led to characterizations of the Nash, Kalai-Smorodinsky and Egalitarian solutions.

Notation: Given P, $Q \in \mathcal{P}$ with $P \subset Q$ and $x \in \mathbb{R}^Q$, x_P denotes its projection on $\mathbb{R}^P$. Similarly, if $A \subseteq \mathbb{R}^Q$, A_P is its projection on $\mathbb{R}^P$.

6.1 Consistency and the Nash solution

We start with the independence axiom. Informally, it says that the desirability of a compromise should be unaffected by the departure of some of the agents with their payoffs. To be more precise, let $Q \in \mathcal{P}$ and $T \in \Sigma_0^Q$, consider some point $x \in T$ as the candidate compromise for T. Assume that it has been accepted by the subgroup P', and let us imagine its members leaving the scene with the understanding that they will indeed receive their payoffs $x_{P'}$. Now, let us reevaluate the situation from the viewpoint of the group $P = Q \backslash P'$ of remaining agents. It is natural to think as the set $\{y \in \mathbb{R}^P | (y, x_{Q\backslash P}) \in T\}$ obtained from points of T by giving the agents in P' the payoffs $x_{P'}$, as the feasible set for P. Let us denote it by $r_P^x(T)$. Geometrically, $r_P^x(T)$ is the "slice" of T through x by a plane parallel to the coordinate subspace relative to the group P. If this set is a well-defined member of Σ_0^P, does the solution recommend the utilities x_P? If yes, and this coincidence always occurs, the solution is *Consistent* (Figure 13a).

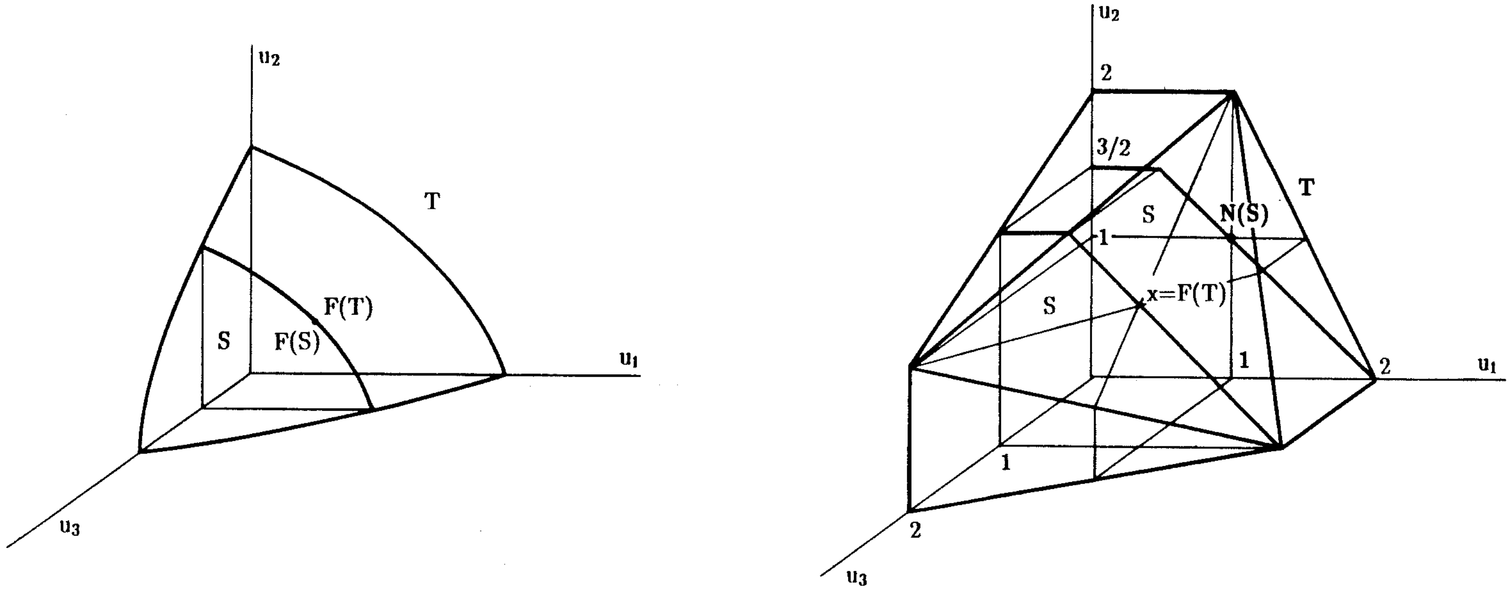
Consistency: Given P, $Q \in \mathcal{P}$ with $P \subset Q$, if $S \in \Sigma_0^P$ and $T \in \Sigma_0^Q$ are such that $S = r_P^x(T)$, where $x = F(T)$, then $x_P = F(S)$.

Consistency is satisfied by the Nash solution (Harsanyi, 1959) but not by the Kalai-Smorodinsky solution, nor by the Egalitarian solution. Violations are usual for the Kalai-Smorodinsky solution but rare for the Egalitarian solution; indeed, on the class of strictly comprehensive problems, the Egalitarian solution does satisfy the condition, and if this restriction is not imposed, it still satisfies the slightly weaker condition obtained by requiring $x_P \leqq F(S)$ instead of $x_P = F(S)$. Let us call this weaker condition *Weak Consistency.* The Lexicographic Egalitarian solution satisfies *Consistency.*

By substituting this condition for *contraction independence* in Nash's classic theorem (Theorem 1), we obtain another characterization of the Nash solution.

Theorem 18 (Lensberg 1988) The Nash solution is the only solution on Σ_0 satisfying *Pareto-Optimality*, *Anonymity*, *Scale Invariance*, and *Consistency.*

Proof: (Figure 13b) It is straightforward to see that N satisfies the four axioms. Conversely, let F be a solution on Σ_0 satisfying the four axioms. We only show that F coincides with N on Σ_0^P if $|P| = 2$. Let $S \in \Sigma_0^P$ be



(a) (b)

Figure 13: ***Consistency*** **and the Nash solution.** (a) The axiom of *Consistency*: the solution outcome of the "slice" of T by a plane parallel to the coordinate subspace relative to agents 1 and 2 through the solution outcome of T, $F(T)$, coincides with the restriction of $F(T)$ to that subspace. (b) Characterization of the Nash solution (Theorem 18.)

given. By *Scale Invariance*, we can assume that S is normalized so that $N(S) = (1,1)$.

In a first step, we assume that $PO(S) \supset [(3/2,1/2),(1/2,3/2)]$. Let $Q \in \mathcal{P}$ with $P \subset Q$ and $|Q| = 3$ be given. Without loss of generality, we take $P = \{1,2\}$ and $Q = \{1,2,3\}$. (In Figure 13b, $S \equiv cch\{(2,0),(1/2,3/2)\}$). Now, we translate S by the third unit vector, we replicate the result twice by having agents 2, 3 and 1, and then agents 3, 1 and 2 play the roles of agents 1, 2, and 3 respectively; finally, we define $T \in \Sigma_0^Q$ to be the convex and comprehensive hull of the three sets so obtained. Since $T = cch\{(1,2,0),(0,1,2),(2,0,1)\}$ is invariant under rotations of the agents, by *Anonymity*, $F(T)$ has equal coordinates, and by *Pareto-Optimality*, $F(T) = (1,1,1)$. But, since $r_P^{(1,1,1)}(T) = S$, *Consistency* gives $F(S) = (1,1) = N(S)$, and we are done.

In a second step, we only assume that $PO(S)$ contains a non-degenerate segment centered at $N(S)$. Then, we may have to introduce more than one additional agent and repeat the same construction by replicating the problem faced by agents 1 and 2 many times. If the order of replication is sufficiently large, the resulting T is indeed such that $r_P^{(1,\ldots,1)}(T) = S$ and we conclude as before. If S does not contain a non-degenerate segment centered at $N(S)$, a continuity argument is required.

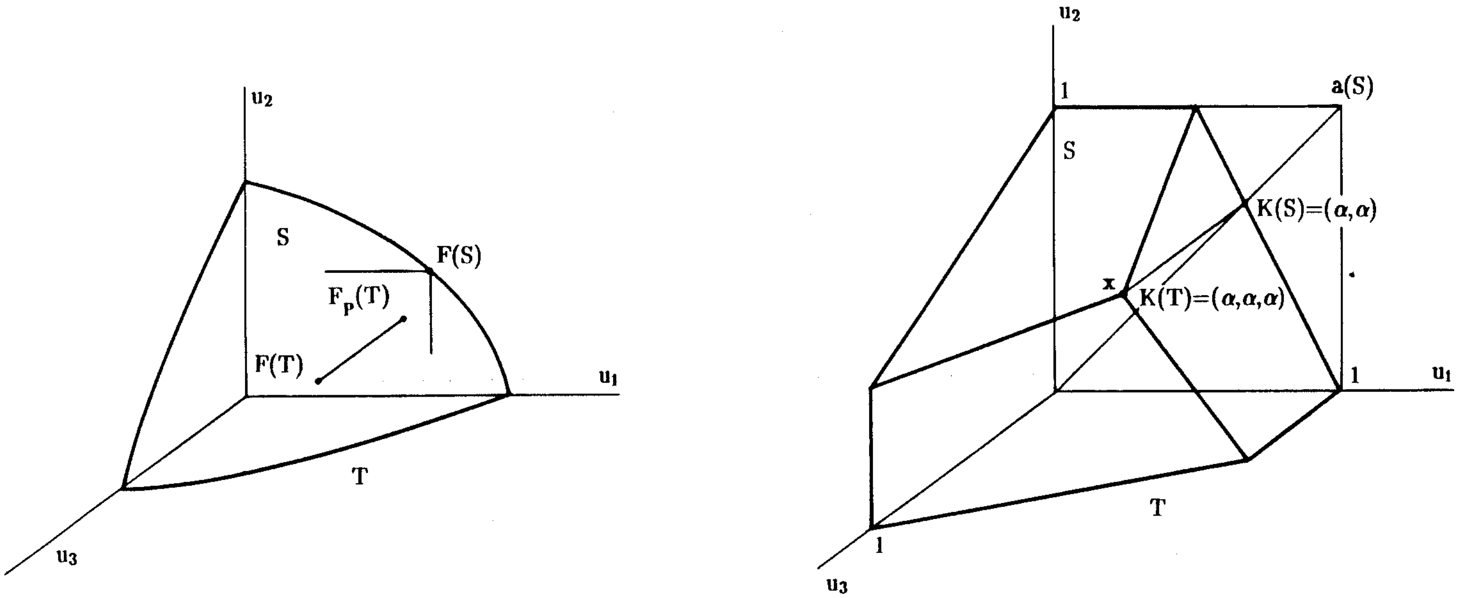
Q.E.D.

Bibliographic note: The above proof requires having access to groups of arbitrarily large cardinalities, but the Nash solution can still be characterized when the number of potential agents is bounded above, by adding *Continuity* (Lensberg 1988). Unfortunately, two problems may be close in the Hausdorff topology and yet sections of those problems through two points that are close by, parallel to a given coordinate subspace, may not be close to each other. A weaker notion of continuity recognizing this possibility can however be used to obtain a characterization of the Nash solution (Lensberg 1985), even if the number of potential agents is bounded above (Thomson 1985b). Just as in the classic characterization of the Nash solution, *Pareto-Optimality* turns out to play a very minor role here: without it, the only additional admissible solution is the disagreement solution (Lensberg and Thomson 1988).

Deleting *Symmetry* and *Scale Invariance* from Theorem 18, the following solutions become admissible: For each $i \in \mathbb{N}$, let $f_i : \mathbb{R}_+ \to \mathbb{R}$ be an increasing function such that for each $P \in \mathcal{P}$, the function $f^P : \mathbb{R}^P_+ \to \mathbb{R}$ defined by $f^P(x) = \sum_{i \in P} f_i(x)$ be strictly quasi-concave. Then, given $P \in \mathcal{P}$ and $S \in \Sigma_0^P$, $F^f(S) \equiv argmax\{f^P(x)|x \in S\}$. These **separable additive** solutions F^f are the only ones to satisfy *Pareto-Optimality*, *Continuity*, and *Consistency* (Lensberg 1987; Young 1988 proves a variant of this result).

6.2 Population Monotonicity and the Kalai-Smorodinsky solution

Instead of allowing some of the agents to depart with their payoffs, we will now imagine them to leave empty-handed, without their departure affecting the opportunities of the agents that remain. Do all of these remaining agents gain? If yes, the solution will be said to be *Population Monotonic.* Conversely, and somewhat more concretely, think of a group of agents $P \in \mathcal{P}$ dividing goods on which they have equal rights. Then new agents come in, who are recognized to have equally valid rights on the goods. This requires



(*a*) (*b*)

Figure 14: *Population Monotonicity* and the Kalai-Smorodinsky solution. (*a*) The axiom of *Population Monotonicity*: the projection of the solution outcome of the problem T onto the coordinate subspace pertaining to agents 1 and 2 is dominated by the solution outcome of the intersection of T with that coordinate subspace. (*b*) Characterization of the Kalai-Smorodinsky solution on the basis of *Population monotonicity* (Theorem 19).

that the goods be redivided. *Population Monotonicity* says that none of the agents initially present should gain. Geometrically (Figure 14a), this means that the projection of the solution outcome of the problem involving the large group onto the coordinate subspace relative to the smaller group of remaining agents is Pareto-dominated by the solution outcome of the intersection of the large problem with that subspace.

Population Monotonicity: For all P, $Q \in \mathcal{P}$ with $P \subset Q$, if $S \in \Sigma_0^P$ and $T \in \Sigma_0^Q$ are such that $S = T_P$, then $F(S) \geqq F_P(T)$.

The Nash solution does not satisfy this requirement but both the Kalai-Smorodinsky and Egalitarian solutions do. In fact, characterizations of those two solutions can be obtained with the help of this condition:

Theorem 19 (Thomson 1983c) The Kalai-Smorodinsky solution is the only solution on Σ_0 satisfying *Weak Pareto-Optimality*, *Anonymity*, *Scale Invariance*, *Continuity*, and *Population Monotonicity*.

Proof: (Figure 14b) It is straightforward to see that K satisfies the five axioms. Conversely, let F be a solution on Σ_0 satisfying the five axioms. We only show that F coincides with K on Σ_0^P if $|P| = 2$. So let $S \in \Sigma_0^P$ be given. By *Scale Invariance*, we can assume that S is normalized so that $a(S)$ has equal coordinates. Let $Q \in \mathcal{P}$ with $P \subset Q$ and $|Q| = 3$ be given. Without loss of generality, we take $P = \{1,2\}$ and $Q = \{1,2,3\}$. (In the Figure $S = cch\{(1,0),(1/2,1)\}$ so that $a(S) = (1,1)$.) Now, we construct $T \in \Sigma_0^Q$ by replicating S in the coordinates subspaces $\mathbb{R}^{\{2,3\}}$ and $\mathbb{R}^{\{3,1\}}$, and taking the comprehensive hull of S, its two replicas and of the point $x \in \mathbb{R}^Q$ of coordinates all equal to the common value of the coordinates of $K(S)$. Since all agents enter symmetrically in the definition of T and $x \in PO(T)$, it follows from *Anonymity* and *Weak Pareto-Optimality* that $x = F(T)$. Now, note that $T_P = S$ and $x_P = K(S)$ so that by *Population Monotonicity*, $F(S) \geqq K(S)$. Since $|P| = 2$, $K(S) \in PO(S)$ and equality holds.

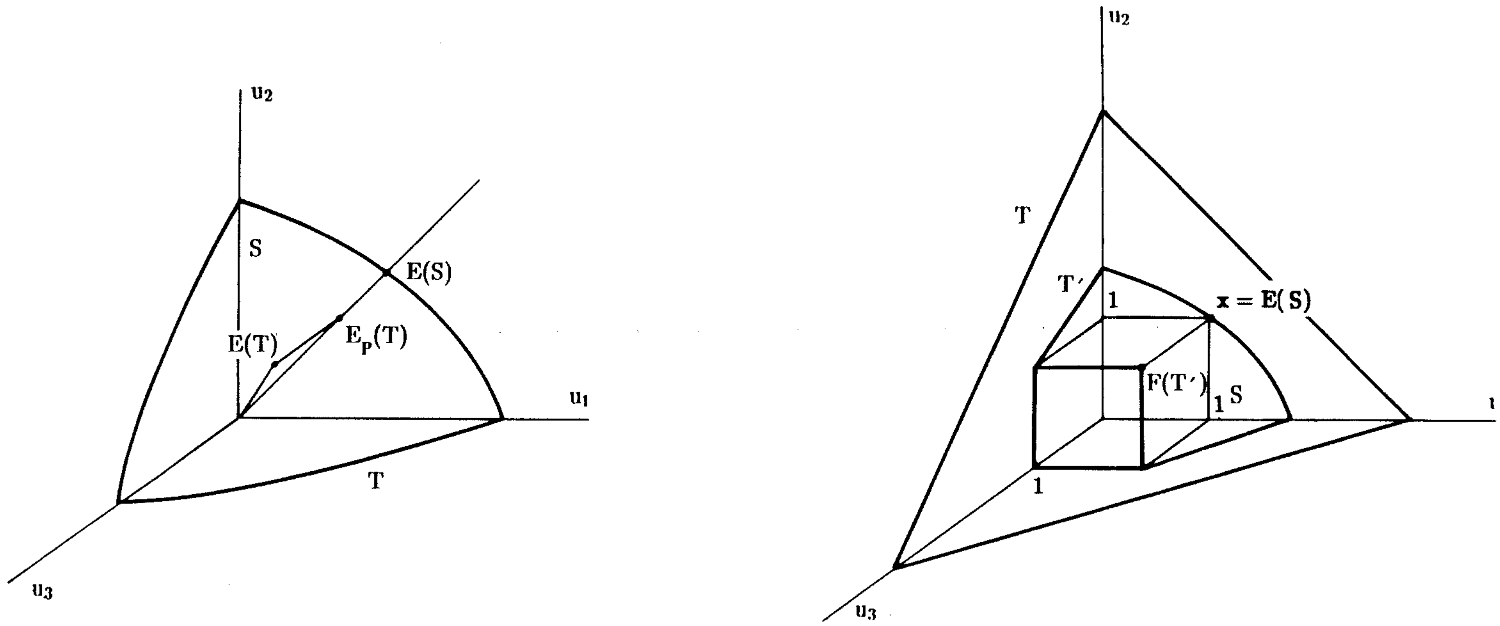
To prove that F and K coincide for problems of cardinality greater than 2, one has to introduce more agents and *Continuity* becomes necessary. Q.E.D.

Bibliographic note: Solutions in the spirit of the solutions E^α described after Theorem 20 below satisfy all of the axioms of Theorem 19 except for *Weak Pareto-Optimality*. Without *Anonymity*, we obtain certain generalizations of the Weighted Kalai-Smorodinsky solutions (Thomson, 1983a). For a clarification of the role of *Scale Invariance*, see the next result.

6.3 Population Monotonicity and the Egalitarian solution

All of the axioms used in the next theorem have already been discussed. Note that the theorem differs from the previous one only in that *Contraction Independence* is used instead of *Scale Invariance.*

Theorem 20 (Thomson 1983d) The Egalitarian solution is the only solution on Σ_0 satisfying *Weak Pareto-Optimality*, *Symmetry*, *Contraction Independence*, *Continuity*, and *Population Monotonicity.*



(a) (b)

Figure 15: *Population Monotonicity* and the Egalitarian solution. (a) The Egalitarian solution satisfies *Population Monotonicity.* (b) Characterization of the Egalitarian solution on the basis of *Population monotonicity* (Theorem 20).

Proof: It is easy to verify that E satisfies the five axioms (see Figure 15a for *Population Monotonicity*). Conversely, let F be a solution on Σ_0 satisfying the five axioms. To see that $F = E$, let $P \in \mathcal{P}$ and $S \in \Sigma_0^P$ be given. Without loss of generality, suppose $E(S) = (1, ..., 1)$ and let $\beta \equiv max\{\sum_{i \in P} x_i | x \in S\}$. Now, let $Q \in \mathcal{P}$ be such that $P \subset Q$ and $|Q| \geqq \beta + 1$; finally, let $T \in \Sigma_0^Q$ be defined by $T = \{x \in \mathbb{R}_+^Q | \sum_{i \in Q} x_i \leqq |Q|\}$. (In Figure 15b, $P = \{1, 2\}$ and $Q = \{1, 2, 3\}$.) By *Weak Pareto-Optimality* and *Symmetry*, $F(T) = (1, ..., 1)$. Now, let $T' \equiv cch\{S, F(T)\}$. Since $T' \subset T$ and $F(T) \in T'$, it follows from *Contraction Independence* that $F(T') = F(T)$. Now, $T'_P = S$, so that by *Population Monotonicity*, $F(S) = F_P(T') = E(S)$. If $E(S) \in PO(S)$ we are done. Otherwise we conclude by *Continuity.* Q.E.D.

Bibliographic note: Without *Weak Pareto-Optimality*, the following family of **Truncated Egalitarian solutions** become admissible: let $\alpha \equiv \{\alpha^P | P \in \mathcal{P}\}$ be a list of non-negative numbers such that for all P, $Q \in \mathcal{P}$ with $P \subset Q$, $\alpha^P \geqq \alpha^Q$; then, given $P \in \mathcal{P}$ and $S \in \Sigma_0^P$, let $E^\alpha(S) \equiv \alpha^P(1, ..., 1)$ if this point belongs to S and $E^\alpha(S) = E(S)$ otherwise (Thomson 1984b). The

Monotone Path solutions encountered earlier, appropriately generalized, satisfy all the axioms of Theorem 20, except for *Symmetry*: let $G \equiv \{G^P | P \in \mathcal{P}\}$ be a list of monotone paths such that $G^P \subset \mathbb{R}^P_+$ for all $P \in \mathcal{P}$ and for all P, $Q \in \mathcal{P}$ with $P \subset Q$, the projection of G^Q onto $\mathbb{R}^P$ be contained in G^P. Then, given $P \in \mathcal{P}$ and $S \in \Sigma^P_0$, $E^G(S)$ is the maximal point of S along the path G^P (Thomson, 1983a, 1984b).

6.4 Other implications of Consistency and Population Monotonicity

The next result involves considerations of both *Consistency* and *Population Monotonicity*.

Theorem 21 (Thomson 1984c:) The Egalitarian solution is the only solution on Σ_0 satisfying *Weak Pareto-Optimality*, *Symmetry*, *Continuity*, *Population Monotonicity* and *Weak Consistency*.

In order to recover full optimality, the extension of *individual monotonicity* to the variable population case can be used.

Theorem 22 (Lensberg 1985a, 1985b) The Lexicographic Egalitarian solution is the only solution on Σ_0 satisfying *Pareto-Optimality*, *Symmetry*, *Individual Monotonicity*, and *Consistency*.

6.5 Opportunities and Guarantees

Consider a solution F satisfying *Weak Pareto-Optimality*. When new agents come in without opportunities enlarging, as described in the hypotheses of *Population Monotonicity*, one of the agents originally present will lose.

We propose here a way of quantifying these losses and of ranking solutions on the basis of the extent to which they prevent agents from losing too much.

Formally, let P, $Q \in \mathcal{P}$ with $P \subset Q$, $S \in \Sigma^P_0$, and $T \in \Sigma^Q_0$ with $S = T_P$. Given $i \in P$, consider the ratio $F_i(T)/F_i(S)$ of agent i's final to initial utility: let $\alpha_F^{(i,P,Q)} \in \mathbb{R}$ be the greatest number α such that $F_i(T)/F_i(S) > \alpha$ for all S, T as just described. This is the **guarantee offered to i by F when he is initially part of P and P expands to Q** : agent i's final utility is guaranteed to be at least $\alpha_F^{(i,P,Q)}$-times his initial utility. If F satisfies *Anonymity*, then the number depends only on the cardinalities of P and $Q\backslash P$, denoted by m and n respectively, and we can write it as α_F^{mn}:

$$\alpha_F^{mn} \equiv \inf\{\frac{F_i(T)}{F_i(S)}|S \in \Sigma_0^P, T \in \Sigma_0^Q, P \subset Q, S = T_P, |P| = m, |Q\backslash P| = n\}$$

We call the list $\alpha_F \equiv \{\alpha_F^{mn}|m,\, n \in \mathbb{N}\}$ the **guarantee structure of** F.

We now proceed to compare solutions on the basis of their guarantee structures. Solutions offering greater guarantees are of course preferable. The next theorem says the Kalai-Smorodinsky is the best from the viewpoint of guarantees. In particular it is strictly better than the Nash solution.

Theorem 23 (Thomson and Lensberg 1983) The guarantee structure α_K of the Kalai-Smorodinsky solution is given by $\alpha_K^{mn} = 1/(n+1)$ for all $m,\, n \in \mathbb{N}$. If F satisfies *Weak Pareto-Optimality* and *Anonymity*, then $\alpha_K \geqq \alpha_F$. In particular, $\alpha_K \geq \alpha_N$.

Note that solutions could be compared in other ways. In particular, protecting individuals may be costly to the group to which they belong. To analyze the trade-off between protection of individuals and protection of groups, we introduce the coefficient

$$\beta_F^{mn} \equiv \inf\{\sum_{i\in P} \frac{F_i(T)}{F_i(S)}|S \in \Sigma_0^P, T \in \Sigma_0^Q, P \subset Q, S = T_P, |P| = m, |Q\backslash P| = n\}$$

and we define $\beta_F \equiv \{\beta_F^{mn}|m,\, n \in \mathbb{N}\}$ as the **collective guarantee structure of** F. Using this notion, we find that our earlier ranking of the Kalai-Smorodinsky and Nash solutions is reversed.

Theorem 24 (Thomson 1983b) The collective guarantee structure β_N of the Nash solution is given by $\beta_N^{mn} = n/(n+1)$ for all $m, n \in \mathbb{N}$. If F satisfies *Weak Pareto-Optimality* and *Anonymity*, then $\beta_N \geqq \beta_F$. In particular, $\beta_N \geq \beta_K$. Also, $\beta_K = \alpha_K$.

> **Bibliographic note:** Theorem 23 says that the Kalai-Smorodinsky solution is best in a large class of solutions. However, it is not the only one to offer maximal guarantees and to

satisfy *Scale Invariance* and *Continuity* (Thomson and Lensberg 1983). Similarly, the Nash solution is not the only one to offer maximal collective guarantees and to satisfy *Scale Invariance* and *Continuity* (Thomson 1983b).

Solutions can alternatively be compared on the basis of the opportunities for gains that they offer to individuals (and to groups). Solutions that limit the extent to which individuals (or groups) can gain in spite of the fact that there may be more agents around while opportunities have not enlarged, may be deemed preferable. Once again, the Kalai-Smorodinsky solution performs better than any solution satisfying *Weak Pareto Optimality* and *Anonymity* when the focus is on a single individual, but the Nash solution is preferable when groups are considered. However, the rankings obtained here are less discriminating (Thomson 1987b).

Finally, we compare agent i's percentage loss $F_i(T)/F_i(S)$ to agent j's percentage loss $F_j(T)/F_j(S)$, where both i and j are part of the initial group P. Let

$$\epsilon_F^{mn} \equiv \inf\{\frac{F_j(T)/F_j(S)}{F_i(T)/F_i(S)} | S \in \Sigma_0^P, T \in \Sigma_0^Q, P \subset Q, S = T_P, |P| = m, |Q\backslash P| = n\}$$

and $\epsilon_F \equiv \{\epsilon_F^{mn} | (m,n) \in \mathbb{N}\backslash 1) \times \mathbb{N}\}$.

Here, we would of course prefer solutions that prevent agents from being too differentially affected. Again, the Kalai-Smorodinsky solution performs the best from this viewpoint.

Theorem 25 (Chun and Thomson 1989) The relative guarantee structures ϵ_K and ϵ_E of the Kalai-Smorodinsky and Egalitarian solutions are given by $\epsilon_K^{mn} = \epsilon_E^{mn} = 1$ for all $(m,n) \in (\mathbb{N}\backslash 1) \times \mathbb{N}$. The Kalai-Smorodinsky solution is the only solution on Σ_0 to satisfy *Weak Pareto-Optimality, Anonymity, Scale Invariance* and to offer maximal relative guarantees. The Egalitarian solution is the only solution on Σ_0 to satisfy *Weak Pareto-Optimality, Anonymity, Contraction Independence* and to offer maximal relative guarantees.

6.6 Replication and Juxtaposition

Now, we consider the somewhat more special situations where the preferences of the new agents are required to bear some simple relation to those of the agents originally present, such as when they are exactly opposed or exactly in agreement. There are several ways in which opposition or agreement of preferences can be formalized. To each such formulation corresponds a natural way of writing that a solution respects the special structure of preferences.

Given a group P of agents facing the problem $S \in \Sigma_0^P$, introduce for each $i \in P$, n_i additional agents "of the same type" as i and let Q be the enlarged group. Given any group P' with the same composition as P (we write $comp(P') = comp(P)$), define the problem S' faced by P' to be the copy of S in $\mathbb{R}^{P'}$ obtained by having each member of P' play the role played in S by the agent in P of whose type he is. Then, to construct the problem T faced by Q, we consider two extreme cases. One case formalizes a situation of maximal compatibility of interests among all the agents of a given type:

$$S^{max} \equiv \cap\{S^{P'} \times \mathbb{R}^{Q \setminus P'} | P' \subset Q, comp(P') = comp(P)\}$$

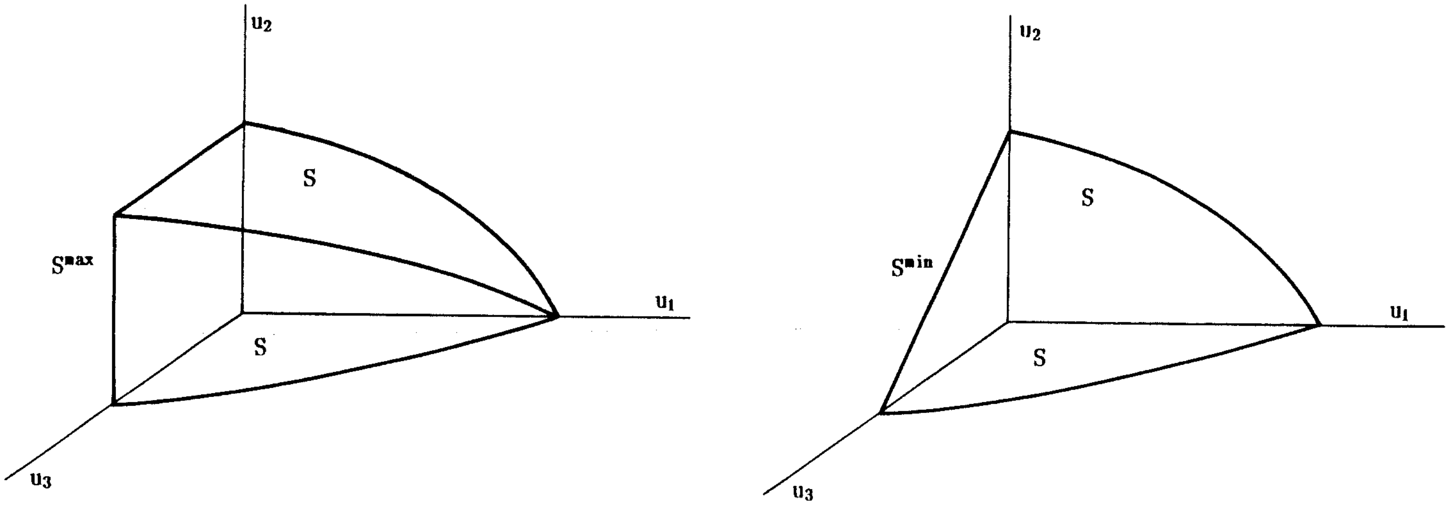
The other formalizes the opposite:

$$S^{min} \equiv cch\{S^{P'} | P' \subset Q, comp(P') = comp(P)\}$$

These two notions are illustrated in Figure 16 for an initial group of 2 agents and one additional agent (agent 3) being introduced to replicate agent 2.

Theorem 26 (based on Kalai 1977a) In S^{max}, all of the agents of a given type receive what the agent they are replicating receives in S if either the Kalai-Smorodinsky solution or the Egalitarian solution is used. However, if the Nash solution is used, all of the agents of a given type receive what the agent they are replicating would have received in S under the application of the Weighted Nash solution with weights proportional to the orders of replication of the different types.

Theorem 27 (Thomson 1984a, 1986) In S^{min}, the sum of what the agents of a given type receives under the replication of the Nash, Kalai-Smorodinsky, and Egalitarian solutions is equal to what the agent they are replicating receives in S under the application of the corresponding weighted solution for weights proportional to the orders of replication.



(a) (b)

Figure 16: Two notions of replication. (*a*) Maximal compatibility of interests. (*b*) Minimal compatibility of interests.

7 Applications to Economics

Solutions to abstract bargaining problems, most notably the Nash solution, have been used to solve concrete economic problems, such as management-labor conflicts, on numerous occasions; in such applications, S is the image in utility space of the possible divisions of a firm's profit, and d the image of a strike. Problems of fair division have also been analyzed in that way; given some bundle of infinitely divisible goods Ω, S is the image in utility space of the set of possible distributions of Ω, and d is the image of the 0 allocation (perhaps, of equal division). Alternatively, each agent may start out with a share of Ω, his endowment, and choosing d to be the image of the initial allocation may be more appropriate.

Under standard assumptions on utility functions, the resulting problem (S, d) satisfies the properties typically required of admissible problems in the axiomatic theory of bargaining. Conversely, given $S \in \Sigma_0^n$, it is possible to find exchange economies whose associated feasible set is S (Billera and Bixby 1973).

When concrete information about the physical alternatives is available, it is natural to use it in the formulation of properties of solutions. For instance, expansions in the feasible set are often the result of increases in resources or improvements in technologies. The counterpart of *strong monotonicity*, (which says that such an expansion would benefit all agents) would be that all agents benefit from greater resources or better technologies. How well-behaved are solutions on this domain? The answer is that when there is only one good, solutions are better behaved than on abstract domains, but as soon as the number of goods is greater than 1, the same behavior should be expected of solutions on both domains (Chun and Thomson 1988).

The axiomatic study of solutions to concrete allocation problems is currently an active area of research. Many of the axioms that have been found most useful in the abstract theory of bargaining have now been transposed for this domain and their implications analyzed. Early results along those lines are characterizations of the Walrasian solution (Binmore 1987) and of egalitarian-type solutions (Roemer 1986a, 1988) and Nieto (1992). For a recent contribution, see Klemisch-Ahlert and Peters (1994).

8 Strategic Considerations

Analyzing a problem (S, d) as a strategic game requires additional structure: strategy spaces and an outcome function have somehow to be associated with (S, d). This can be done in a variety of ways. We limit ourselves to describing formulations that remain close to the abstract model of the axiomatic theory. This brief section is only meant to facilitate understanding of the relation between the axiomatic models and the strategic models.

Consider the following game: each agent demands a utility level for himself; the outcome is the vector of demands if it is in S and d otherwise. The set of Nash (1951) equilibrium outcomes of this **game of demands** is $PO(S) \cap I(S, d)$ (to which should be added d if $PO(S) \cap I(S, d) = WPO(S) \cap I(S, d)$), a typically large set, so that this approach does not help in reducing the set of outcomes significantly. However, if S is known only approximately (replace its characteristic function by a **smooth** function), then as the degree of approximation increases, the set of equilibrium outcomes of the resulting **smoothed game of demands** shrinks to $N(S, d)$ (Nash 1950, Harsanyi 1956, Zeuthen 1930, Crawford 1980, Anbar and Kalai 1978, Binmore 1987, and Anbarcı1992, 1993a. Calvo and Gutiérrez 1994 analyze yet a different game).

If bargaining takes place over time, agents take time to prepare and communicate proposals, and the worth of an agreement reached in the future is discounted, a **sequential game of demands** results. Its equilibria (here some perfection notion has to be used) can be characterized in terms of the Weighted Nash solutions when the time period becomes small: it is $N^{\delta}(S,d)$ where δ is a vector related in a simple way to the agents' discount rates (Rubinstein 1982, Binmore 1987. Livne 1987 contains an axiomatic treatment).

Imagine now that agents have to justify their demands: there is a family $\mathcal{F}$ of "reasonable" solutions such that agent i can demand $\bar{u}_i$ only if $\bar{u}_i = F_i(S,d)$ for some $F \in \mathcal{F}$. Then strategies are in fact elements of $\mathcal{F}$. Let F^1 and F^2 be the strategies chosen by agents 1 and 2. If $F^1(S,d)$ and $F^2(S,d)$ differ, eliminate from S all alternatives at which agent 1 gets more than $F_1^1(S,d)$ and agent 2 gets more than $F_2^2(S,d)$; one can argue that the truncated set S^1 is the relevant set over which to bargain; so repeat the procedure: compute $F^1(S^1,d)$ and $F^2(S^1,d)$ ... If, as $\nu \to \infty$, $F^1(S^\nu,d)$ and $F^2(S^\nu,d)$ converge to a common point, take that as the solution outcome of this induced **game of solutions**. For natural families $\mathcal{F}$, convergence does occur for all F^1 and $F^2 \in \mathcal{F}$, and the only equilibrium outcome of the game so defined is $N(S,d)$ (van Damme 1986, Chun 1984 studies a variant of the procedure).

Thinking now of solutions as normative criteria, note that in order to compute the desired outcomes, the utility functions of the agents will be necessary. Since these functions are typically unobservable, there arises the issue of **manipulation**. To the procedure is associated a game of misrepresentation, where strategies are utility functions. What are the equilibria of this game? In the game so associated with the Nash solution when applied to a one-dimensional division problem, each agent has a dominant strategy which is to pretend that his utility function is linear. The resulting outcome is equal division (Crawford and Varian 1979). If there is more than one good and preferences are known ordinally, a dominant strategy for each agent is a least concave representation of his preferences (Kannai 1977). When there are an increasing number of agents, only one of whom manipulates, the gain that he can achieve by manipulation does not go to zero, although the impact on each of the others vanishes; only the first of these conclusions holds, however, when it is the Kalai-Smorodinsky solution that is being used (Thomson 1994). In the multi-commodity case, Walrasian allocations are obtained at equilibrium, but there are others (Sobel 1981, Thomson 1984d).

Rather than directly using information sent by agents about their utility functions, one should of course recognize strategic behavior and design games that produce the right outcomes in spite of it. Supposing that some solution has been selected as embodying society's objectives, does there exist a game whose equilibrium outcome always yields the desired utility allocations? If yes, the solution is **implementable**. The Kalai-Smorodinsky solution is implementable by stage games (Moulin 1984). Howard (1992) establishes the implementability of the Nash solution. Recent work indicates that much can be achieved by such games. Other implementation results are due to Anbarcı(1990) and Bossert and Tan (1992).

References

Anant, T.C.A., K. Basu, and B. Mukherji (1990), "Bargaining Without Convexity: Generalizing the Kalai-Smorodinsky Solution," **Economics Letters 33**, 115-119.

Anbar, D. and E. Kalai (1978), "A One-Shot Bargaining Problem," **International Journal of Game Theory 7,** 13-18.

Anbarcı, N. (1988), "Three Essays on Axiomatic and Non-Cooperative Bargaining Theory," University of Iowa Ph.D. Thesis.

—— (1989), "The Kalai-Smorodinsky Solution with Time Preferences", **Economics Letters 31**, 5-7.

—— (1991), "The Nash Solution and Relevant Expansions" **Economics Letters 36**, 137-140.

—— (1992), "Final-Offer Arbitration and Nash's Demand Game", University of Buffalo Discussion Paper.

—— (1993a), "Non-Cooperative Foundations of the Area Monotonic Solution," **Quaterly Journal of Economics 108**, 245- 258.

—— (1993b), "Bargaining with Finite Number of Alternatives: Strategic and Axiomatic Approaches", University of Buffalo Discussion Paper.

—— (1993), "Simple Characterizations of the Nash and Kalai- Smorodinsky Solutions", University of Buffalo Discussion Paper.

—— and J.F. Bigelow (1988), "The Area Monotonic Solution to the Cooperative Bargaining Problem," **Mathematical Social Sciences**, forthcoming.

—— and —— (1993), "Non-dictatorial, Pareto-monotonic, Cooperative Bargaining: an Impossibility Result", **European Journal of Political Economy 9**, 551-558.

—— and G. Yi (1992), "A Meta-Allocation Mechanism in Cooperative Bargaining", **Economics Letters 20**, 176-179.

Arrow, K., (1965), **Aspects of the Theory of Risk-Bearing,** Yrjö Jahnsson Foundation, Helsinki.

Billera, L.F. and R.E. Bixby (1973), "A Characterization of Pareto Surfaces," **Proceedings of the American Mathematical Society 41**, 261-267.

Binmore, K.G. (1984), "Bargaining Conventions," **International Journal of Game Theory 13**, 193-200.

—— (1987), "Nash Bargaining Theory I, II, III," in **The Economics of**

Bargaining (K. Binmore and P. Dasgupta, eds.), Basil Blackwell, New York, 27-46, 61-76, and 239-256.

Blackorby, C., W. Bossert and D. Donaldson, (1992), "Generalized Ginis and Cooperative Bargaining Solutions", University of Waterloo Working Paper 9205, forthcoming in **Econometrica**.

Bossert, W. (1990), "Disagreement Point Monotonicity, Transfer Responsiveness, and the Egalitarian Bargaining Solution," University of British Columbia mimeo.

—— (1992a), "Monotonic Solutions for Bargaining Problems with Claims", **Economics Letters** 39, 395-399.

—— (1992b), "Rational Choice and Two-Person Bargaining Solutions", University of Waterloo mimeo, forthcoming in the **Journal of Mathematical Economics**.

—— (1993), "An Alternative Solution to Bargaining Problems with Claims," **Mathematical Social Sciences 25**, 205-220.

—— , E. Nosal and V. Sadanand (1992), "Bargaining under Uncertainty and the Monotonic Path Solutions", University of Waterloo mimeo.

—— and G. Tan, (1992) "A Strategic Justification of the Egalitarian Bargaining Solution", University of Waterloo Working paper 92-12.

Brito, D.L., A.M. Buoncristiani and M.D. Intriligator (1977), "A New Approach to the Nash Bargaining Problem," **Econometrica 45**, 1163-1172.

Brunner, J.K. (1992), "Bargaining with Reasonable Aspirations", University of Linz mimeo.

Butrim, B.I. (1976), "A Modified Solution of the Bargaining Problem," (in Russian), **Zhurnal Vychistitel'noi Matematiki Matematicheskoi Fiziki 16**, 340-350.

Calvo, Emilio (1989), "Solucion de Alternativea No Dominadas Iguales", University of Bilbao Doctoral Dissertation.

—— and E. Gutiérrez, (1993), "Extension of the Perles-Maschler Solution to N-Person Bargaining Games", University of Bilbao Discussion Paper.

—— and —— (1994), "Comparison of bargaining solutions by means of altruism indexes", University of Bilbao Discussion Paper.

Cao, X. (1981), "Preference Functions and Bargaining Solutions", IEEE 164-171.

Chun, Y. (1984), "Note on 'The Nash Bargaining Solution is Optimal,'" University of Rochester Mimeo.

—— (1986), "The Solidarity Axiom for Quasi-Linear Social Choice Problems," **Social Choice and Welfare 3**, 297-310.

—— (1987b), "Axioms Concerning Uncertain Disagreement Points for 2-Person Bargaining Problems," University of Rochester Working Paper 99.

—— (1988a), "The Equal-Loss Principle for Bargaining Problems," **Economics Letters 26**, 103-106.

—— (1988b), "Nash Solution and Timing of Bargaining," **Economics Letters 28**, 27-31.

—— (1989), "Lexicographic Egalitarian Solution and Uncertainty in the Disagreement Point," **Zeitschrift für Operations Research 33**, 259-266.

—— (1991), "Transfer Paradox and Bargaining Solutions," Seoul National University Mimeo.

—— (1990), "Minimal Cooperation in Bargaining," **Economics Letters 34**, 311-316.

—— and H.J.M. Peters (1988), "The Lexicographic Egalitarian Solution," **Cahiers du CERO 30**, 149-156.

—— and —— (1989a) "Lexicographic Monotone Path Solutions," **O.R. Spektrum 11**,

—— and —— (1991) "The Lexicographic Equal-Loss Solution," **Mathematical Social Sciences, 22**, 151-161.

—— and W. Thomson (1988a), "Monotonicity Properties of Bargaining Solutions When Applied to Economics," **Mathematical Social Sciences 15**, 11-27.

—— and —— (1988b), "Bargaining Problems with Claims," **Mathematical Social Sciences**, forthcoming.

—— and —— (1989), "Bargaining Solutions and Relative Guarantees," **Mathematical Social Sciences 17**, 285-295.

—— and —— (1990a), "Bargaining with Uncertain Disagreement Points," **Econometrica 58**, 951-959.

—— and —— (1990b), "Egalitarian Solutions and Uncertain Disagreement Points," **Economics Letters 33**, 29-33.

—— and —— (1990c), "Nash Solution and Uncertain Disagreement Points," **Games and Economic Behavior** 2, 213-223.

Conley, J., R. McLean and S. Wilkie, "The Duality of Bargaining Theory and Multi-Objective Programming, and Generalized Choice Problems",

Bellcore mimeo, March 1994.
—— and S. Wilkie (1991a), "The Bargaining Problem Without Convexity," **Economics Letters** 36, 365-369.
—— and —— (1991b), "An Extension of the Nash Bargaining Solution to Non-Convex Problems : Characterization and Implementation", University of Illinois mimeo.
Crawford, V. (1980), "A Note on the Zeuthen-Harsanyi Theory of Bargaining," **Journal of Conflict Resolution 24**, 525-535.
—— and H. Varian (1979), "Distortion of Preferences and the Nash Theory of Bargaining," **Economics Letters 3**,
Crott, H.W. (1971), "Experimentelle Untersuchung zum Verhandlungs verhaltern" in **Kooperativen Spielen, Zeitschrift fur Sozialpsychologie 2**, 61-74.
van Damme, E. (1986), "The Nash Bargaining Solution is Optimal," **Journal of Economic Theory 38**, 78-100.
Dasgupta, P., and E. Maskin (1988), "Bargaining and Destructive Powers," mimeo.
Dekel, E. (1982), Harvard University, Untitled Mimeo.
Demange, G. (1984), "Implementing Efficient Egalitarian Equivalent Allocations," **Econometrica 52**, 1167-1177.
Foster, D. and R.V. Vohra (1988), "Consistency and the Nash Bargaining Solution," University of Maryland Mimeo.
Freimer M. and P.L. Yu (1976), "Some New Results on Compromise Solutions for Group Decision Problems," **Management Science 22**, 688-693.
Furth, D. (1990), "Solving Bargaining Games by Differential Equations," **Mathematics of Operations Research 15**, 724-735.
Gaertner, W., and M. Klemisch-Ahlert (1991), "Gauthier's Approach to Distributive Justice and Other Bargaining Solutions" in **Contractarianism and Rational Choice** (P. Vallentyne, ed), Cambridge University Press, 162-176.
Green, J. (1983), "A Theory of Bargaining with Monetary Transfers," Harvard Institute of Economic Research Discussion Paper.
—— and J.J. Laffont (1988), "Contract Renegotiation and the Underinvestment Effect," Harvard University Mimeo.
Gupta, S. and Z. Livne (1988), "Resolving a Conflict Situation with a Reference Outcome: An Axiomatic Model," **Management Science 34**, 1303-1314.

Harsanyi, J.C. (1955), "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility," **Journal of Political Economy 63**, 309-321.

—— (1956), "Approaches to the Bargaining Problem Before and After the Theory of Games: A Critical Discussion of Zeuthen's, Hicks', and Nash's Theories," **Econometrica 24**, 144-157.

—— (1958), "Notes on the Bargaining Problem," **Southern Economic Journal 24**, 471-476.

—— (1959), "A Bargaining Model for the Cooperative n-Person Games," in **Contributions to the Theory of Games IV, Annals of Mathematical Studies 40** (A.W. Tucker and R.D. Luce, eds) Princeton University Press, Princeton, NJ, 325-355.

—— (1961), "On the Rationality Postulates Underlying the Theory of Cooperative Games," **Journal of Conflict Resolution 5**, 179-196.

—— (1963), "A Simplified Bargaining Model for the n-Person Cooperative Game," **International Economic Review 4**, 194-220.

—— (1965), "Bargaining and Conflict Situations in the Light of a New Approach to Game Theory," **American Economic Review 55**, 447-457.

—— (1977), **Rational Behavior and Bargaining Equilibrium in Games and Social Situations**, Cambridge: Cambridge University Press.

—— and R. Selten (1972), "A Generalized Nash Solution for Two-Person Bargaining Games with Incomplete Information," **Management Science 18**, 80-106.

Heckathorn, D. (1980), "A Unified Model for Bargaining and Conflict," **Behavioral Science 25**, 261-284.

—— and R. Carlson (1983), "A Possibility Result Concerning n-Person Bargaining Games," University of Missouri Technical Report 6.

Herrero, C. (1993), "Endogenous Reference Points and the Adjusted Proportional Solution for Bargaining Problems with Claims", University of Alicante mimeo.

—— and M.C. Marco, (1993), "Rational Equal-Loss Solutions for Bargaining Problems", **Mathematical Social Sciences 26**, 273-286.

Herrero, M.J. (1989), "The Nash Program: Non-Convex Bargaining Problems," **Journal of Economic Theory 49**, 266-277

Howe, R.E. (1987), "Sections and Extensions of Concave Functions," **Journal of Mathematical Economics 16**, 53-64.

Huttel, G. and W.F. Richter (1980), "A Note on "an Impossibility Result Concerning n-Person Bargaining Games"," University of Bielefeld Working Paper 95.

Imai, H. (1983), "Individual Monotonicity and Lexicographic Maxmin Solution," **Econometrica 51**, 389-401, erratum in **Econometrica 51**, 1603.

Iturbe, I. and J. Nieto (1992), "Stable and Consistent Solutions on Economic Environments", mimeo.

Jansen, M.J.M. and S.H. Tijs (1983), "Continuity of Bargaining Solutions," **International Journal of Game Theory 12**, 91-105.

Kalai, E. (1977a), "Nonsymmetric Nash Solutions and Replications of Two-Person Bargaining," **International Journal of Game Theory 6**, 129-133.

—— (1977b), "Proportional Solutions to Bargaining Situations: Interpersonal Utility Comparisons," **Econometrica 45**, 1623-1630.

—— (1985), "Solutions to the Bargaining Problem," Chapter 3 in **Social Goals and Social Organization**; Essays in memory of E. Pazner (L. Hurwicz, D. Schmeidler and H. Sonnenschein, eds.), Cambridge University Press, 77-105.

—— and R.W. Rosenthal (1978), "Arbitration of Two-Party Disputes Under Ignorance," **International Journal of Game Theory 7**, 65-72.

—— and M. Smorodinsky (1975), "Other Solutions to Nash's Bargaining Problem," **Econometrica 43**, 513-518.

Kannai, Y. (1977), "Concavifiability and Constructions of Concave Utility Functions," **Journal of Mathematical Economics 4**, 1-56.

Kihlstrom, R.E. and L.J. Mirman (1974), "Risk Aversion with Many Commodities," **Journal of Economic Theory 8**, 361-388.

—— and A.E. Roth (1982), "Risk Aversion and the Negotiation of Insurance Contracts," **Journal of Risk and Insurance 49**, 372-387.

——, —— and D. Schmeidler (1981), "Risk Aversion and Solutions to Nash's Bargaining Problem," in **Game Theory and Mathematical Economics**, (O. Moeschlin and D. Pallaschke, eds.), North-Holland, 65-71.

Klemisch-Ahlert, M. (1991), "Independence of the Status Quo ? A Weak and a Strong Impossibility Result for Social Decisions by Bargaining" **Journal of Economics 53**, 83-93.

—— (1992a), "Distributive Justice and Risk Sensitivity", **Theory and Decision 32**, 303-318.

—— (1992b), "Distributive Effects Implied by the Path Dependence of the Nash Bargaining Solution", University of Osnabruck mimeo.
—— (1993), "Bargaining when Redistribution is Possible", University of Osnabrück Discussion Paper.
Kohlberg, E., M. Maschler and M.A. Perles (1983), "Generalizing the Super-Additive Solution for Multiperson Unanimity Games," Discussion Paper, forthcoming.
de Koster, R., H.J.M. Peters, S.H. Tijs and P. Wakker (1983), "Risk Sensitivity, Independence of Irrelevant Alternatives and Continuity of Bargaining Solutions," **Mathematical Social Sciences 4**, 295-300.
Lahiri, S. (1990), "Threat Bargaining Games with a Variable Population," **International Journal of Game Theory 19**, 91-100.
Lensberg, T. (1985a), "Stability, Collective Choice and Separable Welfare," Ph.D. Thesis, Norwegian School of Economics and Business Administration, Bergen, Norway.
—— (1985b), "Bargaining and Fair Allocation" in **Cost Allocation: Methods, Principles, Applications** (H.P. Young, ed), North-Holland, 101-116.
—— (1987), "Stability and Collective Rationality," **Econometrica 55**, 935-961.
—— , (1988), "Stability and the Nash Solution," **Journal of Economic Theory 45**, 330-341.
—— and W. Thomson (1988), "Characterizing the Nash Bargaining Solution Without Pareto-Optimality," **Social Choice and Welfare 5**, 247-259.
Livne, Z.A. (1986), "The Bargaining Problem: Axioms Concerning Changes in the Conflict Point," **Economics Letters 21**, 131-134.
—— (1987), "Bargaining Over the Division of a Shrinking Pie: An Axiomatic Approach," **International Journal of Game Theory 16**, 223-242.
—— (1988), "The Bargaining Problem with an Uncertain Conflict Outcome," **Mathematical Social Sciences 15**, 287-302.
—— (1989a), "Axiomatic Characterizations of the Raiffa and the Kalai-Smorodinsky Solutions to the Bargaining Problem," **Operations Research 37**, 972-980.
—— (1989b), "On the Status Quo Sets Induced by the Raiffa Solution to the Two-Person Bargaining Problem," **Mathematics of Operations Research 14**, 688-692.
Luce, R.D. and H. Raiffa (1957), **Games and Decisions: Introduction**

and **Critical Survey,** New York: Wiley.
Marco, M.C. (1993a), "Efficient Solutions for Bargaining Problems with Claims", University of Alicante Discussion Paper.
—— (1993b), "An Alternative Characterization of the Extended Claim-Egalitarian Solution", University of Alicante Discussion Paper.
Maschler, M. and M.A. Perles (1981), "The Present Status of the Super-Additive Solution," in **Essays in Game Theory and Mathematical Economics in honor of Oskar Morgenstern** (R. Aumann et al. eds.), 103-110.
McLennan, A. (198), "A General Noncooperative Theory of Bargaining," University of Toronto Discussion Paper.
Moulin, H. (1978), "Implementing Efficient, Anonymous and Neutral Social Choice Functions," Mimeo.
—— (1983), "Le Choix Social Utilitariste," Ecole Polytechnique Discussion Paper.
—— (1984), "Implementing the Kalai-Smorodinsky Bargaining Solution," **Journal of Economic Theory 33**, 32-45.
—— (1988), **Axioms of Cooperative Decision Making**, Cambridge University Press.
Myerson, R.B. (1977), "Two-Person Bargaining Problems and Comparable Utility," **Econometrica 45**, 1631-1637.
—— (1981), "Utilitarianism, Egalitarianism, and the Timing Effect in Social Choice Problems," **Econometrica 49**, 883-897.
Nash, J.F. (1950), "The Bargaining Problem," **Econometrica 28**, 155-162.
—— (1951), "Noncooperative Games," **Annals of Mathematics 54**, 286-295.
—— (1953), "Two-Person Cooperative Games," **Econometrica 21,** 128-140.
von Neumann, J. and O. Morgenstern (1944), **Theory of Games and Economic Behavior,** Princeton: Princeton University Press.
Nielsen, L.T. (1983), "Ordinal Interpersonal Comparisons in Bargaining," **Econometrica 51**, 219-221.
—— (1984), "Risk Sensitivity in Bargaining with More Than Two Participants," **Journal of Economic Theory 32**, 371-376.
Nieto, J. (1992) "The Lexicographic Egalitarian Solution on Economic Environments", **Social Choice and Welfare 9**, 203-212.
O'Neill, B. (1981), "Comparison of Bargaining Solutions, Utilitarianism and

the Minmax Rule by Their Effectiveness," Northwestern University Discussion Paper.

Osborne, M. and A. Rubinstein (1990), **Bargaining and Markets**, Academic Press.

Perles, M.A. (1982), "Nonexistence of Super-Additive Solutions for 3-Person Games," **International Journal of Game Theory 11**, 151-161.

—— and M. Maschler (1981), "The Super-Additive Solution for the Nash Bargaining Game," **International Journal of Game Theory 10**, 163-193.

—— , "Non-Symmetric Super-Additive Solutions," forthcoming.

—— , "Super-Additive Solutions Without the Continuity Axiom," forthcoming.

Peters, H.J.M. (1986a), "Simultaneity of Issues and Additivity in Bargaining," **Econometrica 54**, 153-169.

—— (1986b), **Bargaining Game Theory**, Ph.D. Thesis, Maastricht, The Netherlands.

—— (1987a), "Some Axiomatic Aspects of Bargaining," in **Axiomatics and Pragmatics of Conflict Analysis** (J.H.P. Paelinck and P.H. Vossen, eds.), Gower Press, Aldershot, *U.K*, 112-141.

—— (1987b), "Nonsymmetric Nash Bargaining Solutions," Chapter 12 in **Surveys in Game Theory and Related Topics** (H.J.M. Peters and O.J. Vrieze, eds) CWI Tract 39, Amsterdam.

—— (1992), **Axiomatic Bargaining Game Theory**, Kluwer Academic, The Netherlands.

—— and E. van Damme (1991), "Characterizing the Nash and Raiffa Bargaining Solutions by Disagreement Point Axioms," **Mathematics of Operations Research 16**, 447-461.

—— and M. Klemisch-Ahlert (1994), "An Impossibility Result Concerning Distributive Justice in Axiomatic Bargaining", University of Limburg Discussion Paper.

—— and S. Tijs (1981), "Risk Sensitivity of Bargaining Solutions," **Methods of Operations Research 44**, 409-420.

—— and —— (1983), "Risk Properties of n-Person Bargaining Solutions," Nijmegen University Discussion Paper 8323.

—— and —— (1984a), "Individually Monotonic Bargaining Solutions for n-Person Bargaining Games," **Methods of Operations Research 51**, 377-384.

—— and —— (1984b), "Probabilistic Bargaining Solutions," **Operations Research Proceedings**, Springer-Verlag, Berlin, 548-556.

—— and —— (1985a) "Risk Aversion in n-Person Bargaining," **Theory and Decision 18**, 47-72.

—— and —— (1985b), "Characterization of All Individually Monotonic Bargaining Solutions," **International Journal of Game Theory 14**, 219-228.

—— , —— and R. de Koster (1983), "Solutions and Multisolutions for Bargaining Games," **Methods of Operations Research 46**, 465-476.

—— and P. Wakker (1991), "Independence of Irrelevant Alternatives and Revealed Group Preferences," **Econometrica 59**, 1787-1801.

Pfingsten A.(1991), "Axiomatic Bargaining when Claims do not Always Dominate the Disagreement Point", mimeo

Ponsati, C. and J. Watson, "Multiple Issue Bargaining and Axiomatic Solutions", Stanford University mimeo, 1994.

Pratt J.W. (1964), "Risk Aversion in the Small and the Large," **Econometrica 32**, 122-136.

Raiffa, H. (1953), "Arbitration Schemes for Generalized Two-Person Games," in **Contributions to the Theory of Games II**, **Annals of Mathematics Studies 28**, (H.W. Kuhn and A.W. Tucker, eds), Princeton: Princeton University Press, 361-387.

Ritz, Z. (1985), "An Equal Sacrifice Solution to Nash's Bargaining Problem," University of Illinois, Mimeo.

Roemer, J., (1986a), "Equality of Resources Implies Equality of Welfare," **Quarterly Journal of Economics 101**, 751-784.

—— (1986b), "The Mismarriage of Bargaining Theory and Distributive Justice," **Ethics 97**, 88-110.

—— , (1988) "Axiomatic Bargaining Theory on Economic Environments," **Journal of Economic Theory 45**, 1-31.

—— , (1990), "Welfarism and Axiomatic Bargaining Theory", **Recherches Economiques de Louvain 56**, 287-301.

Rosenthal, R.W. (1976), "An Arbitration Model for Normal-Form Games," **Mathematics of Operations Research** *l*, 82-88.

—— (1978), "Arbitration of Two-Party Disputes Under Uncertainty," **Review of Economic Studies 45**, 595-604.

Roth, A.E. (1977a), "Individual Rationality and Nash's Solution to the Bargaining Problem," **Mathematics of Operations Research 2**, 64-65.

—— (1977b), "Independence of Irrelevant Alternatives, and Solutions to Nash's Bargaining Problem," **Journal of Economic Theory 16**, 247-251.
—— (1978), "The Nash Solution and the Utility of Bargaining," **Econometrica 46**, 587-594, 983.
—— (1979a), "Proportional Solutions to the Bargaining Problem," **Econometrica 47**, 775-778.
—— (1979b), "Interpersonal Comparisons and Equal Gains in Bargaining," Mimeo.
—— (1979c), **Axiomatic Models of Bargaining**, Springer-Verlag, Berlin and New York, No. 170.
—— (1979d), "An Impossibility Result Concerning n-Person Bargaining Games," **International Journal of Game Theory 8**, 129-132.
—— (1980), "The Nash Solution as a Model of Rational Bargaining," in **Extremal Methods and Systems Analysis**, (A.V. Fiacco and K.O. Kortanek, eds.), Springer-Verlag.
—— (1987), "Bargaining Phenomena and Bargaining Theory," **Laboratory Experimentation in Bargaining: Six Points of View**, (A.E. Roth, ed), Cambridge University Press.
—— (1988), "Risk Aversion in Multi-Person Bargaining Over Risky Agreements," University of Pittsburg mimeo.
—— and U. Rothblum (1982), "Risk Aversion and Nash's Solution for Bargaining Games with Risky Outcomes," **Econometrica 50**, 639-647.
Rubinstein, A. (1982), "Perfect Equilibrium in a Bargaining Model," **Econometrica 50**, 97-109.
——, Z. Safra and W. Thomson, "On the interpretation of the Nash Bargaining Solution and its Exension to Non-Expected Utility Preferences", **Econometrica 50**, 639-647.
Safra, Z., L. Zhou and I. Zilcha (1990), "Risk Aversion in the Nash Bargaining Problem with Risky Outcomes and Risky Disagreement Points," **Econometrica 58**, 961-965.
—— and I. Zilcha (1991), "Bargaining Solutions without the Expected Utility Hypothesis", Tel Aviv University mimeo.
Salonen, H. (1985), "A Solution for Two-Person Bargaining Problems," **Social Choice and Welfare 2**, 139-146.
—— (1987), "Partially Monotonic Bargaining Solutions," **Social Choice and Welfare 4**, 1-8.

—— (1992), "A Note on Continuity of Bargaining Solutions", University of Oulu mimeo.

—— (1993), "Egalitarian Solutions for N-Person Bargaining Games", University of Oulu Discussion Paper.

Salukvadze, M.E., (1971a) "Optimization of Vector Functionals I. The Programming of Optimal Trajectories", **Automation and Remote Control 32**, 1169-1178.

—— (1971b), "Optimazation of Vector Functionals II. The Analytic Constructions of Optimal Controls", **Automation ans Remote Control 32**, 1347-1357.

Schelling, T.C. (1959), "For the Abandonment of Symmetry in Game Theory," **Review of Economics and Statistics 41**, 213-224.

Schmitz, N. (1977), "Two-Person Bargaining Without Threats: A Review Note," **Proceedings of the OR Symposium — Aachen 29**, 517-533.

Segal, U. (1980), "The Monotonic Solution for the Bargaining Problem: A Note," Hebrew University of Jerusalem Mimeo.

Shapley, L.S. (1953), "A Value for N-Person Games," in **Contributions to the Theory of Games**, II (H. Kuhn and A.W. Tucker, eds.), Princeton University Press, 307-317.

—— (1969), "Utility Comparison and the Theory of Games," in **La Decision** (G.Th. Guilbaud, ed.), Editions du CNRS, Paris, 251-263.

—— (1984), Oral Presentation at the I.M.A., Minneapolis.

Shubik, M. (1982), **Game Theory in the Social Sciences**, MIT University Press.

Sobel, J. (1981), "Distortion of Utilities and the Bargaining Problem," **Econometrica 49**, 597-619.

Thomson, W. (1980), "Two Characterizations of the Raiffa Solution," **Economics Letters 6**, 225-231.

—— (1981a), "A Class of Solutions to Bargaining Problems," **Journal of Economic Theory 25**, 431-441.

—— (1981b), "Independence of Irrelevant Expansions," **International Journal of Game Theory 10**, 107-114.

—— (1981c), "Nash's Bargaining Solution and Utilitarian Choice Rules," **Econometrica 49**, 535-538.

—— (1983a), "Truncated Egalitarian and Monotone Path Solutions," University of Minnesota Discussion Paper.

—— (1983b), "Collective Guarantee Structures," **Economics Letters 11**, 63-68.

——(1983c), "The Fair Division of a Fixed Supply Among a Growing Population," **Mathematics of Operations Research 8**, 319-326.

—— (1983d), "Problems of Fair Division and the Egalitarian Solution," **Journal of Economic Theory 31**, 211-226.

—— (1984a), "Two Aspects of the Axiomatic Theory of Bargaining," University of Rochester Discussion Paper 6.

—— (1984b), "Truncated Egalitarian Solutions," **Social Choice and Welfare 1**, 25-32.

—— (1984c), "Monotonicity, Stability and Egalitarianism," **Mathematical Social Sciences 8**, 15-28.

—— (1985a), "Axiomatic Theory of Bargaining with a Variable Population: A Survey of Recent Results," Chapter 11 in **Game Theoretic Models of Bargaining**, (A.E. Roth, ed), Cambridge University Press, 233-258.

—— (1985b), "On the Nash Bargaining Solution," University of Rochester Mimeo.

—— (1986), "Replication Invariance of Bargaining Solutions," **International Journal of Game Theory 15**, 59-63.

—— (1987a), "Monotonicity of Bargaining Solutions with Respect to the Disagreement Point," **Journal of Economic Theory 42**, 50-58.

—— (1987b), "Individual and Collective Opportunities," **International Journal of Game Theory 16**, 245-252.

—— (1990), "The Consistency Principle in Economics and Game Theory," in **Game Theory and Applications** (T. Ichiishi, A. Neyman, and Y. Tauman, eds.), Academic Press, 187-215.

—— (1991a), **Bargaining Theory: the Axiomatic Approach**, Academic Press, forthcoming.

—— and T. Lensberg (1983), "Guarantee Structures for Problems of Fair Division," **Mathematical Social Sciences 4**, 205-218.

—— and —— (1989), **The Theory of Bargaining with a Variable Number of Agents**, Cambridge University Press.

—— and R.B. Myerson (1980), "Monotonicity and Independence Axioms," **International Journal of Game Theory 9**, 37-49.

Tijs S. and H.J.M. Peters (1985), "Risk Sensitivity and Related Properties for Bargaining Solutions," Chapter 10 in **Game Theoretic Models of Bargaining**, (A.E. Roth, ed), Cambridge University Press, 215-231.

Valenciano, F. and J.M. Zarzuelo, "On the Interpretation of Nonsymmetric Bargaining Solutions and their Extension to Non-Expected Utility Preferences", University of Bilbao Discussion Paper.

Wakker, P., H. Peters, and T. van Riel (1986), "Comparisons of Risk Aversion, with an Application to Bargaining," **Methods of Operations Research 54**, 307-320.

Young, P., (1988b), "Consistent Solutions to the Bargaining Problem," University of Maryland Discussion Paper.

Yu, P.L. (1973), "A Class of Solutions for Group Decision Problems," **Management Science 19**, 936-946.

Zeuthen, F. (1930), **Problems of Monopoly and Economic Warfare**, London: *G.* Routledge.