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# EQUILIBRIUM, TRADE, AND CAPITAL ACCUMULATION

My work in economic theory is usually referred back to Walras and it might be said that I have devoted myself to the task of finishing the analysis of the competitive economy along the lines that he first laid out. Along with many other authors of the '50's when modern theory that uses mathematics freely was being developed, I proved theorems on the existence and uniqueness of competitive equilibrium, the stability of the tatonnement, and the efficiency of competitive equilibrium. However there were some particular characteristics of my work. It usually departed from well known models of the competitive economy which had been put to use in policy contexts or empirical contexts. For example my first existence theorem was concerned with Frank Graham's model of international trade (1954a) which he had used to criticize the classical trade theory and its policy implications. His model had been criticized because no proof of existence or of uniqueness was supplied. On the other hand, when I proved turnpike theorems for optimal paths of capital accumulation, I dealt first with the simple open Leontief model (1963a) with variable coefficients, and later in developing a theorem that would apply to the neoclassical model with independent industries, and constant returns at the industry level, I used a Leontief model (1963b) with durable capital goods present. In both cases, of course, I went on to models which were more general, but the initial steps were taken in simple models which had been given rather concrete applications.

A second note of my work has been to preserve the vision of the competitive economy which I found in Walras (1974–77). In this vision the basic facts about the economy are the tastes of consumers, the supplies of factors, and the productive processes which are available. The processes are treated as independent and linear, at least on the industry scale. This is in contrast with the Hicksian model (1939) of a set of firms who own idiosyncratic production sets where no provision is made for the entry of new firms. This agrees with my use of Graham's model and the Leontief model which have linear production processes in each industry. Arrow and Debreu (1954) took the other route. Mathematically the models can be shown to be equivalent, but economically I think they are not. The linear model can represent the dissolution and formation of firms in terms of the flow of entrepreneurial factors between activities which may use them in different proportions. This is not a natural development of the Arrow–Debreu model.

I believe my inspiration actually goes back to a hundred years before Walras, to Adam Smith. I first read Adam Smith's *Wealth of Nations* (1776) during one summer when I was attending a small junior college in the university system of Georgia, Middle Georgia College located in Cochran, Georgia, a town no larger than my home town of Montezuma and close by it. I was struck by Smith's description of the competitive economy, which is indeed the Walrasian description in embryo. The price of a commodity is equated to the cost of the factors that enter into its production, and the factors migrate to the activities where they can earn the largest returns. Moreover, production can be extended by the duplication of activities, so from the economywide standpoint activities can be treated as linear. I found Simth in the Harvard classics, so it is possible that my entry into economics depended on the visit of a travelling salesman to my home leading my Mother, who had been a school teacher in her youth, to buy the Harvard classics. In another summer I read Darwin's *Origin of Species* from the same Harvard classics, and the professor who recommended me to Duke for scholarship support was a biologist. Perhaps my preference for economics arose from the pressing economic problems of the time. However the most serious competitor for my major subject was physics, and my decision in favor of economics may have arisen from a tension between doing a scientific subject or a literary one. Economics seemed in prospect to be a fair compromise between them. But in retrospect the scientific side (or at least the mathematical side) seems to have won out!

My plan for this essay is to discuss some of the ideas for which I can claim a degree of originality, the activities model of competitive equilibrium, irreducible economies, minimum income functions, quasidominant diagonals and positive excess demand for tâtonnement stability, nonneutral cycles and comparative advantage, maximal balanced growth and von Neumann equilibria, von Neumann facets, and neighborhood turnpike theorems. These ideas are applied to general equilibrium theory for the competitive economy, international trade theory, and the theory of capital accumulation.

## 1. Existence and the Economy of Activities

A characteristic feature of my treatment of general equilibrium for the competitive economy is the use of a social production set which is a convex cone with vertex at the origin. This means in more familiar terms that production functions show constant returns to scale. In the light of other assumptions, made by Arrow–Debreu as well, of divisibility of goods and convexity of the production sets, one might think constant returns is only to be expected. Indeed one may reason that any tendency to diminishing returns to scale must result from some indivisibility of a productive factor. In that case assuming constant returns is the same type of approximation to reality that is involved in assuming divisibility of goods in the first place. However I believe my motivation comes largely from the vision of Walras as I interpret it, where the basic fact about production is the existence of potential productive activities which may be undertaken by any people possessing the requisite knowledge and able to obtain the additional factors, not the existence of a set of firms given *a priori* who own convex production sets.

This leads to the zero profit condition of competitive equilibrium assumed by Walras and indeed by Marshall as well.

If we assume that demand functions are single valued (that is, functions rather than correspondences) a simple proof of existence is available which can be displayed geometrically. The proof passes from a normalized price vector to the market demand which it implies, then back to the possible production set by projection, and then to the set of normalized prices at which this production is profit maximizing. An initial price may be found which is contained in the set of prices into which it maps. This is a competitive equilibrium price. Figure 3 illustrates the proof.

(Figure 1 about here)

If one wishes to introduce firms into production perhaps it should be done by treating firms as coalitions of entrepreneurs which may dissolve and reform and whose stable configurations lie in the core of a profit game where the profits arise from prices that are competitive equilibria given the set of coalitions. That is, the competitive equilibria would be the strong Nash equilibria where no group of entrepreneurs saw an opportunity to earn larger incomes from forming new firms and trading at the existing prices.

An alternative way to interpret the assumption of constant returns is to use a correspondence between convex cones and convex sets. The Arrow–Debreu model can be mapped into the constant returns model by introducing a specific factor for each firm whose ownership is shared among the stockholders in proportion to their ownership of stock in the firm as assumed in that model. This ploy also has the advantage that the profits are realized as prices of the specific factors in the same way as other prices of goods in the equilibrium. Of course this does not alter the Arrow–Debreu model but it has proved to have some mathematical advantages in exploring the properties of the model.

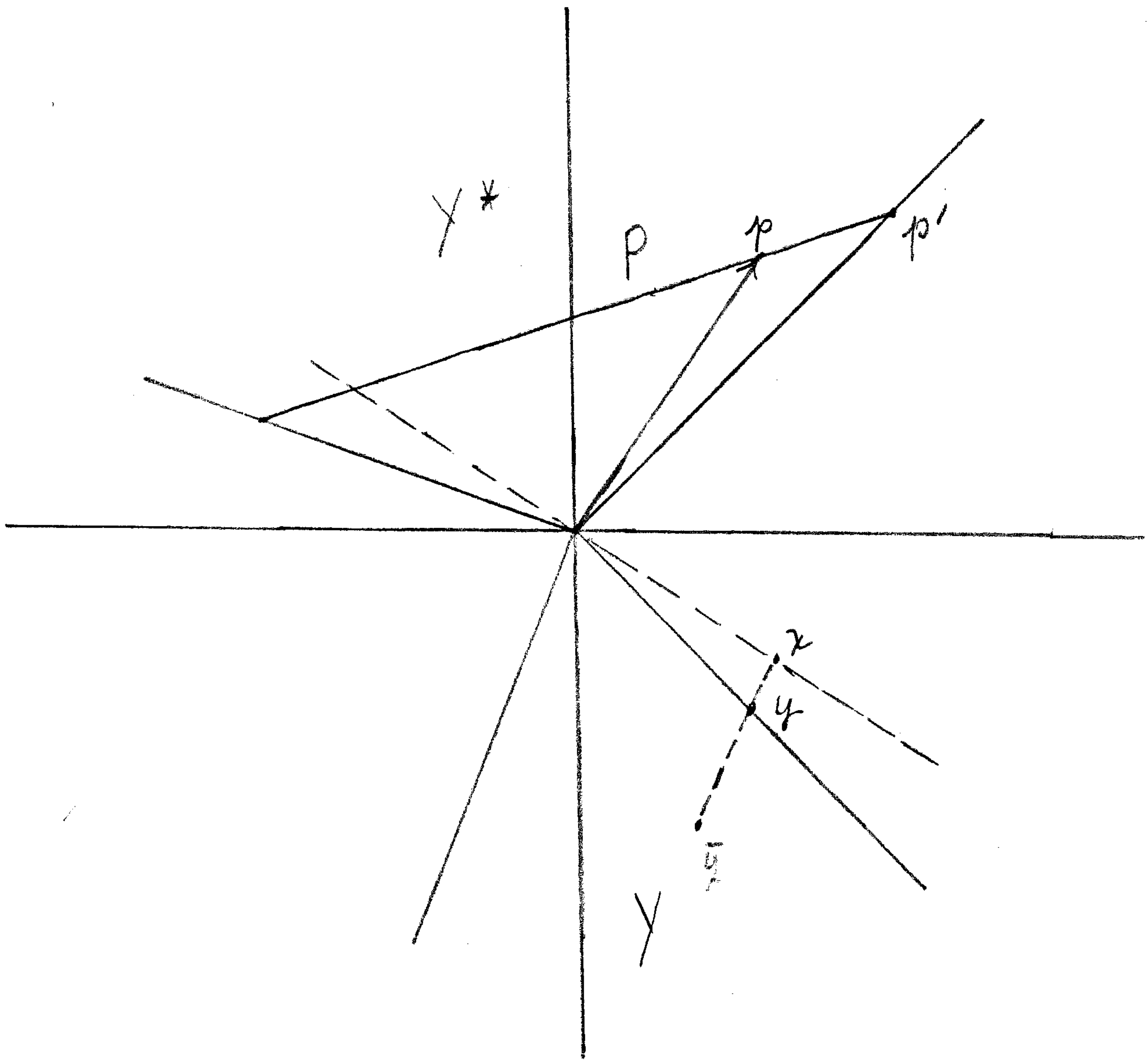


FIGURE 1

Y is the set of outputs. Y* is the set of prices that meet the zero profit condition. P is the set of normalized prices. p is a normalized price vector from P. $x = \Sigma_1^H x^h = \Sigma_1^H f^h(p) = f(p)$ is the market demand at prices p. $\bar{y}$ lies in the interior of Y and y is the projection of x on the boundary of Y from $\bar{y}$. Let $F(p) = g(h(f(p)))$. By Kakutani's fixed point theorem there is $p^* \in F(p^*)$. Then $(p^*, y^*, x^{1*}, \cdots, x^{H*})$ is a competitive equilibrium.

I believe irreducibility and constant returns are the principal characteristics that have attracted attention in my general equilibrium theory. However I also (1956) showed how the existence theory could be extended to models where consumer tastes were allowed to depend on quantities bought by other consumers and on levels of production of the different goods. I did not see how to include external effects on production sets or on possible consumption sets for the general case of both economies and diseconomies in an economically acceptable manner. And in my opinion this has not been done to this day. The difficulty is that the feasible set then need not be simply connected except by an *ad hoc* assumption.

## 2. Existence of Equilibrium and Irreducibility

One of my chief contributions to general equilibrium, following a suggestion from David Gale, is the concept of irreducibility for a competitive economy. This replaces the assumption used by Arrow and Debreu that everyone owns a factor that is always able to increase the output of a good which is always desired in larger quantities by everyone. This assumption implies irreducibility but it is rather implausible. Loosely speaking, irreducibility means that the economy cannot be divided into two groups where one group has nothing to offer the other group which has value for it. This idea was first defined and used in my paper on existence of equilibrium published in 1959 where various generalizations were made of the theory announced by Arrow and Debreu and myself in 1954 and by me in 1956. In addition to replacing the assumption just described, negative prices were allowed, free disposal and irreversibility of production were dispensed with, and a model was developed which allowed for the entry of new firms and the dissolution of old firms, but which could accommodate the Arrow–Debreu model as well. The assumption that the production set has an interior was also dropped. However it was still assumed that each consumer could survive without trading with other consumers. This assumption may be implausible even for the primates who were ancestors of *homo sapiens*, but it seems in any case undesirable to make for modern man. It was first seen from the work of John Moore (1975) that irreducibility made this assumption unnecessary, although he did not call attention to the generalization. He applied a fixed point theorem to a mapping of a set of normalized utility possibility vectors into itself in the manner of Negishi (1960) and Arrow–Hahn (1971). This suggested that the survival assumption for isolated consumers had only been needed because the mappings were defined by demand functions in the commodity and price spaces. I was able to confirm this in my presidential address to the Econometric Society (1982) by showing that demand functions based on the pseudo–utility functions of Shafer with some rather difficult indirect arguments allowed a commodity space approach to the existence proof where survival for isolated individuals is not assumed. The new approach was essential in order to achieve in this paper the further generalization of the existence theorem without individual survival to the case of intransitive preferences, since in the absence of transitivity the utility functions do not exist. Thus a mapping in the space of utility vectors is not available.

Since my 1959 paper there has been some discussion about the meaning of the irreducibility assumption. I made a small amendment in 1961 to take account of an objection raised by Gerard Debreu in private correspondence. When Arrow and Hahn wrote their book, which expounds the basic general equilibrium theory for competitive economies, they chose not to use irreducibility but to introduce a notion of resource relatedness which implies individual survival. Thus resource relatedness does not permit a generalization which only assumes survival by the whole body of consumers. Also in Debreu's paper of 1962 "New Concepts and Techniques for Equilibrium Analysis" individual survival still appears. In that paper the basic theorem establishes the existence of of a quasi–equilibrium and an implication of irreducibility is introduced at the end to convert the quasi–equilibrium into a full equilibrium. In a quasi–equilibrium consumers minimize the cost of reaching the equilibrium level of preference but they do not necessarily maximize preference over the budget set. More recently Hammond (1993) and Ali Khan (1993) analysed and extended the concept of irreducibility in papers delivered to a Rochester conference.

I think the best formulation of the irreducibility assumption for present purposes may be derived from the statement of strong irreducibility in the paper of Boyd and Mckenzie (1993) on existence over an infinite horizon. Index the set of consumers by $\{1,\cdots,H\}$. Let $C^h$ be the set of possible trades for the $h^{th}$ consumer and $P^h$ denote his preference relation over trades. Then $C^h$ is the set of possible consumption bundles less the initial stocks held by the $h^{th}$ consumer. Let $x_I$ denote $\Sigma_{h\in I}\, x_h$. Y is the production set for the economy, which is assumed to be a convex cone with vertex at the origin. I showed (1959) that this is not inconsistent with diminishing returns to scale for each firm in an Arrow–Debreu model once entrepreneurial factors are introduced. The definition is

*An economy is irreducible if, whenever $I_1, I_2$ is a non–trivial partition of $\{1,\cdots,H\}$ and $x_{I_1} + x_{I_2} \in Y$ with $x^h \in C^h$, there are $z_{I_1} + z_{I_2} \in Y$, with $z^h \in C^h$ and $z^h P^h x^h$, for $h \in I_1$, and $z^h \in \alpha C^h$ for some $\alpha > 0$, for $h \in I_2$.*

It is easily seen how this assumption implies that a quasi–equilibrium must be a competitive equilibrium. In a quasi–equilibrium each consumer minimizes the cost of reaching the preference level he is in. This will imply that he maximizes his preference level under his budget constraint provided there is a point in his budget set which costs less than his income. If this condition holds the quasi–equilibrium is a competitive equilibrium. Since the income of consumers arises from the sale of productive services or initial stocks, the value of the consumer's actual trade cannot exceed 0. Thus in a quasi–equilibrium a better trade must cost more than 0. Irreducibility implies that should a feasible allocation place some consumers, those in $I_1$, in the position that their trading sets lie above their budget planes, the consumers whose consumption sets contain points which lie below their budget planes, those in $I_2$, are not satiated in all the commodity bundles that can be obtained from trades that are possible for the consumers in $I_2$ (disregarding the equilibrium trade) together with adjustments in production. However in the quasi–equilibrium the balance condition requires

$$z_{I_1} + z_{I_2} = y.$$

Let the prices of the quasi–equilibrium be represented by the vector p. Because of the zero profit condition for the quasi–equilibrium we have $py = 0$. But $pz_{I_2} > 0$ must hold by the demand condition of equilibrium. However by assumption $pz_{I_1} \geq 0$, so we arrive at a contradiction. This implies that the set $I_1$ is empty. Therefore all consumers must have points in their consumption sets that have value less than 0, if any do. But someone must be in this position because of the survival assumption made for the set of consumers as a whole. In the simplest statement of this assumption there is a point z interior to the production set such that $z = \Sigma\, w^h$ where $w^h \in C^h$ and the sum is over all consumers. Such a point z will satisfy $pz < 0$ by the profit condition. Thus all consumers will maximize their preferences under their budget contraints and the quasi–equilibrium is in fact a competitive equilibrium. This is how the assumption of irreducibility was used in my 1959 article.

Notice that irreducibility does not require that any subset of consumers smaller than the whole body of consumers can survive alone. The very simple example illustrated in Figure 2 shows this. There are two goods, called food and

(Figure 2 about here)

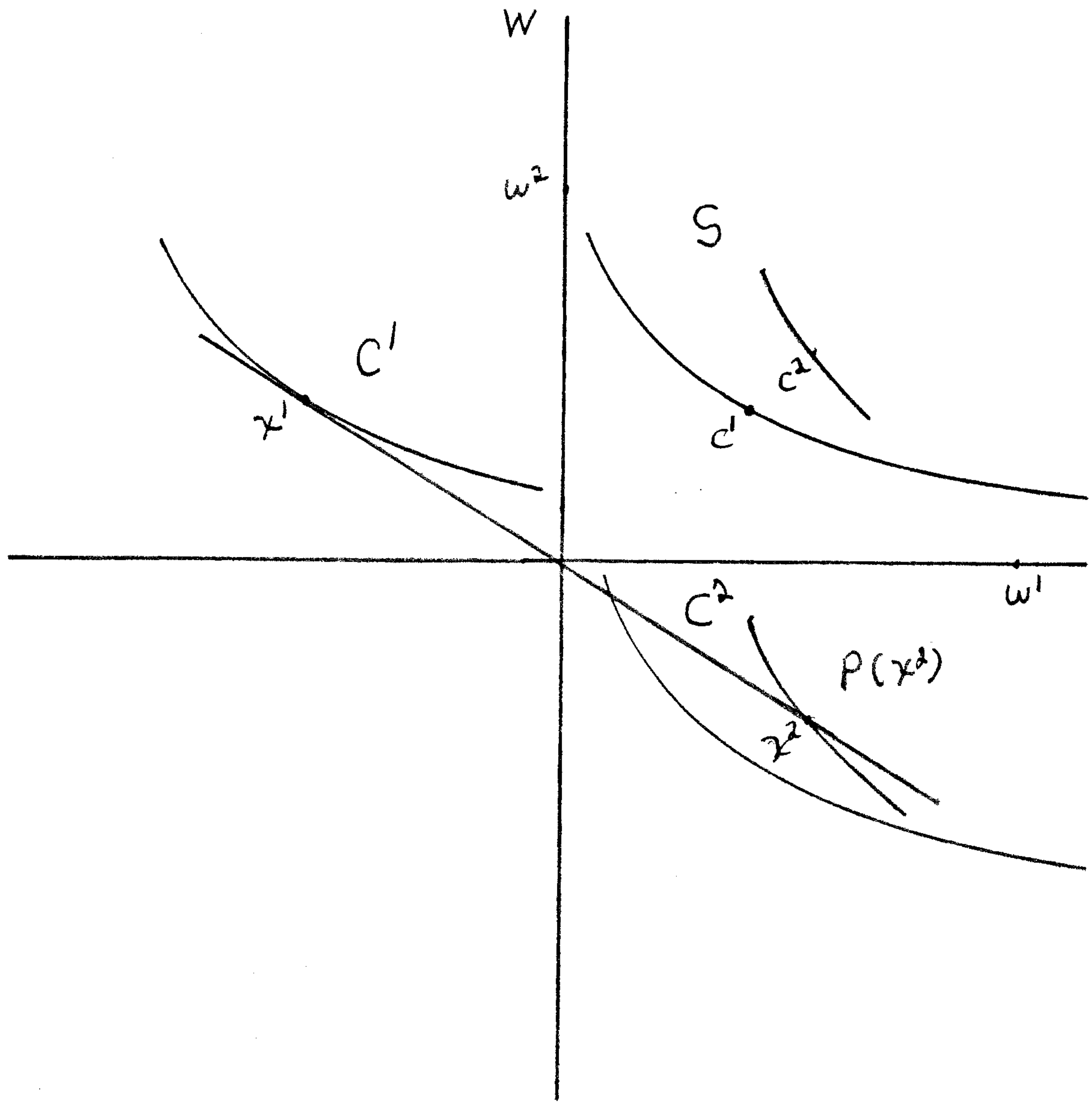


FIGURE 2

S is the possible consumption set for both traders. $C^h$ is a possible trading set. $w^h$ is an endowment. $c^h$ is a consumption for consumer h. $x^h$ is a trade for consumer h. $x^1$ lies on the boundary of $C^1$. $P(x^2)$ is the preferred set of $x^2$. The economy is irreducible.

water. The first consumer has more water than he needs to survive but not enough food. The second consumer has more food than he needs to survive but not enough water. Clearly they either trade or die. The possible consumption sets are translated into possible trading sets by the subtraction of initial stocks. In the trading sets negative quantities represent amounts of goods given up and positive quantities anounts of goods received. However when the first consumer is on the boundary of his trading set and providing all the water possible while surviving, given the quantity of food he is receiving, the second consumer is not satiated in the trade implied by this allocation but would like to receive more water from the first consumer on the same terms. Since the analogous condition prevails when the second consumer is on the boundary of his trading set, the economy is irreducible. Of course an equilibrium exists at an intermediate price atio despite the absence of individual survival. In trading models of this type without production Hammond (1993) has given conditions which are necessary and sufficient for turning a quasi–equilibrium into a competitive equilibrium. These conditions do not imply a cheaper point. Irreducibility is designed to provide cheaper points, but it is important that it also allows us to dispense with the condition that consumers must be able to survive without trading.

The condition of resource relatedness will always provide a cheaper point at a compensated equilibrium if each consumer's trading set contains a point in the negative orthant with negative components for the initial resources and zeros otherwise. This is precisely what is assumed by Arrow and Hahn (1971) in Assumption 2, page 77. It is a strong survival assumption for individual consumers without trade. In this sense irreducibility represents a much weaker assumption for the proof of existence than resource relatedness.

The example we have used did not include production possibilities. However to include them one may replace the trading sets $C^h$ with $C^h - Y$, when there is, for example, a technology for turning water into food. It still may not be possible for consumers to survive alone. The use of irreducibility to turn a quasi–equilibrium into a competitive equilibrium is quite straightforward and easily seen. However its use to dispense with the survival of individual consumers without trade in establishing that an equilibrium exists is quite subtle and cannot be presented in such an intuitive way. There is a good discussion of this proof in Ali Khan (1993).

At the time of my retirement conference in Rochester (1989) Ali Khan correctly pointed out that I had not contributed to existence theory for competitive equilibria when there are an infinite number of commodities. However I had made a practice of introducing the paper of Peleg and Yaari (1970) into my classes. When I came to write up my notes on their paper I thought it would be nice to make use of a consumption set which was not assumed to have the shape of the positive orthant, which I felt to be highly inappropriate for economic reasons. This assumption implies that the lower bound of the possible consumption set lies in this set. For example, the lowest consumption possible for me this period is independent of my consumption in earlier periods. Also there is no substitution on the subsistence boundary of the consumption set in a given period. However I found that this assumption was virtually universal. Kerry Back (1988) was apparently the only exception, but his argument was very laconic and his assumptions were not entirely satisfactory in some respects. In particular, it excluded the neoclassical model of capital accumulation in which. However, in a collaboration with my colleague John H. Boyd III (1993) we were able to prove a more satisfactory theorem using a line of argument suggested by Peleg–Yaari.

It did not turn out to be possible to prove existence in the case of an infinite number of goods without introducing the assumption that isolated consumers can survive, as well as a stronger irreducibility assumption than the finite case requires. The irreducibility assumption may be taken in the following form which is somewhat stronger than in the Boyd and McKenzie paper.

*An economy is strongly irreducible if it is irreducible and, in the definition of irreduciblility, whenever* $x_{I_2}$ *does not lie on the relative boundary of* $C_{I_2}$ *the corresponding value of* $\alpha$ *may be taken to be 1.*

The relative boundary of $C_{I_2}$ is the boundary in the smallest affine subspace or flat containing $C_{I_2}$. This means that the improvement for consumers in $I_1$ may be obtained by increasing the allocation to $I_1$ by some quantity that is large enough to sustain the consumers in $I_2$. Thus it is not enough that the members of $I_1$ may reach a preferred position by adding to their current trade (the negative of) some fraction of a trade that $I_2$ could accept initially. They must reach a preferred position when receiving that trade itself. If there is only one other consumer in the economy the condition is rather severe. However if there are many other consumers it would appear not to make a very significant difference. The reason this strengthening is needed for the proof in the infinite case is that the method of proof uses a generalization of the Debreu–Scarf (1963) theorem. This theorem says that when an economy is replicated indefinitely often an allocation that remains in the core is a competitive equilibrium. The core consists of allocations that no subset of consumers can improve upon using only on their own resources. We need strong irreducibility to prove that replicates of a given consumer receive allocations in the core which are indifferent. Another method of proof might eliminate this embarassment. The need for the survival of consumers without trade may also arise from the line of proof. We combine the theorem of Debreu–Scarf with Scarf's theorem (1967) that the core is not empty for a balanced game. The Scarf argument assumes the ability of every coalition to survive in isolation. In the finite case where the stronger assumption is known not to be needed (Border, 1985), so some suspicions must be aroused that the stronger assumption is not needed for the infinite case either.

## 3. Demand Theory and Minimum Income Functions

I may be forgiven for intruding here a contribution to demand theory which was not itself a major advance but which led to some developments in the hands of Leo Hurwicz and his collaborators. I showed (1957) how the Slutsky equation for the separation of the effect of price change on demand between a pure substitution effect and an income effect and other results in the theory of demand could be derived by the use of properties of the minimum income function. The minimum income is the smallest income, given prices, that allows the consumer to reach a given level of preference, determined by a reference consumption bundle.

I wrote this function as $M_x(p)$. It is illustrated in Figure 4. The matrix

(Figure 3 about here)

of second partial derivatives of $M_x(p)$ (the Hessian matrix) is equal to the substitution matrix (or to the matrix of first derivatives of the compensated demand functions (the Jacobian of those functions)). Since $M_x(p)$ is a concave function this matrix is negative semidefinite, from which the results of pure demand theory follow. Unlike the classical approach this derivation of demand theory avoids the use of the mathematics of determinants and Jacobi's theorem. Also the results do not depend on goods being divisible or utility being a quasiconcave function. Since $M_x(p)$ is concave, it has a well defined Hessian at almost every point in the price space. Hurwicz was in the audience when my paper was presented to the Econometric Society. Later Hurwicz (1971) and Hurwicz and Uzawa (1971) used this approach to demand theory to do some definitive work on the old problem of integrability, the problem that asks when a field of indifference directions can be integrated into a set of indifference curves.

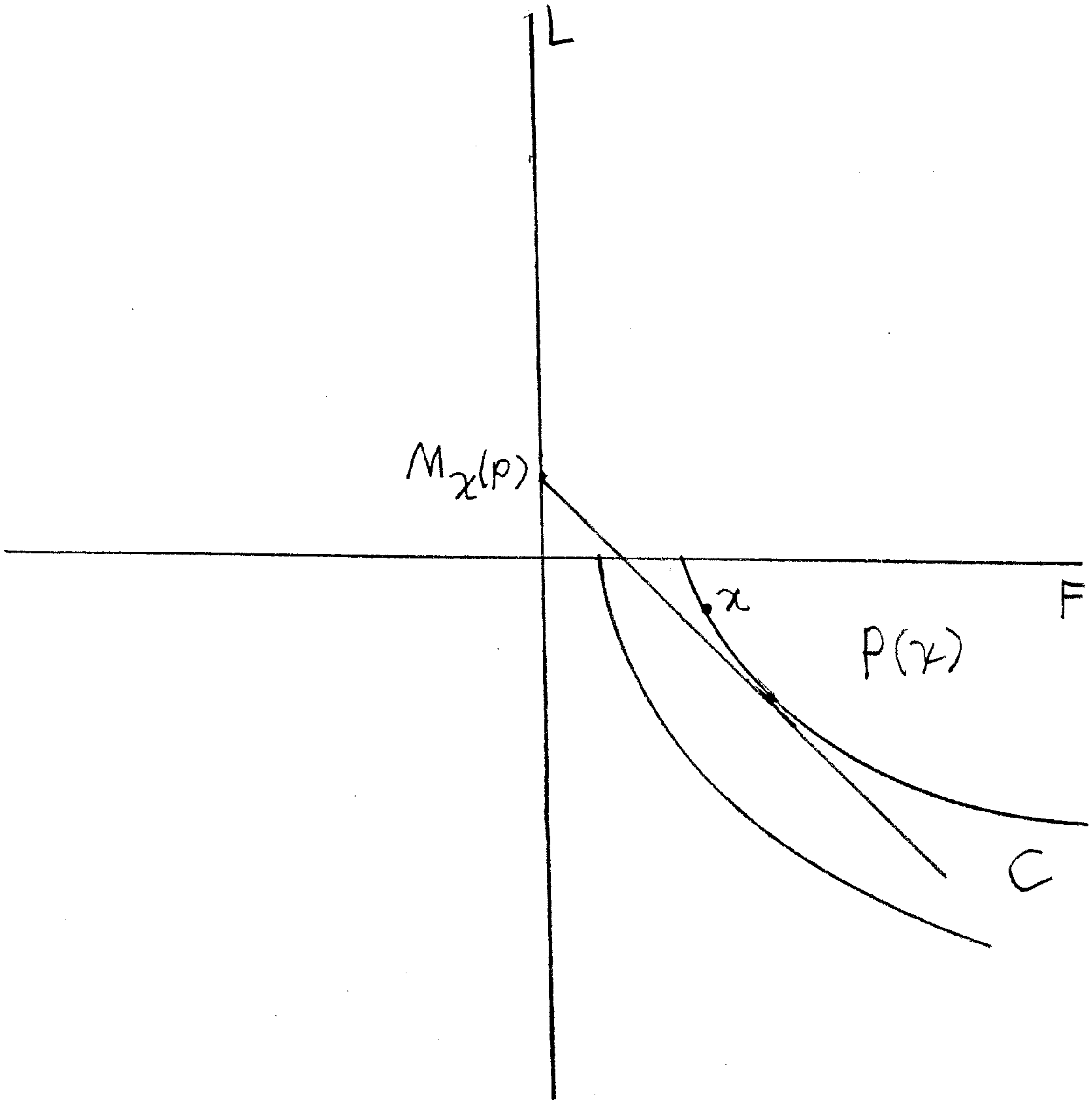


FIGURE 3
The goods are labor (L) and food (F). Labor is numéraire. $M_x(p)$ is the minimum income needed at prices p to reach the preference level given by the consumption vector x.

Today the minimum income function is often referred to as the minimum cost function or the minimum expenditure function, sometimes with the qualifier "minimum" omitted. I do not regard this change of terminology as an improvement.

## 4. Tâtonnement Stability and Positive Excess Demand

In addition to my work on the existence problem I also did a few things on the question of stability of the Walrasian tâtonnement. Interest in this question had been revived by papers of Arrow and Hurwicz (1958) in which for the first time a dynamic stability was proved using meaningful economic assumptions. Although interest has lapsed in recent years, I do not regard the subject as completely obsolete. If equilibrium prices are defined by equality of supply and demand, it would seem to be implied that a dynamic system is present where inequality of supply and demand has consequences for the evolution of prices. Then it is natural to associate an excess of demand over supply with a price rise and an excess of supply over demand with a price fall, as Walras did. If there are unsatisfied traders, as there might be for perishable goods, the excess demand is easy to define and the dynamics are appealing. However, there may also be unsatisfied traders in markets for durable goods if they are not literally auctions. Then for the very short run the evolution of prices may display the dynamics of the tâtonnement, for example, on Marshall's market day (1890). So when expectations are not treated as necessarily correct and the longer term development of prices is analyzed in a sequence of temporary equilibria, there is still a place for tâtonnement stability in the Walrasian sense for the attainment of temporary equilibrium. My work on stability made use of the notions of a quasidominant diagonal and of positive excess demand.

A square matrix has a dominant diagonal by columns if there are positive numbers by which the rows may be multiplied so that the diagonal element in any column is larger in absolute value than the sum of the absolute values of the off–diagonal elements. *Mutatis mutandis* for a dominant diagonal by rows. A square matrix has a quasidominant diagonal by columns if we replace "larger than" by "at least as large as" and require "larger than" for at least one column in each submatrix that is symmetric around the diagonal. It may be proved that a quasidominant diagonal implies a dominant diagonal. Nevertheless we introduce the notion of a quasidominant diagonal since this condition is naturally implied in some economic situations with the prices as multipliers. This is true at equilibrium for columns in Leontief models because of the profit condition. When the gross substitute condition holds, it is true for columns in the Jacobian of market demand functions, omitting the numéraire row and column, by Walras' Law, and for rows in this reduced Jacobian by homogeneity of zero degree.

Let $e_i(p)$ be the excess demand for the $i^{th}$ good when the price vector is p. Most of the theory developed for global tâtonnement stability used the assumption of gross substitutes $(\partial e_i(p)/\partial p_j > 0$ for $i \neq j)$ or weak gross substitutes $(\partial e_i(p) \geq 0$ for $i \neq j)$. Gross substitutes means that an increase in the price of one good with the prices of other goods constant increases the demand for all other goods. Weak gross substitutes means that the demand for other goods does not fall under these conditions. In a paper delivered at the 1959 Stanford conference on mathematical methods in the social sciences (1960a) I used the properties of the gross substitute matrix, whose elements are the rates of change of demand as prices rise, to prove global stability for the simplest tâtonnement dynamics, under the assumption of weak gross substitutes.

Consider the reduced gross substitute matrix where the numéraire is omitted. The assumption of weak gross substitutes implies that the gross substitute submatrix containing only the rates of change of the goods with positive excess demand, with respect to their prices, has a negative dominant diagonal in its columns, and nearly so in its rows. To see the meaning of this condition we may choose units of measurement so that all prices equal one. The dominant diagonal in columns means that the marginal effect on the demand for a given good in excess demand of an increase in its price is larger in absolute value than the sum of the effects on all other goods in excess demand. In rows a slightly modified condition requires that the sum of marginal effects of raising the price of one good in positive excess demand on the demand for this good *at least equals* in absolute value the sum of the marginal effects on the demands for all other goods with positive excess demand. A possible gross substitute matrix is given in Table 1. The column sums equal minus the excess demands, which may be seen by differentiating Walras' Law, and the row sums equal zero by homogeneity. Each column corresponds to

$$\begin{array}{c} \left[\begin{array}{rrr} -.6 & .6 & 0 \\ .2 & -.7 & .5 \\ .2 & 0 & -.2 \end{array}\right] \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \\ \begin{array}{rrr} -.2 & -.1 & .3 \end{array} \quad \end{array}$$

Table 1

a good and each row to the price of a good (arranged in the same order). Any good may be chosen as numéraire, but to have a nontrivial submatrix for the goods in excess demand, excluding the numéraire, choose the third good. In the tâtonnement where the rate of price change is proportional to the excess demand the dominant diagonal implies that the length of the excess demand vector for the goods in excess demand, exclusive of the numéraire, is always falling away from equilibrium. As a consequence the market prices converge to the set of equilibrium prices when they start from any positive initial prices. I first proposed a false conjecture along these lines, when discussing the first Arrow–Hurwicz paper on stability at a meeting of the Econometric Society. I did not immediately recognize the special role of the goods in excess demand. Fortunately I was able to pass a note containing the correct proof to Ken Arrow before the meetings ended.

Formally the simple adjustment model for the tâtonnement is

$$dp_i/dt = k_i e_i(p)$$

where p is the price vector and $k_i$ is some positive constant. Thus the rate of price change is the same for a given level of excess demand however high or low the price of the good may be. (Recall that excess demand will also depend on other prices.) I do not think this is a reasonable assumption. The weakest assumption for the Walrasian tâtonnement is that excess demand causes a price to increase and excess supply causes a price to fall where the rate of change is not specified. I defined (1960b) a Liapounov function V(p) equal to the value of positive excess demand and used it to prove the global stability of this general model of the tâtonnement under the assumption of weak gross substitutes. Formally, let P be the set of goods with nonnegative excess demand. Then

$$V(p) = \Sigma_{i \in P}\ p_i e_i(p).$$

The function V(p) is positive away from equilibrium and zero at equilibrium. Its derivative with respect to time is nonnegative everywhere and it converges to zero over time. This implies that the prices approach the set of equilibria, which I showed to be a convex set under the assumption of weak gross substitutes. I learned later (from Negishi (1962)) that Allais had conjectured in the 1940's that an argument of this type could prove the stability of the tâtonnement.

If our attention is turned to local stability, which I now regard as the central problem, it may be shown that the local stability of any well behaved nonlinear price adjustment mechanism for the Walrasian tâtonnement is locally equivalent to the simple model where the rate of price change for any good is proportional to the level of excess demand for that good. In this setting I believe the most meaningful assumption for stability is that the weak axiom of revealed preference holds in a neighborhood of equilibrium. Other conditions for stability that are recommended by economic theory imply the weak axiom. For example, small net income effects or weak gross substitutes. It has been shown by Anjan Mukherji (1989) that local stability is actually equivalent to the weak axiom or its analog after a transformation of coordinates.

On the other hand, there is the evolution of prices along an equilibrium path over time, of the kind studied in the theory of capital accumulation or in the theory of the equilibrium business cycle, where it is assumed that the future prices are foreseen, at least in a probabilistic sense. If the movement is towards a path which remains constant thereafter, or is represented by a constant probability distribution thereafter, it is natural to speak of stability in this context too.

## 5. The Graham Model: Existence and Comparative Advantage

The other major areas in which I have worked are the theory of international trade and the theory of optimal growth. My initial excursion into existence theory was to prove the existence of equilibrium in Graham's model of international trade (1954a) which I had studied with Graham at Princeton. Graham had us solve, purely by hand, using trial and error, small general equilibrium models of trade with a few countries and a few goods assuming the simplest production and demand functions in each country. While we always found solutions and never more than one for a given model, no one had found a proof that this must be the case, and von Neumann had informed Graham that no analytical solution was to be expected. Actually the problem can be reduced to a nonlinear programming problem under the particular demand functions used by Graham, which are derivable from Cobb–Douglas utility functions and are assumed to be the same for all countries. That is, he assumed that the same proportion of income is spent on all goods in all countries. Under these conditions the world demand maximizes a utility function over the production possibility set, so fixed point methods are not needed. However even a small generalization of these assumptions, say the proportions in which income is spent in different countries differ, will invalidate this approach. My theorem was proved under much more general demand conditions and extended to more general production models as well, so that the application of a fixed point theorem was essential. My initial proof was to smooth the production set and apply Brouwer's fixed point theorem followed by an approximation argument, but having in my hands a discussion paper from the Cowles Commission by the mathematician Morton Slater suggesting that the Kakutani theorem might be useful in economics I saw that this was the perfect theorem for the problem. This paper (1954a) and the Arrow–Debreu paper (1954) were first presented to the meeting of the Econometric Society in Chicago in December 1952. They were written independently.

However, my principal papers in trade theory proper were concerned with comparative advantage (1954b) and factor price equalization (1955). The comparative advantage paper was written in its first version for Tjalling Koopmans in his class on activity analysis at Chicago in 1950. He then urged me to tackle the factor price equalization problem and seemed to be somewhat disappointed when I first proved a theorem on existence of equilibrium. The major point of the comparative advantage paper was to show by elementary arguments that efficient specialization in production in Graham's model is equivalent to the existence of prices at which the profit conditions of competitive equilibrium are satisfied. The sufficiency of the profit conditions for efficiency is rather obvious and very quickly proved, but the implication the other way in the general case requires the use of a theorem on the separation of convex sets by hyperplanes. I also argued that bilateral comparisons of comparative advantage in the classical manner was not adequate to determine whether a specialization in production is efficient. To illustrate this point I used a Graham model summarized in table 2 where each number represents the quantity of the good in the row produced by one unit of labor in the country in the column.

| | A | B | C | D |
|---|---|---|---|---|
| Cloth | 10 | 10 | 10 | 10 |
| Linen | 19 | 20 | 15 | 28 |
| Corn | 42 | 24 | 30 | 40 |

Table 2.

The efficient specializations in production are presented shematically in the diagram of Figure 4 where all countries are specialized to one good at the vertices

(Figure 4 about here)

and countries are moving from one specialization to another along the lines of the diagram. At the points where lines meet each country is producing one good. Within lines some countries produce two goods and substitutions are possible at constant rates between these goods. In the interior of the spaces outlined by the lines substitutions are possible at constant rates between all three goods and countries produce all goods they produce at some vertex of the enclosed space. The metric changes from one enclosed space to another. The lines lie between points of complete specialization which are efficient and where the change of specialization involves only one country. As Richard Rosett pointed out to me when he was still a student of Koopmans at Yale, my original diagram is incorrect and the error illustrates very nicely the dangers of bilateral comparison. I had the specialization which is starred to be A and D in corn, C in cloth, and B in linen. It is now shown correctly to be A and B in corn, C in cloth, and D in linen. It is ironic that this specialization is one vertex of the facet labelled 5 in the diagram which was used correctly in the paper to illustrate the assignment of prices to support an efficient specialization. One of the results proved in the

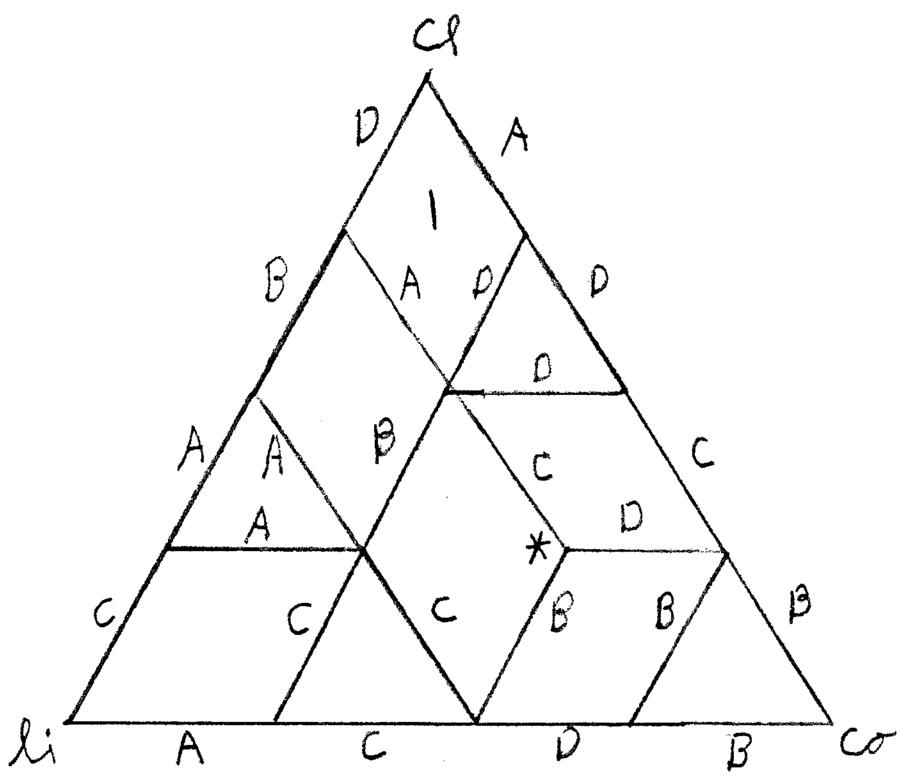


FIGURE 4

At the vertex marked cl only cloth is produced. Similarly only linen is produced at li and only corn at co. In the sector labeled 1 country A is changing from cloth to corn on the right boundary from the vertex and country D is changing from cloth to linen along the left boundary from the vertex. Similarly for other boundaries. In the interior of any sector all countries produce all goods they produce at any vertex of that sector. The diagram is qualitative since the metric changes from one sector to another.

paper is that a specialization is efficient if and only if there is no non–neutral circuit of substitution, that is, a series of substitutions leading to a net increase in world output. The non–neutral circuit in this case is outlined in Figure 5, where solid dots indicate active production processes and circles indicate

(Figure 5 about here)

inactive processes in the inefficient specialization. The vertical lines indicate substitutions in production between processes. A "+" represents an increase in output and a "–" represents a reduction in output. If linen is reduced in output by 1 unit and the circuit is followed the result is

$$1\cdot(10/20)\cdot(30/10)\cdot(28/40) - 1 = 21/20 - 1 = 1/20.$$

Thus the output of linen has risen by 1/20 and all other goods are produced in the same amounts. So the specialization is not efficient. This possibility is not seen from bilateral comparisons, since every country has a comparative advantage in the good it is producing compared with any other country.

The product of ratios used in identifying the non–neutral circuit is in general the product of the terms $a_{sj}/a_{rj}$ where $a_{sj}$ is the amount of the $s^{th}$ good produced by a unit of labor in the $j^{th}$ country and $a_{rj}$ is the amount of the $r^{th}$ good produced by a unit of labor in the $j^{th}$ country. Suppose we consider all such products where the denominator of one ratio and the numerator of the next always have the same first subscript and $a_{rj}$ only appears in a denominator if the $j^{th}$ country is producing the $r^{th}$ good in positive amount. Then for efficiency it must not be possible to form such a product which is greater than 1. Ronald Jones (1961) showed that this condition is equivalent to requiring that an efficient specialization minimize the product of the labor requirements $1/a_{rj}$ over all patterns of specialization which have a given number of countries in each good. Using the direct labor requirements this criterion can be extended to the case where intermediate goods are traded. However the world production frontier must

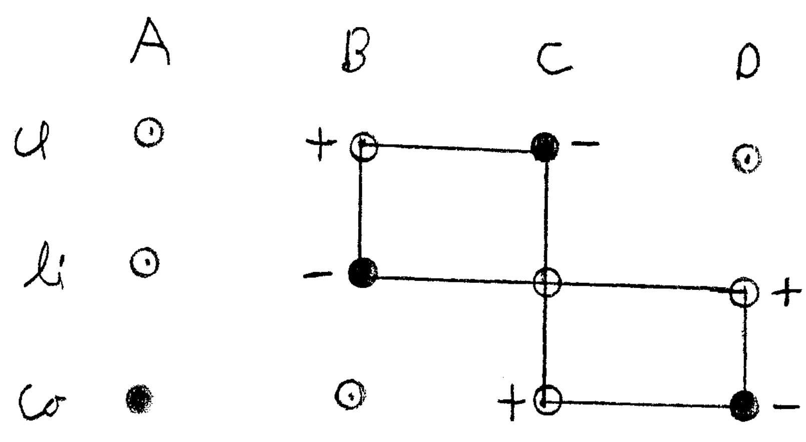


FIGURE 5

The location li,B represents production of linen in country B and similarly for other locations. Solid dots represent activities mistakenly used in an inefficient specialization at vertex * of Figure 4. The lines describe a non–neutral circuit. When the circuit is traversed output increases at locations labelled with + decreases at locations labelled with –. When the changes are complete the specialization is that of vertex * as it appears in Figure 4. This specialization is efficient. There are no non–neutral circuits when it is used.

be intersected with the positive orthant to prevent the inputs of intermediate goods from exceeding their outputs. Then some "efficient" specializations are seen to be infeasible.

## 6. Factor Price Equalization and the Cone of Diversification

The analysis of specialization was done in the Graham model where each country was treated as though it held only one factor of production, which is called labor, and which differs in relative productivity in different lines of production in different countries. At least the analysis depends on the possibility of making substitutions in the production of traded goods without encountering seriously diminishing returns, a possibility that Harry Johnson (1966, p. 697) argued is realistic. On the other hand the theory of factor price equalization involves just the opposite viewpoint in which there are multiple factors of production and these are uniform in quality between countries but differing in quantity. My main contribution to the theory of factor price equalization was the introduction of the cone of diversification (so labelled by John Chipman in his survey article (1965–66)). Given a set of production processes, also assumed to be the same between countries, and a set of goods prices, for any given factor prices there will be one or more least cost combinations of factors for operating a process at unit level, defined as the level where the process produces a unit value of output, which may include several goods. The cone of diversification lies in the factor space and it is the convex cone with vertex at the origin spanned by the least cost combinations for the processes present which meet the zero profit conditions. In equilibrium all other factor combinations on the unit value isoquants must have more than a unit cost. The cone of diversification is illustrated in Figure 6.

(Figure 6 about here)

My central result is that factor price equalization, given the goods prices,

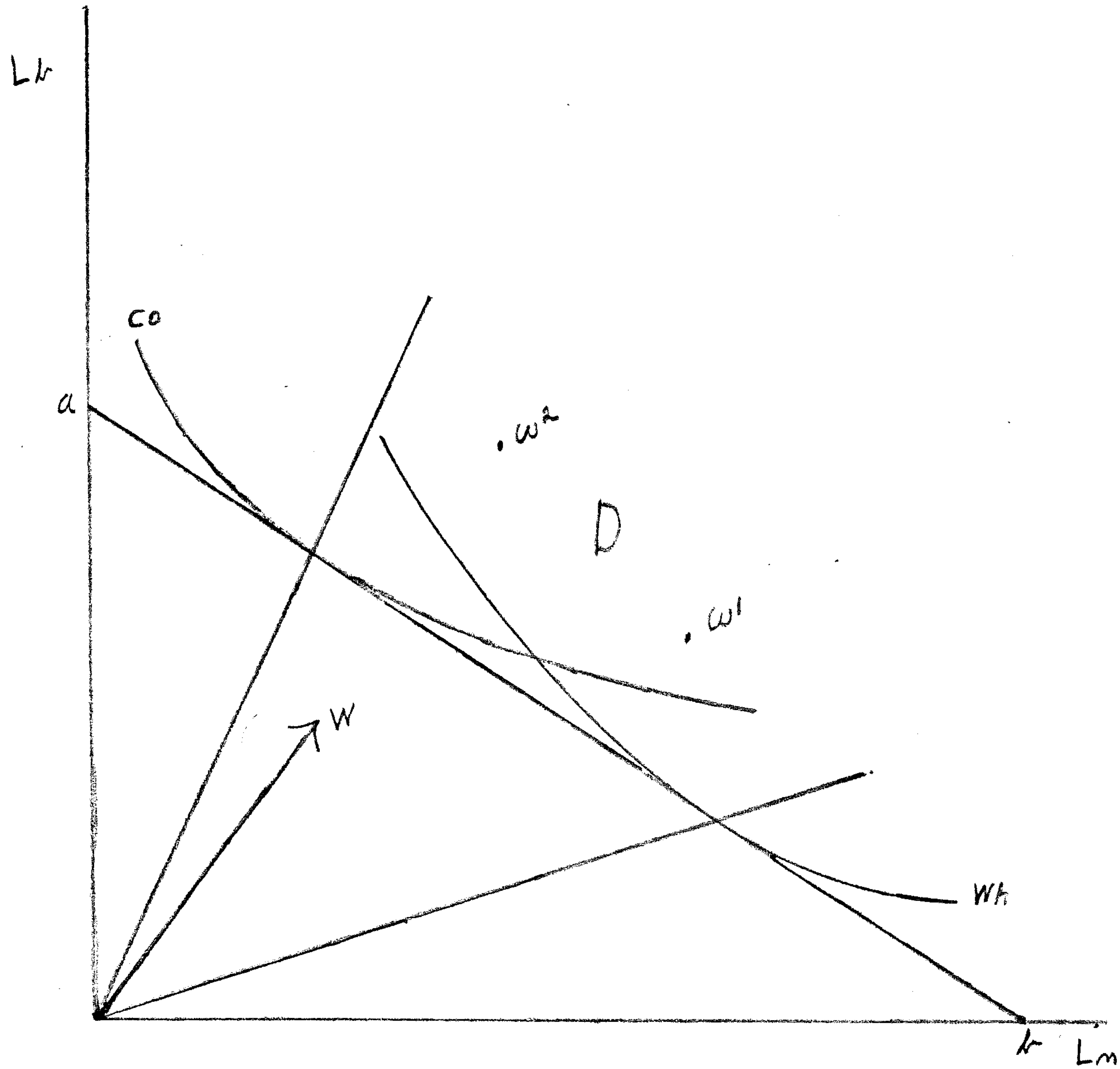


FIGURE 6

Ln labels the land axis and Lb the labor axis. w is the vector of factor prices. Co is a set of minimal factor combinations able to produce $1 worth of corn. Wh is a similar set for wheat. $\omega^1$ and $\omega^2$ are the vectors of factor endowments for countries 1 and 2 respectively. D is the cone of diversification. Since the factor endowments lie in the interior of D the factor price vector w, converted to their respective currencies, must prevail in both countries. That is, at the prices for goods implied by the $1 isoquants there must be factor price equalization.

can only occur if the factor supplies of the countries involved lie within the cone of specialization for some choice of factor prices. Moreover if the factor supplies for some countries lie in the interior of the cone of specialization for some choice of factor prices, factor price equalization for these countries must occur in competitive equilibrium, and these factor prices must prevail in all countries. This result does not depend on smooth production functions nor on an assumption of nonjoint production. However, as usual in this subject, costs of transport are being neglected. Allowance may be made for goods which cannot enter into international trade because transport costs are prohibitive. Then the resource vector should include the resources that remain after the domestic goods have been produced. Of course this is a considerable complication since the resources absorbed by domestic goods will depend both on the prices for traded goods and on the factor prices.

## 7. The von Neumann Model and Turnpike Theory

My contributions to the existence and stability of competitive equilibrium and the theory of international trade were made in the 1950's when economic theory using mathematics was burgeoning. My only contribution to optimality theory in this period was my first substantial article which was published in the *Economic Journal* in 1951. This was not concerned with the perfectly competitive economy but with monopolistic competition. It argued that an efficient distribution of resources among firms could not be achieved in a competitive market unless price and marginal cost were equated everywhere. No system of proportionality would suffice. In the 1960's I moved to Rochester and shifted my attention to the theory of optimal capital accumulation, where the models do not necessarily refer to competitive markets. The analysis of optimal capital accumulation attracted many economists in this period. The subject was started by Dorfman, Samuelson, and Solow (1958), following a conjecture by Samuelson.

I read their book around 1959. Indeed, I can recall deciding to pursue this topic while reading their chapter on the turnpike theorem while waiting for a flight south in the Newark airport. There was also a paper of Samuelson and Solow (1958) which the literature has neglected, although it dealt with a generalized Ramsey model with a proper economic objective in the form of a social utility function.

The modern period began with three papers in the early 60's by Radner (1961), Morishima (1961), and myself (1963). The papers of Radner and Morishima were inspired by a lecture given by Hicks as he traveled from Rochester to Berkeley to Tokyo in 1958. My work was inspired by the theorem in the book of Dorfman, Samuelson, and Solow, where the model was a Leontief model with variable coefficients and two goods. However their proof was somewhat incomplete. I related my model to Morishima's since his paper became available to me before mine was published, although my work was independent of his. These papers were all concerned with a von Neumann model with the objective of reaching the largest possible stock of capital goods in preassigned ratios after a given time. Radner's model assumed a convex social production cone with vertex at the origin with strictly convex cross sections, at least in the neighborhood of the ray along which maximal balanced expansion occurs. This means that expansion at the maximal rate requires that all goods be jointly produced. Thus Radner's model is not consistent with the neoclassical production model. The neoclassical model has independent industries which operate together. In the neoclassical model with more than one industry the production possibility set always has flats on the efficient part of its boundary. Morishima dealt with a Leontief type model without fixed capital, assuming one nonproduced factor of production, a finite number of alternative processes for each industry, and no joint production. I also used a type of Leontief (1941) model where inputs are initial capital stocks and outputs are terminal capital stocks and the input coefficients in each industry are assumed to be continuously variable. My paper (1963a) was delivered to the Econometric Society meeting in St. Louis in December 1960. Radner's paper was scheduled for the same meeting in the preliminary program but I believe he did not take part in the final program.

My method of proof was to appeal to an old theorem of Samuelson and Solow (1953) on the convergence of optimal growth paths to a balanced path. Their theorem was not really appropriate for optimal growth paths since it assumed that higher levels of capital stocks this period imply higher levels of capital stocks next period. However this condition is an implication for levels of prices in the Leontief model when the technology matrix is assumed to be indecomposable. As a consequence prices converge do converge over time. Once convergence has occurred the Samuelson–Solow theorem may be applied to the transformation of capital stocks, now with coefficients which are approximately fixed because of the nearly constant prices, to obtain convergence of capital stocks to a balanced path except for a limited number of initial and terminal periods. The path is traced back from the terminal stocks, so the transitions are from capital stocks in one period to capital stocks in the following period. The Samuelson–Solow assumption is implied for capital stocks on this path since the inverse of the Leontief input–output matrix is a positive matrix. This proof has the feature that one argument (on prices forwards) is used to obtain convergence to a face of the production cone which I later called the von Neumann facet. The facet is generated by the activities meeting the profit conditions at the prices that support growth at the maximal rate. Another argument, on stocks backwards, is used to obtain convergence in the neighborhood of the facet to the maximal balanced growth path lying on it. This method is the characteristic feature of my work on optimal growth problems. In the general model I use the value loss argument due to Radner for the first stage to obtain convergence to a face of the production cone, which I named the von Neumann facet. The argument of the second stage is analogous to the argument for the simple Leontief case.

The paper (1963b) in which the von Neumann facet was explicitly recognized uses a generalized Leontief (1953) model with variable coefficients and durable capital goods. This paper was given at the Econometrics Society summer meeting at Stillwater, Oklahoma, in 1961. The von Neumann facet in this model is characterized as the smallest face of the production cone which contains a von Neumann ray. A von Neumann ray, which need not be unique, is the locus of paths that achieve the maximal rate of balanced growth of the capital stocks possible in the technology. The analogy of the role of this path to the role of an express highway led Samuelson to call it a turnpike. Ever since the theorems proving asymptotic convergence for optimal paths of capital accumulation have been termed "turnpike theorems". Like the previous papers, with the notable exception of the Samuelson–Solow paper (1958), the objective is pure capital accumulation over a given time interval. Radner's idea of value loss for paths off the von Neumann ray is used. However, the loss is no longer associated with the von Neumann ray in the production cone but rather with the flat in the efficiency boundary of the production cone which properly contains the von Neumann ray. The dimension of the flat is equal to the number of industries, that is, the number of goods that are produced. It is shown that the rate of increase in the value of the input–output vector on an optimal path, valued at the prices supporting the von Neumann facet, cannot fall short of the rate of increase for paths on the facet by an $\epsilon > 0$ for more than a fixed number of periods. This forces convergence of the optimal path to the facet, in angular distance. Indeed the optimal path can not stay out of a given angular neighborhood of the von

Neumann facet for more than a finite number of periods, however long the path might be.

The continuation of the argument is to show that a path converging to the von Neumann facet must converge to a path that is permanently on the facet. Then the turnpike result will follow if the technology spanning the facet requires that any path remaining on the facet permanently must converge to a unique von Neumann equilibrium. A von Neumann equilibrium is a balanced path of expansion that is supported by prices in the sense that each process in use earns zero profits and all other processes earn zero or negative profits. This is analogous to the argument of the second stage of the St. Louis paper. The convergence to a von Neumann facet can also be proved for the general von Neumann model described by Kemeny, Morgenstern, and Thompson (1956) in which several von Neumann equilibria with different rates of balanced growth are possible. Kemeny, Morgenstern, and Thompson considered only those equilibria whose outputs have positive value at the supporting prices. In a paper of mine (1967) given to a conference in Cambridge, England, in 1963, of the International Economic Association I proved that the growth rate of such an equilibrium is the supremum of the rates of balanced growth in which some particular set of goods can participate. The associated von Neumann facets are the only faces of the production cone which have the turnpike property. That is, the property that optimal paths which are long enough can be made to spend an arbitrarily large fraction of the accumulation time arbitrarily near a von Neumann facet. This property gives the economic significance of the requirement introduced by Kemeny, Morgenstern, and Thompson that the output should have positive value in the von Neumann equilibrium.

## 8. The Ramsey Model and Turnpike Theory

Of course, the objective of maximal accumulation of capital goods in preassigned ratios has limited interest to economists. The von Neumann model with its maximal balanced growth equilibria is chiefly important because it led to models with alternative processes and joint production where the objective is defined in terms of utility for consumers. A model of capital accumulation whose objective is based on consumer utility was introduced by Ramsey (1928) in a one sector economy a few years before von Neumann presented his multisectoral model. The fusion of the von Neumann and Ramsey models was done by McFadden (1967), Gale (1967), and myself (1968) in papers presented to a summer workshop at Stanford in 1965 which was sponsored by the Mathematical Social Science Board. In the previous summer in a workshop at Rochester I had introduced the idea of a reduced model in which an initial stock of capital and a final stock in each period is associated with the maximal sonsumer utility *per capita* achievable over the period.

For finite paths define an optimal path as one that achieves the largest sum of *per capita* utility over the entire program. Consider the one period utility function $u(k_{t-1},k_t)$ defined on the initial and terminal stocks of the $t$*th* period. If this function is assumed to be concave the set of input–output vectors which lie below its graph, called its epigraph, will be convex. For any point on the upper boundary of the epigraph there is a hyperplane which contains the point and has the epigraph lying below it. The vector is perpendicular to this hyperplane and may be treated as a price vector. The length of the price vector may be chosen so that utility has the price 1.

A facet here is the set of points in the epigraph which are contained in a flat part of its boundary. They lie in a supporting hyperplane of the epigraph. The von Neumann facet is the smallest flat in the boundary that contains all the points of maximal sustainable utility *per capita*. At a point of maximal sustainable utility the terminal capital stocks must be at least as large as the initial stocks and the utility must be as large as possible given this condition. A von Neumann facet is represented in Figure 7. In this case the graph of the utility function is a lined surface, so the von Neumann facet is a line segment. The figure represents a projection of the facet on the input–output space for capital stocks. My theorem says that an optimal path from any initial stocks which are large enough to allow a path starting there to

(Figure 7 about here)

reach a point of maximal sustainable utility will converge to the von Neumann facet, in the sense that given any neighborhood of the facet an optimal path cannot remain outside this neighborhood more than a certain fixed number of periods, however long the path may be. Again if the facet is stable in the sense that any path that remains on the facet indefinitely must converge to a point of maximal sustainable utility, called by me a von Neumann point, an optimal path also converges to a the von Neumann point for all but a finite number of periods.

Von Weizsäcker (1965) and Atsumi (1965) in the same issue of The Review of Economic Studies introduced a criterion for optimality over an infinite horizon which made it unnecessary to select a terminal stock in considering optimal paths. Atsumi used this criterion to prove a turnpike theorem for infinite optimal paths in a two sector model. Gale extended the turnpike result to a multisector model using the assumption that utility defined on capital stocks is strictly concave near the point of maximum sustainable utility. In loose terms the criterion is that a path from given initial stocks is optimal if the utility sum for finite initial segments of the path permanently overtakes the utility sum from an alternative path from the same initial stocks, after a time T that depends on the alternative path. More exactly, if $\{k_t\}$ is the optimal path and $\{k_t'\}$ is the alternative, for an arbitrary $\epsilon > 0$ there is a time T such that

$$\Sigma_{t=1}^{\mathcal{T}} u_t(k_{t-1}', k_t') - \Sigma_{t=1}^{\mathcal{T}} u_t(k_{t-1}, k_t) < \epsilon,$$

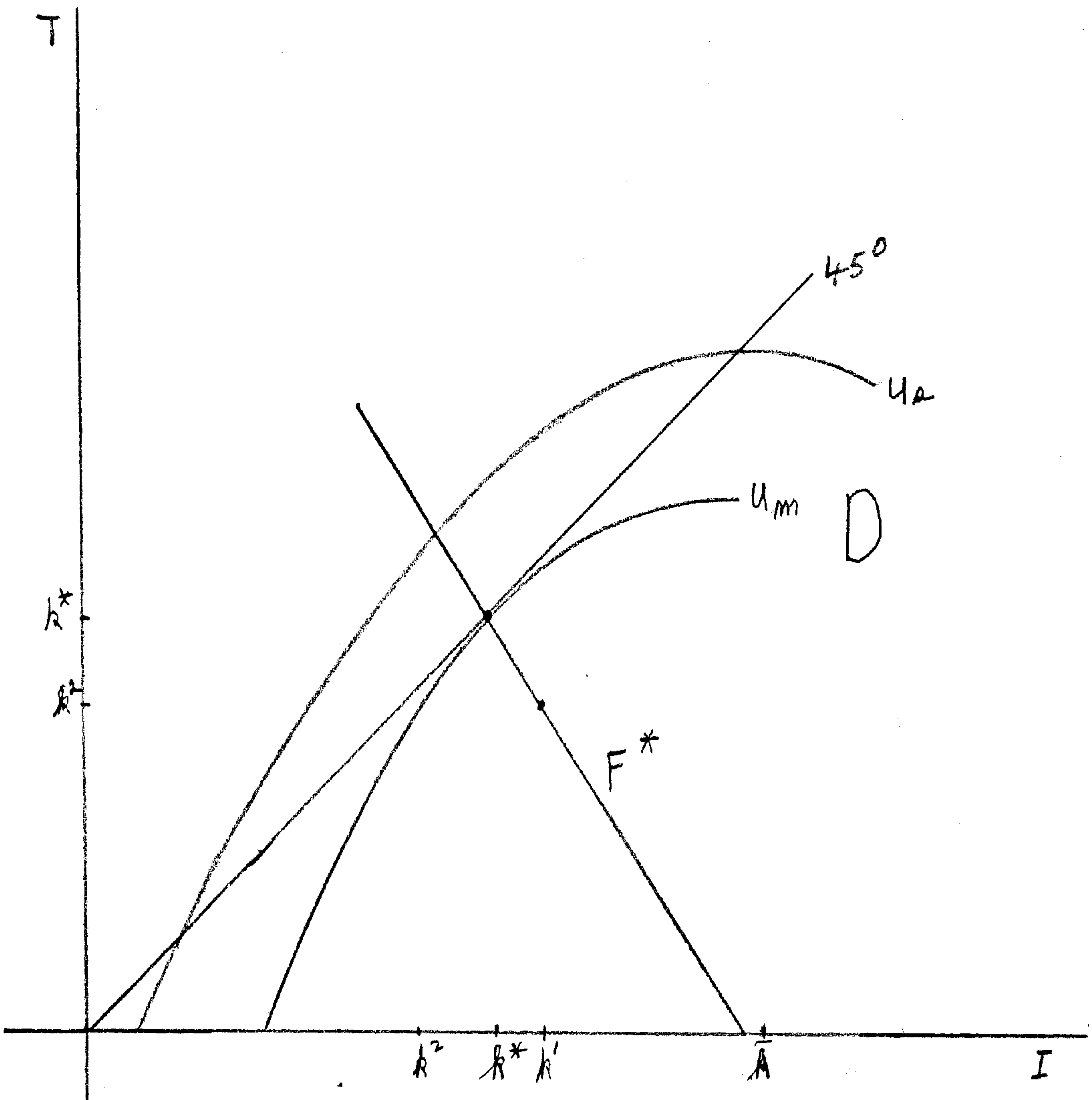


FIGURE 7

I labels the axis of initial capital stocks and T the axis of terminal capital stocks. The input–output combinations on $u_s$ can only provide subsistence levels of utility for consumers. Those on $u_m$ provide the maximum sustainable level of utility. k* is the capital stock of the path of maximal sustainable utility. $\bar{k}$ is the maximal sustainable stock. $k^1$ to $k^2$ is a path that cannot be continued on F* which is the von Neumann facet. Since there are no paths on F* which can be continued indefinitely except the balanced path at k*, the facet F* is said to be stable. If F* made an angle of less than 45 degrees with the horizontal axis to the left all paths would converge to k* and F* would again be stable. Only facets with nontrivial cyclic paths fail to be stable.

for all $\tau$ greater than T. The utility sum for all later times along the alternative path does not exceed the utility sum along the optimal path by as much as $\epsilon$. The turnpike result is convergence to the balanced path along which maximum sustainable utility is achieved, using strict concavity of the utility function so that the path is unique and the von Neumann facet is trivial.

In subsequent years the turnpike theorems for infinite paths in the multisector Ramsey model were generalized in various ways. I showed in my Fisher–Schultz lecture to the Econometric Society (1976, see McKenzie and Yano (1980) for some corrections) that the proper setting for the turnpike theorems in general models, where it is not assumed that the reduced utility function is constant over time, is convergence of optimal paths to each other, provided one path is reachable from the other and utility is strictly concave. This removes the special position of balanced paths none of which may be optimal when utility and production functions vary over time.

I gave (1982) a simple proof for this result, adapting an argument given by Jeanjean (1974) for the stationary stochastic model. It uses the fact that the chord of a concave function lies beneath the graph of the function. Put utility along the first optimal path equal to 0 in every period. We are free to do this since adding an arbitrary constant to utility in each period makes no difference to the relative sizes of utility sums along different paths. Let K be the set of stocks from which this path can be reached in finite time. Then the optimal paths from any initial stock in K will have a utility sum greater than $-\infty$. Also assume that the initial stock of the first path is relative interior to K. The fact that the first path can be reached from the starting point of the second implies that the finite utility sums along the second path are uniformly bounded below. But strict concavity implies that a path halfway between the two does better than the average of the two. Suppose the paths do not converge and the concavity of the utilities is uniformly bounded from zero. Then the gain of the midpath is bounded above 0. Thus the finite utility sums along the midpath are unbounded. Since the initial stock of the first path is interior to K it may be expressed as a convex combination of the midpath and some third path in K, both of which have their utility sums bounded below. By concavity of u this implies that one of the original paths also has unbounded utility sums over finite periods. However these sums are bounded by the normalization. This contradiction implies that the paths must converge. Note that the argument does not require either path to be stationary and the utility function may depend on t. However discounting is excluded, since then uniform strict concavity is lost. It is astonishing that we took so long to find this simple argument although we were quite conscious of the fact that we had no direct proof of convergence. We used a dual argument which is a roundabout way through support prices. I recall William Brock early predicting to me in conversation that some day a primal proof would be found because the convergence properties really had nothing to do with duality.

## 9. The Neighborhood Turnpike Theorem

The von Neumann facet plays a crucial role in the multisector Ramsey model once the assumption of strict concavity of the reduced utility function is dropped. This is true for the model in which future utility is discounted as well as for the model in which it is not. Indeed, Strict concavity is inconsistent with the neoclassical model. It is not the lack of strict concavity for consumer's utility that causes the problem but the lack of strictly convex cross sections in the production cone.

The dimensionality of the von Neumann facet may be seen intuitively in the case of a Leontief model with capital coefficients where there may be alternative activities in each industry and each industry produces a single net output (gross outputs include the capital stocks which are not completely consumed in the period). In a Leontief model labor is only unproduced factor. Integrated activities may be defined whose inputs are labor and initial stocks and whose net outputs are consumption goods and additions to the initial stocks entering the activity. If the total labor input to the set of such integrated activities producing capital goods is held constant, while activity levels are changed by shifting labor between the integrated activities, there is no effect on the output of consumption goods and services. The utility level is constant while the inputs of initial stocks and the outputs of terminal stocks are changing. Suppose the stocks from which these changes are made are the stocks of an optimal stationary path and the changes are confined to integrated activities which operate at positive levels. Then by the argument of the last paragraph all the resulting points on the graph of the utility function are in the von Neumann facet. This follows from the fact that the same prices are supporting, together with the definition of the von Neumann facet as the smallest facet of the epigraph containing the points of maximum sustainable utility.

The generalized Leontief case is discussed at length in Takahashi (1985). If there are n capital goods the dimension of the von Neumann facet associated with any optimal stationary path will be n–1. Thus if we are considering a model with a single good in addition to labor, the facet is trivial unless the utility function for consumption is linear in some direction around the consumption bundle on the optimal stationary path. In other words, the example in Figure 8 requires linearities other than those arising from variations of activity levels.

The first turnpike theorems with discounting were proved independently by Cass and Shell (1976) and by José Scheinkman (1976). These assumed strict concavity and differentiability of the utility functions so that the von Neumann facets are single points on the graph of the utility function which correspond to optimal stationary paths. When the discount factor is near enough to 1 an optimal stationary path lies in a small neighborhood of the path of optimal sustainable utility. The optimal stationary path is also unique (see Brock (1973)) and globally stable. Sheinkman's method of proof is to show that an optimal path must approach close to the path of optimal sustainable utility at least once, which brings it within the basin of stability of the optimal stationary path for $\rho$ near 1.

Turnpike theory has been extended to quasistationary models where differentiability is not assumed (1982, 1983). However the basic theorem with discounting proves convergence of the optimal path to a neighborhood of the von Neumann facet rather than to the facet itself. The size of the neighborhood depends on the discount factor and converges to the facet as the discount factor approaches 1. The utility function in the t*th* period is $\rho^t u(x,y)$ with $0 < \rho < 1$. The proof uses support prices for optimal paths. If $k_t$, $t = 1,2,\cdots$, is an optimal path support prices $p_t$ satisfy the relation

$$u_t(k_t,k_{t+1}) + p_{t+1}k_{t+1} - p_tk_t \geq u_t(x,y)+p_{t+1}y - p_tx,$$

for all input–output vectors (x,y) of initial and terminal stocks which are consistent with the technology. Weitzman (1973) found these prices for any model where the utility sum $\Sigma_{\tau=t}^{\tau=\infty} u_\tau(k_{\tau-1},k_\tau)$ is finite whether or not the model is quasi–stationary. His results were extended (1974) to paths for which this utility sum is finite only after utility is normalized relative to an optimal path. It is important that these prices support the value function as well as the periodwise utility function. The value function $V_t(x)$ has capital stocks at the start of the $t^{th}$ period as its argument. Utility in each period along an optimal path is set equal to 0. In the quasi–stationary model a stationary optimal path may be chosen for the normalization. Suppose any infinite path from x is valued at the largest number that its finite utility sums do not fall below permanently.

Then the value $V_t(x)$ is the smallest number that is not exceeded by the value of any path from x. Support of the value function means that the following relation is satisfied.

$$V_t(k_t) - p_t k_t \geq V_t(y) - p_t y,$$

for all y from which there are infinite paths of accumulation. Of course $V_t(k_t) = 0$ for all t along the optimal path. This allows the definition of a Liapounov function $L_t(y)$. The Liapounov function $L_t$ is nonnegative everywhere, positive, decreasing at a point which is not on the von Neumann facet, and zero on the von Neumann facet. In the quasistationary case of discounted utility it may be shown under rather general assumptions that a stationary optimal path $\{\bar{k}_t\}$ exists where $\bar{k}_t = k^\rho$ for all t. This proof is illustrated in Figure 8. The proof uses an assumption that a capital stock exists

(Figure 8 about here)

which can be expanded in one period by a factor at least as great as $\rho^{-1}$. A stationary optimal path is supported by prices $\bar{p}_t = \rho^t q^\rho$. Let $\{k_t\}$ be an optimal path of accumulation and let $\{p_t\}$ be prices that support $\{k_t\}$ in the sense that both the periodwise utility and the value function are supported.

Put $q_t = \rho^{-t} p_t$. Then it is possible to use the Liapounov function

$$L_t(y) = (q_t - q^\rho)(k_t - k^\rho)$$

to prove a convergence property for $k_t$. The result was proved assuming that the reduced utility function is strictly concave in the neighborhood of a von Neumann facet $F(k^\rho)$ which contains the optimal stationary path with capital stock $k^\rho$. This will be true in the neoclassical model if each capital good appears in some activity in use which comes from a process whose production cone has a strictly convex and smooth cross section at the activity used for generating $F(k^\rho)$. However it is clear that by an approximation process the strict concavity can be removed. The convergence is not asymptotic, as it is in the case of a discount

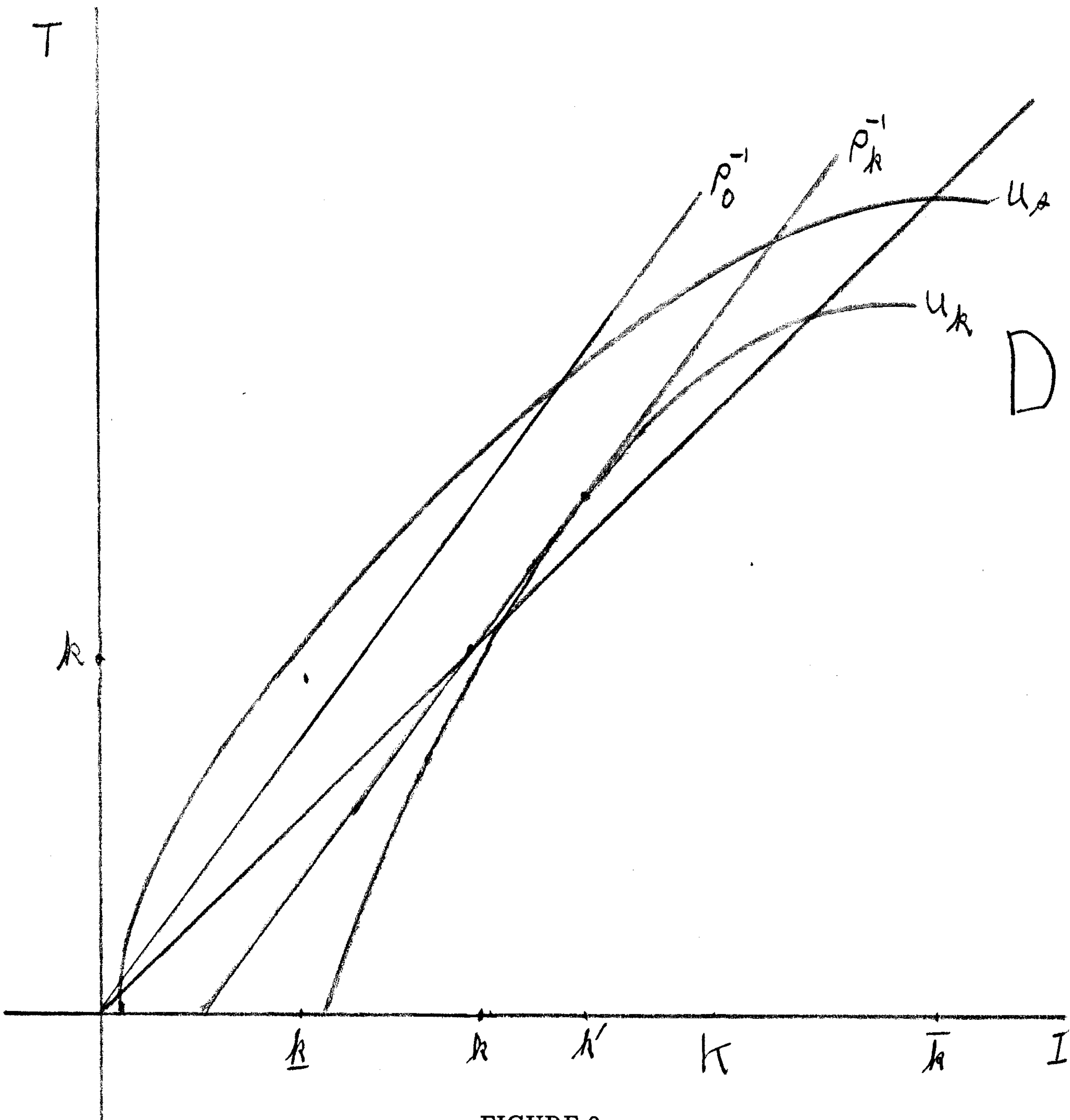


FIGURE 8

The line labelled $\rho_0^{-1}$ has slope $\rho^{-1}$ and passes through the origin. $\underline{k}$ is a capital stock that can be expanded in a ratio larger than $\rho^{-1}$. $\rho_k^{-1}$ is a line with slope $\rho^{-1}$ that passes through the point (k,k). $u_k$ is the highest level of utility for any point in the feasible set D that lies above the line $\rho_k^{-1}$. This utility is achieved at a point with initial stock k′. This procedure defines a mapping of the set K of capital stocks less than or equal to $\bar{k}$ into convex subsets of K. There is a fixed point by Kakutani's theorem which is an optimal stationary path for the discount factor $\rho$.

factor equal to 1. Also the neighborhood of convergence depends on $\rho$. Define a neighborhood $N(\epsilon)$ of $F(k^{\rho})$ as all k such that the distance of $(k_t, k_{t-1})$ from $F(k^{\rho})$ is less than $\epsilon > 0$. It may be shown that for any $\epsilon > 0$ there is a discount factor $\bar{\rho}$, less than one, and a time T such that $k_t$ lies in $N(\epsilon)$ for all $t > T$ for any $\rho$ greater than $\bar{\rho}$ and less than 1. I call this a neighborhood turnpike theorem (1983).

It is worth remarking that the neighborhood turnpike theorem is consistent with optimal paths which are cyclic or even chaotic. Thus the many recent results which find optimal paths of this type, for example, of Benhabib and Nishimura (1985), of Boldrin and Montruchio (1986), Majumdar and Mitra (1994), and Nishimura, Sorger, and Yano (1994), do not conflict with the neighborhood turnpike theorem. Indeed Nishimura, Sorger, and Yano point out that in their model as $\rho \to 1$ the chaotic attractor converges to an optimal stationary path. The cyclic or chaotic paths will eventually be confined to a neighborhood of a von Neumann facet and the extent of the neighborhood will depend on $\rho$. There is a turnpike or asymptotic result no matter how far $\rho$ $(> \bar{\rho})$ is from 1, although if it is too far away the neighborhood may be too large to be of interest.

By introducing differentiability near k(1) and assuming certain regularity properties there, asymptotic convergence to the optimal balanced paths $k(\rho)$ can be proved for $\rho$ sufficiently near to 1. In particular there should be no cyclic paths on the von Neumann facet for $\rho = 1$, and the Hessian at k(1) should be negative definite on the complement of the von Neumann facet for k(1) in a small neighborhood of k(1).

A further step in this line of research was the demonstration by Bewley (1982) and, following him, Yano (1984) that turnpike theory can be extended to competitive equilibria with consumers who have diverse tastes so long as their utilities are separable over time and they have the same discount factors. This theory uses the approach that Negishi (1960) used to prove the existence of competitive equilibrium. Negishi shows that at a competitive equilibrium an appropriately weighted sum of the utility functions of the consumers is maximized. Bewley and Yano apply this theorem to intertemporal utilities. Coles (1985) introduces the von Neumann facet into the equilibrium theory. Marimon (1989) generalized the results of Bewley and Yano in the equilibrium theory to allow for uncertainty. Others have carried the program to certain cases where utility can be recursive (Lucas and Stokey, 1984), so discount rates are endogenous and may differ for different consumers. Since the weighted sum of utilities may be regarded as belonging to a representative consumer, this line of argument supports the use of a representative consumer in dynamic models like those of Kydland and Prescott (1982) and the other real business cycle theorists. Thus one application of the analysis of optimal growth paths is the theory of the equilibrium business cycle, the cycle which can appear in an economy where markets are complete and Pareto optimality is attained. In such a theory the asymptotic properties of optimal growth paths would be realized if the reduced utility function of the representative consumer satisfies the conditions of a turnpike theorem. Although it is extremely unlikely that the required conditions would be satisfied exactly, this does not exclude the possibility that a significant approximation to actual dynamics will be observed in some realistic conditions. On the other hand, David Cass (1991) has shown persuasively that incomplete financial markets will lead to indeterminacy of the intertemporal equilibrium even under conditions where traders are agreed on future prices, so the extent to which the equilibrium approach to business cycles is applicable is an empirical question.

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