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1 Introduction

We address the problem of allocating indivisible goods and some amount of an infinitely divisible good among agents with equal rights on these resources. Our specific objective is to investigate the implications of the following requirement on allocation rules: when the preferences of some of the agents change, all agents with fixed preferences are affected in the same direction: they all (weakly) gain, or they all (weakly) lose. This condition is a particular application of a general principle of solidarity discussed in Thomson (1990b) under the name of "replacement principle". The principle pertains to situations where one or several components of the environment in which the agents find themselves change. Together with efficiency, it says that when the change is beneficial and no one in particular deserves any credit for it, all agents should gain. When the change is hurtful, but no one is responsible, it says that all agents should lose: in general, and whether or not efficiency is imposed, there should be domination between the list of welfare levels initially chosen and the list of welfare levels chosen after the replacement.

Here, we consider the possibility that the preferences of some of the agents change, and we limit the welfare comparisons to the "relevant" agents, that is, the agents whose preferences are fixed. We refer to the condition as "welfare-domination under preference-replacement". The property was introduced and analyzed by Moulin (1987) in the context of quasi-linear binary social choice under the name of "agreement". It was also studied by Thomson (1992, 1993) in the context of resource allocation when preferences are single-peaked, in private good economies and in public good economies respectively. The condition formulated by Sprumont (1992), which he names "solidarity", can be understood as an instance of the replacement principle when two of the components defining the problem at hand change simultaneously (resources as well as preferences). Another study of the way allocation rules respond to changes in preferences is due to Fleurbaey (1992).

We look for solutions that satisfy this property and in addition select envy-free allocations, that is, allocations such that no agent prefers what someone else receives to what he receives. Our first result is negative: in general, there is no selection from the no-envy solution satisfying *welfare-domination under preference-replacement*. This result is reminiscent of the negative conclusion obtained by Tadenuma and Thomson (1991b) concerning the existence of selections from the no-envy solution satisfying the requirement of weak population-monotonicity, which says that the arrival of additional agents, resources being kept fixed, should affect all agents initially present in the same direction.[1]

Our second result pertains to the case of a single object; think of a prize that has to be awarded to one of several agents. Then, *welfare-domination under preference-replacement* is compatible with no-envy. However, it is compatible in a unique way: there is only one selection from the no-envy solution satisfying the property. This is the solution that selects the worst envy-free allocation for the winner of the prize. This solution was characterized in earlier work by Tadenuma and Thomson (1993) on the basis of consistency, the requirement that the desirability of an allocation be unaffected by the departure of some of the agents with their assigned bundles[2], and alternatively on the basis of weak population-monotonicity.

We close this introduction by noting that all of our results hold on the smaller class of quasi-linear economies.

2 The model

We consider economies with a finite number of agents among whom are to be distributed an equal number of indivisible goods, or "objects", and some amount of an infinitely divisible good, called "money", each agent receiving at most one object. Let $N = \{1, 2, ..., n\}$ be the set of agents, A the set of objects, and $M \in \mathbf{R}$ the amount of money. We assume that $|N| = |A|$.[3] Each agent $i \in N$ is equipped with a preference relation on $A \times \mathbf{R}$, denoted by R_i, P_i denoting the strict preference relation associated with R_i and I_i the indifference relation. As in most of the literature on the subject, we let the amount of money received by an agent be positive or negative; we might want m_0 to be negative when, for example, the cost of providing the objects has to be covered by the agents. Each preference relation R_i is assumed

[1] We refer to this requirement as "weak" population-monotonicity in order to distinguish it from the requirement that all agents initially present be affected negatively by the arrival of newcomers (Thomson, 1983). This condition is too strong in some models and the weak version, first explored by Chun (1986) in the context of quasi-linear social choice, can be met more generally.

[2] In the last few years, this condition has been the object of an extensive literature, reviewed in Thomson (1995).

[3] Given a finite set X, $|X|$ denotes the number of elements of X.

to be reflexive, transitive, and complete, and to satisfy the following two properties. The first one is strict monotonicity with respect to money:

(1) for all $\alpha \in A$, for all m_0, $m_1 \in \mathbf{R}$ with $m_1 > m_0$, we have $(\alpha, m_1) P_i (\alpha, m_0)$

Second is a property that states the general possibility of compensation: given any bundle and any object, there is some amount of money, which together with the object defines a second bundle indifferent to the first. Formally,

(2) for all α, $\beta \in A$, and for all $m_0 \in \mathbf{R}$, there is $m_1 \in \mathbf{R}$ such that $(\alpha, m_0) I_i (\beta, m_1)$.

Let $\mathcal{R}$ be the class of admissible preference relations. The symbol R denotes a list $(R_i)_{i \in N} \in \mathcal{R}^n$ of preference relations. Since the set of agents, the set of objects, and the amount of money are all kept fixed, an ***economy*** is simply denoted by such a list. Let $\mathcal{R}_{ql}$ be the subclass of $\mathcal{R}$ of quasi-linear preferences, that is, preferences such that if two bundles are judged indifferent to each other, then adding to each of them the same amount of money creates two bundles that are also judged indifferent to each other. Given the monotonicity and compensation assumptions, in order to specify such a preference relation, we only need one "indifference set".

A ***feasible allocation*** is a pair $z = (\sigma, m)$ where $\sigma : N \to A$ is a bijection and m is a vector in $\mathbf{R}^n$ satisfying $\sum_{i \in N} m_i = M$: for each $i \in N$, $\sigma(i) \in A$ designates the object assigned to agent i and m_i the amount of money he receives. Agent i's bundle, $(\sigma(i), m_i)$, is also denoted by z_i. Let $\boldsymbol{Z}$ be the set of feasible allocations.

We would like to be able to make recommendations for all admissible economies. A ***solution*** is a correspondence φ that associates with each economy $R \in \mathcal{R}^n$ a nonempty subset of Z, denoted by $\boldsymbol{\varphi(R)}$. A solution provides for each economy a set of feasible allocations regarded as desirable for the economy. A familiar example is the following:

The Pareto solution, $\boldsymbol{P}$: For all $R \in \mathcal{R}^n$, $P(R) = \{z \in Z :$ there is no $z' \in Z$ such that $z'_i R_i z_i$ for all $i \in N$, and $z'_i P_i z_i$ for at least one $i \in N\}$.

We are particularly interested in solutions satisfying the following fundamental notion of equity: no agent should prefer the bundle of any other agent to his own.

The no-envy solution, F: (Foley, 1967) For all $R \in \mathcal{R}^n$, $F(R) = \{z \in Z :$ for all $i, j \in N$, $z_i R_i z_j\}$.

Under our assumptions, the set of envy-free allocations is non-empty (Alkan, Demange and Gale, 1991), and any envy-free allocation is efficient (Svensson, 1983).[4] For other existence results, see Svensson (1983), Maskin (1987) and Aragones (1991). See also Tadenuma and Thomson (1993) for a proof in a special case.

Although the no-envy concept is very attractive intuitively, the set of envy-free allocations may be quite large, and in these situations the no-envy solution does not make a recommendation that is precise enough. This is what motivated Alkan, Demange and Gale (1991), Aragones (1992), Tadenuma and Thomson (1991a,b, 1993) and Alkan (1994) to search for well-behaved selections from the no-envy solution, and it is in part what prompted our undertaking the research on which we report here.

Our main requirement in the present study is the following application of the general idea of solidarity: replacing the preferences of one of the agents by some other admissible preferences affects all other agents in the same direction. For a formal statement, let φ be a single-valued, or essentially single-valued,[5] solution. Also, given $R \in \mathcal{R}^n$, $i \in N$, and $R'_i \in \mathcal{R}$, let (R'_i, R_{-i}) denote the list R after the replacement of R_i by R'_i.

Welfare-domination under preference-replacement: For all $R \in \mathcal{R}^n$, for all $i \in N$, for all $R'_i \in \mathcal{R}$, then either (i) for all $j \in N\backslash\{i\}$, $\varphi_j(R) R_j \varphi_j(R'_i, R_{-i})$ or (ii) for all $j \in N\backslash\{i\}$, $\varphi_j(R'_i, R_{-i}) R_j \varphi_j(R)$.

Obviously, for this condition to have any force, there should be at least 3 agents. In what follows we therefore assume that $n \geq 3$. Also, the condition could be strengthened by requiring that its conclusion applies to the replacement of the preferences of several agents at one time.

[4]This property holds as long as the number of objects is less than or equal to the number of agents.

[5]A solution is "essentially single-valued" if for every economy in its domain, all agents are indifferent between any two allocations that it selects.

As noted earlier, *welfare-domination under preference-replacement* first appears in Moulin (1987) in an analysis of a class of binary social choice problems.

3 The general case

Our first result is that even on the restricted class of quasi-linear economies, *welfare-domination under preference-replacement* is incompatible with no-envy.

Theorem 1 *Suppose that $n \geq 4$. Then, there is no selection from the no-envy solution satisfying* welfare-domination under preference-replacement.

Proof: Let $N = \{1, 2, 3, 4\}$, $A = \{\alpha, \beta, \gamma, \delta\}$, $M = 0$, and $R \in \mathcal{R}_{ql}$ be such that

$(\alpha, 0)I_1(\beta, 0)I_1(\gamma, 4)I_1(\delta, 4)$
$R_2 = R_1$
$(\alpha, 0)I_3(\beta, 0)I_3(\gamma, -4)I_3(\delta, -4)$
$R_4 = R_3$

Note that all agents consider objects α and β to be equivalent, the same holding for objects γ and δ. Let $\varphi \subseteq F$ and $z = (\sigma, m) \in \varphi(e)$. By quasi-linearity of preferences, and since $F \subseteq P$, it follows that at z agents 1 and 2 receive objects α and β, and agents 3 and 4 receive objects γ and δ; moreover, $m_1 = m_2$ and $m_3 = m_4$. Clearly, $m_1 + m_3 = 0$. Without loss of generality, suppose that $m_1 \geqq 0$.

Let $R'_3 \in \mathcal{R}_{ql}$ be defined by $(\alpha, 0)I'_3(\beta, 0)I'_3(\gamma, 1)I'_3(\delta, 1)$. Let $z' = (\sigma', m') \in \varphi(R'_3, R_{-3})$. Again, since $\varphi \subseteq F \subseteq P$, it follows that at z', agents 1 and 2 still receive objects α and β, and agents 3 and 4 still receive objects γ and δ; moreover, $m'_1 = m'_2$ and $m'_3 = m'_4$, so that $m'_1 + m'_3 = 0$ also. Then, for agent 3 not to envy either agent 1 or agent 2, we need $m'_3 > m'_1$. Hence, $m'_3 > 0 \geqq m_3$ and $m'_1 = m'_2 < m_1 = m_2$. Therefore, when agent 3's preferences change, agent 4 is made better-off and agents 1 and 2 are made worse-off, in violation of *welfare-domination under preference-replacement.*

To handle the case $n > 4$ it suffices to introduce additional agents of the type specified previously.[6]

[6] For completeness, it would be useful to have an example for the three-person case, to which we suspect that the result does extend.

Q.E.D.

4 Single prize

Although we have so far required the numbers of agents and objects to be the same, one possible application of our model is to the problem of allocating a number of "actual" objects that is less than the number of agents. The equality of the numbers of agents and objects assumed previously is obtained in a straighforward way by thinking that each of the agents not receiving one of the actual objects is attributed a "null" object instead. If the number of actual objects is at least 2, the impossibility results still holds. We consider in this section the simple case in which there is a single actual object to be assigned to one of the agents, the remaining $n-1$ agents being assigned null objects. We denote the actual object by α and each of the null objects by ν, so that $A = \{\alpha, \nu, \ldots, \nu\}$. An example is a prize that only one of the agents can receive. Alternatively, the actual object may be a chore that one of the agents will have to perform. In spite of this second interpretation of the model, we will refer for convenience to the agent who receives the actual object as the "winner" and to the others as the "losers". Note that at an envy-free allocation, all losers receive the same amount of money.

This case was examined by Tadenuma and Thomson (1993) who characterized the solution that selects the envy-free allocation most favorable to the winner on the basis of consistency on the one hand, and on the basis of weak population-monotonicity on the other. More precisely, let φ^* denote the (essentially single-valued) solution that associates with each economy its set of envy-free allocations such that the winner is indifferent between his bundle and the common bundle of the losers. Given $R \in \mathcal{R}^n$ and $z \in F(R)$, we will refer to the winner as agent w, to this bundle as $z_w = (\alpha, m_w)$, and to his preference relation as R_w; similarly, we will designate the losers' common bundle by $z_\ell = (\nu, m_\ell)$. We can now formally define the solution φ^*. Given $R \in \mathcal{R}^n$,

$$\varphi^*(R) = \{z \in F(R) : z_w I_w z_\ell\}$$

Our next result is a characterization of the solution φ^* on the basis of *welfare-domination under preference-replacement.*

Theorem 2 *Case of a single prize:* $A = \{\alpha, \nu, \ldots, \nu\}$. *The solution* φ^* *is the only selection from the no-envy solution satisfying* welfare-domination under preference-replacement.

Proof: It is easy to see that the solution φ^* does satisfy the properties listed in the theorem. Conversely, let $\varphi \subseteq F$ be a solution satisfying *welfare-domination under preference-replacement.* We assume by contradiction that there is $R \in \mathcal{R}^n$ such that $\varphi(R) \neq \varphi^*(R)$. This means that for some $z \in \varphi(R)$, $z_w P_w z_\ell$. To simplify the notation, we assume that the winner is agent 1. Let $a > 0$ be such that $z_1 = (\alpha, m_1) I_1 (\nu, m_\ell + a)$. Also, for each $i \in N$, let $c_i \in \mathbf{R}$ be such that $z_\ell I_i (\alpha, c_i)$. Finally, let i^* be such that $c_{i^*} \leqq c_i$ for all $i \in N \backslash \{1\}$. To simplify the notation, we take $i^* = 2$.

We now change the preferences of agent 2 from R_2 to R'_2 such that $z_\ell = (\nu, m_2) I'_2 (\alpha, c'_2)$ for some $c'_2 \in]m_1 - (n-1)a, m_1[$.

Let $z' \in F(R'_2, R_{-2})$. We consider two cases.

Case 1: The winner at z' is agent 1. Then $m'_1 < m_1$. Indeed, if $m'_1 \geqq m_1$, then $m'_2 \leqq m_2$ and we have $(\alpha, m'_1) R'_2 (\alpha, m_1) P'_2 (\nu, m_2) R'_2 (\nu, m'_2) = z'_2$, so that agent 2 envies agent 1 at z', in contradiction with $z' \in F(R')$. But if $m'_1 < m_1$, then $m'_\ell > m_\ell$, so that agent 1 is worse-off and agents 3,..,n are better-off from the change in agent 2's preferences, in contradiction with *welfare-domination under preference-replacement.*

Case 2: The winner at z' changes. Then $m'_w < m_w$. Indeed, if $m'_w \geqq m_w = m_1$, then $m'_\ell \leqq m_\ell$ and we have $z'_w = (\alpha, m'_1) R_1 (\alpha, m_1) P_1 (\nu, m_\ell) R_1 (\nu, m'_1) = z'_1$ so that agent 1 envies the new winner. Therefore $m'_w < m_w$ as claimed, and $m'_\ell > m_\ell$. Now, we claim that the new winner is agent 2; otherwise, $z'_\ell = (\nu, m'_\ell) P_w (\nu, m_\ell) R_w (\alpha, m_w) P_w (\alpha, m'_w) = z'_w$ and the new winner envies the losers. If the winner is agent 2, for him not to envy the losers, we need $m'_2 = m'_w \geqq c'_2$. Otherwise, $z'_\ell = (\nu, m'_\ell) P'_2 (\nu, m_\ell) I'_2 (\alpha, c'_2) P'_2 (\alpha, m'_2) = z'_2$. Since $c'_2 > m_1 - (n-1)a$, we have $m'_1 = m'_\ell < m_\ell + a$ and agent 1 is worse-off. Since $m'_\ell > m_\ell$ we also obtain that agents 3,...,n are better-off. Altogether, we obtain a contradiction with *welfare-domination under preference-replacement.* Q.E.D.

The characterization of Theorem 2 still holds on the subdomain of economies with quasi-linear preferences, and in fact for that domain, the proof is a little simpler. It suffices to change the preferences of agent 2 in such a way that $z_\ell I_2'(\alpha, m_1 - a/2)$. By quasi-linearity of preferences, we deduce that agent 1 remains the winner, and only Case 1 is relevant: the amount of money that he should receive has to decrease, so that he is made worse-off; this means that the amount of money associated with the losers' bundle increases, so that agents 3,...,n are made better-off. This is in contradiction with *welfare-domination under preference-replacement.*

5 Conclusion and open questions

We have considered the problem of fair allocation in the context of economies with indivisible goods and investigated the implications of the requirement on solutions that changes in the preferences of an agent affect all other agents in the same direction. We have shown that in general there is no selection from the no-envy solution satisfying this requirement, but that in the case of a single prize (or chore), it can be met; however, in that case, it can be met in only one way: there is a unique solution satisfying it, and this solution is one that has been shown in previous studies of the problem to be most desirable.

In the face of our impossibility result for the general case, it is natural to ask whether appealing weakenings of *welfare-domination under preference-replacement* can be formulated that would be met more generally. Also, the question arises whether the choice of distributional requirements other than no-envy would lead to positive results. The answer to that second question is yes. Consider the requirement of egalitarian-equivalence: an allocation is ***egalitarian-equivalent*** (Pazner and Schmeidler, 1978) if there is some reference bundle that all agents find indifferent to their assigned consumptions. It is easy to check that the standard selections from the egalitarian-equivalence and Pareto solution obtained by requiring the reference bundle to contain a fixed object do satisfy *welfare-domination under preference-replacement.*[7]

An alternative distributional requirement is that each agent receive a consumption that he prefers to what he would be attributed at an efficient and

[7]For a number of models, similarly defined selections from the egalitarian-equivalence and Pareto solution satisfy the property. Note that the solution φ^* is egalitarian-equivalent. However, the compatibility of no-envy and egalitarian-equivalence does not hold in the case of more than one object (Thomson, 1990a).

equal treatment allocation in a hypothetical economy made up of agents with preferences identical to his. This defines the agent's "identical-preferences lower bound". The criterion that each agent be placed above his lower bound was proposed by Moulin (1990) for production economies, and it has been the object of further analysis by Maniquet (1994) and Fleurbaey and Maniquet (1994). In the context of economies with indivisible goods it has been extensively studied by Bevia (1992, 1993). Generalizing an observation due to Moulin (1990), Bevia shows that any envy-free allocation meets this criterion. Two open questions are the following: does the impossibility of Theorem 1 persist when the search is widened from the class of selections from the no-envy solution to the class of solutions satisfying the criterion? And, does the characterization of Theorem 2 still hold for that wider class?

We can offer partial anwers to both questions. For quasi-linear economies it is negative. Indeed, consider the solutions defined as follows. For each agent, identify his identical-preferences lower bound welfare level. In a quasi-linear economy, the assignment of objects is essentially unique at all efficient allocations.[8] Then, it is possible unambiguously to define the monetary surplus available when all the lower bounds are met. Now, given an arbitrary vector of weights, selects the allocation(s) at which this surplus is distributed according to these weights. Any solution so defined satisfies all the desired requirements. By choosing equal weights, we obtain a solution that in addition satisfies the requirement of anonymity (the rule is invariant under renaming of agents). This family of examples shows that the impossibility of Theorem 1, which as we noted holds on the subdomain of quasi-linear economies, does not extend when no-envy is replaced by the requirement of identical-preferences lower bound. It also shows that in the one-object case, the uniqueness of Theorem 2, which also holds on the quasi-linear domain, does not extend either. This is because the solution φ^* is not a member of the family just described.

[8]We used this fact in the proof of Theorem 1.

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