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COMPETITIVE EQUILIBRIUM TURNPIKE II

This paper is an extension of my paper The Competitive Equilibrium Turnpike. The earlier paper asserts that the combination of the turnpike theorem for optimal growth paths and the existence theorem for competitive equilibria provides a competitive equilibrium turnpike theorem when the discount factor is close enough to 1. However the proof of this result is not given. The proof requires returning to the proof of the turnpike theorem itself and showing how the proof can be carried through despite the changing weights on individual utilities used to define the welfare function as the equilibrium changes. At each step the results must hold uniformly over the variation of weights. It is not sufficient to show that the competitive equilibrium may be used to define an optimal growth problem where the turnpike theorem may be applied.

In The Competitive Equilibrium Turnpike we consider a Malinvaud model E_m of an economy with an infinite horizon satisfying assumptions which imply that a competitive equilibrium exists. This is a period model with separability between periods in both production and consumption. Let $C_t^h \subset \mathbb{R}^n$ be the possible consumption of the h*th* consumer in the t*th* period. We assume that $C_t^h = C$ for all t. Let $\mathbf{C}^h = \Sigma_{t=1}^{\infty} C_t^h$. Bold type is used for symbols that represent infinite sequences. In E_m the utility of a consumer's consumption path $\mathbf{z}^h = (z_1, z_2, \cdots) \in \mathbf{C}^h$ is given by $U^h(\mathbf{z}^h) = \Sigma_{t=1}^{\infty} \rho^t u(z_t^h)$ where ρ is the discount factor $0 < \rho < 1$. We assume that ρ is the same for all consumers. Let $\gamma = (\gamma^1, \cdots, \gamma^H)$ where $\gamma^h \geq 0$. We define a social welfare function $W(\mathbf{z}, \gamma, \rho)) =$ maximum $\Sigma_{h=1}^{H} \gamma^h U^h(\mathbf{z}^h)$ over all $\mathbf{z}^h \in \mathbf{C}^h$ with $\mathbf{z} = \Sigma_{h=1}^{H} \mathbf{z}^h$. The welfare weights γ^h are normalized so that $0 \leq \gamma^h \leq 1$, $\Sigma_{h=1}^{H} \gamma^h = 1$. We consider a competitive equilibrium path $\mathbf{x} = (x_1, x_2, \cdots)$ where $x_t = \Sigma_{h=1}^{H} x_t^h$ and the discount factor on utility for all consumers is ρ. Choose the welfare weight γ^h equal to the marginal utility of income for the h*th* consumer. They may be made to sum to 1 by choosing the length of the equilibrium price vector. It is proved that $W(\mathbf{x},\gamma,\rho)$ maximizes the welfare function over all $\mathbf{x} \in \mathbf{Y} \cap \mathbf{C}$ where $\mathbf{Y}$ is the social production set and the initial stocks are $k_0 = -x_0$. In the Malinvaud model $\mathbf{Y} = \Sigma_{t=1}^{\infty} \mathbf{Y}^t$, where $\mathbf{Y}^t$ is the production set for the t*th* period. That is, $\mathbf{y} \in \mathbf{Y}^t$ implies $y_{t-1} \leq 0$, $y_t \geq 0$ and all other components equal to 0. Let Y^t be the set (y_{t-1}, y_t) for $\mathbf{y} \in \mathbf{Y}^t$. We assume that Y^t, considered to lie in $\mathbb{R}^{2n}$, is equal to Y for all t.

In the Competitive Equilibrium Turnpike an optimal growth model is derived for the economy E_m. Let $F(k,k') = \{z|\ (-k,z+k') \in Y \text{ and } z \in C\}$, that is, the set of consumption levels feasible in one period when the initial stocks are k and the terminal stocks are k′. F(k,k′) is convex and closed from the convexity and closedness of Y and C. Define $w(k,k',\gamma)$ = maximum $\Sigma_{h=1}^{H} \gamma^h w^h(z^h)$ over all z^h such that $\Sigma_{h=1}^{H} z^h = z \in F(k,k')$. Then if $\mathbf{x}$ is a competitive equilibrium path with $\mathbf{k} = (k_0,k_1,\cdots)$ as the path of capital stocks, $W(\mathbf{x},\gamma,\rho) = \Sigma_{t=1}^{\infty} \rho^t w(k_{t-1},k_t,\gamma)$. Since $W(\mathbf{x},\gamma,\rho) \geq W(\mathbf{z},\gamma,\rho)$ for any feasible choice of $\mathbf{z}$ consistent with the initial stocks k_0 it follows that $\Sigma_{t=1}^{\infty} \rho^t w(k_{t-1},k_t,\gamma) \geq \Sigma_{t=1}^{\infty} \rho^t w(k'_{t-1},k'_t,\gamma)$ for any feasible choice of $(k'_t)_{t=0}^{\infty}$ with $k'_0 = k_0$. In other words $\mathbf{k} = (k_t)_{t=1}^{\infty}$ is an optimal path of capital accumulation when the welfare function is $W(\mathbf{x},\gamma,\rho)$. We define a value function for capital stocks k by $V(k,\gamma,\rho) = \Sigma_{t=1}^{\infty} \rho^t w(k_{t-1},k_t,\gamma)$ when $\mathbf{k}$ is an optimal path for initial stocks $k_0 = k$ given ρ and γ. We will often consider welfare functions V and value functions W where $\gamma \in \gamma(\rho)$ and $\gamma(\rho)$ is the set of γ consistent with some competitive equilibrium path for the discount factor ρ. We will also write $\gamma(\rho)$ for an arbitrary selection from the set $\gamma(\rho)$ when this will not cause confusion. There may be many competitive equilibrium paths with the discount factor ρ and there may be many values of γ for a given competitive equilibrium path.

Let S_g be the subspace of $\mathbb{R}^n$ spanned by the coordinate axes for storable goods. To simplify the notation statements about capital stock vectors will be understood to be relative to the subspace S_g without explicit mention. Let D be the set of (k,k′) such that F(k,k′) is not empty. In the present setting the assumptions for the turnpike theorem in McKenzie (1982) may be given in the following form.

M1. The function w is concave and continuous.

M2. There is $\zeta > 0$ such that $(k_{t-1},k_t) \in D$ and $|k_{t-1}| \geq \zeta$ implies $|k_t| < \xi|k_{t-1}|$, for $0 < \xi < 1$.

M3. If $(k_{t-1},k_t) \in D$, then $(k'_{t-1},k'_t) \in D$ for all $k'_{t-1} \geq k_{t-1}$ and $0 \leq k'_t \leq k_t$. Moreover $w(k'_{t-1},k'_t) \geq w(k_{t-1},k_t)$.

M4. There is $(\bar{k}_{t-1},\bar{k}_t) \in D$ and $\bar{\rho} < 1$ such that $\bar{\rho}\bar{k}_t > \bar{k}_{t-1}$.

Let Σ be the set of all k such that $(k,k) \in D$, that is, the set of sustainable stocks. Then Σ is compact. Let $\Delta(\gamma)$ be the set of $k \in \Sigma$ such that $w(k,k,\gamma) \geq \bar{v} = w(\bar{k}_{t-1},\bar{k}_t,\gamma)$.

M5. The function $w(k,k,\gamma)$ is uniformly strictly concave for $k \in \Sigma$ for all γ.

Given γ define the optimal stationary stock k^* by $w(k^*,k^*,\gamma) \geq w(k,k,\gamma)$ for all $k \in \Sigma$. Assumption M5 implies that k^* is unique.

M6′. The optimal stationary stock k^* is expansible. Also $\Delta(\gamma)$ lies in the relative interior of Σ for all γ.

Define a *nontrivial* stationary optimal path for a given γ as an optimal path $k_t = k^\rho$, all t, which satisfies the condition that $w(k^\rho,k^\rho,\gamma) \geq w(k',k'',\gamma)$ for all $(k',k'') \in D$ such that $\rho k'' - k' \geq (\rho-1)k^\rho$. Note for $\bar{\rho} \geq \rho \leq 1$ the set of vectors satisfying $\rho k'' - k' \geq (\rho-1)k^\rho$ always includes $(\bar{k}_{t-1},\bar{k}_t)$ from Assumption M4. So $w(k^\rho,k^\rho,\gamma) \geq w(\bar{k}_{t-1},\bar{k}_t,\gamma)$ and $(k^\rho,k^\rho) \in \Delta(\gamma)$. These paths are proved to exist in McKenzie (1982).

It was shown in The Competitive Equilibrium Turnpike that Assumption

M6′ implies

M6. Let $k_t = k^\rho$, $t = 0,1,\cdots$, be a non–trivial stationary optimal path for $\bar{\rho} < \rho < 1$. Let $(k,k') \in D$. Then there is $\eta > 0$ and $\epsilon > 0$ such that $|k - k^\rho| < \eta$ implies that $(k,k+\epsilon e_g) \in D$.

These assumptions are consistent with the assumptions used to prove existence of equilibrium. From them a turnpike theorem for optimal growth paths may be proved. If the welfare weights γ are given in advance the turnpike theorem gives a convergence result when the discount factor ρ approaches 1. However if the welfare function is derived from a competitive equilibrium the choice of γ is limited by the particular competitive equilibrium. A welfare function derived from a competitive equilibrium will be written $W(\mathbf{x}(\rho),\gamma(\rho),\rho)$, where ρ may be chosen independently but both $\mathbf{x}$ and γ must be consistent with a competitive equilibrium which has ρ as the discount factor. This means that the corresponding optimal capital accumulation path $\mathbf{k}(\gamma(\rho))$ depends on ρ both directly as a discount factor applied to w and indirectly from the requirement that γ must be a selection from $\gamma(\rho)$. Thus we must repeat some of the arguments used to prove the turnpike theorem for optimal growth in the more general context of competitive equilibrium. In the following theorem write k^ρ for $k^\rho(\gamma(\rho))$.

Theorem. Assume that Assumptions M1 – M6′ hold in the Malinvaud economy E_m. Then a competitive equilibrium path $(p,y,x^1,\cdots,x^h)$ from a sufficient stock $k_0 = \Sigma_{h=1}^{H} k_0^h$ defines an optimal growth program for the objective function $W(\mathbf{x},\gamma(\rho))$ for any ρ with $1 > \rho > \bar{\rho}$. Given an ϵ–ball $S_\epsilon(k^\rho)$ about k^ρ, there are $\rho' > \bar{\rho}$ and T such that $k_t(\rho) \in S_\epsilon(k^\rho)$ for all $t > T$ and all ρ with $\rho' < \rho < 1$.

It was proved in McKenzie (1994) that an optimal growth program is given by a competitive equilibrium. In order to prove that the optimal growth program has a turnpike property for ρ near 1 we must first bound the value function $V(k,\gamma(\rho),\rho)$ as $\rho \to 1$ along a sequence ρ^s, $s = 1,2,\cdots$. This result us to bound the value of the Liapounov function which is used in the proof of the main theorem. From the Boyd–McKenzie paper (1993) there is at least one competitive equilibrium for each value of ρ. The admissible $\gamma(\rho)$ are restricted to equilibria consistent with ρ.

Let $k_t = k^\rho(\gamma)$, all t, be a stationary optimal path of capital accumulation in the model of capital accumulation derived from the welfare function that uses utility weights γ and discount factor ρ. From Lemma 4 of McKenzie (1982) there is $p^\rho(\gamma) \in \mathbb{R}^n$ such that

$$(1) \qquad w(k^\rho(\gamma),k^\rho(\gamma)) + (1-\rho^{-1})p^\rho(\gamma)k^\rho(\gamma) \geq w(k,k') + p^\rho(\gamma)k' - \rho^{-1}p^\rho(\gamma)k,$$

for all $(k,k') \in D$. In order to prove that $V(k_0,\gamma(\rho),\rho)$ is bounded above we will need to prove that the prices $p^\rho(\gamma)$ are uniformly bounded for for ρ near 1 and $\gamma = \gamma(\rho)$ where $\gamma(\rho)$ are utility weights derived from competitive equilibria. Write $k^\rho(\gamma)$ for $k^\rho(\gamma(\rho))$.

Lemma 1. Let $k_t(\gamma(\rho),\rho) = k^\rho(\gamma)$, all $t \geq 0$, be an optimal stationary path of capital accumulation for the welfare function based on utility weights $\gamma(\rho)$. Let $p^\rho(\gamma)$ be support prices for $k^\rho(\gamma)$. Then $p^\rho(\gamma)$ is uniformly bounded as $\rho \to 1$.

Proof. The argument of Lemma 5 of McKenzie (1982) applies. Suppose there is a subsequence ρ^s (preserve notation) such that $\rho^s \to 1$ and $|p^{\rho^s}(\gamma^s)| \to \infty$ as $s \to \infty$. Let $\gamma = \gamma(\rho)$ in (1). Divide (1) through by $|p^{\rho^s}(\gamma^s)|$ and consider a further subsequence (preserve notation) for which $p^{\rho^s}(\gamma^s)/|p^{\rho^s}(\gamma^s)| \to p$. From (1) we obtain $0 \geq p(k' - k)$ for all $(k,k') \in D$ where $p \geq 0$, $p \neq 0$. This contradicts Assumption 4. Since the choice of the subsequence is arbitrary this proves the lemma. □

Lemma 2. If k_0 is sufficient and the capital accumulation path $\mathbf{k}(\gamma(\rho))$ corresponds to a competitive equilibrium allocation $\mathbf{x}(\gamma(\rho))$ then $V(k_0,\gamma(\rho),\rho)$ is bounded as $\rho \to 1$.

Proof. Let $k^\rho(\gamma)$ be the capital stock of a nontrivial optimal stationary path. Let $\mathbf{k}(\gamma(\rho))$ be a path of capital accumulation which is consistent with a competitive equilibrium path from initial stocks k_0 when the discount factor is ρ. We will suppress $\gamma(\rho)$ in the expressions for w and k. Given ρ subtract $w(k^\rho,k^\rho)/H\gamma^h(\rho)$ from each u^h so that $w(k^\rho,k^\rho) = 0$. This has no effect on the comparison of paths for the particular ρ. Then multiplying (1) through by ρ^t gives

$$(2) \qquad \rho^t p^\rho k^\rho - \rho^{t-1} k^\rho \geq w(k_{t-1}(\rho),k_t(\rho)) + \rho^t p^\rho (k_t(\rho) - \rho^{t-1} k_{t-1}(\rho)).$$

Summing (2) from $t = 1$ to $t = \infty$, together with the definition of V, gives

$$(3) \qquad V(k_0,\gamma(\rho),\rho) = \Sigma_{t=1}^{T} \rho^t w(k_t(\rho),k_{t+1}(\rho)) \leq p^\rho(k_0 - k^\rho).$$

Thus we have $V(k_0,\gamma(\rho),\rho) < p^\rho k_0$. Then $V(k_0,\gamma(\rho),\rho)$ is uniformly bounded above as $\rho \to 1$ since p^ρ is bounded as $\rho \to 1$ by Lemma 1.

To show that $V(k_0,\gamma(\rho),\rho)$ is bounded below note that k sufficient implies that there is a path $\{k^t\}$, $t = 0,1,\cdots,T$, where k_T is expansible. Suppose $y > k_0$. In the following the argument $\gamma(\rho)$ for w and k is omitted. By Gale's lemma (McKenzie (1982)) there is an infinite path $\{k_t\}$, $t = T,T+1,\cdots$, where $k_{T+\tau} = \alpha^\tau k_T + (1-\alpha^\tau)k^\rho$, with $0 < \alpha < 1$, and $w(k_{T+\tau},k_{T+\tau+1}) \geq \alpha^\tau w(k_T,k_{T+1}) + (1-\alpha^\tau)w(k^\rho,k^\rho) = \alpha^\tau w(k_T,k_{T+1})$, since $w(k^\rho,k^\rho)$ is put equal to 0. Then $\Sigma_{t=T}^{\infty} \rho^t w(k_t,k_{t+1},\gamma(\rho)) \geq (\rho^T/(1-\rho\alpha))w(k_T,k_{T+1},\gamma(\rho))$. Suppose there were a sequence $\rho^s \to 1$ for which $w(k_T,k_{T-1},\gamma(\rho^s)) \to -\infty$. Since γ lies in a compact set there is a subsequence for which $\gamma(\rho^s)$ (retain notation) converges to $\bar{\gamma}$. Since the $w(k_T,k_{T+1},\gamma(\rho^s))$ is a continuous function of γ this implies that

$w(k_T,k_{T+1},\overline{\gamma})$ is not well defined contradicting the fact that $(k_T,k_{T+1}) \in D$. Thus no such sequence exists and $w(k_T,k_{T+1},\gamma(\rho^s))$ is bounded below as $\rho \to 1$. The utility accumulated in the first T periods is $\Sigma_{t=1}^{T} \Sigma_{h=1}^{H} w(k_{t-1},k_t,\gamma(\rho^s))$, which is finite from $(k_{t-1},k_t) \in D$ for $t = 1,\dots,T$. Since $(\rho^T/(1-\rho\alpha)) \to 1/(1-\alpha)$ we have that $V(k_0,\gamma(\rho),\rho)$ is bounded below uniformly as $\rho \to 1$. □

Let $k_t'(\rho) = 1/2\ (k_t(\rho) + k^\rho)$. Define the *utility gain* in period t by $g_t(\mathbf{k}(\rho),k^\rho) = w(k_{t-1}'(\rho),k_t'(\rho)) - 1/2(w(k_{t-1}(\rho),k_t(\rho)) + w(k^\rho,k^\rho))$. Concavity of v implies $g_t(\mathbf{k}(\rho),k^\rho) \geq 0$. The utility gain is relative to k^ρ which may be arbitrarily chosen from the set of k^ρ consistent with ρ and $\gamma(\rho)$. Also $\mathbf{k}(\rho)$ depends on ρ and $\gamma(\rho)$. Recall that $\gamma(\rho)$ is derived from the marginal utilities of income consistent with a competitive equilibrium where ρ is the discount factor. For notational simplicity these relations are not always explicitly recognized. Define the Liapounov function $G_t(\mathbf{k}(\rho),k^\rho) = \Sigma_{\tau=1}^{\infty} \rho^\tau g_{t+\tau}(\mathbf{k}(\rho),k^\rho)$. Then $G_{t+1}(\mathbf{k}(\rho),k^\rho) - G_t(\mathbf{k}(\rho),k^\rho) = \Sigma_{\tau=1}^{\infty} \rho^\tau g_{t+\tau+1} - \Sigma_{\tau=1}^{\infty} \rho^\tau g_{t+\tau}$ where the arguments of the g functions are omitted. Thus omitting the arguments $(k(\rho),k^\rho)$ of both G and g functions we have

(4) $$G_{t+1} - G_t = (\rho^{-1} - 1)G_t - g_{t+1}.$$

For any $\delta > 0$ we may choose ρ close enough to 1 to give $(\rho^{-1} - 1)G_0 < \delta$ provided that $G_0(\mathbf{k}(\rho),k^\rho)$ is bounded as $\rho \to 1$. This is proved in

***Lemma* 3**. $G_0(\mathbf{k}(\rho),k^\rho)$ is bounded for ρ near 1, $\rho < 1$.

Proof. Write $V^\rho(k)$ for $V(k,\gamma(\rho),\rho)$. Then

$$G_0(\mathbf{k}(\rho),k^\rho) = V^\rho(1/2(k_0 + k^\rho)) - 1/2\ (V^\rho(k_0) + V^\rho(k^\rho)).$$

We note that $G_0 \geq 0$ follows from $g_t \geq 0$. Thus boundedness below is immediate and only boundedness above needs to be proved. $V^\rho(k_0)$ is bounded below by Lemma 2 since k_0 is sufficient. Also $V^\rho(1/2(k + k^\rho))$ is bounded above by the first part of the proof of Lemma 2. Since $w(k^\rho,k^\rho,\gamma(\rho)) = 0$ by the normalization at $\gamma(\rho)$ we have that $V^\rho(k^\rho) = 0$ as well. Therefore G_0 is bounded above. □

In order to show that the path of a competitive equilibrium must enter an arbitrary ϵ–neighborhood of k^ρ for ρ sufficiently close to 1 we must prove that the Liapounov function $G_t(\mathbf{k}(\rho),k^\rho)$ decreases by at least $\delta > 0$ each period that the path is outside the ϵ–neighborhood.

Since $G_0(\mathbf{k}(\rho),k^\rho)$ is bounded as $\rho \to 1$ by Lemma 2, for any $\delta > 0$ we may choose ρ near enough to 1 so that $(\rho^{-1} - 1)G_0 < \delta$. A first step to establish G_t as a Liapounov function is to show for any ϵ that there is $\delta > 0$ such that the left side of (4) when $t = 0$ is less than $-\delta$ when $|k_0 - k^\rho| > \epsilon$. However this follows since $|(k_0,k_1(\rho)) - (k^\rho,k^\rho)| > \epsilon$ implies $g_1(k(\rho),k^\rho) > 2\delta$ for some $\delta > 0$ by uniform strict concavity of $w(k,k,\gamma)$ over all $k \in \Sigma$ and $\gamma \in S_H$, the unit simplex in $\mathbb{R}^H$. Thus $G_1 - G_0 < -\delta$ may be guaranteed for some $\delta > 0$ when k_0 is outside the ϵ–neighborhood of k^ρ by choosing ρ near enough to 1.

The argument continues by induction. Suppose $G_{t+1} - G_t < -\delta$ and $(\rho^{-1} - 1)G_t < \delta$, relations which have been established for $t = 0$. Then $(\rho^{-1}-1)G_{t+1} < (\rho^{-1}-1)G_t < \delta$. On the other hand if $|(k_{t+1}(\rho) - k^\rho)| > \epsilon$ holds then, as in the argument above, by uniform strict concavity $g_{t+1} > 2\delta$ holds, so using (4) again we have $G_{t+1} - G_t < -\delta$. In other words $G_{t+1} - G_t < -\delta$ continues to hold for $t \geq 0$ so long as $k_t(\rho)$ remains outside an ϵ–neighborhood of k^ρ. By summing these inequalities we then obtain

$$G_T(\mathbf{k}(\rho),k^\rho) \leq G_0(\mathbf{k}(\rho),k^\rho) - T\delta, \tag{5}$$

if $k_t(\rho)$ is outside the ϵ–neighborhood of k^ρ from $t = 0$ until $t = T$. Since G_T is nonnegative by its definition the inequality (5) forces $k^t(\rho)$ eventually to enter the ϵ–neighborhood to avoid contradiction. Note that if the argument holds for ρ' it holds uniformly for all ρ such that $\rho' \leq \rho < 1$. Also given ρ with $\rho' \leq \rho < 1$ all nontrivial stationary optimal paths k^ρ must lie in the ϵ–neighborhood of any one of them.

We have shown that paths cannot stay outside any neighborhood U of (k^ρ,k^ρ) indefinitely. However to complete the proof we must show that once a path has entered U there is a neighborhood W which it cannot leave. For this purpose another lemma is needed. In the lemma it is understood that the welfare function is defined by ρ and $\gamma(\rho)$.

Lemma 4. If $\mathbf{k}(\rho)$ is an optimal path from k_0 then for any $\delta > 0$ there is $\epsilon > 0$ such that $|(k_0,k_1(\rho)) - (k^\rho,k^\rho)| < \epsilon$ implies $G_0(\mathbf{k}(\rho),k^\rho) < \delta$, uniformly for $\bar{\rho} \leq \rho < 1$ and all nontrivial stationary paths for ρ.

Proof. By the definition of $G_0$and the feasibility of the intermediate path

(6) $$G_0(\mathbf{k}(\rho),k^\rho) \leq w(1/2\ (k_0 + k^\rho),1/2\ (k_1(\rho) + k^\rho)) \\ -1/2\ (w(k_0,k_1(\rho)) + w(k^\rho,k^\rho)) \\ + \rho V^\rho(1/2\ (k_1(\rho) + k^\rho)) - 1/2\ \rho(V^\rho(k_1(\rho)) + V^\rho(k^\rho).$$

It is implied by Assumption 2 and the definition of k^ρ that $w(k^\rho,k^\rho) \geq w(\bar{x},\bar{y})$ for all k^ρ, $\bar{\rho} \leq \rho < 1$. Thus $k^\rho \in \Delta(\gamma(\rho))$. By Assumptions M5 and M6′ we have that $\Delta(\gamma)$ is relative interior to the set Σ of sustainable stocks and w is uniformly concave over Σ and all $\gamma \in S_H$. This implies that $w(k,k,\gamma)$ is uniformly continuous with respect to k over all $k \in \Delta(\gamma)$ and all $\gamma \in S_H$. Suppose that $(k_0',k_1') \to (k^\rho,k^\rho)$ implies $V^\rho(k_1') \to V^\rho(k^\rho)$. Then from (6) it follows that G_0 is small for $(k_0',k_1'(\rho))$ near (k^ρ,k^ρ). However, Assumption M6′ implies that k^ρ is uniformly expansible for ρ with $\bar{\rho} < \rho < 1$ and $\gamma \in S_H$. Expansibility of k^ρ and free disposal imply that $(k^\rho,k_1'(\rho)) \in D$ for $k_1'(\rho)$ near k^ρ. This implies $w(k^\rho,k^\rho) + \rho V^\rho(k^\rho) \geq w(k^\rho,k_1'(\rho)) + \rho V^\rho(k_1'(\rho))$. Therefore $V^\rho(k_1'(\rho)) \leq V^\rho(k^\rho) + \epsilon$ may be assured for any assigned $\epsilon > 0$ for any ρ, $\bar{\rho} < \rho < 1$, and any k^ρ by bringing $(k^{0\prime},k^{1\prime}(\rho))$ near to (k^ρ,k^ρ).

This argument may be repeated for a switch from k_0' to k^ρ. Let Σ' be a compact set contained in the relative interior of Σ having $\Delta(\gamma(\rho))$ in its interior relative to Σ (see Berge (1963), p. 68). Then by the proof of Lemma 9 of The

Competitive Equilibrium Turnpike k is uniformly expansible over all $k \in \Sigma'$ for all ρ and $\gamma(\rho)$ with $\bar{\rho} < \rho < 1$. Then k^ρ may be reached from k_0' near k^ρ. This gives $V^\rho(k^\rho) \leq V^\rho(k_1'(\rho)) + \epsilon$ for any assigned $\epsilon > 0$ when $(k_0', k_1'(\rho))$ is sufficiently near (k^ρ, k^ρ). Thus $V^\rho(k_1'(\rho)) \to V^\rho(k^\rho)$ as needed. □

We may now complete the proof of the Theorem. Choose an arbitrary $\epsilon > 0$. We have shown that there is ρ' such that the optimal path from a sufficient k may be brought within ϵ of any k^ρ for any ρ and $\gamma(\rho)$ where $\rho' < \rho < 1$. To complete the proof we must show that given any $\epsilon' > 0$ it is possible to choose ϵ so small that once the path has become within ϵ of k^ρ it must lie within $\epsilon' > 0$ of k^ρ thereafter. Choose $\eta < \rho G_0(\gamma(\rho), \rho)$. By strict concavity, Assumption 1, given ϵ' there is η so small that $|(k^{t+1}(\rho), k^{t+2}(\rho)) - (k^\rho, k^\rho)| > \epsilon'$ implies

$$G_{t+1} \geq \rho g_{t+2} \geq \rho^{-1}\eta \tag{7}$$

uniformly for all k^ρ and all ρ with $\bar{\rho} \leq \rho < 1$. Relation (4) implies

$$G_{t+1} \leq \rho^{-1} G_t. \tag{8}$$

By Lemma 4 for any $\eta > 0$ there is an $\epsilon > 0$ such that $|(k_t(\rho), k_{t+1}(\rho)) - (k^\rho, k^\rho)| \leq \epsilon$ implies $G_t < \eta$. Inserting this bound for G_t into (8) gives a contradiction of (7). Therefore, if $(k_t(\rho), k_{t+1}(\rho))$ lies in the ϵ–neighborhood of (k^ρ, k^ρ), (7) cannot hold and $(k_{t+1}(\rho), k_{t+2}(\rho))$ must lie within ϵ' of (k^ρ, k^ρ). If $(k_{t+1}(\rho), k_{t+2}(\rho))$ is not within ϵ of (k^ρ, k^ρ), we have

$$G_{t+2} < G_{t+1} < \rho^{-1} G_t < \rho^{-1}\eta$$

Therefore $\rho g_{t+3} < \rho^{-1}\eta$ and the inequality for G_{t+2} analogous to (7) cannot hold and (k_{t+2}, k_{t+3}) lies within ϵ' of (k^ρ, k^ρ). The first part of the proof implies that G_τ decreases for $\tau > t + 1$ until $(k_\tau(\rho), k_{\tau+1}(\rho))$ is within ϵ of (k^ρ, k^ρ) once more. Therefore $(k^\tau(\rho), k^{\tau+1}(\rho))$ cannot leave the neighborhood of (k^ρ, k^ρ) defined by ϵ' as $\tau \to \infty$. □

This completes the proof of the Theorem. For ρ sufficiently near 1 it is also possible to prove a neighborhood turnpike theorem for the von Neumann facet F(ρ) when F(ρ) is not trivial. Moreover this result may be extended to a neighborhood theorem for an optimal stationary stock k(ρ) when k(ρ) is unique and F(ρ) is stable. Finally with some conditions of negative definiteness for the Hessian matrix of u the stability can be extended to asymptotic convergence of the optimal path to k(ρ). See McKenzie (1982).

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