The Existence of Steady States in Multisector Capital Accumulation Models with Recursive Preferences

Boyd, John H. III

Working Paper No. 392
November 1994

The Existence of Steady States in Multisector Capital Accumulation Models with Recursive Preferences

John H. Boyd III

Rochester Center for Economic Research
Working Paper No. 392

The Existence of Steady States in Multisector Capital Accumulation Models with Recursive Preferences

John H. Boyd III

November 1994

Abstract. This paper proves the existence of a non-trivial stationary optimal path in a multisectoral capital accumulation model with recursive preferences. The reduced-form recursive preferences are represented by an aggregator function. I introduce a new form of δ-normality that is appropriate for use with recursive preferences. Under some mild conditions on the aggregator, non-trivial steady states exist when the technology is bounded and δ-normal.

Email: `boyj@troi.cc.rochester.edu`

Department of Economics, University of Rochester, Rochester, NY 14627.

I thank Robert Becker, Lionel McKenzie, Karl Shell and Tomoichi Shinotsuka for helpful discussions concerning this project and their comments on earlier drafts of this paper.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. Introduction

Steady states play an important role in the analysis of optimal growth models. The Euler equations may be linearized about the steady states. Their stability properties inform us about the stability of the optimal paths. Moreover, the linearization is often useful in numerical approximations to the optimal paths. In models with additively separable utility, the issue of existence of steady states has long been settled. It was first broached in multisector models by Sutherland (1970). The first real existence results were obtained by Hansen and Koopmans (1972) in a type of von Neumann model, and by Peleg and Ryder (1974) in a consumption-based model. Cass and Shell (1976) handled the continuous-time case. Flynn (1980) and McKenzie (1982, 1986) independently established the current theory for reduced-form models. This analysis has been further enriched by Kahn and Mitra (1986), using purely primal methods, and by Becker and Foias (1986), who use Ky Fan's Inequality instead of the Kakutani Theorem.

Models with more general recursive preferences are a different story. If there is one sector, the existence of steady states follows from appropriate Inada conditions and the intermediate value theorem. If a steady state exists, a local analysis of the optimal path is sometimes possible (e.g., Epstein 1987a, b). Unfortunately, conditions that guarantee existence of steady states in general multisector models have previously been unknown.

This paper proves the existence of a non-trivial stationary optimal path in a reduced-form multisectoral capital accumulation model with recursive preferences. The key to the proof is a new form of δ-normality that is appropriate for use with recursive preferences. Under some mild conditions on the aggregator, non-trivial steady states exist when the technology is bounded and δ-normal. The actual proof is akin to those in McKenzie (1982, 1986) and Khan and Mitra (1986).

Section Two sets up the basic model. Section Three proves the existence of steady states. Some concluding remarks are in Section Four.

2. Multi-Sector Capital Accumulation Models

The reduced-form model is described by a technology set $\mathbf{T} \subset \mathbb{R}_+^{2m}$ obeying (T1)–(T5) and an aggregator function $W\colon \mathbf{T} \times \mathbb{R}_+ \to \mathbb{R}_+$ obeying (A1)–(A6) below. Given a technology set $\mathbf{T}$ and initial capital stock k_0, a path $\mathbf{k} = \{k_t\}_{t=0}^{\infty}$ is *feasible* from k_0 if $(k_{t-1}, k_t) \in \mathbf{T}$ for $t = 1, 2, \ldots$. Define the shift operator S by $S\mathbf{k} = (k_1, k_2, \ldots)$. The utility function is defined as the unique function U that obeys the recursion $U(\mathbf{k}) = W(k_0, k_1, U(S\mathbf{k}))$ for all feasible $\mathbf{k}$. The function W is referred to as the *aggregator*.[1] The aggregator allows us to express lifetime utility as a function of current inputs, current outputs and future utility. By recursively applying W, we may generate the lifetime utility function. An *optimal path* from k_0 is a feasible path that maximizes utility over all feasible paths from k_0. A stock k is a *steady state* if $(k, k, \ldots)$ is optimal from k.

Technology Assumptions. A set $\mathbf{T} \subset \mathbb{R}_+^{2m}$ is a bounded Malinvaud technology set if:

T1) $\mathbf{T}$ is a closed set. (closure)

T2) $\mathbf{T}$ is a convex set. (convexity)

T3) There is $(a, b) \in \mathbf{T}$ with $b \gg 0$. (productivity)

T4) If $(a, b) \in \mathbf{T}$ and $a' \geq a$, $0 \leq b' \leq b$, then $(a', b') \in \mathbf{T}$. (free disposal)

T5) There is a $\beta > 0$ such that $\|b\| < \|a\|$ whenever $(a, b) \in \mathbf{T}$ with $\|a\| > \beta$. (boundedness)

When $(a, b) \in \mathbf{T}$, we interpret a as the input stock and b as the output stock, once consumption has been subtracted. These assumptions on technology: closure, convexity,

[1] See Koopmans (1960) for a discussion of the properties of preferences that yield such a utility function. The preferences considered here are somewhat more general. The converse problem of deriving a utility function U from the aggregator W is discussed in Lucas and Stokey (1984) and Boyd (1990).

productivity, free disposal and boundedness, are all fairly standard. Neither the inaction postulate ($0 \in \mathbf{T}$) nor the no free lunch postulate ($(0,b) \in \mathbf{T}$ implies $b = 0$) are needed. When $\mathbf{T}$ satisfies boundedness, the technology is bounded in the sense that all feasible paths are bounded. If $(k, k, \dots)$ is feasible, boundedness implies $\|k\| \leq \beta$. Any steady state k must have $\|k\| \leq \beta$, and any feasible path starting at k_0 with $\|k_0\| \leq \beta$ obeys $\|k_t\| \leq \beta$. Boundedness does not imply the set $\mathbf{T}$ is bounded.

Let $\mathbf{U}$ be a closed interval in $\mathbb{R} \cup \{-\infty\}$.

Aggregator Assumptions. The aggregator $W\colon \mathbf{T} \times \mathbf{U} \to \mathbf{U}$ obeys:

A1) $W(a,b,u)$ is finite whenever $(a,b) \in \operatorname{int} \mathbf{T}$ and $u > -\infty$. If $-\infty \in \mathbf{U}$, then $W(a,b,-\infty) = -\infty$ for all $(a,b) \in \mathbf{T}$.

A2) W is both upper semicontinuous and lower semicontinuous on $\mathbf{T} \times \mathbf{U}$.[2]

A3) $W(a,b,u)$ is concave in (a,b) for each $u \in \mathbf{U}$.

A4) $W_3 \equiv dW/du$ exists and is continuous in (a,b,u) whenever $W(a,b,u) > -\infty$. Moreover, $0 < \delta^- \leq W_3 \leq \delta^+ < 1$ for some δ^- and δ^+.

A5) If $a' \geq a$, $0 \leq b' \leq b$ and $u' \geq u$ then $W(a',b',u') \geq W(a,b,u)$. (monotonicity)

A6) Let $u \in \mathbf{U}$. Each W^n defined inductively by $W^n(\mathbf{k}) = W(k_0, k_1, W^{n-1}(S\mathbf{k}))$ and $W^0(\mathbf{k}) = u$ is concave in $\mathbf{k}$.

Assumptions (A1) and (A2) are meant to include commonly used cases, such as the additively separable aggregator $W(a,b,u) = \log(f(a) - b) + \delta u$ in the case $m = 1$. Here the technology set is $\mathbf{T} = \{(a,b) \in \mathbb{R}^2_+ : b \leq f(a)\}$ with f a smooth, increasing, concave production function with $f(0) = 0$ and $f'(a) < a$ for $a > \beta$. This example also illustrates that continuity of W with respect to (a,b) does not follow from (A4). Assumption (A6) amounts to assuming that the derived utility function U is concave. The concavity assumption (A3) and monotonicity (A6) are standard in reduced-form models.

[2] This means W is continuous as an extended real-valued function.

There are also non-separable examples that satisfy (A1)–(A6). Again take the one-sector case with production function f above. Let $W(a,b,u) = -1 + e^{-v(f(a)-b)}u$ with $u \in \mathbf{U} = \mathbb{R}_-$, where v is a smooth, increasing, concave function with $v(0) > 0$. Properties (A1)–(A3) and (A5) are obvious. Set $\delta^- = e^{-v(\beta)}$ and $\delta^+ = e^{-v(0)}$ to satisfy (A4). To see that (A6) holds, examine

$$W^n(\mathbf{k}) = -\sum_{t=1}^{n} \exp\left(-\sum_{\tau=1}^{t-1} v(f(k_{\tau-1}) - k_\tau)\right) + u \exp\left(-\sum_{t=1}^{n} v(f(k_{t-1}) - k_t)\right).$$

Now W^n is concave because the mapping $x \mapsto -e^{-x}$ is concave and each of the sums $\sum_{\tau=1}^{t-1} v(f(k_{\tau-1}) - k_\tau)$ is concave.

Let $u \in \mathbf{U}$ and set $B_0 = \sup \mathbf{U}$ and $B_1 = \sup\{W(a,b,u) : (a,b) \in \mathbf{T} \text{ and } \|a\| \le \beta\}$. Let $B_2 = \max\{u, B_1\}$ and $B = \min\{B_0, (1-\delta^+)^{-1}(B_2 - \delta^+ u)\}$. Note $B \ge u$. By (A4),

$$\begin{aligned} W(a,b,B) &\le W(a,b,u) + \delta^+(B-u) \\ &\le B_2 + \delta^+(B-u) \le B \end{aligned}$$

for all $(a,b) \in T$ with $\|a\| \le \beta$. Confining our attention to feasible paths starting at some k_0 with $\|k_0\| \le \beta$, define W^n as above using $u = B$. We have seen that $W^1(\mathbf{k}) \le B = W^0(\mathbf{k})$. An easy induction shows $W^n(\mathbf{k}) \le W^{n-1}(\mathbf{k})$ for all n. Define the utility function $U(\mathbf{k}) = \inf_n W^n(\mathbf{k})$.

The Lipschitz condition implicit in (A4) implies that basing W^n on u' rather than u makes a difference of less than $(\delta^+)^n|u - u'|$. As a result, the infimum of the W^n does not depend on our starting point u. The function U is recursive as $U(\mathbf{k}) = \inf W^n(\mathbf{k}) = \inf W(k_0, k_1, W^{n-1}(S\mathbf{k})) = W(k_0, k_1, \inf W^{n-1}(S\mathbf{k})) = W(k_0, k_1, U(S\mathbf{k}))$.[3] The infimum is upper semicontinuous in the product topology. Condition (A6) guarantees that U is concave.

[3] This construction is based on the "partial summation" method of Boyd (1990).

When $(k,k) \in \mathbf{T}$ with $\|k\| \leq \beta$, define $\Phi(k) = U(k,k,\dots)$. Note that if $W(k,k,u) > -\infty$ for some $u \in \mathbf{U}$ then $\Phi(k) > -\infty$. Of course Φ is upper semicontinuous because U is.

We also need a joint assumption on W and $\mathbf{T}$. Even in the additive case, productivity must be sufficient to overcome discounting if there is to be a steady state. Define $\delta_k = W_3(k,k,\Phi(k))$. This is the discount factor on the constant path $(k,k,\dots)$. Let $\delta = \inf \delta_k \geq \delta^- > 0$.

δ-NORMALITY ASSUMPTION. The reduced-form model $(\mathbf{T}, W)$ is δ-normal if there is a pair $(\bar{a},\bar{b}) \in \mathbf{T}$ such that $0 \ll \bar{a} \leq \delta\bar{b}$ and $\Phi(\bar{a}) > \Phi(0)$, where we interpret $\Phi(0)$ as $-\infty$ if $(0,0) \notin \mathbf{T}$.

Here $\Phi(\bar{a})$ makes sense as $(\bar{a},\bar{a}) \in T$ by free disposal.

3. The Existence of Steady States

In this section, we will assume $(\mathbf{T}, W)$ obeys (T1)–(T5), (A1)–(A6) and δ-normality. Define $\mathbf{T}_1 = \{a \in \mathbb{R}^m_+ : (a,b) \in \mathbf{T}$ for some b and $\|a\| \leq \beta\}$ and $\mathbf{K} = \{k \in \mathbb{R}^m_+ : (k,k) \in \mathbf{T}$ and $\Phi(k) \geq \Phi(\bar{a})\}$. We will obtain a steady state in $\mathbf{K}$. Of course, $\|k\| \leq \beta$ whenever $k \in \mathbf{K}$, so $\mathbf{K} \subset \mathbf{T}_1$. Note that $\mathbf{K}$ is compact and convex. It also contains $\bar{a}$ by free disposal and monotonicity from $(\bar{a},\bar{b})$. Even if $(0,0) \in \mathbf{T}$, $0 \notin \mathbf{K}$ because $\Phi(0) < \Phi(\bar{a})$. As a result, any steady state in $\mathbf{K}$ is automatically non-trivial.

For $k \in \mathbf{T}_1$, $g(k)$ denote the closest point in $\mathbf{K}$ to k. This defines a continuous function from $\mathbf{T}_1$ to $\mathbf{K}$. The mappings $k \mapsto \delta_{g(k)}$ and $k \mapsto \Phi(g(k))$ extend δ_k and Φ_k. Abusing notation, we also denote the extensions by δ_k and $\Phi(k)$.

For $k \in \mathbf{T}_1$, define $\phi(k) = \{(a,b) \in \mathbf{T} : a \leq (1-\delta_k)k + \delta_k b\}$. Because $(\bar{a},\bar{a}) \in \phi(k)$, $\phi(k)$ is non-empty and $\phi\colon \mathbf{T}_1 \twoheadrightarrow \mathbf{T}$ defines a correspondence. If $(a,b) \in \phi(k)$ with $\|a\| > \beta$, $\|a\| \leq (1-\delta_k)\|k\| + \delta_k\|b\| \leq (1-\delta_k)\|k\| + \delta_k\|a\|$ by boundedness. But then, $\|a\| \leq \|k\| \leq \beta$. It follows that $\|a\| \leq \beta$ for all $(a,b) \in \phi(k)$. But then $\|b\| \leq \beta$ too, as we could otherwise take $a' \geq a$ with $\|b\| > \|a'\| > \beta$ and $(a', b) \in \mathbf{T}$. This would violate boundedness. Therefore $\phi(k)$ is compact and convex for every $k \in \mathbf{T}_1$.

A stock $k \in \mathbf{K}$ is a *recursive golden rule stock* if $(k,k) \in \mathbf{T}$ and $W(k,k,\Phi(k)) \geq W(a,b,\Phi(k))$ for all $(a,b) \in \phi(k)$. Notice that 0 cannot be a recursive golden rule as $0 \notin \mathbf{K}$. When W is additively separable, with $W(a,b,u) = v(a,b) + \delta u$, this definition reduces to the usual definition of a non-trivial discounted golden rule.

Given a stock $k \in \mathbf{T}_1$, the *implicit programming problem* is to solve

$$J(k) = \sup\{W(a,b,\Phi(k)) : (a,b) \in \phi(k)\}.$$

This problem has a solution because $\phi(k)$ is compact and $W(a,b,\Phi(k))$ is upper semicontinuous in (a,b). Because $(\mathbf{T}, W)$ is δ-normal, not only is $(\bar{a}, \bar{b}) \in \phi(k)$ for all $k \in \mathbf{K}$, but $J(k) \geq W(\bar{a}, \bar{b}, \Phi(k))$. Since $\Phi(k) \geq \Phi(\bar{a})$, it follows that $J(k) \geq W(\bar{a}, \bar{b}, \Phi(\bar{a})) \geq W(\bar{a}, \bar{a}, \Phi(\bar{a})) = \Phi(\bar{a}) > -\infty$ by free disposal and monotonicity. We may examine the set of solutions $\mu(k)$. More formally, the correspondence $\mu(k)$ is given by

$$\mu(k) = \{(a,b) \in \phi(k) : W(a,b,\Phi(k)) = J(k)\}.$$

Note that $\mu(k)$ is also non-empty, compact and convex for each $k \in \mathbf{K}$. Thus a stock $k \in \mathbf{K}$ is a recursive golden rule if and only if $(k,k) \in \mu(k)$.

We will only show the existence of non-trivial golden rules when the technology is both bounded and δ-normal. As is well-known, discounted golden rules may not exist if the technology is not bounded, even in the one-sector case. This occurs in models that generate balanced growth. If the technology is not sufficiently productive, all optimal paths may converge to 0. The δ-normality condition rules out cases where all optimal paths converge to 0, and is a generalization of the requirement that $\delta f'(0+) > 1$ in the one-sector model.

The proof of existence of golden rules proceeds as follows. We start with a preliminary lemma that shows Φ and δ_k are continuous on $\mathbf{T}_1$. We next show ϕ has a lower semicontinuity property in Lemma 2. We then use that fact to show $\mu(k)$ is a closed correspondence in Lemma 3.[4] Finally, we project μ onto the input space $\mathbf{T}_1$. This new correspondence ψ on $\mathbf{T}_1$ is closed with compact and convex values. The Kakutani Fixed Point Theorem yields a fixed point of ψ, and that fixed point is a recursive golden rule. Finally, we show that if k is a recursive golden rule, the path $(k, k, \dots)$ is optimal from k.

LEMMA 1. *The function Φ and the mapping $k \mapsto \delta_k$ are continuous on $\mathbf{T}_1$.*

PROOF. Because g is continuous, it is enough to show δ_k and $\Phi(k)$ are continuous on $\mathbf{K}$. Let $\epsilon > 0$ and $k \in \mathbf{K}$. Set $u = \Phi(k)$ in the definition of W^n. I first claim that for $k' \in \mathbf{K}$, $|W^n(\mathbf{k}') - \Phi(k)| \le (1-\delta^+)^{-1}|W(k', k', \Phi(k)) - W(k, k, \Phi(k))|$ where $\mathbf{k}' = (k', k', \dots)$. This is established by induction. When $n = 1$, it follows because $(1 - \delta^+)^{-1} \ge 1$ and $W(k', k', \Phi(k)) = W^1(\mathbf{k}')$. Now suppose it holds for n. Then

$$\begin{aligned}
|W^{n+1}(\mathbf{k}') - \Phi(k)| &= |W(k', k', W^n(\mathbf{k}')) - \Phi(k)| \\
&\le |W(k', k', W^n(\mathbf{k}')) - W(k', k', \Phi(k))| + |W(k', k', \Phi(k)) - \Phi(k)| \\
&\le \delta^+|W^n(\mathbf{k}') - \Phi(k)| + |W(k', k', \Phi(k)) - \Phi(k)| \\
&\le \frac{\delta^+}{1-\delta^+}|W(k', k', \Phi(k)) - \Phi(k)| + |W(k', k', \Phi(k)) - \Phi(k)| \\
&\le (1-\delta^+)^{-1}|W(k', k', \Phi(k)) - W(k, k, \Phi(k))|.
\end{aligned}$$

The claim now follows by induction.

Because $W(k, k, \Phi(k)) > -\infty$, (A2) implies we may pick $\eta > 0$ with $|W^n(\mathbf{k}') - \Phi(k)| < \epsilon/2$ for $\|k' - k\| < \eta$. Letting $n \to \infty$ shows $|\Phi(k') - \Phi(k)| < \epsilon$ for $\|k' - k\| < \eta$.

Now $\delta_k = W_3(k, k, \Phi(k))$. By (A4) and the continuity of Φ, δ_k is also continuous. □

[4] A correspondence ϕ is *closed* if whenever $x_n \to x$ and $y_n \to y$ with $y \in \phi(x_n)$, we have $y \in \phi(x)$.

LEMMA 2. *Suppose $(x,y) \in \phi(k)$ with $(x,y) \neq (\bar{a},\bar{b})$. If $\epsilon > 0$, there is α with $0 < \alpha < \min\{1, \epsilon/(\|(x,y) - (\bar{a},\bar{b})\|)\}$ and $\eta > 0$ such that $((1-\alpha)x + \alpha\bar{a}, (1-\alpha)y + \alpha\bar{b}) \in \phi(k')$ whenever $\|k' - k\| < \eta$.*

PROOF. Let $z = (1-\alpha)x + \alpha\bar{a}$ and $w = (1-\alpha)y + \bar{b}$. By convexity of $\mathbf{T}$, $(z,w) \in \mathbf{T}$. Then define

$$q(k') = (1-\alpha)(\delta_{k'} - \delta_k)y + [(1-\delta_{k'})k' - (1-\alpha)(1-\delta_k)k] + \alpha(\delta_{k'} - \delta)\bar{b}.$$

Note that q is continuous in k' and that $q(k) = \alpha(1-\delta_k)k + \alpha(\delta_k - \delta)\bar{b} \geq 0$. There are now two cases to consider. First, if $\delta_k > \delta$, $q(k) \gg 0$. It follows that $q(k') \gg 0$ for k' near k. Second, if $\delta_k = \delta$, the first and third terms of $q(k')$ are always non-negative, and the middle term is non-negative when k' is sufficiently near k. Either way, we may choose $\eta > 0$ with $q(k') \geq 0$ for $\|k' - k\| < \eta$.

Using α to form a convex combination of $x \leq (1-\delta_k)k + \delta_k y$ and $\bar{a} \leq \delta\bar{b}$, we find

$$z \leq (1-\delta_{k'})k' + \delta_{k'}w - q(k') \leq (1-\delta_{k'})k' + \delta_{k'}w$$

for $\|k' - k\| < \eta$. It follows that $(z,w) \in \phi(k')$ for $\|k' - k\| < \eta$. □

LEMMA 3. *The correspondence $\mu\colon \mathbf{K} \twoheadrightarrow \mathbf{T}$ is a closed correspondence.*

PROOF. Suppose $(a_n, b_n) \in \mu(k_n)$ with $(a_n, b_n) \to (a,b)$ and $k_n \to k$. Then $(a,b) \in \mathbf{T}$ by closure and $a \leq (1-\delta_k)k + \delta_k b$, so $(a,b) \in \phi(k)$. Since W and Φ are upper semicontinuous and W is non-decreasing in u, $J(k) \geq W(a,b,\Phi(k)) = \limsup W(a_n, b_n, \Phi(k_n)) = \limsup J(k_n)$. We need only show $W(a,b,\Phi(k)) = \limsup W(a_n, b_n, \Phi(k_n)) \geq J(k)$ to complete the proof.

Note that $W(a,b,\Phi(k)) \geq \limsup J(k_n) \geq W(\bar{a},\bar{b},\Phi(k))$. Suppose $J(k) > W(a,b,\Phi(k))$. Then $J(k) > W(\bar{a},\bar{b},\Phi(k))$. Take $(x,y) \in \phi(k)$ with $W(x,y,\Phi(k)) = J(k)$. By Lemma 2, we can construct a subsequence of the k_n (also labeled k_n) and $\alpha_n < 1/n$ with $(x_n, y_n) \in \phi(k_n)$ where $x_n = (1-\alpha_n)x + \alpha_n \bar{a}$ and $y_n = (1-\alpha_n)y + \alpha_n \bar{b}$.

Now $W(x_n, y_n, \Phi(k_n)) \leq J(k_n)$ because $(x_n, y_n) \in \phi(k_n)$. Furthermore,

$$W(x_n, y_n, \Phi(k_n)) \geq (1-\alpha_n)W(x, y, \Phi(k_n)) + \alpha_n W(\bar{a}, \bar{b}, \Phi(k_n))$$

by concavity. Rearranging and using $J(k_n) \geq W(x_n, y_n, \Phi(k_n))$, we obtain

$$\alpha_n[W(x, y, \Phi(k_n)) - W(\bar{a}, \bar{b}, \Phi(k_n))] \geq W(x, y, \Phi(k_n)) - J(k_n)$$

Let $\epsilon > 0$. We know

$$[W(x, y, \Phi(k_n)) - W(\bar{a}, \bar{b}, \Phi(k_n))] \to [J(k) - W(\bar{a}, \bar{b}, \Phi(k))] > 0$$

by Lemma 1 and the continuity of W in u. For n large, we then have $\epsilon > [W(x, y, \Phi(k_n)) - J(k_n)]$. But $W(x, y, \Phi(k_n)) \to J(k)$, so $\epsilon/2 + J(k_n) > J(k)$ for n large. It follows that $J(k) \geq \limsup J(k_n) \geq \liminf J(k_n) \geq J(k)$, and so $W(a, b, \Phi(k)) = J(k)$. This shows μ is closed. □

THEOREM 1. *Suppose (A1)–(A6), (T1)–(T5) and δ-normality hold. A recursive golden rule exists.*

PROOF. Define a correspondence $\psi\colon \mathbf{T}_1 \twoheadrightarrow \mathbf{T}_1$ by $\psi(k) = \{a \in \mathbf{T}_1 : (a, b) \in \mu(k)$ for some $b\}$. We know ψ maps into $\mathbf{T}_1$ because ϕ and μ are bounded by β. It is clear that ψ has convex, compact, non-empty values. It is also closed. Let $k_n \to k$ and $a_n \to a$ with $a_n \in \psi(k_n)$. Take b_n with $(a_n, b_n) \in \mu(k_n)$. Now $\|a_n\| \leq \beta$ and $\|b_n\| \leq \beta$. By passing to a subsequence, we may assume $b_n \to b$. But then $(a, b) \in \mu(k)$ because μ is closed. Hence ψ is closed.

Now apply the Kakutani Fixed Point Theorem to obtain k with $k \in \psi(k)$. There is a y with $(k, y) \in \mu(k)$. Then $k \leq (1 - \delta_k)k + \delta_k y$, which implies $k \leq y$. By free disposal $(k, k) \in \mathbf{T}$, and $W(k, k, \Phi(k)) \geq W(k, y, \Phi(k)) = J(k)$ by monotonicity. It then follows that $J(k) = W(k, k, \Phi(k)) \geq W(a, b, \Phi(k))$ for all $(a, b) \in \phi(k)$. Moreover, $\Phi(k) = J(k) \geq \Phi(\bar{a}) > -\infty$. Thus $k \in \mathbf{K}$, and so is a recursive golden rule. □

The final step is to show that any recursive golden rule yields a stationary optimal program.

THEOREM 2. *Suppose (A1)–(A6), (T1)–(T5) and δ-normality hold. If k is a recursive golden rule, the stationary path $(k, k, \dots)$ is optimal from k.*

PROOF. By δ-normality, $\phi(k)$ has an interior. Now $(a, b) \mapsto W(a, b, \Phi(k))$ is concave on $\phi(k)$, and has a maximum at (k, k). Thus there is a vector (p, q) which supports both W and $\phi(k)$. In other words, $W(a, b, \Phi(k)) \leq W(k, k, \Phi(k)) + p(a - k) + q(b - k)$ for all $(a, b) \in \mathbf{T}$, and $(p, q) = \lambda(p', q') + \sum_{i=1}^{m} \lambda_i(e_i, -\delta_k e_i)$ for some $\lambda, \lambda_i \geq 0$ where (p', q') supports $\mathbf{T}$.[5] That is, $p'a + q'b \leq pk + qk$ for all $(a, b) \in \mathbf{T}$.

Set $u = \Phi(k)$ in the definition of W^n and let $\mathbf{k} = (k, k, \dots)$. Then $W^n(\mathbf{k}) = U(\mathbf{k}) = \Phi(k)$. Because W^n is concave, and W is continuously differentiable and non-decreasing in the third argument, we have

$$W^n(\mathbf{x}) - W^n(\mathbf{k}) \leq p(x_0 - k) + \sum_{t=1}^{n-1} (\delta_k p + q)\delta_k^{t-1}(x_t - k) + \delta_k^n q(x_n - k).$$

Recalling that $x_0 = k$, and taking the limit as $n \to \infty$ yields

$$U(\mathbf{x}) \leq U(\mathbf{k}) + \sum_{t=1}^{\infty} (\delta_k p + q)\delta_k^{t-1}(x_t - k).$$

[5] Here e_i denotes the i^{th} standard basis vector of $\mathbb{R}^m$.

Now $\delta_k p + q = \lambda\delta_k p' + \lambda q' + \sum_{i=1}^m \lambda_i(\delta_k e_i - \delta_k e_i) = \lambda\delta_k p' + \lambda\delta_k q'$. Consider the convex combination $(1-\delta_k)^{-1}\sum_{t=1}^\infty \delta_k^{t-1}(x_{t-1}, x_t) \in \mathbf{T}$.[6] By the support property of (p', q'),

$$\sum_{t=1}^\infty \delta_k^{t-1}(p'x_{t-1} + q'x_t) \leq \sum_{t=1}^\infty \delta_k^{t-1}(p'k + q'k)$$

Since $\mathbf{x}$ is feasible from $x_0 = k_0$, we can rewrite to obtain $\sum_{t=1}^\infty \delta_k^{t-1}(\delta_k p' + q)(x_t - k) \leq 0$. But $\delta_k p + q = \delta_k p' + q'$, so $U(\mathbf{x}) \leq U(\mathbf{k})$. Thus $\mathbf{k}$ is optimal from k. □

4. Conclusion

As we see above, stationary optimal programs will exist in a wide variety of multisector models with recursive utility. This provides a firm ground for local analysis of the steady states.

A still broader existence theorem might be possible in the recursive case. One important difference between recursive models and additively separable models is that the discount factor can adjust in recursive models. As a result, it is common in one-sector recursive models to have steady states even when the economy can grow at a high rate. In contrast, this only occurs in the additive case under special circumstances. The general question of existence of steady states in multisector models with unbounded technology remains open, both in the recursive setting, and in the special case of additive utility.

[6] This construction has been used in a primal context by Dechert and Nishimura (1983) and Khan and Mitra (1986).

References

Robert A. Becker and Ciprian Foias (1986), A minimax approach to the implicit programming problem, *Econ. Letters* **20**, 171–175.

John H. Boyd III (1990), Recursive utility and the Ramsey problem, *J. Econ. Theory* **50**, 326–345.

David Cass and Karl Shell (1976), The structure and stability of competitive dynamical systems, *J. Econ. Theory* **12**, 31–70.

W. Davis Dechert and Kazuo Nishimura (1983), A complete characterization of optimal growth paths in an aggregated model with a non-concave production function, *J. Econ. Theory* **31**, 332–354.

Larry G. Epstein (1987a), The global stability of efficient intertemporal allocations, *Econometrica* **55**, 329–355.

Larry G. Epstein (1987b), A simple dynamic general equilibrium model, *J. Econ. Theory* **41**, 68–95.

James Flynn (1980), The existence of optimal invariant stocks in a multi-sector economy, *Rev. Econ. Stud.* **47**, 809–811.

T. Hansen and Tjalling C. Koopmans (1972), On the definition and computation of a capital stock invariant under optimization, *J. Econ. Theory* **5**, 487–523.

M. Ali Khan and Tapan Mitra (1986), On the existence of a stationary optimal stock for a multi-sector economy: A primal approach, *J. Econ. Theory* **40**, 319–328.

Tjalling C. Koopmans (1960), Stationary ordinal utility and impatience, *Econometrica* **28**, 287–301.

Robert E. Lucas, Jr. and Nancy L. Stokey (1984), Optimal growth with many consumers, *J. Econ. Theory* **32**, 139–171.

Lionel W. McKenzie (1982), A primal route to the turnpike and Liapounov stability, *J. Econ. Theory* **26**, 194–209.

Lionel W. McKenzie (1986), Optimal economic growth, turnpike theorems and comparative dynamics, *in* "Handbook of Mathematical Economics, III" (Kenneth J. Arrow and Michael D. Intriligator, eds.), North-Holland, New York.

Bezalel Peleg and Harl E. Ryder, Jr. (1974), The modified golden rule of a multi-sector economy, *J. Math. Econ.* **1**, 193–198.

W. R. S. Sutherland (1970), On optimal development in a multi-sectoral economy: The discounted case, *Rev. Econ. Studies* **37**, 585–588.