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*This paper will be published in a collection of papers dedicated to the memory of Yasuo Uekawa, who was the eighth Japanese to enter the graduate program of the University of Rochester after its beginning in 1957. He entered in 1963. He was the fifth Japanese to receive the Ph. D. degree in economics from Rochester. He received his degree in 1966. After his graduation he was always a loyal supporter of Rochester economics, sending us many good students and providing generous hospitality to Rochester faculty and graduates who visited Japan, indeed who were sometimes able to visit Japan through his efforts. He was a leader in bringing our Japanese graduates together as an informal and then formal group. His affection for Rochester and his friendship for his colleagues here will always be remembered by those who enjoyed it. We are proud of his contributions to international economics and the position he earned in the ranks of Japanese economists. I am happy to contribute this small paper to his memory.

THE COMPETITIVE EQUILIBRIUM TURNPIKE

It was first shown by Truman Bewley (1982) that it is possible to combine the Ramsey turnpike theorem and the theorem on existence for a competitive equilibrium over an infinite horizon to provide a turnpike theorem for the competitive equilibrium. His result was generalized and extended by Yano (1984a, 1984b, and 1985), Marimon (1989), Coles (1989), Epstein (1987), and Lucas and Stokey (1984). The theorem which I will prove in this paper is close to the theorem of Yano (1984a). It dispenses with differentiability which is assumed by Bewley and uses a convex cone for the social production set rather than a collection of firms owning convex production sets. My purpose is to present a proof which exploits a theorem on existence for an economy with an infinite horizon (Boyd and McKenzie (1993)) and a theorem on turnpikes for an optimal growth problem (McKenzie (1982). Thus attention is focused on how these theorems can be combined to give the result. This may show more clearly the relation between the assumptions needed for existence and those needed for optimal growth and the further steps required to establish the competitive turnpike theorem.

Since the turnpike theorem was proved for a model in which separability is assumed both for consumer utility and for production, the model in which the competitive turnpike theorem is proved will not have the generality of the infinite horizon economy of Boyd and McKenzie. In their existence theorem neither type of separability appeared. We will also make the assumption that future utility is discounted by a constant factor which is the same for all consumers. Separability for consumption and constant discounting has been partially relaxed in optimal growth models by Lucas and Stokey (1984) and Epstein (1987). However we will not pursue this generalization here.

The existence theorem of Boyd and McKenzie (1993) is specialized to an economy with the production sector used by Malinvaud (1953) to discuss efficiency for infinite programs. The model is in discrete time and production has the property of separability between time periods. We also introduce a consumption sector with separability between time periods. Thus the economy has the Markov property that the possibilities of production and consumption in any period depend only on the state of the economy in that period. The state of the economy is given by the stock of capital, the technology, and the consumption possibility sets. The technology set and the consumption possibility sets are constant over time. On the consumption side this may reflect a set of families for which the distribution of abilities and tastes do not change from one generation to another. On the production side capital accumulation may occur. However the social production set is assumed to be a convex cone which is constant over time. This production set may still represent an economy of firms if entrepreneurial factors are introduced (McKenzie (1959)).

The commodity space is $s^n = \Pi_{t=0}^{\infty} \mathbb{R}^n(t)$, the space of sequences of n–vectors. This space is given the product topology where each $\mathbb{R}^n$ has the l_1 topology ($|x| = \Sigma_{i=1}^{n} |x_i|$). Separability in the production sector is recognized by setting $Y = \Sigma_{t=1}^{\infty} \Upsilon_t$ where Υ_t contains vectors of the form $y^t = (0,\cdots,0,-k_{t-1},v_t,0,\cdots)$. The vector $-k_{t-1} \in \mathbb{R}^n_-$ represents inputs of capital stocks at the beginning of the t*th* period and $v_t \in \mathbb{R}^n$ represents inputs and outputs during the t*th* period including terminal capital stocks and inputs of goods and services supplied by consumers. Let the set Y_t be the set of all $y(t) = (-k_{t-1},v_t) \in \mathbb{R}^n_- \times \mathbb{R}^n$ for which there is $y^t \in \Upsilon_t$ with $y^t = (0,\cdots,0,-k_{t-1},v_t,0,\cdots)$. The capital stocks $-k_0$ are inputs for the production processes of the first period. Outputs $v_t = k_t + x_t$ where $k_t \in \mathbb{R}^n_+$ are terminal stocks and $x_t \in \mathbb{R}^n$ are goods and services either taken by consumers or provided by consumers.

Possible consumption sets are $C^h = \Sigma_{t=0}^{\infty} \mathcal{C}_t^h$ where $\mathcal{C}_t^h$ contains vectors of the form $x^{ht} = (0,\cdots,0,x_t^h,0,\cdots)$. $\mathcal{C}_0^h = \{(x_0^h,0,\cdots)\}$ with $x_0^h = -k_0^h$. Thus $\mathcal{C}_0^h$ represents the provision of initial stocks. Let C_t^h be the set of all $x_t^h \in \mathbb{R}^n$ such that there is $x^{ht} \in \mathcal{C}_t^h$ with $x^{ht} = (0,\cdots,0,x_t^h,0,\cdots)$. Negative components of x_t^h are quantities of goods and services provided by the hth consumer during the period and positive components of x_t^h are quantities of goods and services received by the consumer during the period. We assume that some components of x_t^h are necessarily not positive. These are included in $x_t^{1h} \in \mathbb{R}_-^{n_1}$ and the remaining components are included in $x_t^{2h} \in \mathbb{R}^{n_2}$ where $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with $n_1,n_2 \neq 0$ and $n_1 + n_2 = n$. Then we may write $x_t^h = (x_t^{1h},x_t^{2h})$. We will say that $y \in s^n$ is feasible when $y = \Sigma_{t=0}^{\infty} y^t = \Sigma_{t=0}^{\infty} x^t = x$, where $y^t \in Y_t$ and $x^t = \Sigma_{h=1}^{H} x^{ht}$ with $x^{ht} \in \mathcal{C}_t^h$.

The initial capital stocks are allocated among consumers, that is, $k_0 = \Sigma_{h=1}^{H} k_0^h$. Subsequent capital stocks need not be explicitly allocated since only their values are relevant and the value of a consumer's holding of capital stocks is implied by the value of his initial stocks and the spending he does in earlier periods. We abstract from timing during the period of the use of consumption goods and intermediate products. This comes from modeling the competitive economy in discrete time. A periodwise utility function u_t^h is defined on the periodwise consumption set C_t^h for $t = 1,2,\cdots$. A preference relation P^h is defined on consumption streams by zP^hx if $\Sigma_{t=1}^{\infty} u_t^h(z_t) > \Sigma_{t=1}^{\infty} u_t^h(x_t)$. R^h and I^h are defined in terms of P^h in the usual way. The use of a periodwise consumption possibility set and this definition of the preference relation is the meaning of separability in consumption.

In this paper we will say that an economy is *irreducible* if, whenever the consumers are divided into two subsets and the allocation to consumers is producible, there is another producible allocation in which everyone in the first subset is better off, and the allocation to the second subset is a positive multiple of some consumption bundle which is sufficient for their survival. We will say that an economy is *strongly irreducible* if it is irreducible and, in the definition of irreduciblility, whenever the second subset contains only one consumer his allocation lies within his possible consumption set. The Malinvaud economy E_m is given by $(Y_t, C^1_{t-1}, \cdots, C^H_{t-1}, \rho, u^1, \cdots, u^H)$. Let $e = (e_0, e_1, \cdots)$, where $e_t = (1, \cdots, 1) \in \mathbb{R}^n$ for all t. Let $C_t = \Sigma^H_{h=1} C^h_t$. We make the following assumptions.

1. The periodwise production set $Y_t \subset \mathbb{R}^n_- \times \mathbb{R}^n$, $t = 1,2,\cdots$, is a closed convex cone with vertex at the origin, and $Y_t = Y_s$, all s and t. If $y(t) = (-k_{t-1}, v_t) \in Y_t$ then $k'_{t-1} \geq k_{t-1}$ and $v'_t \leq v_t$ implies $(-k'_{t-1}, v'_t) \in Y^t$.

2. Let $(-k_{t-1}, v_t) \in Y_t$, where $v_t = k_t + x_t$. Then $k_{t-1} = 0$ and $x^1_t = 0$ implies $v_t = 0$. Also there is $\zeta > 0$ such that $|k_{t-1}| > \zeta$ and $x_t \in C_t$ implies $|k_t| < \gamma |k_{t-1}|$ for $\gamma < 1$.

3. The periodwise consumption set $C^h_t \subset \mathbb{R}^n$ is convex, closed, and bounded below by $\bar{z}$ for all h. Also $C^h_t = C^h_s$ for all h, and s,t.

4. The utility function $u^h_t = \rho^t u^h$, where u^h is a real valued function on C^h_t, which is concave, continuous, and bounded. Also if $x_t \in C^h_t$ and $z_t > x_t$ then $z_t \in C^h_t$ and $u^h(z_t) > u^h(x_t)$.

5. The economy E_m is strongly irreducible.

6. There is $\bar{w}^h \in C^h - Y$, for all h, with $\bar{w}^h \leq 0$. Moreover, for some $\delta > 0$, $\Sigma^H_{h=1} \bar{w}^h = \bar{w} < -\delta e$. For any $x^h \in C^h - Y$ there is τ_0 such that for any $\tau > \tau_0$ there is $\alpha > 0$ with $(x^h_0, \cdots, x^h_\tau, \alpha\bar{w}^h_{\tau+1}, \alpha\bar{w}^h_{\tau+2}, \cdots) \in C^h - Y$.

Define $U^h(z) = \Sigma^\infty_{t=1} \rho^t u^h(z_t)$ for $z \in C^h$. We first prove

<u>Lemma</u> 1. The function U^h is concave, continuous, and bounded on C^h.

Proof. The fact that u^h is uniformly bounded with respect to t on C^h_t and that $\rho < 1$ implies that U^h is well defined and bounded on C^h. Concavity is an immediate consequence of the concavity of u^h. For continuity it is sufficient to show that for any open neighborhood V of $v = U^h(z)$ there is an open neighborhood Z of z relative to C^h in the topology of s^n such that $z' \in Z$ implies $U^h(z') \in V$. We may assume that V contains all $v' \in \mathbb{R}$ such that $|v - v'| < \epsilon > 0$. In the product topology all but a finite number of the factors of an open set must be unrestricted. Choose small open sets W_t of z_t relative to C_t^h for $1 < t < T$. Then the open set $Z = \mathbb{R}_+^n \times \Pi_{t=1}^T W_t \times \Pi_{t>T} C_t^h$ is a neighborhood of z. The W_t and T may be chosen so that $z' \in Z$ implies that $\Sigma_{t=1}^T \rho^t |(u^h(z_t') - u^h(z_t)| < \epsilon/2$ and $\Sigma_{t=T+1}^{\infty} \rho^t b < \epsilon/2$ where b is an upper bound on $|u^h(w_t)|$ for $w_t \in C_t^h$ for all t. It is implied by the continuity and boundedness of u and the fact that $\gamma < 1$ that these choices can be made and define an open set Z of z relative to C^h that maps into V under U^h. □

We may now prove that a competitive equilibrium exists.

<u>Theorem 1</u>. The economy E_m has a competitive equilibrium.

Proof. Existence of competitive equilibrium follows from Theorem 3 in Boyd and McKenzie (1993) if we can show that Assumptions 1 – 7 in that paper are implied by Assumptions 1 – 6 above. The Assumptions for the existence theorem are

B–M.1. Y is a closed convex cone with vertex at the origin that contains no straight lines.

B–M.2. For each $\bar{y} \in s^n$ the set $\{y \in Y | \ y \geq \bar{y}\}$ is bounded.

B–M.3. C^h is convex, closed, and bounded below by $\bar{z} \in l_\infty$.

B–M.4. For all h the correspondence P^h is convex valued and, relative to C^h, open valued with open lower sections. The preference relation P^h is irreflexive and transitive. The weakly preferred set $R^h(x^h)$ is the closure of

$P^h(x^h)$ for all $x^h \in C^h$, unless $P^h(x^h) = \phi$.

B–M.5. Let $x^h \in C^h$. If $z_t^h \geq x_t^h$, with strict inequality for some t, then $z \in P^h(x^h)$.

B–M.6. The economy is strongly irreducible.

B–M.7. For all h, there is $\bar{x}^h \in C^h - Y$ with $\bar{x}^h \leq 0$. Moreover, $\bar{x} = \Sigma_{h=1}^H \bar{x}^h < 0$ and $\bar{x}_t = \bar{x}_s$ for all s and t. For any x^h, let $z^h \in R^h(x^h) - Y$ and $\delta > 0$. Then there is a τ_0 such that for each $\tau > \tau_0$, there is an $\alpha > 0$ with $(z_0^h+\delta e_0, z_1^h, \cdots, z_\tau^h, \alpha\bar{x}_{\tau+1}^h, \cdots) \in R^h(x^h) - Y$.

The first part of Assumption B–M.1 is implied by Assumption 1. For the second part note that $y \in Y$ and $y \neq 0$ implies that $y(t) \in Y_t$ and $y(t) \neq 0$ for some t. Then Assumption 2 requires that either $-k_0 \neq 0$, or $x_t^1 \neq 0$ for some t. In either case –y fails to satisfy the conditions of membership in Y. Thus Y cannot contain a straight line. The assumption that Y_t is constant is only needed for the optimal growth theorem. It could be replaced by a boundedness assumption but this would complicate the application of the turnpike theorem from McKenzie (1982). By a classical boundedness argument we have that k_0 given and x_1 bounded below implies that v_1 is bounded above. Thus k_1 is bounded. Then by induction v_t is bounded in every period. Thus y_t is bounded in every period. This is boundedness of y in the topology of s^n. Thus Assumption B–M.2 holds. Assumption B–M.3 is immediate by Assumption 3. Again the constancy assumption of Assumption 3 can be replaced by a boundedness assumption that would complicate the argument. Assumption B–M.5 is immediate from the last part of Assumption 4. Assumption B–M.4 is established in

<u>Lemma</u> 2. P^h is open valued and has open lower sections in s^n and R^h is the closure of P^h in s^n.

Proof. Suppose zP^hx holds. Then $U^h(z) > U^h(x)$. Since U^h is continuous by Lemma 1, there is a neighborhood W of z relative to C^h such that $z' \in W$ implies $U^h(z') > U^h(x)$. Since z is arbitrary $P^h(x)$ is open. A similar argument shows that the lower section of P^h at x is also open. Since x is arbitrarily chosen from C^h we have established that P^h is open valued and has open lower sections. The continuity of U^h and the fact that C^h is closed imply that $R^h(x)$ is closed. Since there are preferred points in every neighborhood of z for any $z \in C^h$ by monotonicity, R^h is the closure of P^h for any $x \in C^h$. Convexity and transitivity are obvious. □

Assumption 5 is the same as Assumption B–M.6. A strengthened form of the first part of Assumption B–M.7 is contained in Assumption 6. For the second part we may add δe_1 to x_1^h rather than δe_0 to z_0^h without weakening the arguments of Boyd–McKenzie. Then the modified version of the second part of Assumption B–M.7 is implied by monotonicity and the definition of P^h. Indeed, let Δu_1^h be an increase in u_1^h made possible by the addition of δe_1 to the consumption vector x_1^h. Choose τ so that $\Sigma_{t=\tau}^{\infty} \rho^t b < \Delta u_1^h$. Then B–M.7 follows. This completes the proof of Theorem 1. □

Let $p = (p_0, p_1, \cdots)$ with $p_t \in \mathbb{R}^n$. A *competitive equilibrium* for the economy E_m is given by a list $(p, y, x^1, \cdots, x^H)$. In Boyd–McKenzie competitive equilibrium is defined by the following conditions.

B–M.I. $px^h \le 0$ and $z \in P^h(x^h)$ implies $pz > 0$.

B–M.II. $y \in Y$, $py = 0$ and $\lim\sup_{\tau \to \infty} pz(\tau) \le 0$ for any $z \in Y$.

B–M.III. $\Sigma_{h=1}^{H} x^h = y$.

In the case of E_m it is convenient to restate the equilibrium conditions in a slightly modified form. Let $pz = \Sigma_{t=0}^{\infty} p_t z_t$ when the sum exists as a finite number or $+\infty$.

I. $px^h \le 0$, and $U^h(z) > U^h(x^h)$ implies $pz > 0$.

II. $y(t) \in Y_t$ and $(p_{t-1}, p_t) \cdot y(t) = 0$, for all t. Also $(p_{t-1}, p_t) \cdot z(t) \le 0$ for all $z(t) \in Y_t$.

III. $\Sigma_{h\in I}\, x^h = y$.

The definition of P^h in terms of U^h implies that I is equivalent to B–M.I as defined in Boyd–McKenzie. It follows from separability in production and the definition of Y that II implies B–M.II in E_m. Finally III is unchanged from B–M.III.

Let $H^h(m) = \{z \mid z \in C^h \text{ and } pz \le m\}$. It is proved in Boyd and McKenzie that an equilibrium p lies in l_1^∞ and is nonnegative. Since C^h is bounded below, pz is well defined as a finite number or $+\infty$. See Lemma 8 of Boyd and McKenzie. Thus $H^h(m)$ is well defined. By Condition I the competitive equilibrium consumption stream x^h for the h*th* consumer satisfies the condition that $U^h(x^h)$ maximizes $U^h(z)$ over $H^h(0)$ for $z \in C^h$. Let $g(m) = \sup U^h(z)$ over $H^h(m)$. The supremum is finite since $U^h(z)$ is bounded over C^h by Lemma 1. Let $I = \{m \mid \text{there is } z \in C^h \text{ and } pz \le m\}$. The subgraph G of g(m) is defined by $G = \{(v,m) \mid v \le g(m)\}$. The concavity of g implies that G is convex. Consider the point $(v^*,0)$ in $\mathbb{R}^2$ where $v^* = g(0) = U^h(x^h)$. This is a boundary point of G and therefore by a separation theorem (Berge (1963), p. 163) there is a vector (γ^h,μ) such that $\gamma^h v - \mu m \le \gamma^h v^*$ for all $(v,m) \in G$. Monotonicity implies that $\mu = 0$ is not possible. Monotonicity also implies that m is unbounded above. Therefore $\mu > 0$ must hold. On the other hand irreducibility implies that a cheaper point at equilibrium prices p exists in C^h. That is, there is $m < 0$ with $m \in I$. Therefore $\gamma^h = 0$ is excluded. We may choose (γ^h,μ) so that $\mu = 1$. Then letting $v = g(m)$ we have $g(0) \ge g(m) - (1/\gamma^h)m$. In other words

(1) $$U^h(x^h) \ge U^h(z^h) - (1/\gamma^h)pz^h, \text{ for any } z^h \in C^h,$$

We refer to $1/\gamma^h$ as the marginal utility of wealth for the h*th* consumer. Let $\gamma = (\gamma^1,\cdots,\gamma^H)$ and write $\gamma = \gamma(\rho)$.

Define a social welfare function W, for $z \in C$ and a discount factor $\rho < 1$, by $W(z,\gamma(\rho)) = \text{maximum } \Sigma_{h=1}^{H} \gamma^h U^h(z^h)$ over all $z^h \in \bar{C}^h$ such that $\Sigma_{h=1}^{H} z^h = z$.

Lemma 3. At the competitive equilibrium the welfare function $W(x,\gamma(\rho)) = \Sigma_{h=1}^{H} \gamma^h U^h(x^h)$.

Proof. Suppose not. Then there is another allocation $\{z^h\}$ of x such that $\Sigma_{h=1}^{H} \gamma^h U^h(z^h) > \Sigma_{h=1}^{H} \gamma^h U^h(x^h)$. Multiply the inequalities (1) by γ^h and sum. Since $\Sigma_{h=1}^{H} z^h = \Sigma_{h=1}^{H} x^h$ we have $\Sigma_{h=1}^{H} pz^h = 0$. Thus $\Sigma_{h=1}^{H} \gamma^h U^h(z^h) \leq \Sigma_{h=1}^{H} \gamma^h U^h(x^h)$. This is a contradiction, so no such allocation $\{z^h\}$ exists. □

Lemma 4. At the competitive equilibrium $W(x,\gamma(\rho))$ maximizes the welfare function over all $y \in Y$.

Proof. The profit condition II of competitive equilibrium requires $(p_{t-1},p_t)\cdot z(t) \leq 0$ for all $z(t) \in Y_t$ when p are the equilibrium prices. However summing (1) over h we find that $W(z,\gamma(\rho)) > W(x,\gamma(\rho))$ implies $pz > 0$. Since $Y = \Sigma_{t=1}^{\infty} Y_t$, this is inconsistent with the profit condition. Thus no such z exists. □

Define the correspondence Γ mapping C into $\Pi_{h=1}^{H} C^h$ by $\Gamma(z) = \{(z^1,\cdots,z^H)| \ z^h \in C^h \text{ for all h and } \Sigma_{h=1}^{H} z^h = z\}$. We first prove

Lemma 5. Γ is lower semicontinuous in the product topology.

Proof. Let V be an arbitrary open neighborhood of $\tilde{z} \in \tilde{C} = \Pi_{h=1}^{H} C^h$ in the product topology relative to $\tilde{C}$. We must show that there is an open neighborhood U of $z \in C = \Sigma_{h=1}^{H} C^h$, relative to C, such that $z' \in U$ implies that $\Gamma(z') \cap V$ is not empty (Berge (1963) p. 109). If $\tilde{z}$ is an element of the open set V then for a finite number of indices ht we have $z_t^h \in V_t^h$, where V_t^h is an open set properly contained in C_t^h and for the remaining indices $V_t^h = C_t^h$. Similarly the open set U has a finite number of indices t such that $z \in U$ implies

$z_t \in U_t$ where U_t is an open set properly contained in C_t while for the remaining indices $U_t = C_t$. We must select the U_t so that $z_t' \in U_t$ implies there is $\tilde{z}' \in \Gamma(z)$ with $z_t^{h\prime} \in V_t^h$ for all t. For t such that $V_t^h \neq C_t^h$ we may assume with no loss of generality that $V_t^h = \{z_t^{h\prime} \in C_t^h | \; |z_t^{h\prime} - z_t^h| < \epsilon > 0\}$ for an appropriate ϵ. For a given t let $U_t = \{z_t' \in C_t | \; |z_t' - z_t| < \delta > 0\}$. Each $z_t' \in U_t$ can be expressed by $\Pi_{h=1}^H z_t^{h\prime} \in \Gamma(z_t')$. Moreover if $|z_t' - z_t|$ is reduced by a factor α then the $|z_t^{h\prime} - z_t^h|$ may be reduced by the same factor. Since the C_t^h are bounded below, the $z_t^{h\prime}$ are bounded and it is possible to choose δ so small that for every $z_t' \in U_t$ it follows that $\Gamma(z_t') \cap V_t^h \neq \phi$. This construction may be repeated for each t for which $V_t^h \neq C_t^h$. Then it will hold for $z \in U$ that $\Gamma(z) \cap V \neq \phi$ and therefore Γ is lower semicontinuous (compare Berge (1963), p. 109). □

Lemma 6. W is concave and continuous on C.

Proof. Consider $W(z,\gamma(\rho)) = \Sigma_{h=1}^H \gamma^h U^h(z^h)$ and $W(z',\gamma(\rho)) = \Sigma_{h=1}^H \gamma^h U^h(z^{h\prime})$. Let $z'' = \alpha z + (1-\alpha)z'$, $0 \leq \alpha \leq 1$. By Lemma 1, U^h is concave for all h. Since $W(z'',\gamma(\rho)) \geq \Sigma_{h=1}^H \gamma^h U^h(z^{h\prime\prime}) \geq \alpha W(z,\gamma(\rho) + (1-\alpha)W(z',\gamma(\rho))$ it follows that W is concave.

Consider $z^s \to z$, $s = 1,2,\cdots$, where z^s and z lie in C. Suppose $W(z^s,\gamma(\rho)) = \Sigma_{h=1}^H \gamma^h U^h(z^{hs})$. Since z^{hs} is bounded below by $\bar{z}$ and $z^s = \Sigma_{h=1}^H z^{hs} \to z$ it follows that z^{hs} is bounded. Since C^h is closed a subsequence $\{z^{hs}\}$ (retain notation) converges to a limit $z^h \in C^h$ for all h. By Lemma 1, U^h is continuous. Therefore $\Sigma_{h=1}^H \gamma^h U^h(z^{hs})$ converges to $\Sigma_{h=1}^H \gamma^h U^h(z^h)$. Let $\tilde{w}$ be an arbitrary element of $\Gamma(z)$. Suppose $W(z,\gamma(\rho)) = \Sigma_{h=1}^H \gamma^h U^h(w^h) > \Sigma_{h=1}^H \gamma^h U^h(z^h)$. Since Γ is lower semicontinuous by Lemma 5 there is a sequence $w^{hs} \to w^h$ where $w^{hs} \in C^h$ and $\Sigma_{h=1}^H w^{hs} = z^s$. Continuity of U^h implies $U^h(w^{hs}) \to U^h(w^h)$. However $\Sigma_{h=1}^H \gamma^h U^h(w^{hs}) \leq \Sigma_{h=1}^H \gamma^h U^h(z^{hs})$. Therefore $\Sigma_{h=1}^H \gamma^h U^h(w^h) \leq$

$\Sigma_{h=1}^{H} \gamma^h U^h(z^h)$. This contradicts the property assumed for $\tilde{w}$, so no such $\tilde{w}$ exists and $W(z,\gamma(\rho)) = \Sigma_{h=1}^{H} \gamma^h U^h(z^h)$. Therefore W is continuous over C. □

The definition of Y as the sum of the Y_t implies that if $x \in Y$ then for each t there is $(-k_{t-1}, x_t + k_t) \in Y_t$. Let $F(k_{t-1}, k_t) = \{z_t | (-k_{t-1}, z_t + k_t) \in Y_t$ and $z_t \in C_t\}$. Define $w(k_{t-1}, k_t) =$ maximum $\Sigma_{h=1}^{H} \gamma^h u^h(z_t^h)$ for $\Sigma_{h=1}^{H} z_t^h = z_t \in F(k_{t-1}, k_t)$. These definitions are independent of t since Y_t and C_t are constant. The maximum exists at some $x_t \in F(k_{t-1}, k_t)$ since $F(k_{t-1}, k_t)$ is compact by the proof of Lemma 1 in Boyd–McKenzie and u^h is continuous by Assumption 4 for all h and t. Then $x \in Y \cap C$ and $W(x,\gamma(\rho)) \geq W(z,\gamma(\rho))$ for all $z \in Y \cap C$. The fact that $\Sigma_{h=1}^{H} \gamma^h u^h(x_t^h) = w(k_{t-1}, k_t)$ implies that $W(x,\gamma(\rho)) = \Sigma_{t=1}^{\infty} \rho^t w(k_{t-1}, k_t)$. Moreover $W(x,\gamma(\rho)) \geq W(z,\gamma(\rho))$ implies that $\Sigma_{t=1}^{\infty} \rho^t w(k_{t-1}, k_t) \geq \Sigma_{t=1}^{\infty} \rho^t w(k'_{t-1}, k'_t)$ for any choice of $(k'_t)_{t=0}^{\infty}$, with $k'_0 = k_0$, for which $w(k'_{t-1}, k'_t)$ is well defined. This is the condition for a path of capital accumulation to be optimal. In order to prove that a path of capital accumulation for a competitive equilibrium exists and satisfies a turnpike theorem we must look for a set of assumptions that are consistent with both the assumptions used that imply the existence of a competitive equilibrium and the assumptions that imply a turnpike for optimal paths.

We will use the turnpike theorem proved in McKenzie (1982). This is a neighborhood turnpike theorem which implies that an optimal path enters a neighborhood of a stationary optimal path and remains there. In the following discussion we will sometimes use small latin letters without the t subscript to represent vectors of $\mathbb{R}^n$ rather than vectors of s^n. The context should distinguish these uses. Let $D_t \subset \mathbb{R}_+^n \times \mathbb{R}_+^n$ be a convex set which contains the combinations of initial stocks k and terminal stocks k′ that are consistent with the production and consumption sets of the *tth* period. That is, $D_t = \{(k,k') |$ there is $z \in C_t$ and $(-k, k'+z) \in Y_t\}$. Since C_t and Y_t are constant this definition is independent of t, so we may put $D_t = D$ for all t. Since services cannot be stored capital stocks k_t lie in the subspace S_g of $\mathbb{R}^n$ spanned by the coordinate axes for goods. Thus k_t is an n–vector whose services coordinates are equal to 0. By assumption S_g has dimension less than n. Then D is contained in the nonnegative orthant of $S_g \times S_g$ which is a proper subspace of $\mathbb{R}^n \times \mathbb{R}^n$.

We will say that a convex set S is *relative interior* to another convex set S′ if S lies in the interior of S′ relative to the smallest affine subspace that contains S′. A concave function f(z) defined on a closed convex set S is said to be *closed* if when y is a relative boundary point of S we have $f(y) = \limsup f(z)$ as $z \to y$. (See Fenchel (1953), p. 58). The assumptions for the turnpike theorem in McKenzie (1982) may be written in our present notation as

M1. $D_t = D$ and $w_t = \rho^t w(k_{t-1},k_t)$ for $0 < \bar{\rho} < \rho < 1$, for all t. The function w is concave and closed.

M2. There is $\zeta > 0$ such that $|k_{t-1}| \geq \zeta$ implies for any $(k_{t-1},k_t) \in D$, $|k_t| < \gamma|k_{t-1}|$, where $\gamma < 1$.

M3. If $(k_{t-1},k_t) \in D$, then $(k'_{t-1},k'_t) \in D$ for all $k'_{t-1} \geq k_{t-1}$ and $0 \leq k'_t \leq k_t$, where $w(k'_{t-1},k'_t) \geq w(k_{t-1},k_t)$.

M4. There is $(\bar{k}_{t-1},\bar{k}_t) \in D$ such that $\overline{\rho k}_t > \bar{k}_{t-1}$.

Let Δ be the set of all $(k,k) \in S_g \times S_g$ such that $k \geq 0$ and $|k| \leq \zeta$. Let $\bar{\Delta}$ be the set of $(k,k) \in \Delta \cap D$ such that $w(k,k) \geq \bar{w} = w(\bar{k}_{t-1},\bar{k}_t)$.

M5. The utility function w is strictly concave in $\bar{\Delta}$.

Define a *nontrivial* stationary optimal path as an optimal path $k_t = k^\rho$, all t, which satisfies the condition that $w(k^\rho,k^\rho) \geq w(k',k'')$ for all (k',k'') such that $\rho k'' - k' \geq (\rho-1)k^\rho$. Note that the set of vectors satisfying this condition always includes $(\bar{k}_{t-1},\bar{k}_t)$ from Assumption M4. So $w(k^\rho,k^\rho) \geq w(\bar{k}_{t-1},\bar{k}_t)$ and $(k^\rho,k^\rho) \in \bar{\Delta}$. These paths are proved to exist in McKenzie (1982). Let e_g be the projection of e on the S_g subspace.

M6. Let $k_t = k^\rho$, $t = 0,1,\cdots$, be a non–trivial stationary optimal path for $\bar{\rho} < \rho < 1$. Let $(k,k') \in D$. Then there is $\eta > 0$ and $\epsilon > 0$ such that $|k - k^\rho| < \eta$ implies that $(k,k+\epsilon e_g) \in D$.

In McKenzie (1982) a further assumption is stated: "If $(k,k') \in D$ and $|k| < \zeta < \infty$, then there is $\zeta < \infty$ such that $|k'| < \zeta$." However this assumption is easily seen to be an implication of Assumptions 2 and 3. Let $\Sigma = \Delta \cap D$. If $(k,k) \in \Sigma$ we say that k is a *sustainable* stock. If unbounded terminal stocks are possible with any given initial stocks then they are possible with larger initial stocks, given free disposal. But stocks of sufficiently large size are not sustainable. Therefore no given initial stocks in a period can support indefinitely large terminal stocks for that period.

Let k_g be the projection of k on the S_g subspace. Define a stock k to be *expansible* if there is $(k,k') \in D$ where $k'_g > k_g$. Define a stock k to be *sufficient* if there is a finite path $\{k_t\}$, $t = 0,1,\cdots,T$, such that $k_0 = k$ and k_T is expansible. Let $k_t = k^\rho$, $t = 0,1,\cdots$, be a non–trivial stationary optimal path for $\bar{\rho} < \rho < 1$. The turnpike theorem is

<u>Theorem</u> <u>2</u>. Let $\{k_t(\rho)\}$, $t = 0,1,\cdots$, be an optimal path with $k_0(\rho)$ sufficient. Then given an ϵ–ball $S_\epsilon(k^\rho)$ about k^ρ, there are ρ' and T such that $k_t(\rho) \in S_\epsilon(k^\rho)$ for all $t > T$ and all ρ with $\rho' < \rho < 1$.

We will establish prove Theorem 2 on the basis of our Assumption 1–6. This will be done if we can show that our Assumptions 1–6 imply Assumptions M1–M6. The first part of Assumption M1 is implied by Assumptions 1 and 4 given the definitions of D and $w(k_{t-1},k_t)$. The second part of Assumption M1 is established in

<u>Lemma</u> 7. The utility function $w(k_{t-1},k_t)$ is concave and closed.

Proof. Recall that $w(k_{t-1},k_t)$ = maximum $\Sigma_{h=1}^H u^h(z_t^h)$ over all z_t^h, $h = 1,\cdots,H$ which satisfy $z_t^h \in C^h$, $\Sigma_{h=1}^H z_t^h = z_t$, and $(-k_{t-1},z_t+k_t) \in Y_t$.

The concavity of u^h and the convexity of Y_t imply concavity of w. Let $(k^s_{t-1},k^s_t) \to (k_{t-1},k_t)$ along a line segment. Then concavity of w implies that $w(k_{t-1},k_t) \leq \lim\sup w(k^s_{t-1},k^s_t)$. Therefore $w(k_{t-1},k_t) \leq \lim\sup (k'_{t-1},k_t')$ as $(k'_{t-1},k'_t) \to (k_{t-1},k_t)$ along any path. However Y and C^h closed and u^h continuous implies that $w(k_{t-1},k_t)$ cannot be less that $\lim\sup w(k^s_{t-1},k^s_t)$ along any path. Thus $w(k_{t-1},k_t) = \lim\sup (k'_{t-1},k_t')$ as $(k'_{t-1},k_t') \to (k_{t-1},k_t)$. □

Assumption M2 is the same as the second part of Assumption 2. Assumption M3 follows from free disposal in capital stocks which is the second part of Assumption 1 and monotonicity of the utility function which is the second part of Assumption 4.

To establish Assumptions M4 and M5 we must prove that stocks expansible by a factor $\bar{\rho}^{-1}$ exist for some choice of $\bar{\rho} < 1$.

<u>Lemma</u> <u>8</u>. In the Malinvaud economy E_m there is $\bar{\rho} < 1$ and $(k_{t-1},k_t) \in D$ such that $k_t > \bar{\rho}^{-1} k_{t-1}$.

Proof. By Assumption 6 there is $y \in Y$ and $z \in C$ with $y - z > \delta e > 0$. This means that it is possible to increase each capital stock k_t, all $t > 0$, in every period by $\delta e_g > 0$, where $e_g \in S_g$ has all goods components equal to 1 and other components equal to 0. Thus D is not empty. By convexity the average initial capital stocks k_a and terminal capital stocks k'_a over the period of accumulation from 1 to T also give an element $(k_a,k_a') \in D$. Since the capital stocks are uniformly bounded along the path, k_a and k'_a are arbitrarily close for large T. Since the terminal stock k_t of any period can be increased by more than δe_g, it follows that k'_a can be increased by δe_g. Since k_a can be made arbitrarily close to k'_a by choice of T it follows that, for any ratio $\sigma < (\kappa+\delta)/\kappa$ where κ is an upper bound on capital stocks, there is a capital stock which is expansible in a ratio arbitrarily close to σ. Thus $\bar{\rho} < 1$ may be chosen so that $\bar{\rho}^{-1} < \sigma$. □

Lemma 8 implies that Assumption M4 will hold for an appropriate choice of $\bar{\rho}$. By Assumption M3 which has been established and Lemma 8 we have that $\bar{\Delta}$ is not empty. Moreover $\bar{\Delta}$ is closed since the C^h are closed and Y is closed. Since $\bar{\Delta}$ is bounded by Assumption 2, it belongs to a compact set. Thus it is compact (Berge (1963), p.68). The maximum of w(k,k) on $\bar{\Delta}$ is achieved by the continuity of w. Let this maximum be attained at $(\bar{k},\bar{k})$. Then $\bar{k}$ is the stock of the unique nontrivial stationary optimal path for $\rho = 1$. Assumptions M5 and and M6 must be made explicitly in addition to the Assumptions 1 – 6 since they involve notions which are special to the reduced utility function $w(k_{t-1},k_t)$. Recall that $\Sigma = \Delta \cap D$. Note that $\bar{\Delta} \subset \Sigma \subset \Delta$. It may be shown that Assumption M6 is implied by the following alternative assumption.

M6′. The optimal stationary stock $\bar{k}$ is expansible. Also $\bar{\Delta}$ lies in the relative interior of Σ.

<u>Lemma</u> <u>9</u>. Assumption M6′ implies Assumption M6.

Proof. We first show that M6′ implies that stocks in $\bar{\Delta}$ are uniformly expansible. Since $\bar{k}$ is expansible by Assumption M6′, there is ϵ' such that $(\bar{k},\bar{k} + \epsilon' e_g) \in D$ The convexity of D implies that Σ is convex. Let Σ′ be a compact set contained in the relative interior of Σ with $\bar{\Delta}$ in the relative interior of Σ′. Let Σ′ be a compact set contained in the relative interior of Σ and containing Σ′′ in its relative interior (see Berge (1963), p.68). Since $\Sigma'' \subset$ relative interior Σ′ and $(\bar{k},\bar{k}) \in \bar{\Delta}$, any $(k,k) \in \bar{\Delta}''$ may be expressed as $(k,k) = \alpha(k',k') + (1-\alpha)(\bar{k},\bar{k})$ for some α with $0 \leq \alpha < 1$ and some $(k',k') \in \Sigma\backslash\Sigma'$. Then $(k,k+(1-\alpha)\epsilon' e_g) = \alpha(k',k') + (1-\alpha)(\bar{k},\bar{k}+\epsilon' e_g) \in D$. Let α_k satisfy these relations for the stock k. If there were a sequence $(k^s,k^s) \in \Sigma''$ with $\alpha_{k^s} \to 1$ it would follow that Σ′′ and Σ\Σ′ have a point in common contradicting the fact that Σ′′ lies in the relative interior of Σ′. Therefore we may select an $\alpha < 1$ valid for all $k \in \Sigma''$. For this α choose $\epsilon = (1-\alpha)\epsilon'$. Then $(k,k+\epsilon e_g) \in D$, or k is uniformly expansible for $(k,k) \in \Sigma''$. Let $|z - k^\rho| < \delta > 0$. Recall that $(k^\rho,k^\rho) \in \overline{\Delta}$. For sufficiently small δ we have $(z,z) \in \Sigma''$. Then by uniform expansibility $(z,z+\epsilon e_g) \in D$. □

Let k^ρ be a nontrivial optimal stationary path for the welfare function $W(k_0,\gamma(\rho))$. We may now prove the competitive equilibrium turnpike theorem.

<u>Theorem</u> <u>3</u>. Assume that Assumptions 1 – 6, M5, and M6 or M6′ hold in the Malinvaud economy E_m. Then there is a choice of $\overline{\rho} < 1$ such that a competitive equilibrium path $(p,y,x^1,\cdots,x^h)$ from a sufficient stock $k_0 = \Sigma_{h=1}^H k_0^h$ defines an optimal growth program for the objective function $W(x,\gamma(\rho))$ for any ρ with $1 > \rho > \overline{\rho}$. Given an ϵ–ball $S_\epsilon(k^\rho)$ about k^ρ, there are $\rho' > \overline{\rho}$ and T such that $k_t(\rho) \in S_\epsilon(k^\rho)$ for all $t > T$ and all ρ with $\rho' < \rho < 1$.

Proof. The existence of a competitive equilibrium path is provided by Theorem 1. The competitive equilibrium path is an optimal path for the welfare function $W(z,\gamma(\rho))$ by Lemma 4, given the initial stock of capital k_0. It should be noted that the utility weights γ^h depend on the distribution $\{k_0^h\}$ of the initial stock. Then Theorem 2 provides the conclusion of Theorem 3. □

The convergence of the capital stock vector k_t of the competitive equilibrium to a neighborhood of a capital stock vector k^ρ does not obviously imply the convergence of the consumption vectors x_t to a neighborhood of consumption vectors. However we will be able to show that such a convergence is implied to an neighborhood which may be arbitrarily small of the set consisting of all the consumption vectors z_t with $z_t = \Sigma_{h=1}^H z_{h=1}^H$, $(-k^\rho,z_t+k^\rho) \in Y_t$, and $z_t \in C_t$ that satisfy $\Sigma_{h=1}^H u^h(z_t^h) = w(k^\rho,k^\rho)$. We will need

<u>Lemma</u> <u>10</u>. If (k_{t-1},k_t) is relative interior to D the correspondence F is continuous at (k_{t-1},k_t).

Proof. If $(-k^s_{t-1},x^s_t+k^s_t) \in Y_t$ and $x^s_t \in C_t$ and $(-k^s_{t-1},x^s_t+k^s_t)$ converges to $(-k_{t-1},x_t+k_t)$, then $(-k_{t-1},x_t+k_t) \in Y$ and $x_t \in C_t$. Thus F is closed. Since it is also compact valued it is upper semicontinuous. We must show that F is lower semicontinuous when the interiority condition is met. Let $S_\epsilon(x_t)$ be a ball of radius ϵ about x_t where $x_t \in F(k_{t-1},k_t)$. We must show that, for a sufficiently small $\delta > 0$, for every $(k'_{t-1},k'_t) \in S_\delta(k_{t-1},k_t)$ it follows that there is $x'_t \in S_\epsilon(x_t)$ with $x_t' \in F(k'_{t-1},k_t')$. Let x'_t be the element of $F(k'_{t-1},k'_t)$ closest to x_t. Let the Euclidean distance between x'_t and x_t equal η. Choose the largest α with $0 \leq \alpha \leq 1$ so that $\alpha\eta < \epsilon$. Then $x''_t = \alpha x'_t + (1-\alpha)x_t \in S_\epsilon(x_t)$. But $(k''_{t-1},k''_t) = \alpha(k'_{t-1},k'_t) + (1-\alpha)(k_{t-1},k_t) \in D$. By convexity of D and concavity of F we have $x''_t \in F(k''_{t-1},k''_t)$. Since x'_t is bounded by classic arguments given Assumptions 2 and 4, α is bounded above 0 as (k'_{t-1},k'_t) ranges over $S_\delta(k_{t-1},k_t)$. Let α^* be a lower bound of α over $S_\delta(k_{t-1},k_t)$. Then the set of (k''_{t-1},k''_t) for $\delta < \alpha^*\epsilon$ is an open neighborhood U of (k_{t-1},k_t) relative to D. Moreover by construction the points $x''_t \in F(k''_{t-1},k''_t)$ for any $(k''_{t-1},k''_t) \in U$ lie in $S_\epsilon(x_t)$. Therefore F is lower semicontinuous at (k_{t-1},k_t). □

We define a correspondence φ which maps D into $\mathbb{R}^n$ by $y_t \in \varphi(k_{t-1},k_t)$ if y_t realizes $w(k_{t-1},k_t)$. Recall that $w(k_{t-1},k_t)$ equals maximum $\Sigma^H_{h=1}\, u^h_t(z^h_t)$ over $z_t \in F(k_{t-1},k_t)$ where $z_t = \Sigma^H_{h=1}\, z^h_t$.

<u>Lemma</u> <u>11</u>. The correspondence φ is upper semicontinuous at $(k_{t-1},k_t) \in$ relative interior D.

Proof. The correspondence F is a continuous at (k_{t-1},k_t) by Lemma 10. Also $w(k_{t-1},k_t)$ is continuous in the interior of D since it is concave by Lemma 7 (Fenchel (1953), p.75). Therefore by the same argument used in Lemma 6 $\varphi(k_{t-1},k_t)$ is upper semicontinuous (this is the theorem of the maximum (see Berge (1963), p. 116)). □

Lemma 11 allows us to prove the convergence of the consumption vectors along a competitive equilibrium path.

Theorem 4. Make the assumptions of Theorem 3. Let (k_{t-1}, k_t), $t = 1,2,\cdots$, be the capital stock vectors of a competitive equilibrium path. Let $W(z,\gamma(\rho))$ be the welfare function for which this path is an optimal path of accumulation. Then there is a choice of $\bar{\rho} < 1$ in the economy E_m such that a competitive equilibrium path $(p,y,x^1,\cdots,x^h)$ from a sufficient stock $k_0 = \Sigma_{h=1}^H k_0^h$ defines an optimal growth program for the objective function $W(x,\gamma(\rho))$ for any ρ with $1 > \rho > \bar{\rho}$. For any $\epsilon > 0$ let $S_\epsilon(\varphi(k^\rho,k^\rho))$ be the set of vectors z_t which lie within ϵ of $\varphi(k^\rho,k^\rho)$. Then there are $\rho' < \rho < 1$ and T such that $x_t \in S_\epsilon(\varphi(k^\rho,k^\rho))$ for all $t > T$ and all ρ with $\rho' < \rho < 1$.

Proof. By Theorem 3 given any δ and ρ with $\rho' < \rho < 1$ we may chose T so that $k_t \in S_\delta(k^\rho)$ for $t > T$. Assumptions 6 or 6′ imply that $(k^\rho,k^\rho) \in$ relative interior D. Then φ is upper semicontinuous at $\varphi(k^\rho,k^\rho)$ by Lemma 11. This implies for any open neighborhood V of $\varphi(k^\rho,k^\rho)$ there is an open neighborhood U of (k^ρ,k^ρ) in the relative interior of D such that $(k_{t-1},k_t) \in U$ implies that $\varphi(k_{t-1},k_t) \in V$. Therefore for any $\epsilon > 0$ we may choose δ small enough that $S_\epsilon(\varphi(k^\rho,k^\rho))$ contains all $\varphi(k'_{t-1},k'_t)$ for k'_{t-1} and k'_t in $S_\delta(k^\rho)$. It follows for $t > T$ and $\rho' < \rho < 1$ that x_t must lie in $S_\epsilon(\varphi(k^\rho,k^\rho))$. □

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