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Clarence Anthony Bush, Ph.D.

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Ford Visiting Assistant Professor Department of Economics
and
The Frederick Douglass Institute for African and African-American Studies

University of Rochester

## ABSTRACT

If contemporaneous errors of two or more singular systems of equations are correlated, the optimal use of information requires that the multiple systems be jointly (simultaneously) estimated. The estimation procedure relies on the generalized inverse of the covariance matrix of the combined system, where adding-up restrictions on parameters of each respective system are imposed. If autoregressive errors are assumed, restrictions on autoregressive parameters are imposed across systems, and lags of one system may appear in another system. The algorithm of Dhrymes (forthcoming), which is used to estimate the parameters of one singular system of equations, can be analogously applied to estimate parameters of a combined system of two or more singular systems of equations.

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## Introduction

This paper addresses the problem of jointly estimating two or more singular systems of equations. The basic assumptions of the paper are that the contemporaneous errors of the various systems are correlated and that each singular system of equations has its' own distinct specification. The applicability of the general framework and algorithm of Dhrymes (1984, revised 1988, Forthcoming) is extended to the problem of jointly estimating multiple singular systems of equations. In the joint estimation problem, exclusion restrictions and adding-up restrictions are imposed within and across systems. The estimation algorithm relies on a more expansive covariance matrix, which is singular, and its' generalized inverse.

## The Literature

Singular systems of equations arise in a variety of economic applications. The consumer allocation problem is an example, where the sum of the regressands (expenditures on commodities) must equal the value of a regressor (total expenditure) at each observation. This implies that the contemporaneous errors are linearly dependent. The linear dependency of errors means that the covariance matrix of the contemporaneous errors will be singular [1].

In modeling the consumer allocation problem, Stone (1954), Pollak and Wales (1969), Deaton (1975), Lewis (1989), Andrikopoulos et al. (1990), and others have relied upon the linear expenditure system. In addition, the Almost Ideal Demand System ("AIDS") of Deaton and Muellbauer (1980), was used by Ray (1980), Blanciforti and Green (1983), Mergos and Donatos (1989), Bush (1990), Conrad and Schroder (1991), and Taube and MacDonald (1991).

Singular systems of equations result from estimating asset demand and factor shares. Using an adaptation of the AIDS, Zietz and Weichert (1988) estimated asset demand in a singular system of equations. Adams (1991) examines the demand for assets within an extended linear expenditure system. The estimation of factor shares associated with a translog production function, as analyzed by Berndt and Savin (1975), involves singular systems of equations.

In a singular system of $n$ equations, the standard econometric procedure is to estimate $n-1$ equations after dropping the $n$th equation from the system. When autoregressive errors are present, Berndt and Savin (1975) identify and impose restrictions that are implied by the adding-up condition on parameters of the autoregressive process. Invoking the "invariance" result of Barten (1969), maximum likelihood estimation is then employed.

Dhrymes and Schwarz (1987a) demonstrate that the unrestricted Barten estimator depends on the auxiliary positive parameter $k$ unless all equations contain the same set of variables, and, thus, Barten's "invariance" result is problematic [2]. Dhrymes (forthcoming) presents a comprehensive and symmetric estimation approach. The estimation procedure relies on the generalized inverse of the contemporaneous covariance matrix, and all restrictions on parameters and the autoregressive process are imposed. Using the data and model of Berndt and Savin (1975), Bush (1990) empirically demonstrates that the estimation procedure of Dhrymes improves computational accuracy which is reflected in smaller estimated standard errors.

Throughout the literature, there is little or no discussion of the problem of jointly estimating multiple singular systems of equations, when contemporaneous errors of multiple systems are correlated. In their analysis of pooling international consumption data, Pollack and Wales (1987) pooled data and estimated a Quadratic Expenditure System ("QES"), where a subset of parameters were identical across countries and other parameters were country specific. In a footnote, Pollack and Wales suggest possible efficiency gains "when account is take of nonzero disturbance between countries." However, no analysis or estimation procedure is presented when contemporaneous errors of several systems are correlated.

## Motivation

In this section, several areas of research that could lead to estimation of a system of singulars systems of equations are presented. Kim (1988) estimated the demand for education within a system of equations. [3] A variation on Kim's work would investigate the demand for education of representative consumers from two or more socioeconomic groupings. The demand for education of individual (group) $p$, $p = 1, \ldots, n$, would be estimated within a unique system of expenditure equations. Each individual's (group's) system of expenditure equations is characterized by a singular contemporaneous covariance matrix. In addition, the explanatory variables of individual (group) $p$'s system and the contemporaneous errors of individual $p$'s system are independent. However, we assume that the contemporaneous errors from the system of individual $p$ are correlated with the contemporaneous errors from the system of individual $q$. Correlation of contemporaneous errors between systems could be a result of externalities or a variety of other factors.

The empirical literature on charitable giving focuses on the estimation of income and price elasticities for charitable giving through a single equation framework. In this literature, estimates of the extent that government funding crowds out private giving are presented [4]. Since pecuniary and nonpecuniary donations are embedded in the consumer allocation problem, price elasticities, income elasticities, and crowd out effects can be estimated within a broader system of expenditure equations. With appropriate assumptions and the assumption that the contemporaneous errors from the system of expenditure or share equations of individual (group) $p$ are correlated with the contemporaneous errors from the system of individual (group) $q$, joint estimation of systems would more efficiently use information.

In telecommunications, Companies such as AT&T, MCI, or Sprint require empirical models and forecasts on the spectrum of their telecommunication services. As an example, AT&T's mananagement has requested forecasts of Software Defined Network, MTS, Megacom 800, Accunet 1. 5, and Mulitiquest 900 service, for various customers. Ford's demand for these five AT&T services could constitute a singular system of equations. A system of expenditure equations can be constructed for the expenditures of General Motors on AT&T's services, and another singular system of equations would exist for Chrysler. Assuming that the contemporaneous errors among the three expenditure systems are correlated and that each system has a unique specification, joint estimation of the three systems implies efficient use of information.

## Framework

To estimate systems of singular systems of equations, we build on the notation and intuition of Dhrymes (1984, 1988, Forthcoming). For intuition, the consumer allocation problem is considered, where the sum of regressands (expenditures on commodities) must equal the value of a regressor (total expenditure) at each observation. Let $p$ denote the system of the $p$th individual, $p = 1, \ldots, n$. Let $n$ denote the total number of individuals or systems.

Let $y_{t\cdot}^{(p)}$ be a $1 \times m_p$ row vector of goods consumed by individual $p$ at time $t$, where $m_p$ is the number of goods in the $p$th system. For the $p$th system, at time

$t$, the row vector of explanatory variables is $x_{t\cdot}^{(p)}$ which is $1 \times l_p$. In addition, we require that total expenditure $x_{tl_p}$ be in the $l_p$th position of $x_{t\cdot}^{(p)}$. The matrix of parameters $B_p$ is $l_p \times m_p$. Let $u_{t\cdot}^{(p)}$ be a row vector of contemporaneous errors on the $p$th system, where $u_{t\cdot}^{(p)}$ is $1 \times m_p$. $\{u_{t\cdot}^{(p)}, t = 1, 2, \dots\}$ is a sequence of i. i. d random vectors, where $E[u_{t\cdot}^{(p)\prime}] = 0$ and $Cov(u_{t\cdot}^{(p)\prime}) = \Omega^{(p)}$. Finally, let $e_{m_p}$ be a $m_p \times 1$ vector of ones.

The $p$th singular system can be written as:

$$y_{t\cdot}^{(p)} = x_{t\cdot}^{(p)} B_p + u_{t\cdot}^{(p)}$$

where $p = 1, \dots, n$.

Adding-up restrictions imply $y_{t\cdot}^{(p)} e_{m_p} = x_{tl_p} \Rightarrow$

$$x_{t\cdot}^{(p)} B_p e_{m_p} = x_{tl_p}$$

This implies

$$B_p e_{m_p} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1_{l_p} \end{bmatrix}.$$

In addition, the covariance matrix $\Omega^{(p)}$ is singular, since $u_{t\cdot}^{(p)} e_{m_p} = 0$.

Asumption A. We assume that the contemporaneous errors of systems $p = 1, \dots, n$ are correlated. Specifically, $E[u_{ti}^{(p)}] = 0$; $E[u_{ti}^{(p)} u_{tj}^{(p)}] = \omega_{ij}^{(p)}$; $E[u_{ti}^{(p)} u_{tj}^{(q)}] = \omega_{ij}^{(pq)}$; and $E[u_{t+\tau,i}^{(p)} u_{tj}^{(q)}] = 0$.

In this framework, exclusion restrictions are permitted, where the $i$th equation of the $p$th system may not contain all explanatory variables from the $p$th system. Since the $p$th system is $y_{t\cdot}^{(p)} = x_{t\cdot}^{(p)} B_p + u_{t\cdot}^{(p)}$,

$Y^{(p)} = (y_{t\cdot}^{(p)})$; $X^{(p)} = (x_{t\cdot}^{(p)})$; and $U^{(p)} = (u_{t\cdot}^{(p)})$, $t = 1, \ldots, T$.

If the $i$th equation of the $p$th system does not contain all variables,

$$y_{\cdot i}^{(p)} = X_i^{(p)} \beta_{\cdot i}^{(p)} + u_{\cdot i}^{(p)} \tag{1.1}$$

The $i$th column of $Y^{(p)}$ is $y_{\cdot i}^{(p)}$, and $u_{\cdot i}^{(p)}$ is the $i$th column of $U^{(p)}$ . The explanatory variables appearing in the $i$th equation of system $p$ are contained in $X_i^{(p)}$ . In the $p$th system, the parameter vector of the $i$th equation is $\beta_{\cdot i}^{(p)}$, where $\beta_{\cdot i}^{(p)}$ is of dimension $G_i^{(p)} \times 1$ and where $G_i^{(p)}$ is the number of explanatory variables in the $i$th equation of the $p$th system.

Assumption B. We assume that some of the $n$ singular systems of equations may share common explanatory variables, i. e. , an explanatory variable in system $p$ may appear in other systems. However, each system has a unique specification.

Under Assumption B, define $X$ to be $T \times G$ matrix such that any column of $X$ is an explanatory vector in at least one of the $n$ singular systems. Clearly, $G \geq G_i^p$ for every $i$ and $p$.

Convention 1. (i) Since each system must satisfy its adding-up restrictions, the elements of $x_{t\cdot}$ are arranged such that $x_{tl_1}^{(1)}, \ldots, x_{tl_n}^{(n)}$ appear in the last $n$ positions of $x_{t\cdot}$ (e. g. , total expenditures of individuals, $p = 1, \ldots, n$, appear in the last $n$ positions of $x_{t\cdot}$).

(ii) $x_{tl_p}^{(p)}$ must appear in the $k+p$th position, $p = 1, \ldots, n$ of $x_{t\cdot}$.

$$x_{t\cdot} = \begin{bmatrix} x_{t1} & \ldots & x_{tk} & x_{tk+1} = x_{tl_1}^{(1)} & x_{tk+2} = x_{tl_2}^{(2)} & \ldots & x_{tk+n} = x_{tl_n}^{(n)} \end{bmatrix}$$

## Continuity of Systems and Adding-up

Using $x_{t\cdot}$, the $p$th system is

$$y_{t\cdot}^{(p)} = x_{t\cdot}\bar{B}_p + u_{t\cdot}^{(p)}$$

where $\bar{B}_p$ is a matrix of parameters and is $G \times m_p$.

Convention 2. In $\bar{B}_p$, parameters corresponding to $x_{tl_p}^{(p)}$ (i. e. , total expenditure in the $p$th system) shall appear in the $k+p$th row of $\bar{B}_p$, where $x_{tl_p}^{(p)}$ is in the $k+p$th position of $x_{t\cdot}$.

To preserve the specification of the $i$th equation in the $p$th system, define $S_{i1}^{(p)}$ to be a permutation of $G_i^{(p)}$ of the columns of $I_G$ so that $X_i^{(p)} = XS_{i1}^{(p)}$. The dimension of $S_{i1}^{(p)}$ is $G \times G_i^{(p)}$, where $G_i^{(p)}$ is the number of explanatory variables appearing in the $i$th equation of the $p$th system. Thus, through the application of $S_{i1}^{(p)}$ the specification of the $i$th equation in the $p$th system is preserved as in (1.1).

Now, $S_1^{(p)} = \text{diag}(S_{11}^{(p)}, \ldots, S_{m_p1}^{(p)})$, and $S_1^{(p)}$ has dimension $m_pG \times (\Sigma_{i=1}^{m_p} G_i^{(p)})$. Also, $vec(\bar{B}_p) = S_1^{(p)}\beta^{(p)}$, where $\beta^{(p)}$ is a vector of parameters in the $p$th system. The dimension of $\beta^{(p)}$ is $(\Sigma_{i=1}^{m_p} G_i^{(p)}) \times 1$. By Assumption B, $S_1^{(p)} \neq S_1^{(q)}$.

The adding-up condition implies $u_{t.}^{(p)} e_{m_p} = 0$, and $y_{t.}^{(p)} e_{m_p} = x_{tl_p}^{(p)}$ which implies

$$x_{t.} \bar{B}_p e_{m_p} = x_{tl_p}^{(p)}$$

This implies

$$\bar{B}_p e_{m_p} = \begin{bmatrix} 0 \\ \vdots \\ 0_k \\ 0_{k+1} \\ \vdots \\ 1_{k+p} \\ 0 \\ \vdots \\ 0_G \end{bmatrix}$$

Thus,

$$(e'_{m_p} \otimes I_G) S_1^{(p)} \beta^{(p)} = \begin{bmatrix} 0 \\ \vdots \\ 0_k \\ 0_{k+1} \\ \vdots \\ 1_{k+p} \\ 0 \\ \vdots \\ 0_G \end{bmatrix}$$

where $p = 1, \dots, n$.

## A System of Singular Systems of Equations

Writing the entire system we have

$$\begin{aligned} y_{t\cdot} &= [y_{t\cdot}^{(1)}, \dots, y_{t\cdot}^{(n)}] \\ x_{t\cdot} &= [x_{t1}, \dots, x_{tk}, x_{tl_1}^{(1)}, \dots, x_{tl_n}^{(n)}] \\ u_{t\cdot} &= [u_{t\cdot}^{(1)}, \dots, u_{t\cdot}^{(n)}] \end{aligned}$$

The dimension of $y_{t\cdot}$ is $1 \times \Sigma_{p=1}^{n} m_p$, and $u_{t\cdot}$ is the same dimension as $y_{t\cdot}$.

Now,

$$[y_{t\cdot}^{(1)} \vdots \dots \vdots y_{t\cdot}^{(n)}] = [x_{t\cdot}\bar{B}_1 \vdots x_{t\cdot}\bar{B}_2 \vdots \dots \vdots x_{t\cdot}\bar{B}_n] + [u_{t\cdot}^{(1)} \vdots \dots \vdots u_{t\cdot}^{(n)}]$$

Let $B = [\bar{B}_1 \vdots \bar{B}_2 \vdots \ldots \vdots \bar{B}_n]$, where $B$ is $G \times \Sigma_{i=1}^{n} m_p$. Thus,

$$y_{t\cdot} = x_{t\cdot} B + u_{t\cdot}, \tag{1.2}$$

where $\{u'_{t\cdot},\ t = 1, 2, 3, \ldots\}$ is a sequence of i.i.d random vectors with mean zero. (This demonstrates that the notation of Dhrymes can be used in the analysis of systems of singular systems of equations. )

System wide adding-up implies, $u_{t\cdot} e = 0$, where $e = (e'_{m_1}, e'_{m_2}, \ldots, e'_{m_n})'$. Under Assumption A,

$$E[u'_{t\cdot} u_{t\cdot}] = \begin{bmatrix} u_{t1}^{(1)} \\ \vdots \\ u_{tm_1}^{(1)} \\ u_{t1}^{(2)} \\ \vdots \\ u_{tm_2}^{(2)} \\ \vdots \\ u_{t1}^{(n)} \\ \vdots \\ u_{tm_n}^{(n)} \end{bmatrix} \begin{bmatrix} u_{t1}^{(1)} & \ldots & u_{tm_1}^{(1)} & u_{t1}^{(2)} & \ldots & u_{tm_2}^{(2)} & \ldots & u_{t1}^{(n)} & \ldots & u_{tm_n}^{(n)} \end{bmatrix}$$

This implies

$$E[u'_{t.}u_{t.}] = \begin{bmatrix} \omega_{11}^{(1)} & \dots & \omega_{1m_1}^{(1)} & \omega_{11}^{(12)} & \dots & \omega_{1m_2}^{(12)} & \dots & \omega_{11}^{(1n)} & \dots & \omega_{1m_n}^{(1n)} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ \omega_{m_1 1}^{(1)} & \dots & \omega_{m_1 m_1}^{(1)} & \omega_{m_1 1}^{(12)} & \dots & \omega_{m_1 m_2}^{(12)} & \dots & \omega_{m_1 1}^{(1n)} & \dots & \omega_{m_1 m_n}^{(1n)} \\ \omega_{11}^{(21)} & \dots & \omega_{1m_1}^{(21)} & \omega_{11}^{(2)} & \dots & \omega_{1m_2}^{(2)} & \dots & \omega_{11}^{(2n)} & \dots & \omega_{1m_n}^{(2n)} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ \omega_{m_2 1}^{(21)} & \dots & \omega_{m_2 m_1}^{(21)} & \omega_{m_2 1}^{(2)} & \dots & \omega_{m_2 m_2}^{(2)} & \dots & \omega_{m_2 1}^{(2n)} & \dots & \omega_{m_2 m_n}^{(2n)} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ \omega_{11}^{(n1)} & \dots & \omega_{1m_1}^{(n1)} & \omega_{11}^{(n2)} & \dots & \omega_{1m_2}^{(n2)} & \dots & \omega_{11}^{(n)} & \dots & \omega_{1m_n}^{(n)} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ \omega_{m_n 1}^{(n1)} & \dots & \omega_{m_n m_1}^{(n1)} & \omega_{m_n 1}^{(n2)} & \dots & \omega_{m_n m_2}^{(n2)} & \dots & \omega_{m_n 1}^{(n)} & \dots & \omega_{m_n m_n}^{(n)} \end{bmatrix} = \Omega$$

which implies $\Omega e = 0 \Rightarrow \Omega$ is singular. In addition, $y_{t.}e = \Sigma_{p=1}^{n} x_{tl_p}^{(p)}$ Thus,

$$x_{t.}Be = \Sigma_{p=1}^{n} x_{tl_p}^{(p)}$$

This implies

$$\begin{bmatrix} x_{t.}\bar{B}_1 & x_{t.}\bar{B}_2 & \dots & x_{t.}\bar{B}_n \end{bmatrix} \begin{bmatrix} e_{m_1} \\ e_{m_2} \\ \vdots \\ e_{m_n} \end{bmatrix} = \Sigma_{p=1}^{n} x_{tl_p}^{(p)}$$

This implies

$$x_{t.}\bar{B}_1 e_{m_1} + x_{t.}\bar{B}_2 e_{m_2} + \dots + x_{t.}\bar{B}_n e_{m_n} = \Sigma_{p=1}^{n} x_{tl_p}^{(p)}$$

Since we have $x_{t\cdot}\bar{B}_p e_{m_p} = x_{tl_p}^{(p)},\ p = 1, \ldots, n$, then,

$$\bar{B}_p e_{m_p} = \begin{bmatrix} 0 \\ \vdots \\ 0_k \\ 0_{k+1} \\ \vdots \\ 1_{k+p} \\ 0 \\ \vdots \\ 0_G \end{bmatrix} = \bar{r}_p$$

The adding-up condition implies

$$(e_{m_p}' \otimes I_G) S_1^{(p)} \beta^{(p)} = \bar{r}_p$$

Consolidating across systems we have

$$\begin{bmatrix} (e'_{m_1} \otimes I_G)S_1^{(1)} & 0 & \dots & 0 \\ 0 & (e'_{m_2} \otimes I_G)S_1^{(2)} & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & (e'_{m_n} \otimes I_G)S_1^{(n)} \end{bmatrix} \begin{bmatrix} \beta^{(1)} \\ \beta^{(2)} \\ \vdots \\ \beta^{(n)} \end{bmatrix} = \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \vdots \\ \bar{r}_n \end{bmatrix}.$$

Now, let

$$R_1 = \begin{bmatrix} (e'_{m_1} \otimes I_G)S_1^{(1)} & 0 & \dots & 0 \\ 0 & (e'_{m_2} \otimes I_G)S_1^{(2)} & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & (e'_{m_n} \otimes I_G)S_1^{(n)} \end{bmatrix}$$

Let

$$\beta = \begin{bmatrix} \beta^{(1)} \\ \beta^{(2)} \\ \vdots \\ \beta^{(n)} \end{bmatrix}$$

Define

$$r_1 = \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \vdots \\ \bar{r}_n \end{bmatrix}$$

The adding-up conditions imply $R_1\beta = r_1$. The dimension of $R_1$ is $nG \times (\Sigma_{p=1}^{n}\Sigma_{i=1}^{m_p}G_i^{(p)})$. The dimension of $\beta$ is $(\Sigma_{p=1}^{n}\Sigma_{i=1}^{m_p}G_i^{(p)}) \times 1$, and matrix $r_1$ is $nG \times 1$

## Estimating

Given that $\Omega$ is singular and given the restrictions on $B$, our problem is to estimate

$$y_{t.} = x_{t.}B + u_{t.}$$

Although, $R_1, r_1$ and $\Omega$ reflect restrictions and properties of systems of singular systems of equations, the form taken by the first order conditions and the algorithm for deriving estimates follows the derivation of Dhrymes [5].

Thus, the problem is

$$\text{minimize}\, L \;=\; (y-(I\otimes X)S\beta)'(\Omega_g\otimes I_T)(y-(I\otimes X)S\beta)+2\lambda_1'(R_1\beta-r_1)$$

with respect to $\beta$ and $\lambda$.

Now, $\lambda$ is a vector of lagrangian multipliers and $vec(B) = S\beta$, where $S = \text{diag}(S_1^{(1)}, S_1^{(2)}, \ldots, S_1^{(n)})$. Letting $\theta = (\beta', \lambda')'$,

$$\frac{\partial L}{\partial \theta} = 0$$

This implies

$$\begin{bmatrix} S'(\Omega_g \otimes X'X)S & R_1' \\ R_1 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} S'(\Omega_g \otimes X')y \\ r_1 \end{bmatrix} \tag{1.3}$$

## Autoregressive Errors

In this section we derive the estimation procedure when autoregressive errors are present in systems of singular systems of equations. For the system as a whole, we replace Assumption A with:

Assumption AA (i) $u_{t\cdot} = u_{t-1\cdot}H + \zeta_{t\cdot}$, where $u_{t\cdot} = [u_{t\cdot}^{(1)}, \ldots, u_{t\cdot}^{(n)}]$. $H$ is a matrix of parameters, and the dimension of $H$ is $(\Sigma_p^n m_p) \times (\Sigma_p^n m_p)$

(ii) $\zeta_{t\cdot} = [\zeta_{t\cdot}^{(1)}, \ldots, \zeta_{t\cdot}^{(n)}]$, and $\{\zeta_{t\cdot}', \ t = 1, 2, 3 \ldots\}$ is a sequence of i.i.d random vectors. In addition, the contemporaneous errors of systems $p = 1, \ldots, n$ are correlated. Specifically, $E[\zeta_{ti}^{(p)}] = 0$; $E[\zeta_{ti}^{(p)}\zeta_{tj}^{(p)}] = \sigma_{ij}^{(p)}$; $E[\zeta_{ti}^{(p)}\zeta_{tj}^{(q)}] = \sigma_{ij}^{(pq)}$; and $E[\zeta_{t+\tau,i}^{(p)}\zeta_{tj}^{(q)}] = 0$.

Now,

$$E[\zeta_{t\cdot}'\zeta_{t\cdot}] = \begin{bmatrix} \sigma_{11}^{(1)} & \ldots & \sigma_{1m_1}^{(1)} & \sigma_{11}^{(12)} & \ldots & \sigma_{1m_2}^{(12)} & \ldots & \sigma_{11}^{(1n)} & \ldots & \sigma_{1m_n}^{(1n)} \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots \\ \sigma_{m_1 1}^{(1)} & \ldots & \sigma_{m_1 m_1}^{(1)} & \sigma_{m_1 1}^{(12)} & \ldots & \sigma_{m_1 m_2}^{(12)} & \ldots & \sigma_{m_1 1}^{(1n)} & \ldots & \sigma_{m_1 m_n}^{(1n)} \\ \sigma_{11}^{(21)} & \ldots & \sigma_{1m_1}^{(21)} & \sigma_{11}^{(2)} & \ldots & \sigma_{1m_2}^{(2)} & \ldots & \sigma_{11}^{(2n)} & \ldots & \sigma_{1m_n}^{(2n)} \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots \\ \sigma_{m_2 1}^{(21)} & \ldots & \sigma_{m_2 m_1}^{(21)} & \sigma_{m_2 1}^{(2)} & \ldots & \sigma_{m_2 m_2}^{(2)} & \ldots & \sigma_{m_2 1}^{(2n)} & \ldots & \sigma_{m_2 m_n}^{(2n)} \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots \\ \sigma_{11}^{(n1)} & \ldots & \sigma_{1m_1}^{(n1)} & \sigma_{11}^{(n2)} & \ldots & \sigma_{1m_2}^{(n2)} & \ldots & \sigma_{11}^{(n)} & \ldots & \sigma_{1m_n}^{(n)} \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots \\ \sigma_{m_n 1}^{(n1)} & \ldots & \sigma_{m_n m_1}^{(n1)} & \sigma_{m_n 1}^{(n2)} & \ldots & \sigma_{m_n m_2}^{(n2)} & \ldots & \sigma_{m_n 1}^{(n)} & \ldots & \sigma_{m_n m_n}^{(n)} \end{bmatrix} = \Sigma$$

Adding-up restrictions imply $u_{t\cdot}e = 0$. Thus, $u_{t-1\cdot}e = 0$, and $\zeta_{t\cdot}e = 0$. Now, $e'E[\zeta_{t\cdot}'\zeta_{t\cdot}] = e'\Sigma = 0$. This implies $\Sigma e = 0$, and $\Sigma$ is singular. In addition, $u_{t-1\cdot}ce = u_{t-1\cdot}He$ which implies the row sums of H must be equal.

Convention 3. In the $p$th system, the lags of the $i$th equation appear in the $(\Sigma_{k=0}^{p-1} m_k) + i$ column of $H$, where $m_0 = 0$. (Cross system lags are not excluded)

Let $h_{i\cdot}$ denote the $i$th row of $H$. Thus,

$$\begin{aligned} h_{1\cdot}e &= h_{(\sum_{p=1}^{n} m_p)\cdot}e \\ h_{2\cdot}e &= h_{(\sum_{p=1}^{n} m_p)\cdot}e \\ \vdots &= \vdots \\ h_{[(\sum_{p=1}^{n} m_p)-1]\cdot}e &= h_{(\sum_{p=1}^{n} m_p)\cdot}e \end{aligned}$$

This implies

$$[I_{[(\Sigma_{p=1}^{n} m_p)-1]} \vdots - e_{[(\Sigma_{p=1}^{n} m_p)-1]}]He = 0$$

where $[I_{[(\Sigma_{p=1}^{n} m_p)-1]}$ is an identity matrix of dimension $(\Sigma_{p=1}^{n} m_p) - 1$, and where $e_{[(\Sigma_{p=1}^{n} m_p)-1]}$ is a column vector of ones.

Let

$$I^*_{[(\Sigma_{p=1}^{n} m_p)-1]} = [I_{[(\Sigma_{p=1}^{n} m_p)-1]} \vdots - e_{[(\Sigma_{p=1}^{n} m_p)-1]}]$$

The dimension of $I^*_{[(\Sigma_{p=1}^{n} m_p)-1]}$ is $[(\Sigma_{p=1}^{n} m_p) - 1] \times \Sigma_{p=1}^{n} m_p$. Thus, the condition that the row sums of $H$ be equal is made effective through

$$I^*_{[(\Sigma_{p=1}^{n} m_p)-1]}He = 0$$

Let $\bar{m}_i^{(p)}$ be the number of lags in the $i$th equation of the $p$th system. Define $S_{i2}^{(p)}$ to be a permutation of $\bar{m}_i^{(p)}$ of the columns of $I_{[\Sigma_{p=1}^{n} m_p]}$. In the $p$th system, $\gamma_i^{(p)}$ is a vector containing the lags of the $i$th equation not a prior known to be zero. Then,

$$h_{\cdot[(\Sigma_{k=0}^{p-1} m_k)+i]} = S_{i2}^{(p)} \gamma_i^{(p)}$$

Define

$$S_2 = \text{diag}(S_{12}^{(1)}, \ldots, S_{m_1 2}^{(1)}, S_{12}^{(2)}, \ldots, S_{m_2 2}^{(2)}, \ldots, S_{12}^{(n)}, \ldots, S_{m_n 2}^{(n)})$$

Now, $S_2$ is dimension $(\Sigma_{p=1}^n m_p)^2 \times (\Sigma_{p=1}^n \Sigma_{i=1}^{m_p} \bar{m}_i^{(p)})$.

Thus, the adding-up restrictions on the autoregressive process imply

$$I^*_{[(\Sigma_{p=1}^n m_p)-1]} He = 0$$

This implies

$$(e'_{[\Sigma_{p=1}^n m_p]} \otimes I^*_{[(\Sigma_{p=1}^n m_p)-1]}) S_2 \gamma = 0, \ \text{vec}(H) = S_2 \gamma$$

where $\gamma = (\gamma^{(1)'}, \ldots, \gamma^{(n)'})'$. In the $p$th system $\gamma^{(p)}$ is a vector of nonzero lag parameters. The dimension of $\gamma$ is $(\Sigma_{p=1}^n \Sigma_{i=1}^{m_p} \bar{m}_i^{(p)}) \times 1$.

Define

$$R_2 = (e'_{[\Sigma_{p=1}^n m_p]} \otimes I^*_{[(\Sigma_{p=1}^n m_p)-1]}) S_2$$

where $R_2$ is dimension $[(\Sigma_{p=1}^n m_p) - 1] \times (\Sigma_{p=1}^n \Sigma_{i=1}^{m_p} \bar{m}_i^{(p)})$.

Let $r_2 = 0$, where $r_2$ is dimension $[(\Sigma_{p=1}^n m_p) - 1] \times 1$.

Thus adding-up retriction on $H$ imply

$$R_2 \gamma = r_2$$

When autoregressive errors are present, our problem is to estimate

$$y_{t\cdot} = y_{t-1\cdot} H + X_{t\cdot} B - X_{t-1\cdot} BH + \zeta_{t\cdot} \tag{1.4}$$

subject to the adding-up contraints within and across systems.

By construction of the problem, joint estimation of two or more singular systems of equations is completely analogous to the estimation of one singular system of equations as presented by Dhrymes [6]. Therefore, when system (1.4) is considered subject to adding-up contraints and subject to Assumption AA, Assumption B, Convention 1, 2, and 3, then the algorithm for estimating one singular system of equations can be applied to the problem of jointly estimating multiple singular systems of equations.

To demonstrate this, we know $\Sigma e = 0$ Now, let $D_1$ be a matrix whose columns are the (orthonormal) characteristic vectors that correspond to the nonzero roots of $\Sigma$. The diagonal matrix containing the nonzero roots of $\Sigma$ (in decreasing order of magnitude) is denoted $\Lambda_1$.

Now, $\Sigma = D_1 \Lambda_1 D_1'$. Let $P'P = \Lambda_1^{-1} \Rightarrow \Lambda_1 = P^{-1}P'^{-1}$. We transform the error $\zeta_{t\cdot}$ such that

$$\begin{aligned} \zeta_{t\cdot}^{**} &= \zeta_{t\cdot} D_1 P' \Rightarrow \\ E[\zeta_{t\cdot}^{**'} \zeta_{t\cdot}^{**}] &= I \Rightarrow \\ \text{minimize } \Sigma_{t=1}^{T} \zeta_{t\cdot}^{**} \zeta_{t\cdot}^{**'} &= \Sigma_{t=1}^{T} \zeta_{t\cdot} \Sigma_g \zeta_{t\cdot}' \end{aligned}$$

subject to adding-up restrictions.

Thus we

$$\text{minimize } L = (y-w)'(\Sigma_g \otimes I_T)(y-w) + 2\lambda_1'(R_1\beta - r_1) + 2\lambda_2'(R_2\gamma - r_2)$$

where

$$
\begin{aligned}
y &= vec(Y) \\
W &= Y_{-1}H + XB - X_{-1}BH \\
w &= vec(W)
\end{aligned}
$$

The vectors of lagrangian multipliers are $\lambda_1$ and $\lambda_2$

The first order conditions can be written as

$$
\begin{bmatrix} P_{11} & P_{12} & R_1' & 0 \\ P_{21} & P_{22} & 0 & R_2' \\ R_1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} d_{.1} \\ d_{.2} \\ d_{.3} \\ d_{.4} \end{bmatrix}
$$

where

$$
\begin{aligned}
P_{11} &= S_1'[(I \otimes X') - (H \otimes X_{-1}')](\Sigma_g \otimes I_T)[(I \otimes X) - (H' \otimes X_{-1})]S_1 \\
P_{12} &= S_1'[(I \otimes X') - (H \otimes X_{-1}')](\Sigma_g \otimes I_T)(I \otimes Y_{-1})S_2 \\
P_{21} &= S_2'[I \otimes (Y_{-1} - X_{-1}B)'](\Sigma_g \otimes I_T)(I \otimes X)S_1 \\
P_{22} &= S_2'[I \otimes (Y_{-1} - X_{-1}B)'](\Sigma_g \otimes I_T)[I \otimes (Y_{-1} - X_{-1}B)]S_2 \\
d_{.1} &= S_1'[(I \otimes X') - (H \otimes X_{-1}')](\Sigma_g \otimes I_T)y \\
d_{.2} &= S_2'[I \otimes (Y_{-1} - X_{-1}B)'](\Sigma_g \otimes I_T)y \\
d_{.3} &= r_1 \\
d_{.4} &= r_2
\end{aligned}
$$

These first order conditions have the same form as the first order conditions taken from the problem of estimating one singular system of equations. Thus, the algorithm from the problem of estimating one singular system of equations can be applied to the problem of jointly estimating multiple singular systems of equations.

## 1.1 Conclusion

When contemporaneous errors from multiple singular systems of equations are correlated, joint estimation of systems uses information more efficiently. To jointly estimate muitiple systems, the generalized inverse of a covariance matrix, which reflects contemporaneous correlation between errors of the various systems, is incorporated within an estimation algorithm. In addition, the estimation algorithm of Dhrymes that is used to estimate the parameters from one singular system of equations is applicable to the joint estimation of multiple singular systems of equations. Finally, adding-up restrictions are imposed within and across systems.

# FOOTNOTES

1. Ernst R. Berndt and N. Eugene Savin, "Estimation and Hypothesis Testing in Singular Equation Systems With Autoregressive Disturbances," *Econometrica* 43 (September 1975): 937-957.

2. The discussion of Barten's approach is taken from the paper of Dhrymes and Schwarz. See Phoebus J. Dhrymes and Samuel Schwarz, "On the Invariance of Estimators for Singular Systems of Demand Equations," *Greek Economic Review* 9 (1987): 88-108.

3. H. Youn Kim, "The Consumer Demand for Education," *The Journal of Human Resources* 23 (Spring 1988): 173-192.

4. Bruce Robert Kingma, "An Accurate Measurement of Crowd-out Effect, Income Effect, and Price Effect for Charitable Contributions," *Journal of Political Economy* 97 (1989): 1197 -1207.

5. Phoebus J. Dhrymes and Samuel Schwarz, "On the Existence of Generalized Inverse Estimators in a Singular System of Equations," *Journal of Forecasting* 6 (1987): 181-192.

6. P. J. Dhrymes, Autoregressive Errors in Singular Systems of Equations, 1984; revised 1988, Discussion Paper No. 257, Department of Economics, Columbia University, New York.

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