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William Thomson*

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*Department of Economics, University of Rochester, Rochester, NY 14627. I am grateful to James Schummer for his very helpful comments, and the NSF for its support, under grant SES9212557.

Contents

1 Introduction

When a firm goes bankrupt, how should its liquidation value be divided among its creditors as a function of the claims they hold against it?

This essay is an introduction to the literature devoted to the formal analysis of such ***bankruptcy problems***. The objective of this literature is to identify well-behaved methods, or ***rules***, of associating with each bankruptcy problem a division between the creditors of the net worth of the firm.

We will present several rules that are commonly used in practice or discussed in the theoretical literature, formulate a number of appealing properties that one may want rules to enjoy, compare the rules on the basis of these properties, and search for rules satisfying the greatest number of the properties together. Our methodology will therefore be mainly axiomatic. This methodology underlies most of the developments on which we report here, and this survey will provide an illustration of the increasingly important role it has played in recent years in the design of allocation rules. However, we will also discuss a variety of strategic analyses of bankuptcy problems.

The best-known rule is the proportional rule: awards are chosen proportional to claims. In fact, proportionality is often taken as the definition of fairness for this class of problems. We will challenge this position and start from more elementary considerations. Is there any reason to believe the proportional rule to be superior to others? An important source of inspiration for the research described here is the Talmud, in which several numerical examples are discussed, and recommendations are made that conflict with proportionality. Are these recommendations rationalizable by means of well-behaved rules, and if there are several such rules, do grounds exist for preferring one of them to the others? Are there yet other rules that deserve to be considered?

This survey is organized as follows. First, we introduce a number of important rules. Then we show how a number of them can be obtained by applying solution concepts developed in the theory of cooperative games. These concepts originate in the theory of bargaining and in the theory of coalition form games. Next, we formulate a variety of properties of rules, first in a setting in which the number of claimants is fixed, then in a richer framework in which this number is allowed to vary. We continue with a presentation of several models of bankruptcy as a non-cooperative game.

Finally, we consider extensions of the model, "dual" situations where the amount to divide is more than sufficient to cover the claims — this is the problem of surplus sharing — and models where the feasible set is specified in utility space.

We close this introduction by noting that the problem of identifying well-behaved taxation rules is formally identical to that of identifying bankruptcy rules, and all of the results that we present here can be reinterpreted in the context of taxation.

2 Bankruptcy rules

First, we formally introduce the class of problems that we will study and the necessary notation. The ***net worth*** $\boldsymbol{E}$ of a bankrupt firm has to be divided among a group of agents N, $\boldsymbol{c_i}$ being the ***claim*** of agent $i \in N$ and $c = (c_i)_{i \in N}$ the vector of claims. We designate by n the cardinality of N. Initially, we take N to be a finite subset of the set of natural numbers $\mathbb{N}$. Later on, we extend the model so as to cover situations in which the population of claimants varies, and we generalize the model and the notation accordingly.

Definition A ***bankruptcy problem*** is a pair $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$ with $\sum c_i \geq E$.[1] Let $\mathcal{B}^N$ denote the class of these problems.

An application of the model is to estate division: a man dies and the amounts he bequeathed separately to his heirs are found to add up to more than the worth of the estate. How should the estate be divided?

A pair (c, E) as above can alternatively be interpreted as a ***tax assessment problem***: then, N is a group of taxpayers with incomes given by the coordinates of c, and who among themselves must cover the cost E of a project. Although the mathematical models are identical, and the axioms that we will find relevant to the analysis of bankruptcy problems and taxation problems are essentially the same, the appeal of each particular axiom may of course depend on the application. In what follows, we will mainly

[1]We denote by $\mathbb{R}_+^N$ the cartesian product of n copies of $\mathbb{R}_+$ indexed by the members of N. A summation without explicit bound should be understood to be carried out over all agents.

think of bankruptcy. Our model is indeed a more accurate description of the actual situation faced by bankruptcy courts. By contrast, the issue of taxation is rarely specified by first stating an amount to be collected, perhaps due to the uncertainty pertaining to the taxpayers' incomes. In most cases, taxation schedules are published first, and the amount collected falls wherever it may, depending upon the realized incomes.[2]

A ***bankruptcy rule*** is a function defined on the class of bankruptcy problems that associates with each problem in the class a division of the net worth of the firm between the claimants. This division is interpreted as a recommendation for that problem.[3]

Definition A bankruptcy rule, or simply a ***rule***, is a function that associates with every bankruptcy problem $(c, E) \in \mathcal{B}^N$ a vector $x \in \mathbb{R}^N_+$ whose coordinates add up to E: $\sum x_i = E$.

In the statement of the axioms in Sections 4 and 5, we designate a generic solution by the letter F.

2.1 Bankruptcy rules

The basic notation being out of the way, we can proceed with a presentation of two intriguing problems discussed in the Talmud. The Talmud specifies only a few numerical examples, but the desire to understand these examples has provided much of the impetus underlying the theoretical efforts described in these pages.

The ***contested garment problem***: two men disagree over the ownership of a garment, worth 200. The first man claims half of it (100) and the other claims it all (200). Assuming both claims to be made in good faith, how should the worth of the garment be divided among them? The Talmud recommends 50 for the first one and 150 for the second (Baba Metzia 2a).[4]

[2]However, we should note that a number of conditions that we will use later have been first considered in the context of taxation.

[3]We will limit ourselves to the search for *single-valued* solutions, since for this model, in contrast with a number of models commonly studied, a great variety of interesting solutions have that property.

[4]All references to the relevant passages of the Talmud and of the secondary literature are taken from O'Neill (1982), Aumann and Maschler (1985), and Dagan (1994).

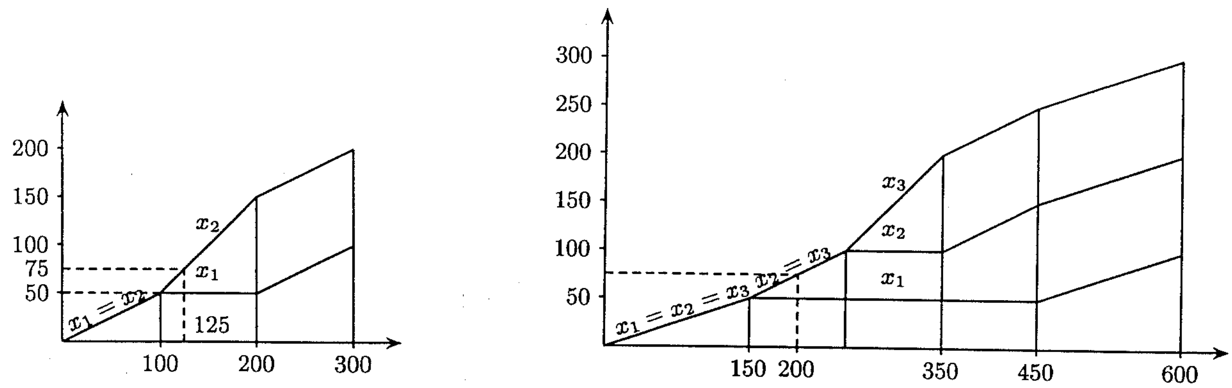


Figure 1: ***The Talmudic rule.*** The value of the estate is measured horizontally and the awards vertically. (a) The two-claimant case. Claims are $(c_1, c_2) = (100, 200)$. The Talmud considers the case when the estate is worth 200 and recommends the division $(50, 150)$. (b) The three-claimant case. Claims are $(c_1, c_2, c_3) = (100, 200, 300)$. If the estate is worth 100, the Talmud recommends equal division, $(100/3, 100/3, 100/3)$; if it is worth 200, it recommends $(50, 75, 75)$; if it is worth 300, it recommends proportional division, $(50, 100, 150)$.

The ***estate division problem:*** a man has three wives whose marriage contracts specify that in case of his death they should receive 100, 200 and 300 respectively. The man dies and his estate is found to be worth only 100. How should the amount be divided among the wives? The Talmud recommends equal division. If the estate is worth 300, the Talmud recommends proportional division, but if it is worth 200, it recommends (50, 75, 75)! (Kethubot 93a; the author of this Mishna is Rabbi Nathan.)

To clarify the mystery posed by the numbers given as resolutions of these problems, we should first of all find a general and natural formula that generates them. Consider the following algorithm proposed by Aumann and Maschler (1985) for the general n-person case (see Figure 1 for the two problems of the Talmud): the first units of the estate are divided equally until each claimant has received an amount equal to half of the smallest claim; then the claimant with the smallest claim does not receive anything for a while; instead, any additional unit is divided equally among all others until each of them has received an amount equal to half of the second smallest claim ... the algorithm proceeds in this way until a value of the estate equal to $\sum c_i/2$; then, each claimant has received half of her claim; for values of the estate greater than $\sum c_i/2$, awards are computed in a symmetric way by starting from a value of the estate equal to the sum of the claims, (in which case each claimant receives her claim), and considering shortfalls of increasing size. Initial shortfalls are divided equally until all claimants experience a loss equal to half of the smallest claim; the loss incurred by the claimant with the smallest claim stops at that point. Any additional shortfall is born equally by the others, until their common loss is equal to half of the second smallest claim ... The process continues until a value of the estate equal to $\sum c_i/2$. It is a simple matter to check that Aumann and Maschler's proposed method, applied to the two problems in the Talmud, does yield the numbers given there. Henceforth, we will call it the ***Talmudic rule***.[5] We will also refer to the restriction of this method to the two-claimant case as the ***contested garment rule***. As we will see, there are other ways of generalizing this rule to the case of more than two claimants.

Talmudic rule, T: For all $(c, E) \in \mathcal{B}^N$ and for all $i \in N$,

1. If $(1/2)\sum c_i \geq E$, then $T_i(c, E) = \min\{c_i/2, \lambda\}$, where λ is chosen so that $\sum \min\{c_i/2, \lambda\} = E$
2. If $(1/2)\sum c_i \leq E$, then $T_i(c, E) = c_i - \min\{c_i/2, \lambda\}$, where λ is chosen so that $\sum[c_i - \min\{c_i/2, \lambda\}] = E$.

In actual practice, the most commonly used rule is the ***proportional rule***, the rule for which awards are proportional to claims.

Proportional rule, P: For all $(c, E) \in \mathcal{B}^N$, $P(c, E) = \lambda c$, where λ is chosen so that $\sum \lambda c_i = E$.

A version of the proportional rule is obtained by making awards proportional to the claims truncated by the value of the estate:

Truncated-claims proportional rule, P^t: For all $(c, E) \in \mathcal{B}^N$ and for all $i \in N$, $P_i^t(c, E) = \lambda \min\{c_i, E\}$, where λ is chosen so that $\sum \lambda \min\{c_i, E\} = E$ (that is, $\lambda = E/\sum \min\{c_i, E\}$ if $c \neq 0$ and $E \neq 0$, and $\lambda = 0$ otherwise).

[5] Of course, the Talmud not offering any example for the case $E \geq \sum c_i/2$ when $n \geq 3$, we can only speculate as to what it would have recommended then. However, we find the sort of considerations that led Aumann and Maschler (1985) to the interpolation and extrapolation they define very compelling, and this is why we refer to the rule they propose as the Talmudic rule. Moreover, the formula is also consistent with another numerical example in the Talmud (Aumann and Maschler, 1985).

The following rule, which always makes equal awards, will be useful mainly as a benchmark. The idea of "equality" underlies many theories of economic justice. The question is what exactly should be equated, especially when agents are not identical. Here, agents differ in their claims and the objective is precisely to take proper account of these differences.

Equal award rule, *EA*: For all $(c, E) \in \mathcal{B}^N$, $EA(c, E) = (E/n, \ldots, E/n)$.

An undesirable consequence of the rule being independent of claims is that some agents may receive more than their claims. The next rule preserves the spirit of egalitarianism but it does not suffer from this problem. It awards the same amount to all agents, adjustments being made to ensure that no agent receives more than his claim. It has been advocated by many authors, including Maimonides (12th Century). Although this departure from egalitarianism does not seem to go far enough in recognizing differences in claims, we will nevertheless show that appealing axiomatic justifications for the rule can be provided. Moreover it is an "ingredient" in the definition of other interesting rules defined later:

Constrained equal award rule,[6] ***CEA*:** For all $(c, E) \in \mathcal{B}^N$ and for all $i \in N$, $CEA_i(c, E) = \min\{c_i, \lambda\}$, where λ is chosen so that $\sum \min\{c_i, \lambda\} = E$.

The rule proposed by Pineles (1861) can be understood as resulting from a "double" application of the constrained equal award rule: first, the rule is applied to the division to the lesser of two amounts, the estate and half of the sum of the claims. If the estate is greater than half of the sum of the claims, the rule is applied again to divide the remainder.

Pineles' rule, *Pin*: For all $(c, E) \in \mathcal{B}^N$ and for all $i \in N$, $Pin_i(c, E) = CEA_i(c, E)$ if $\sum c_i/2 \geq E$, and $Pin_i(c, E) = c_i/2 + CEA_i(c/2, E - \sum c_i/2)$ otherwise.

A dual of the constrained equal award rule focuses on the losses claimants incur (what they do not receive), as opposed to what they receive (the partial compensations awarded to them). It too is suggested by Maimonides (Aumann and Maschler, 1985).

[6] In the context of taxation, this rule is known as ***head taxation***.

Constrained equal loss rule[7]**, *CEL*:** For all $(c, E) \in \mathcal{B}^N$ and for all $i \in N$, $CEL_i(c, E) = \max\{0, c_i - \lambda\}$, where λ is chosen so that $\sum \max\{0, c_i - \lambda\} = E$.[8]

The next rule requires first calculating the "minimal right of each agent". This is the amount that is left over if every other agent receives his claim, or zero if that leftover is negative. Then, the rule selects the allocation at which each claimant receives his minimal right (these payments are feasible), the remainder being divided proportionately to the minimum of the remainder and his minimal right. For all $(c, E) \in \mathcal{B}^N$ and for all $i \in N$, let $m_i(c, E) = \max\{E - \sum_{j \in N \setminus \{i\}} c_j, 0\}$ be the minimal right of claimant i. Also, let $m(c, E) = (m_i(c, E))_{i \in N}$.

Adjusted[9] **proportional rule, *A*:** (Curiel, Maschler and Tijs, 1988) For all $(c, E) \in \mathcal{B}^N$ and for all $i \in N$, $A_i(c, E) = m_i(c, E) + P(c - m(c, E), E - \sum m_i(c, E)\})$.

Another method is formulated by O'Neill (1982) as a generalization of a rule that he attributes to Ibn Ezra (12th Century). The problem discussed by Ibn Ezra is that of dividing an estate worth 120 among four sons whose claims are 30, 40, 60 and 120. He recommends the division 30/4, 30/4 +10/3, 30/4+10/3 +20/2, and 30/4+10/3 +20/2+ 60/1. Rabad (12th Century) suggests the following estate division method, defined for problems such that the net worth of the firm is less than the greatest claim. It gives Ibn Ezra's numbers in his particular application. It is discussed by Aumann and Maschler (1985): when the estate is worth less than the smallest claim, it is divided equally; as its value increases from the smallest to the second smallest claim, the claimant with the smallest claim continues to receive $1/n$ of his claim and the leftover is divided equally among the other claimants. In

[7] In the context of taxation, this rule is known as the ***leveling tax***. Landsburg (1994) considers a version of the solution that does not meet the non-negativity condition on awards.

[8] Dagan, Serrano and Volij (1994) note that the Talmudic rule can be expressed as a function of the constrained equal award and constrained equal loss rules in the following way: $T(c, E) = CEA(c/2, \min\{\sum c_i/2, E\}) + CEL(c/2, \max\{E - \sum c_i/2, 0\})$. This can also be written as $T(c, E) = CEA(c/2, E)$ if $\sum c_i/2 \geq E$ and $CEL(c/2, E - \sum c_i/2)$ otherwise.

[9] Note that ***several*** adjustments are performed.

general, when the value of the estate increases from the kth smallest claim to the $(k+1)$th smallest claim, the agents with the k smallest claims continue to receive the same amounts, and the leftover is divided equally among the other claimants.

O'Neill noted that Ibn Ezra's numbers can be generated as follows. Assume that agents are ordered by claims. Then, do not think of them as claiming an arbitrary part of the estate but instead a specific part, equal division being applied to each part separately among all agents claiming it. Moreover, choose which parts of the estate are claimed by the various agents so as to maximize the part of the estate claimed only by the claimant with the greatest claim, and subject to that, so as to maximize the part of the estate claimed by the second highest claimant but not by anyone with a smaller claim ... and so on.

Minimal overlap rule, *MO*: Claims on specific parts of the estate are arranged so that the size of the estate claimed by exactly $k+1$ claimants is maximized, given that the size of the estate claimed by k claimants is maximized, for $k = 1, \ldots, n-1$. Once claims are arranged in this way, for each part of the estate, equal division prevails among all agents claiming it.

O'Neill shows that the arrangement of the claims solving this exercise is (essentially) unique. When one of the claims is equal to the whole estate, the solution to the exercise consists in nesting the claims. Any claim greater than the whole estate is replaced by the whole estate.

2.2 Rules inspired by concepts of cooperative game theory

Other rules have been inspired by the theory of cooperative games. For this theory to be applicable, we need first to define a formal way of associating with each bankruptcy problem a cooperative game. Two main classes of such games have been studied, bargaining problems and coalition form games.

2.2.1 Bargaining solutions

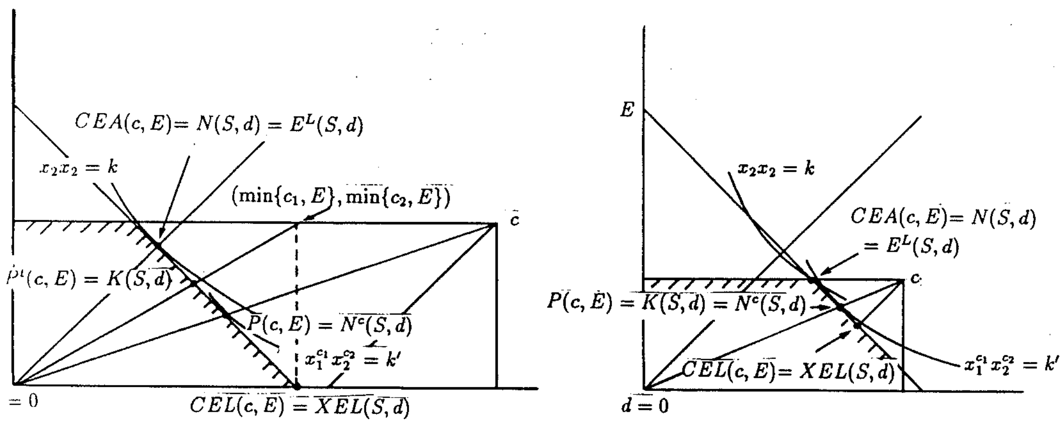
We start with bargaining problems. A ***bargaining problem*** is a pair (S, d), where S is a subset of $\mathbb{R}^N$ and d is a point of S. The set S, called the

feasible set, consists of all the feasible utility vectors attainable by the group N by unanimous agreement, and d, called the ***disagreement point***, is interpreted as the utility vector that obtains if they fail to reach an agreement. A ***bargaining solution*** is a function defined on a class of bargaining problems, which associates with each problem in the class a unique point in the feasible set of the problem. Important solutions are the following. The ***egalitarian solution*** (Kalai, 1977) selects the maximal point of S at which utility gains from d are equal. The ***lexicographic egalitarian solution*** (Imai, 1977) selects the point of S at which these gains are maximal in the lexicographic order.[10] The ***Kalai-Smorodinsky solution*** (Kalai and Smorodinsky, 1975) selects the maximal point of S on the segment connecting d to the ideal point of (S, d), the point whose ith coordinate is the maximal utility agent i can obtain subject to the condition that all other agents receive at least their utilities at d. The ***Nash solution*** (Nash, 1950) selects the point maximizing the product of utility gains from d among all points of S dominating d. Given a vector of weights $\alpha \in \Delta^{n-1}$, the ***weighted Nash solution with weights*** $\boldsymbol{\alpha}$ selects the point of S at which the product $\Pi(x_i - d_i)^{\alpha_i}$ is maximized among all points of S dominating d. The ***extended equal loss solution*** (Bossert, 1993, in a contribution building on Chun, 1988) selects the maximal point of S at which the losses from the ideal point of all agents with a positive utility are equal and the utilities of the others are zero.

To associate a bargaining problem to a bankruptcy problem, the most natural choice is perhaps to take the feasible set to be the set of all non-negative distributions of E dominated by the vector of claims, and to set the disagreement point equal to the origin.[11] This makes good sense if we require that claimants never receive more than their claims (see below). However, solutions satisfying this requirement could be responsive to changes in claims that do not affect the associated bargaining problems as just defined (the proportional solution is an example), and we could argue that too much information is lost in the passage from bankruptcy problems to bargaining

[10] Given $x, y \in \mathbf{R}^n$, $\tilde{x}$ designates the vector obtained from x by rewriting its coordinates in increasing order, $\tilde{y}$ being similarly defined. We say that ***x is lexicographically greater than y*** if, either $[\tilde{x}_1 > \tilde{y}_1]$, or $[\tilde{x}_1 = \tilde{y}_1$ and $\tilde{x}_2 > \tilde{y}_2]$, or more generally, for some $k \in [1, \ldots, n-1]$, $[\tilde{x}_1 = \tilde{y}_1, \ldots, \tilde{x}_{k-1} = \tilde{y}_{k-1}$, and $\tilde{x}_k > \tilde{y}_k]$.

[11] This amounts to assuming that agents have utility functions that are linear in money and normalized so that the utility of zero is zero.



(a) (b)

Figure 2: ***Bankruptcy problems and their associated bargaining problems.*** The shaded area represents the set of feasible award vectors that are dominated by the claims points. This area is taken as the feasible set of the associated bargaining problem. The egalitarian bargaining solution selects the maximal feasible point at which awards are equal. At such a point, the whole estate need not be divided, as is clear in panel (b). The equal loss bargaining solution selects the maximal feasible point at which losses from the ideal point are equal. If there are more than two claimants, such a point need not exist. Key for bargaining solutions; N: Nash solution, K: Kalai-Smorodinsky solution, E^L: lexicographic egalitarian solution, N^c: weighted Nash solution with weights c, XEL: extended equal loss solution.

problems. Another formalization is possible however, in which the claims point remains as separate data. The relevant concept here is then the generalization of the notion of a bargaining problem obtained by adding a claims points (Chun and Thomson (1992) call these problems "bargaining problems with claims". See Section 5 for further discussion.)

Definition Given a bankruptcy problem $(c, E) \in \mathcal{B}^N$, its ***associated bargaining problem*** is the problem $\boldsymbol{B(c, E)}$ whose feasible set is equal to $\{x \in \mathsf{R}^N_+ : \sum x_i = E, x \leq c\}$, and whose disagreement point is the origin.

This operation is illustrated in Figure 2 for two examples. For the first example, the point of equal awards is efficient and for the second it is not.

An alternative specification of the disagreement point is proposed by Dagan and Volij (1993): it is the vector of minimal rights entering the definition of the adjusted proportional solution.

In bargaining theory, the feasible set is allowed to be an arbitrary compact, convex set, but here, we have the special case of a feasible set whose efficient boundary is a subset of a plane normal to a vector of ones. An extension of the model accommodating more general shapes is discussed in Section 5.

If for each bankuptcy problem, the recommendation made by a given rule coincides with the recommendation made by some bargaining solution applied to the associated bargaining problem, we say that the rule ***corresponds*** to the solution. The next lemma describes a number of such correspondences.

Lemma 1 (Various authors) The following correspondences between bankruptcy rules and bargaining solutions hold:

1. The constrained equal award rule and the Nash bargaining solution (Dagan and Volij, 1993).

2. The constrained equal award rule and the lexicographic egalitarian solution.

3. The proportional rule and the weighted Nash solution with weights chosen equal to the claims (Dagan and Volij, 1993).

4. The truncated-claims proportional rule and the Kalai-Smorodinsky solution (Dagan and Volij, 1993).[12]

5. The constrained equal loss rule and the extended equal loss solution.

The recommendations made by various bankruptcy rules and the bargaining solutions to which they correspond are indicated in Figure 2 for two examples.

Although the lemma establishes useful links between the theory of bankruptcy and the theory of bargaining, one should perhaps not attach too much importance to any particular one of them. Indeed, since the bargaining problems associated with bankruptcy problems constitute a very narrow subclass of the class of bargaining problems traditionally studied, it follows that bargaining solutions that give different answers in general often coincide on this subclass. This phenomenon is illustrated by the fact that the constrained equal award rule corresponds to both the Nash solution and the lexicographic egalitarian solution.[13]

2.2.2 Solutions to coalitional form games

We now turn to the richer class of coalitional form games. Such games are formal representations of situations in which all groups, called ***coalitions***, (and not just the group of the whole) can achieve something. Formally, a (transferable utility[14]) ***coalitional form game*** is a list $v = (v(S))_{S \subset N} \in \mathbf{R}^{2^n - 1}$, where for each coalition $S \subset N$, $v(S) \in \mathbf{R}$ is the ***worth*** of S. This number is interpreted as what the coalition can achieve on its own. A ***solution*** associates with every such game v a ***payoff vector***, a point in $\mathbf{R}^N$ whose coordinates add up to $v(N)$.

[12]Dagan and Volij also show that the adjusted proportional rule corresponds to the Kalai-Smorodinsky solution applied to the problem in which the disagreement point is set equal to the vector of minimal rights instead of the origin.

[13]The reader may wonder why a solution that is *scale invariant* (invariant with respect to positive linear transformation, independent agent by agent, of their utilities), such as the Nash solution, coincides with a solution that is based on utility comparisons, such as the lexicographic egalitarian solution. The answer is simply that the subclass of bargaining problems associated with bankruptcy problems is not rich enough for the operation of scale transformation to ever be applicable.

[14]In Section 5, we discuss the non-tranferable utility case.

In order to be able to apply the solutions introduced in the theory of coalitional form games, we need to find a natural procedure of associating with each bankruptcy problem a coalitional form game. The following one was proposed by O'Neill (1982). In the two-claimant case, set the worth of each claimant equal to the net worth of the firm minus the claim of the other agent if this difference is non-negative, and zero otherwise,[15] and set the worth of the grand coalition equal to E. Dividing equally the amount that remains when each claimant is first paid his own worth leads to the following recommendation, which coincides with the recommendation made by the contested garment rule:

$$x_i = \max\{E - x_j, 0\} + \frac{1}{2}[E - \max\{E - x_1, 0\} - \max\{E - x_2, 0\}]$$

If there are more than two agents, set the worth of each coalition equal to the net worth of the firm minus the sum of the claims of the members of the complementary coalition if this difference is non-negative and zero otherwise. This amount is "conceded" by the complementary coalition. It is what the coalition can secure without going to court.

Definition (O'Neill, 1982) Given a bankruptcy problem $(c, E) \in \mathcal{B}^N$, its ***associated coalitional form game*** is the game $\boldsymbol{v(c, E)} \in \mathbb{R}^{2^n-1}$ such that for each $S \subset N$, $v(c, E)(S) = \max\{E - \sum_{N \setminus S} c_i, 0\}$.

Note that as defined here, the worth of each coalition is a somewhat pessimistic assessment of what it can achieve. However, the bias being systematic across coalitions, we might still expect the resulting game to appropriately "summarize" the actual situation.[16]

It is useful to observe that the game $v(c, E)$ is convex[17] (Curiel, Maschler and Tijs, 1987).[18] Therefore, its core[19] is non-empty, and its Shapley value (see below) belongs to it.

[15]Note that we called the minimal right of an agent is simply the worth of the "coalition" consisting only of that agent.

[16]A coalitional form game is a point in a space of considerably greater dimension than a bankruptcy problem. In the passage from bankrupcty problems to coalitional form games, the necessary increase in dimensionality is bound not to be entirely natural.

[17]This means that the contribution of a player to a coalition is always greater than his contribution to any subcoalition.

[18]Conversely however, it is not true that any positive convex game can be generated by a bankruptcy problem.

[19]This is the set of payoff vectors such that no coalition in aggregate receives less than

Just as we saw for bargaining solutions, a number of links exist between bankruptcy rules and solutions to coalitional form games. If for every bankruptcy problem, the recommendation made by a bankruptcy rule coincides with the recommendation made by a solution to coalitional form games when applied to the associated coalitional form game, once again we say that the rule ***corresponds*** to the solution.

The first correspondence that we will describe involves a rule based on the following scenario. Imagine agents arriving one at a time in order to get reimbursed, and suppose that each claim is fully honored until money runs out. The resulting schedule of awards will of course depend on the order in which claimants arrive. To obtain independence, take the average over all orders of the schedules obtained in this way. This proposal was made by O'Neill (1982). It is inspired by the well-known solution to coalitional form games introduced by Shapley (1953) and its "random order" interpretation. There is in fact a formal relation between the two solutions. This relation is described in Lemma 2. The ***Shapley value of player $i \in N$ in the game $v \in \mathbb{R}^{2^n-1}$*** is the expected amount by which his arrival increases the worth of the coalition consisting of all the players that have arrived before him, when all orders are equally likely: Let Π^N be the class of bijections from N into itself. Then, $Sh_i(v) = \frac{1}{n!}\sum_{\pi\in\Pi^N}[v(\{j \in N : \pi(j) < \pi(i)\} \cup i) - v(\{j \in N : \pi(j) < \pi(i)\})$.

Random order rule[20], ***RO*:** For all $(c, E) \in \mathcal{B}^N$ and for all $i \in N$, $RO_i(c, E) = \frac{1}{n!}\sum_{\pi\in\Pi^N} \min\{c_i, \max\{E - \sum_{j\in N,\pi(j)<\pi(i)} c_j, 0\}\}$.

O'Neill (1982) discusses a method of recursive adjustments (under the name of "recursive completion") that produces the Shapley value.[21]

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its worth: more precisely, the core of $v \in \mathbb{R}^{2^n-1}$ is the set of payoff vectors $x \in \mathbb{R}^N$ such that $\sum x_i = v(N)$ and for all $S \subset N$, $\sum_S x_i \geq v(S)$.

[20]This rule coincides with the constrained equal award rule in the two-claimant case (Dagan, 1994).

[21]O'Neill (1982) defines another method of random claims defined as follows: agents randomly make claims on ***specific*** parts of the estate, the total amount claimed by each agent being equal to his claim. For each part of the estate, equal division prevails among all agents claiming it. Unfortunately, this method may not attribute the whole estate (it is not *efficient*, as formally defined below). Moreover, when claims are compatible, it need not give to each agent his claim. Of course, we could take the amounts awarded by this method as a starting point and apply the method iteratively to distribute the leftover.

Another important solution in the theory of coalitional form games is the ***nucleolus*** (Schmeidler, 1969). First, define the dissatisfaction of a coalition at a proposed allocation as the difference between the sum of the awards to its members and its worth. Then the nucleolus is obtained by solving the following sequence of minimization exercises: first, identify the points at which the dissatisfaction of the most dissatisfied coalition is minimized; among the set of minimizers, identify the points at which the dissatisfaction of the second most dissatisfied coalition is minimized[22]... and so on. Lemma 2 describes a formal relationship between this solution and the Talmudic rule.

A relatively new solution to coalitional form games was proposed by Dutta and Ray (1989). It selects the point in the core that is Lorenz-maximal. It too corresponds to a bankruptcy rule, namely the constrained equal award rule.

Consider next the solution to coalitional form games introduced by Tijs (1982) under the name of $\boldsymbol{\tau}$***−value***. It consists in identifying a minimal and a maximal payment for each agent, and in choosing the schedule of awards on the segment connecting the vector of minima to the vector of maxima. Given $v \in \mathbf{R}^{2^n-1}$ and $i \in N$, let $M_i(v) = v(N) - v(N\backslash\{i\})$ and $\mu_i(v) = \max_{S\subseteq N, i\in S}(v(S) - \sum_{j\in S\backslash\{i\}} M_i(v))$. Then, $\tau(v) = \lambda M(v) + (1-\lambda)\mu(v)$, where λ is chosen so as to obtain efficiency.[23]

The next lemma gathers the known correspondences between bankruptcy rules and solutions to coalitional form games.

Lemma 2 (Various authors)
The following correspondences between bankruptcy rules and solutions to coalitional form games hold:

1. The random order rule and the Shapley value (O'Neill, 1982).

2. The Talmudic rule and the nucleolus (Aumann and Maschler, 1985).[24]

3. The constrained equal award rule and the Dutta-Ray solution (Dutta and Ray, 1989).

[22]See our earlier definition of the lexicographic egalitarian solution.
[23]For a convex game v, $M(v)$ indeed dominates $m(v)$.
[24]Lee (1992a) gives a short proof of this result.

4. The adjusted proportional rule and the τ-value (Curiel, Maschler and Tijs, 1988).

Of course, not all bankruptcy rules correspond to some solution to coalitional form games. A necessary and sufficient condition for such a correspondence to exist is that the rule depend only on the truncated claims and the net worth of the firm (Curiel, Maschler and Tijs, 1988).[25]

3 Properties of bankruptcy rules

In this section and the next, we formulate properties of rules and examine how restrictive the properties are. We start with what we consider to be the most natural ones. As we progress, we formulate requirements that we may or may not want to impose depending upon the range of situations to be covered, and depending upon the legal or informational constraints we face.

3.1 Feasibility and efficiency

Feasibility is simply the requirement that the sum of the awards should not exceed the net worth of the firm, and ***efficiency*** the requirement that for every problem, the rule should allocate the entire net worth of the firm. For convenience, we have incorporated efficiency in the definition of a rule. It is obvious that we cannot distribute more than there is, but conceivably, we could distribute less, and depending upon which additional conditions are imposed, we may be satisfied with this.[26]

3.2 Bounds

Next are requirements placing bounds on what each claimant can receive as a function of the data of the problem. First, each agent should receive a non-negative amount. This requirement too is embodied in the definition of a rule since rules take their values in $\mathbb{R}^N_+$.

[25]This means that $F(c, E$ can be written as $\tilde{F}((\min\{c_i, E\})_{i \in N}, E)$ for some function $\tilde{F}$.

[26]It may seem that we should never allocate less than is available, but in other settings this option has proved extremely useful. In the context of public good decision, the so-called Clarke-Groves mechanism succeeds in eliciting truthful information about agents' preferences only because it allows that some of the private good be wasted.

The following requirement provides a natural upper bound on awards: no agent should receive more than his claim. It too is often incorporated in the definition of a rule.

Claims boundedness: For all $(c, E) \in \mathcal{B}^N$ and for all $i \in N$, $F_i(c, E) \leq c_i$.

It turns out that a rule selects a point in the core of the coalitional form game associated with a bankruptcy problem if and only if it satisfies *claims boundedness* (Curiel, Maschler and Tijs, 1988).

Another condition is that each claimant receive at least his "minimal right", a quantity that appears explicitly in the definition of the adjusted proportional rule: recall that this is the difference between the net worth of the firm and the sum of the claims of the other claimants if that difference is non-negative, and zero otherwise.[27]

Respect of minimal rights: For all $(c, E) \in \mathcal{B}^N$ and for all $i \in N$, $F_i(c, E) \geq \max\{E - \sum_{N\setminus\{i\}} c_j, 0\}$.

3.3 Symmetry and related requirements

The next requirement is that the amounts awarded to two agents with equal claims should be equal.

Symmetry: For all $(c, E) \in \mathcal{B}^N$ and for all $i, j \in N$, if $c_i = c_j$, then $F_i(c, E) = F_j(c, E)$.

This requirement may not always be justified. In actual bankruptcy proceedings, some claims may have higher priority than others. In order to allow differential treatment of otherwise identical agents, we can enrich the model and explicitly introduce "priority" parameters. Let a ***bankruptcy problem with priorities*** be a list (c, p, E) where $(c, E) \in \mathcal{B}^N$ and p is a function from N to $[1, \ldots, m]$ where $m \leq |N|$, interpreted as follows: given $i \in N$, $p(i)$ is the ***priority class*** of agent i: two agents in the same priority class are treated differently only to the extent that their claims differ, but agents in different priority classes can be treated differently even though their claims are identical.

[27] Using the language of the theory of cooperative games, this condition could be called "individual rationality".

All rules can be adapted to accommodate priorities by applying them to priority classes in succession. To illustrate, a version of the proportional rule for such problems would make awards proportional to claims within each priority class, but would fully satisfy the claims of higher priority classes before attempting to satisfy the claims of the lower classes. In fact, this is common practice. The role of the different classes of creditors in bankruptcy law is discussed in Aggarwal (1992).

Alternatively, we could define a ***bankruptcy problem with weights*** to be a list (c, α, E), where $(c, E) \in \mathcal{B}^N$ and $\alpha \in \Delta^{n-1}$ is a point in the $(n-1)$-dimensional simplex indicating the ***relative*** priorities (as opposed to the absolute priorities of the previous paragraph), that should be given to agents.

Rules can easily be adapted to this setting too. For instance, to obtain a version of the proportional rule, make awards proportional to claims multiplied by the weights.[28]

A stronger version of *symmetry* is that the rule be invariant under permutations of agents. Recall that Π^N denotes the class of bijections from N into itself.

Anonymity: For all $(c, E) \in \mathcal{B}^N$, for all $\pi \in \Pi^N$, and for all $i \in N$, $F_i(\pi(c), E) = F_{\pi(i)}(c, E)$.

3.4 Order properties

An obvious generalization of *symmetry* is that the rule respects the ordering of claims: if agent i's claim is greater than agent j's claim, he should receive at least as much as agent j.[29]

Order-preservation: For all $(c, E) \in \mathcal{B}^N$ and for all $i, j \in N$, if $c_i \geq c_j$, then $F_i(c, E) \geq F_j(c, E)$.[30]

All of the rules that we have seen satisfy this property but few satisfy the stronger condition of ***strict order-preservation***, which says that if agent

[28] Lee, N-C (1984) discusses a weighted version of the constrained equal award solution.

[29] This property first appears in Aumann and Maschler (1985).

[30] Dagan, Serrano and Volij (1994) add to this requirement the inequality $c_i - F_i(c, E) \geq c_j - F_j(c, E)$, a condition first used by Chun (1989).

i's claim is greater than agent j's claim, then he should receive ***strictly*** more (equality is not permitted any more).

Next is the requirement that claimants with greater claims should receive proportionately less:

Regressivity: For all $(c, E) \in \mathcal{B}^N$ and for all $i, j \in N$, if $c_i > c_j > 0$, then $\frac{F_i(c,E)}{c_i} \leq \frac{F_j(c,E)}{c_j}$.

In the context of taxation, it is mainly the dual condition of "progressivity" that has been used. The underlying motivation is based on the objective of imposing "equal sacrifices" on all agents, under the assumption that they have concave and identical utility functions:

Progressivity: For all $(c, E) \in \mathcal{B}^N$ and for all $i, j \in N$, if $c_i > c_j > 0$ then $\frac{F_i(c,E)}{c_i} \geq \frac{F_j(c,E)}{c_j}$.

The next requirement is that if claims and estate are multiplied by the same positive number, then so should all awards. In situations where minimal guarantees to agents are justified, this is not a reasonable property to impose, but then, as argued by Young (1988), the problem should really be redefined as pertaining to the division of whatever surplus exists after these minimal guarantees have been honored.

Homogeneity: For all $(c, E) \in \mathcal{B}^N$ and for all $\alpha > 0$, $F(\alpha c, \alpha E) = \alpha F(c, E)$.

3.5 Independence, additivity and related properties

In this section, we consider requirements stating the independence of the rule with respect to certain operations performed on the data of the problem.

The first requirement is that a rule should not depend on any part of a claim that is greater than the net worth of the firm: replacing c_i by E if $c_i > E$ should not affect the recommendation.

Invariance with respect to claims truncation:[31] For all $(c, E) \in \mathcal{B}^N$, $F(c, E) = F((\min\{c_i, E\})_{i \in N}, E)$.

[31] By analogy to the condition used in bargaining theory, this condition has been called "independence of irrelevant claims". We prefer our more neutral phrase since it can be legitimately argued that the part of an agent's claim that is above the net worth of the firm is not irrelevant.

If we feel strongly that *invariance with respect to claims truncation* should be imposed, we could of course redefine the domain and only consider bankruptcy problems in which no claim is ever greater than the net worth of the firm. Alternatively, we could define a bankruptcy problem to be a list $(c, E) \in [0,1]^n \times \mathbb{R}_+$, where for each $i \in N$, c_i is interpreted as the ***percentage*** of the net worth claimed by agent i. This restriction might be particularly meaningul in the context of estate division: think of contradictory wills in each of which it the proportion of the estate that some heir should receive is specified.[32]

If a rule is not *invariant with respect to claims truncation*, it can easily be modified so as to satisfy the property by simply replacing the truncated claims in its definition.

Now consider the following situation: after a firm's net worth has been divided among its creditors, its assets are reevaluated and found to be worth more than originally thought (perhaps their market value has changed in the meantime, or new assets are discovered). To deal with the new situation, two options are available: (i) either the first division is cancelled altogether and the rule is applied to the revised problem, or (ii) the rule is applied to the problem of dividing the incremental value of the firm after adjusting the claims down by the amounts received in the first division. The requirement formulated next is that both ways of proceeding should produce the same answers.[33]

Composition: For all $(c, E) \in \mathcal{B}^N$, for all $E' \in \mathbb{R}_+$, if $\sum c_i \geq E' > E$, then $F(c, E') = F(c, E) + F(c - F(c, E), E' - E)$.[34]

The following theorem provides a characterization of the constrained equal award rule on the basis of several of the properties just defined.

Theorem 1 (Dagan, 1994) The constrained equal award rule is the only rule satisfying *symmetry*, *claims boundedness*, *invariance with respect to claims truncation*, and *composition*.

[32] O'Neill calls these problems "simple claims problems".

[33] This condition was introduced in the context of taxation by Young (1987). A condition of *step by step negotiation* in the same spirit was analyzed in the context of bargaining by Kalai (1977).

[34] This is a well-defined problem if F satisfies *claims boundedness*.

A particular form of *composition* is obtained by requiring first that the rule respect minimal rights. Then, if the net worth of the firm is equal to the sum of the minimal rights, the rule can only award his minimal right to each claimant:

Composition from minimal rights: For all $(c, E) \in \mathcal{B}^N$, $F(c, E) = m(c, E) + F(c - m(c, E), E - \sum m_i(c, E))$.[35]

Theorem 2 (Dagan, 1994) The contested garment rule is the only two-claimant rule satisfying *symmetry*, *claims boundedness*, *invariance with respect to claims truncation*, and *composition from minimal rights.*

The next requirement says that the problems of dividing "what is there" and that of dividing "what is not there" should be treated in a symmetric way. The condition was formulated by Aumann and Maschler (1985), who note a number of passages in the Talmud where the idea is central:

Self-duality: For all $(c, E) \in \mathcal{B}^N$, $F(c, E) = c - F(c, \sum c_i - E)$.[36]

An operation associating to each rule its "dual" can easily be defined (this is simply the right-hand side of the formula appearing in the statement of the axiom). To say that a solution is *self-dual* is to say that it coincides with its dual. Many rules are *self-dual*, including the proportional rule, the Talmudic rule, and the adjusted proportional rule. Dagan (1994) notes that *invariance with respect to claims truncation* and *self-duality* together imply *composition from minimal rights.* Aumann and Maschler (1985) observe that the Talmudic rule is the *self-dual* solution that coincides with the constrained equal-award rule on the subdomain of $\mathcal{B}^N$ of problems (c, E) such that $\sum c_i/2 \geq E$. We also have:

Theorem 3 (Young, 1988) The proportional rule is the only rule satisfying *continuity*, *symmetry*, *self-duality*, and *composition.*

The following result pertains to the two-claimant case:

[35]This is a well-defined problem since for all $i \in N$, $c_i \geq m_i(c, E)$.
[36]This is a well-defined problem if F satisfies *claims boundedness.*

Theorem 4 (Dagan, 1994) The contested garment rule is the only two-claimant rule satisfying *invariance with respect to claims truncation*, *claims boundedness*, and *self-duality*.

The next condition states that no group of agents should ever benefit by transfering claims among themselves. In situations in which agents are able to perform operations of this kind, it can be seen as preventing such manipulation.[37] It was analyzed by Chun (1988):

No-advantageous reallocation: For all $(c, E) \in \mathcal{B}^N$, for all $M \subset N$, and for all $c'_M \in \mathbb{R}^M_+$, if $\sum_M c_i = \sum_M c'_i$, then $\sum_M F_i(c, E) = \sum_M F_i(c'_M, c_{N\setminus M}, E)$.[38]

Obviously, in the precense of *efficiency* (which is incorporated in our definition of a rule), this condition is vacuously satisfied for $n = 2$.

A related condition is inspired by the international trade literature. It says that a partial transfer of an agent's claim to the others does not benefit him. This condition was formulated by Chun (1988):

No-transfer paradox: For all $(c, E) \in \mathcal{B}^N_+$, for all $i \in N$, and for all $c' \in \mathbb{R}^N$, if $c'_i < c_i$, and $\sum c'_j = \sum c_j$, then $F_i(c', E) \leq F_i(c, E)$.

Alternatively, we could consider a transfer of claim from one agent to another agent, and require that the former should lose and the latter should gain.

In the statement of the next result, we will use the following requirement, which is often needed for technical reasons, but which intuitively makes much sense. It simply says that small changes in the data of the problem should not lead to large changes in the recommended allocation.

Continuity: For all sequences $\{(c^\nu, E^\nu)\}$ of elements of $\mathcal{B}^N$ and for all $(c, E) \in \mathcal{B}^N$, if $(c^\nu, E^\nu) \to (c, E)$, then $F(c^\nu, E^\nu) \to F(c, E)$.

[37]A condition of this type was used by Gale (1974) and Aumann and Peleg (1974) in the context of allocation in classical exchange economies, and by Moulin (1985) in a study of quasi-linear social choice.

[38]By the notation $(c'_M, c_{N\setminus M})$, we mean the claims vector in which the claim of each $i \in M$ is c'_i, and the claim of each $i \in N\setminus M$ is c_i.

Partial notions of continuity, with respect to the net worth of the firm or with respect to a particular agent's claim, can also be formulated, and in fact often suffice for proofs.[39]

In the next lemma, we do not require rules to take their values in $\mathbf{R}_+^N$ and that $\sum F_i(c, E) = E$. We refer to such functions as "generalized rules".

Lemma 3 (Chun, 1988) A generalized rule F satisfies *anonymity*, *continuity* and *no advantageous reallocation* if there exists a continuous function $g: \mathbf{R}^2 \to \mathbf{R}$ such that for all $(c, E) \in \mathcal{B}^N$ and for all $i \in N$,

$$F_i(c,E) = \frac{c_i}{\sum c_i} E - \frac{1}{\sum c_i}\{(n-1)c_i - \sum_{N\setminus\{i\}} c_j\} g(\sum c_i, E)$$

Also, for $n \geq 3$, if a generalized rule satisfies the three axioms, then it necessarily has that form.[40]

Note that the family described in this lemma includes the proportional and equal award rules (for $g(c,E) = E/n$ and $g(c,E) = 0$ respectively).

A corollary of this result is a characterization of the proportional rule. This characterization pertains to the variable population version of the model and it appears in Section 4.

The next requirement is closely related to *no-advantageous reallocation*:

Linearity: For all $(c, E), (c, E') \in \mathcal{B}^N$ and for all $\lambda \in [0,1]$, $F(c, \lambda E + (1-\lambda)E') = \lambda F(c,E) + (1-\lambda)F(c,E')$.

Lemma 4 (Chun, 1988) A generalized rule F satisfies *anonymity*, *continuity* and *linearity* if and only if there exist continuous functions $h : \mathbf{R}^n \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}$ that are invariant with respect to permutations of their last $n-1$ arguments, and such that for all $(c, E) \in \mathcal{B}^N$ and for all $i \in N$,

$$F_i(c,E) = \frac{E}{n} + \frac{E}{n}[(n-1)h(c_i, c_{-i}) - \sum_{j\in N\setminus\{i\}} h(c_j, c_{-j})] + \frac{1}{n}[(n-1)g(c_i, c_{-i}) - \sum_{j\in N\setminus\{i\}} g(c_j, c_{-j})]$$

where c_{-i} denote the vector c from which the ith coordinate has been removed.

[39] In all the results quoted below, it suffices to assume that the rule is continuous at one point of its domain.

[40] Chun notes that *no-transfer paradox* could be used instead of *continuity*, but then g would not have to be continuous.

The corollary that follows involves the requirements that the generalized rule be in fact a rule, and that it satisfy *claim boundedness.*

Theorem 5 (Chun, 1988) The proportional rule is the only rule satisfying *anonymity*, *claims boundedness*, *continuity*, and *linearity.*

The next requirement pertains to situations in which the amount to divide comes in two parts. It states that dividing the first part first and then dividing the second part yields the same result as consolidating the two parts into one and dividing the result at one time. The implications of this condition were studied by Chun (1988).

Additivity: For all $(c, E) \in \mathcal{B}^N$, and for all $E', E'' \in \mathbb{R}_+$, if $E = E' + E''$, then $F(c, E) = F(c, E') + F(c, E'')$.

This requirement is most appealing in situations in which the vector c is given a broader interpretation than a vector of claims as understood so far, but instead represents notions of rights that are not commensurable with the quantity to divide.[41] Then, the fact that a first estate has already been divided cannot be very meaningfully accompanied by an adjustment of the claims in the division of the second estate.

Lemma 5 (Chun, 1988) A generalized rule F satisfies *anonymity*, *continuity* and *additivity* if there exists a continuous function $h : \mathbb{R}^n \to \mathbb{R}$ that is invariant with respect to permutations of its last $n-1$ arguments, and such that for all $(c, E) \in \mathcal{B}^N$ and for all $i \in N$:

$$F_i(c, E) = \frac{E}{n} + \frac{E}{n}\{(n-1)h(c_i, c_{-i}) - \sum_{N\setminus\{i\}} h(c_j, c_{-j})\}$$

A corollary of this lemma is another characterization of the proportional rule. Here too, it is obtained by requiring that the generalized rule actually be a rule. In fact, it suffices to require that $F(c, E) = E$ if $\sum c_i = E$, or that the generalized rule be *self-dual.* Alternatively, *continuity* can be replaced by *net worth monotonicity* defined in the next section.

[41]Then of course the inequality $\sum c_i \geq E$ has no meaning and it may make more sense to enlarge the class of problems under consideration by dropping it.

3.6 Monotonicity

In this section we formulate monotonicity requirements. Requirements of this type have played an important role in the analysis of other domains and they often have strong implications. In some contexts they are even incompatible with efficiency and very elementary notions of fairness in distribution. In the present context they turn out to be quite weak, and we mainly mention them for completeness.

First is the requirement that if an agent's claim increases, his share should increase.

Claims monotonicity: For all $(c, E) \in \mathcal{B}^N$, for all $i \in N$, and for all $c_i' > c_i$, we have $F_i(c_1, \ldots, c_{i-1}, c_i', c_{i+1}, \ldots, c_n, E) \geq F_i(c, E)$.

Under the same hypotheses as above, we might also want the share of each of the other agents to decrease. Under *efficiency*, this condition implies the previous one.

Strong claims monotonicity: For all $(c, E) \in \mathcal{B}^N$, for all $i \in N$, for all $c_i' > c_i$, and for all $j \in N \backslash \{i\}$, we have $F_j(c_1, \ldots, c_{i-1}, c_i', c_{i+1}, \ldots, c_n, E) \leq F_j(c, E)$.

The next requirement is that when the net worth of the firm increases, each of the claimants gains.[42]

Net worth monotonicity: For all $(c, E) \in \mathcal{B}^N$, for all $E' \in \mathbb{R}_+$, if $\sum c_i \geq E' \geq E$, then for all $i \in N$, we have $F_i(c, E') \geq F_i(c, E)$.

As just noted, these properties are not very restrictive. Indeed, all of the rules that have been considered in the literature satisfy them. However, the stronger versions obtained by requiring that under the same hypotheses, the inequalities appearing in the conclusions be strict, are not satisfied by most of them. "Conditional" versions of these stronger conditions stating that the inequality be strict only for each claimant i whose award is neither 0 nor c_i (this eliminates "corner" situations) are satisfied more generally.

The final condition here is that if the amount to divide increases, agents with greater claims should receive a greater share of the increment (Dagan and Volij, 1994).

[42]The property is used by several authors, including Curiel, Maschler and Tijs (1988) and Young (1988).

Super-modularity: For all $(c, E) \in \mathcal{B}^N$, for all $E' \in \mathbb{R}_+$, if $E' \leq E$, and for all $i, j \in N$, if $c_i \leq c_j$, then $F_i(c, E) - F_i(c, E') \leq F_j(c, E) - F_j(c, E')$.

Here too, a strict version of the property can be formulated. It is easy to see that a *super-modular* rule is *order-preserving.*

4 Variable population

We now consider a richer framework in which the number of claimants involved may vary. We allow problems involving arbitrary, although finite, numbers of claimants. There is a set of "potential" claimants, indexed by the set of natural numbers $\mathbb{N}$, and $\mathcal{N}$ is the set of finite subsets of $\mathbb{N}$. A ***bankruptcy problem*** is obtained by first specifying a set of agents $N \in \mathcal{N}$, then a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_N c_i \geq E$. A ***bankruptcy rule*** is a function defined on the union of all of the $\mathcal{B}^N$, for $N \in \mathcal{N}$, which associates with every $N \in \mathcal{N}$ and every $(c, E) \in \mathcal{B}^N$ a point of $\mathbb{R}^N_+$ whose coordinates add up to E.

4.1 Consistency

The first property is an independence requirement. Consider some problem and apply the chosen solution to it. Consistency says that if some of the claimants leave with their awards, and the situation is reevaluated from the viewpoint of the remaining claimants, the solution should assign to them the same awards as initially. The problem faced by the group of remaining agents is called the ***reduced problem relative to the subgroup and the initial recommendation***. Note that this is indeed a well-defined problem if the solution satisfies *claims boundedness*. Then, in the reduced problem, the sum of the claims is still greater than the amount to divide.[43]

Consistency: For all M, $N \in \mathcal{N}$, for all $(c, E) \in \mathcal{B}^N$, if $M \subset N$ and $(c_M, \sum_M x_i) \in \mathcal{B}^M$, where $x = F(c, E)$, then $x_M = F(c_M, \sum_M x_i)$.

[43]For a survey of the vast and fast-expanding literature devoted to the analysis of the consistency principle, see Thomson (1995b). O'Neill (1982) gives the term *consistency* a different meaning.

Bilateral consistency is the weakening of the condition obtained by considering only subgroups of remaining agents of cardinality 2.

It is clear that the proportional rule is *consistent*, and that so is the constrained equal award rule. What of the Talmudic rule? Let us check with the specific numerical values given in the Talmud (Figure 1). For an estate of 200 in the 3-person case, the amounts awarded to claimants 1 and 2 are 50 and 75 respectively, for a total of 125. Applying the Talmudic rule to divide an estate of 125 between the first two claimants returns the same numbers 50 and 75! In fact, given any value of the estate, and given any pair of claimants $\{i, j\}$, if x denotes the Talmudic solution outcome of the 3-person problem, applying the same rule to divide an estate of $x_i + x_j$ between the pair $\{i, j\}$ yields the settlement (x_i, x_j). This coincidence occurs for all cardinalities. The Talmudic rule is *consistent.*

Aumann and Maschler (1985) show that the Talmudic rule is the only *bilaterally consistent* bankruptcy rule to coincide with the contested garment rule in the two-claimant case.[44] They also establish an interesting connection between the following two procedures: let x be the recommendation for the problem $(c, E) \in \mathcal{B}^N$, where $N \in \mathcal{N}$. Given $M \subset N$, consider the reduced problem $(c_M, \sum_M x_i)$ and its associated coalitional form game $v(c_M, \sum_M x_i)$. Alternatively, calculate the coalitional form game associated with the problem (c, E) and its "reduced game with respect to the subgroup and the payoff vector x" as defined by Davis and Maschler (1965): in this game, the worth of each coalition S is set equal to the maximal surplus obtained when the coalition "cooperates" with a subset S' of the complementary group $N \backslash M$ — this yields $v(S \cup S')$ — and pays the members of S' according to x — for a total of $\sum_{S'} x_i$ (the surplus is then the difference $v(S \cup S') - \sum_{S'} x_i$). The result is that the two ways of proceeding give the same game.

The implications of *consistency* have been described very completely, with very few auxiliary conditions. Consider indeed the following class of rules, introduced by Young (1986).

Parametric rules: Let $f : \mathbf{R}_+ \times [a, b] \to \mathbf{R}_+$, where $[a, b] \subset [-\infty, +\infty]$, be a function that is continuous, is weakly monotonic in its second argument, and

[44] Actually, they establish a slightly stronger result, namely, that even if the rule were allowed to be multivalued, *bilaterally consistency* would imply singlevaluedness, and uniqueness, and coincidence with the contested garment rule in the two-claimant case.

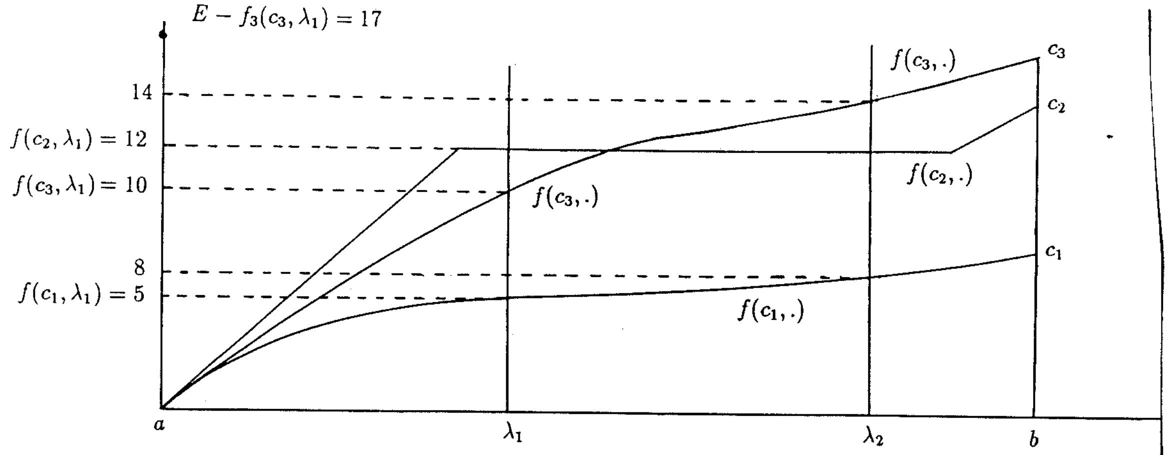


Figure 3: ***Parametric rules are consistent.*** Given the three claims (c_1, c_2, c_3), the parameter λ is given the value λ_1 so that the three awards $f(c_1, \lambda_1)$, $f(c_2, \lambda_1)$, and $f(c_3, \lambda_1)$ add up to the amount to be divided, E. Now, if the amount $E' = E - f(c_3, \lambda_1)$ is to be divided between claimants 1 and 2, the value λ' for which the awards $f(c_1, \lambda')$ and $f(c_2, \lambda')$ add up to E' is, of course, $\lambda' = \lambda_1$, so that claimants 1 and 2 still receive the same amounts after claimant 3 has received $f(c_3, \lambda_1)$.

satisfies $f(c_i, a) = 0$ and $f(c_i, b) = c_i$ for all $c_i \in \mathbf{R}_+$. Then, given $N \in \mathcal{N}$ and $(c, E) \in \mathcal{B}^N$, the ***parametric rule relative to f*** selects the point $x \in \mathbf{R}^N$ such that for some $\lambda \in [a, b]$, $\sum_N x_i = E$ and $x_i = f(c_i, \lambda)$ for all $i \in N$.

It is straightforward to check that all parametric rules are *consistent*.

Figure 3 depicts the graphs of such an f for three possible values of the first argument. The choice of $\lambda = \lambda_1$ produces the distribution $5 + 10 + 12$, and the choice of $\lambda = \lambda_2$ produces the distribution $8 + 14 + 12$. Note that one of the graphs is not strictly increasing and that the graph corresponding to c_3 does not lie entirely above that corresponding to $c_2 < c_3$. At this stage, these are indeed possibilities. It is clear however that they can be ruled out by imposing additional conditions. For instance, for an *order-preserving* rule, the graph corresponding to c_i lies above the graph corresponding to c_j whenever $c_i > c_j$. Also, for a *supermodular* rule, for each value of the parameter λ, the slope of the graph corresponding to c_i is greater than the slope of the graph corresponding to c_j whenever $c_i > c_j$. Figures 4a and 4b give parametric representations of the proportional and Talmudic rules,

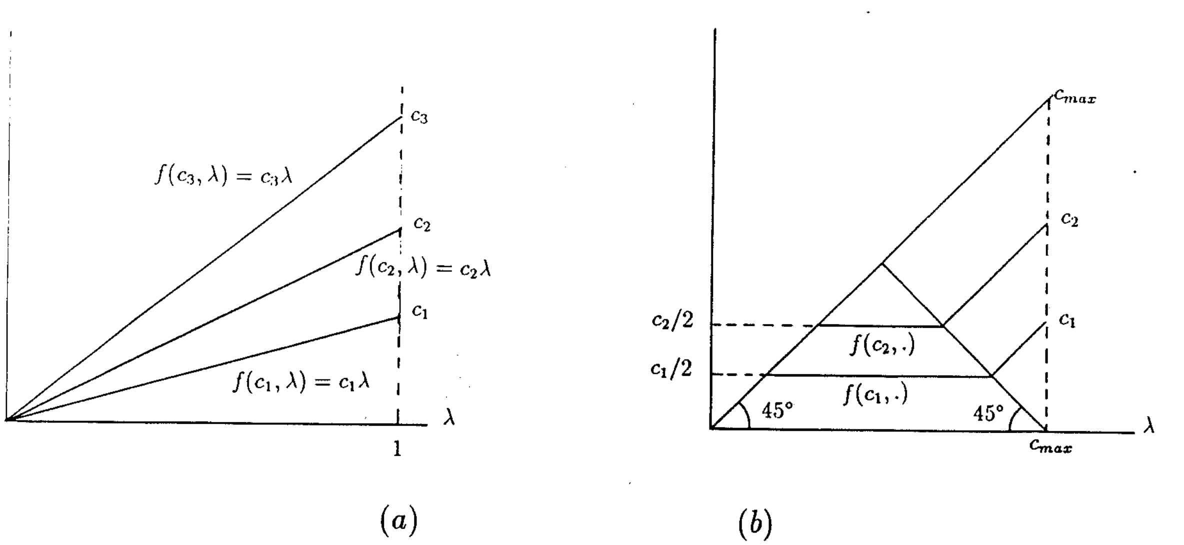


Figure 4: ***Parametric representations of two rules.*** *(a)* Proportional rule: The schedules are straight lines through the origin, of slopes equal to claims. *(b)* Talmudic rule. The schedule relative to claim c_1 follows the 45° line up to the point $(c_1/2, c_1/2)$, continues horizontally until it meets the line of slope - 1 emanating from $(c_{\max}, 0)$, then again follows a line of slope 1.

the latter in the case where an upper bound on claims exists, $c_{\max}$.[45] In all of the results presented in this section, any of the axioms introduced earlier in the fixed population case should be generalized in the obvious way to be applicable to a variable population.

Theorem 6 (Young, 1987b) The parametric rules are the only rules satisfying *symmetry, continuity*, and *consistency.*

It is of interest to note that the proof of this result involves showing that a *continuous* and *consistent* rule is *net worth monotonic.*

In the context of taxation, the following *parametric* rules have also been discussed: for ***Stuart's rule***, $x_i = \max\{c_i - c_i^{1-\lambda}, 0\}$ and for ***Cassel's rule***, $x_i = c_i^2/(c_i + 1/\lambda)$.

Aumann and Maschler propose one more justification for the Talmudic rule based on a consistency argument. Let $N = \{1, \ldots, n\}$ and suppose that claimants are ordered by increasing claims. First, apply the contested garment rule to the two-claimant problem in which the first claimant faces a "composite claimant" whose claim is the sum $c_2 + \cdots + c_n$. The first claimant leaves with his award. Then the second claimant faces the composite claimant

[45]This assumption restricts somewhat the scope of the rule but it permits a very simple (piecewise linear) representation (Chun and Thomson, 1990). See Young (1987b) for a representation without the upper bound.

whose claim is the sum $c_3 + \cdots + c_n$ and the amount to divide is what the first composite claimant received. He leaves with his award. The procedure continues in this way for $n-1$ steps unless a violation of *order preservation* occurs, in which case equal division of what is left is carried out among the members of the composite claimant of that step and the procedure stops.

Chun and Thomson (1990) define a particular member of the parametric family. It is inspired by a rule to the problem of fair division when preferences are single-peaked known as the uniform rule (Sprumont, 1980). Lee (1994) develops a characterization of his weighted generalization of the constrained equal award rule based on *consistency*.[46]

Consider now the following family of solutions:

Equal-sacrifice rules : Let $u : \mathbf{R}_{++} \rightarrow \mathbf{R}$ be a continous and strictly increasing function. Then, given $N \in \mathcal{N}$ and $(c, E) \in \mathcal{B}^N$ with $x > 0$, the ***equal-sacrifice rule relative to u*** selects the point $x \in \mathbf{R}^N$ such that for some $\lambda \geq 0$, and for all $i \in N$, we have $u(c_i) - u(c_i - x_i) = \lambda$.

Theorem 7 (Young, 1988) On the domain of problems where claims are all positive, the equal-sacrifice rules are the only rules satisfying *continuity*, *strict net-worth monotonicity*, *strict order-preservation*, *composition*, and *consistency*. If in addition, *homogeneity* is imposed, then the rule is an equal-sacrifice relative to u such that either $u(x) = ln(x)$ or $u(x) = -c^p$ for $p < 0$.[47]

Within the class of parametric rules, a narrow subclass of great interest can be identified:

Theorem 8 (Young, 1986) A parametric rule satisfies *progressivity, homogeneity*, and *composition* if and only if it can be represented in one of the following ways:

$$
\begin{array}{llll}
f(c_i, \lambda) & = \lambda c_i & 0 \leq \lambda \leq 1 & \\
f_p(c_i, \lambda) & = c_i - c_i/(1 + \lambda c_i^p)^{1/p} & 0 \leq \lambda \leq \infty & p > 0 \\
f_\infty(c_i, \lambda) & = \max\{c_i - 1/\lambda, 0\} & 0 \leq \lambda \leq \infty &
\end{array}
$$

[46]This characterization exploits duality relations between cores, anticores and their reductions.

[47]Then, in the first case, the rule is flat taxation, and in the second, it is a parametric rule of the form $x = c - [c^p + \lambda^p]^{1/p}$, for $\lambda \in [0, \infty[$.

A requirement related to *consistency* can be formulated for situations in which one of the claimants has a claim equal to zero: then, (i) he gets nothing and (ii) deleting him does not change the amounts received by the others. Part (i) corresponds to the condition known in the theory of coalitional form games as the "dummy condition".[48] Part (ii) corresponds to *consistency.* Since its coverage is not as wide as that of the condition that we used under that name, we will refer to it as ***limited consistency.*** The condition amalgamating the two is used in the context of bankruptcy by O'Neill (1982) and Chun (1988).[49]

Dummy: For all M, $N \in \mathcal{N}$, for all $(c, E) \in \mathcal{B}^N$, if $M \subset N$ and $c_i = 0$ for all $i \in N \backslash M$, then $F_i(c, E) = 0$ for all $i \in N \backslash M$.

Limited consistency: Under the hypotheses of *dummy*, if $x = F(c, E)$, then $x_M = F(c_M, \sum_M x_i)$.

We are now ready to present the characterization of the proportional rule announced in Section 3 as a corollary of Lemma 3.

Theorem 9 (Chun, 1988) The proportional rule is the only rule satisfying *anonymity*, *continuity*, *no-advantageous reallocation*, *dummy*, and *limited consistency.*

4.2 Average consistency

Consider a rule that is not *consistent.* Then, for at least one problem and one reduced problem associated with the recommendation and some subgroup made for it by the rule — let this recommendation be denoted x — there is at least one claimant in the subgroup, say claimant i, who receives an amount that is different from what he was initially awarded, x_i. Under *efficiency*, this means that in that reduced problem at least one claimant receives less, and at least one other claimant receives more, than initially decided. Of course, a claimant who receives less in some reduced problem associated with x may receive more in some other reduced problem associated with x. Suppose

[48] It is used in that form in de Frutos (1994).

[49] Chun uses the term "dummy" for the conjunction of what we call *dummy* and *limited consistency.*

however that for each claimant, ***on average***, when all the reduced problems associated with x relative to subgroups to which he belongs are considered, he does receive his component of x. Then, we may be satisfied with x after all. To the extent that the formation of subgroups is a thought experiment anyway, this weaker notion may be quite acceptable.

Average-consistency: For all $N \in \mathcal{N}$, for all $(c, E) \in \mathcal{B}^N$, and for all $i \in N$, $x_i = \frac{1}{(|N|-1)!} \sum_{M \subset N, i \in M} F_i(c_M, \sum_M x_i)$.

This form of *consistency* was studied by Dagan and Volij (1994) who suggested that the averaging be limited to coalitions of size two. We will refer to that version as ***2-average consistency***.[50] Dagan and Volij have in mind situations in which a rule for the two-claimant case has been chosen. Then the idea of *2-average consistency* can be exploited to provide an extension of the rule to all cardinalities as follows: given $N \in \mathcal{N}$, and a problem $(c, E) \in \mathcal{B}^N$, select $x \in \mathbb{R}_+^N$ such that $\sum_N x_i = E$ and for each $i \in N$, $\frac{1}{|N|-1} \sum_{j \in N \setminus \{i\}} F_i(c_i, c_j, x_i + x_j) = x_i$. Questions are whether such an x exists, and if it does, whether it is unique. The following theorem shows that ***no matter what the two-claimant rule is***, both questions have positive answers.

Theorem 10 (Dagan and Volij, 1994) For all two-claimant rules F, for all $N \in \mathcal{N}$, and for all $(c, E) \in \mathcal{B}^N$, there is a unique $x \in \mathbb{R}_+^N$ such that $\sum x_i = E$ and for all $i \in N$, $\frac{1}{|N|-1} \sum_{j \in N \setminus \{i\}} F_i(c_i, c_j, x_i + x_j) = x_i$.

4.3 Merging and splitting agents

We consider next a condition pertaining to the possibility that a group of agents may consolidate their claims and be treated as a single claimant, or conversely that a given claimant divide his claim and be "represented" by several claimants. It says that no such consolidation or division is ever beneficial. It was first used in the present context by O'Neill (1982).[51] Banker

[50] This definition is inspired by an idea analyzed in the context of non-transferable utility games (specifically the class of "hyperplane games"), by Maschler and Owen (1989).

[51] O'Neill uses the name of "strategy-proofness". We refrain from using it here since it is now generally given a different meaning.

(1981) considers the stronger requirement that the merging of two agents does not affect the amounts awarded to the others.[52]

Non-manipulability by merging or splitting: For all $M, N \in \mathcal{N}$, for all $(c, E) \in \mathcal{B}^N$ and $(c', E') \in \mathcal{B}^M$, if $M \subset N$, $E' = E$ and there is $i \in M$ such that $c'_i = c_i + \sum_{N \setminus M} c_j$, and for all $j \in M \setminus \{i\}$, $c'_j = c_j$, then $F_i(c', E) = F_i(c, E) + \sum_{N \setminus M} F_j(c, E)$.

This property is satisfied by the proportional rule and the adjusted proportional rule, but not by any of the other rules that we have seen. Chun (1988) established the following logical relations: *non-manipulability by merging or splitting* implies *no-advantageous reallocation*. *Efficiency*, *anonymity* and *non-manipulability by merging or splitting* together imply *dummy* and *limited consistency*. Finally, *non-manipulability by merging or splitting* implies *no-advantageous reallocation*. We now have:

Theorem 11 (O'Neill, 1982; Chun, 1988) [53] The proportional rule is the only rule satisfying *anonymity*, *continuity*, and *non-manipulability by merging or splitting.*

Theorem 12 (Curiel, Maschler and Tijs, 1988) The adjusted proportional rule is the only rule satisfying *claims boundedness*, *respect of minimal rights*, *symmetry*, and *non-manipulability by merging or splitting.*

In some situations, it may be particularly difficult to merge and in others it is splitting that might not be easy. It is therefore natural to search for rules that are either *non-manipulable by splitting* (if an agent is replaced by two agents whose claims add up to his claim, then the sum of what they receive should be no greater than what he previously received on his own), or *non-manipulable by merging* (if two agents are replaced by one agent whose claim is equal to the sum of their claims, this agent should not receive more than

[52] Banker studies a wider class of problems in which the sum of the claims is not related to the net worth of the firm.

[53] O'Neill imposes *dummy* and *limited consistency*, both of which are shown to be redundant by Chun. Chun derives the conclusion by exploiting the logical relations just stated, and obtains it as a corollary of Theorem 9. Banker (1981) obtains a closely related result based on his strengthening of *non-manipulability by merging or splitting* mentioned earlier.

the sum of what they previously received). These properties were studied by de Frutos (1994), who searched for *consistent* rules satisfying either one of them. Her findings are summarized in the following theorem, which builds on Theorem 8.

Theorem 13 (De Frutos, 1994) If a *continuous* and *consistent* rule is *non-manipulable by merging*, then it is a parametric rule relative to a function f that is concave in its first argument for each value of the parameter λ. If instead, it is *non-manipulable by splitting*, then it is a parametric rule relative to a function f that is convex in its first argument for each value of the parameter λ.

4.4 Population-monotonicity

The monotonicity property that is relevant in the context of a variable number of agents is that if the number of claimants increases, but the amount to divide stays the same, all agents initially present should lose.[54]

Population-monotonicity: For all $N, M \in \mathcal{N}$ with $M \subset N$, for all $(c, E) \in \mathcal{B}^N$, then $F_M(c, E) \leqq F(c_M, E)$.

Like all of the monotonicity properties formulated above for the fixed population case, this condition is rather weak: as before, all of the rules that have been studied in the literature satisfy it, although here too, the stronger version obtained by requiring that the losses incurred by the agents initially present be positive is considerably more restrictive. A conditional version of this stronger requirement, obtained by applying it only to agents whose initial awards are neither zero nor equal to their claims, can be met much more generally.

5 Non-cooperative models

Here, we present a variety of non-cooperative models superimposed on our basic bankruptcy problem.

In the game formulated by O'Neill (1982) each agent chooses particular parts of the estate adding up to no more than his claim, and the outcome

[54]For a survey of the literature on "population-monotonicity", see Thomson (1995a).

function is such that any part that is claimed by several agents is divided equally among them. Therefore, the less overlap exists between what an agent claims and what others claim, the more he receives. The following theorem collects the basic facts about this game, Γ^O. Nash-equilibria exist and interestingly the distribution of claims at equilibrium is a "dual" of O'Neill's extension of Ibn Ezra's method:[55]

Theorem 14 (O'Neill, 1982) For each bankruptcy problem $(c, E) \in \mathcal{B}^N$, the game Γ^O has at least one Nash equilibrium. Any Nash equilibrium is such that the part of the estate that is claimed by all claimants is minimized; subject to that, the part that is claimed by $n-1$ claimants is minimized..., and so on.

In the game defined by Chun (1989), agents propose rules instead of vectors of awards. Apart from *efficiency*, rules are required to satisfy *order preservation*, *claims boundedness*, and *regressivity*. A sequential procedure is defined as follows: the various rules proposed by all the agents are applied to the problem at hand and the claim of each agent is replaced by the maximal amount awarded to the agent by any one of them. The rules are applied to the problem so revised and a second revision is performed... The outcome function is defined by taking the limit point of this process, if it exists.[56] Chun shows that existence is guaranteed, and that in this game of rules, Γ^C, if the agent with the smallest claim announces the constrained equal award rule, then for any agent, the sequence of awards calculated by his announced rule converges to what he receives under the application of the constrained equal award rule. A consequence of this result is the following characterization of the unique equilibrium outcome of the game Γ^C.

Theorem 15 (Chun, 1989) For each bankruptcy problem $(c, E) \in \mathcal{B}^N$, the game Γ^C has a unique Nash equilibrium outcome, which is the allocation selected by the constrained equal award rule.

Sonn (1994) studies a game of demands similar to the game originally formulated by Chae and Yang (1988) (in their extension of Rubinstein, 1982)

[55]To solve the non-uniqueness problem, O'Neill first shows that the set of equilibrium payoffs is a simplex, and he then suggests selecting its center.

[56]This game is inspired by a similar procedure developed by van Damme (1986) for bargaining problems.

for bargaining problems and characterizes its subgame perfect equilibria. In this game, agent 1 proposes an amount to agent 2. If player 2 accepts, he leaves with it and player 1 then proposes an amount to player 3, who again has the choice of leaving with it. If at some point, a player rejects the offer made to him, the next stage starts with his making offers to the next player in line, player 1 being moved to the end of the line. The game continues until only one player is left. Let Γ^S denote this game. The constraint is imposed on offers that no agent be ever offered an amount greater than his claim or the amount that remains to be distributed. In the proof of the following result, consistency and monotonicity properties of certain solutions to the bargaining problem play an important role.

Theorem 16 (Sonn, 1992) For each bankruptcy problem $(c, E) \in \mathcal{B}^N$, as the discount factor of future utilities goes to one, the limit of equilibrium allocations of the game Γ^S converges to the outcome selected by the constrained equal award rule.

Serrano (1993) makes use of the consistency of the nucleolus as a solution to coalition form games to construct a sequential game whose subgame perfect equilibrium outcome is the nucleolus of the associated coalitional form game. Dagan, Serrano and Volij (1993) extend this result to the class of *net worth monotonic* and *consistent* rules. Assume that a two-claimant rule F has been selected. In the n-claimant game Γ^F that they define, the claimant with the highest claim proposes a division of the estate, and each of the other claimants can (i) either accept his proposed share, in which case he leaves with it, or (ii) rejects it, in which case he leaves with what the two-claimant rule would recommend for him in the problem that the proposer and him would face if they had to divide the sum of the amounts that the proposer proposed for himself and for the agent. The proposer leaves with the difference between the estate and the sum of the amounts that the accepters accepted and the adjusted amounts the rejecters took. Then the game is played again among all the rejecters. For the statement of the next theorem, we need the concept of an ***almost strictly net-worth monotonic*** rule: it is a rule such that if the net worth increases, then any agent who is not already receiving his claim receives strictly more.

Theorem 17 (Dagan, Serrano and Volij, 1994) Let F be a *net worth monotonic*, *consistent* and *supermodular* rule. Then, for each bankruptcy problem

$(c, E) \in \mathcal{B}^N$, the game Γ^F has a unique subgame perfect equilibrium outcome, at which every agent receives what the *consistent* extension of F recommends. The equilibria are coalition-proof[57] if and only if the rule is *almost strictly net worth monotonic.*[58]

The outcome function as specified is not feasible out of equilibrium, but it can be made feasible without the result being affected.

Note that calculating the outcome requires the planner's knowledge of the claims. In a follow-up contribution, Dagan, Serrano and Volij (1994) study the case when claims are unknown to the planner and can be strategically misrepresented. They impose the natural restriction that only downward misrepresentation is possible. They construct a game form that implements any *consistent* and *strictly claims-monotonic* rule in subgame perfect equilibrium.

In the game defined by Corchón and Herrero (1995), agents propose allocations that are required to satisfy *claims boundedness.* The proposals are combined, by means of a "compromise function", so as to produce a final outcome. The authors establish necessary and sufficient conditions on a two-claimant rule for it to be implementable in dominant strategies: the rule should be strictly increasing in each claim, and the amount received by each agent should be expressable as a function of his claim and the difference between the net worth and the claim of the other agent. Implementation can be achieved by a simple averaging of proposals. For the n-person case, the results are largely negative however, at least when the averaging method is used.

Landsburg (1994) studies a problem of manipulation in which manipulation is costly. The cost of misrepresenting one's claim is given by a function having the property that the greater the extent of the manipulation, the greater is the cost incurred. In the special case in which the claims add up to the estate, he finds that there is a single rule giving agents the incentive to report truthfully. It is the equal loss rule.

In Sertel's (1992) game, the strategic opportunity of an agent is to transfer a fraction of his claim to the other player (there are two players), payoffs being calculated by applying the Nash bargaining solution to a certain bargaining problem associated with the bankruptcy problem. He shows that at

[57]Bernheim, Peleg and Whinston (1987).

[58]For a *strictly net worth monotonic* rule, the identity of the proposer is immaterial.

equilibrium the two players receive the awards the contested garment rule would select.

In summary, we see that a number of the solutions that we had arrived at on the basis of axiomatic considerations have been provided additional support by taking the non-cooperative route.

6 Extensions of the basic model

In this section we briefly discuss extensions of the model to surplus-sharing and situations where utility is non-transferable.

6.1 Generalization to surplus-sharing

Estate division problems can be generalized in different ways. First, as O'Neill (1982) notes, the number of documents in which amounts are bequeated need not be equal to the number of heirs. Also, in each document, more than one heir may be named. Alternatively, each document may specify a complete division of the estate among all the heirs.

Problems closely related to bankruptcy problems are the surplus-sharing problems studied by Moulin (1985a). Such a problem is a pair $(c, E) \in \mathbf{R}_+^N \times \mathbf{R}_+$ where c_i is interpreted as the investment in a joint venture made by agent $i \in N$ and $E \geq \sum c_i$. The amount $E - \sum c_i$ is the ***surplus*** generated by this venture. How should it be divided among the investors? Moulin characterizes one-parameter families of surplus-sharing methods that contain as particular cases equal sharing and proportional sharing. One of the auxiliary axioms he uses is *homogeneity* (see above). Pfingsten (1991) describes how the class of admissible rules enlarges when *homogeneity* is dropped. Chun (1987) studies the implications of monotonicity conditions for this model, and analyses it from a non-cooperative viewpoint (Chun, 1988).

An even more general class of problems consists of pairs $(c, E) \in \mathbf{R}_+^N \times \mathbf{R}_+$, in which no restriction is imposed on the value of E as compared to $\sum c_i$. This class includes both bankruptcy and surplus sharing problems as particular cases. Solutions defined for it can be easily obtained by piecing together bankruptcy rules and surplus-sharing rules.

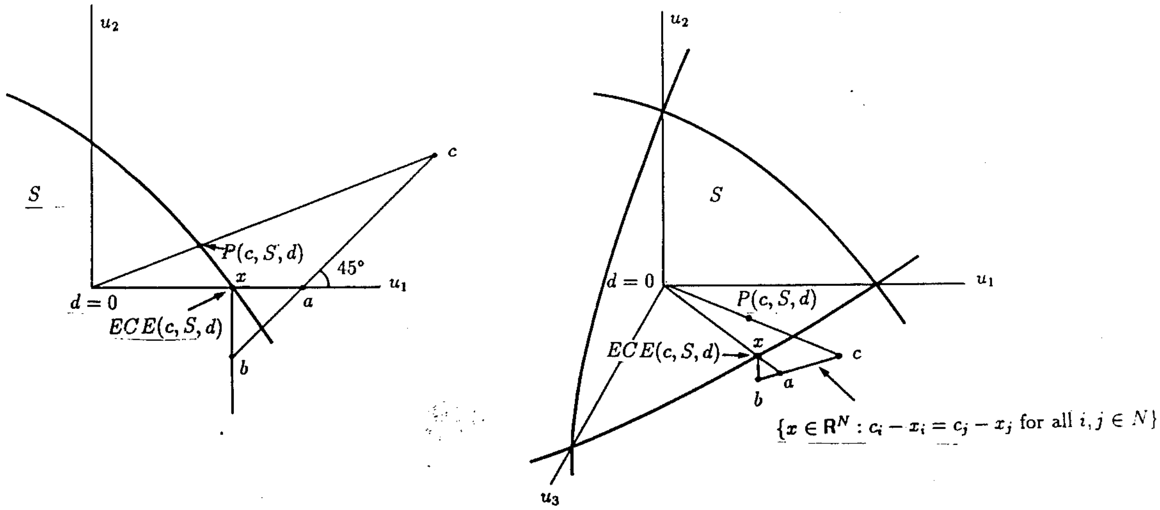
Finally, we could consider the class just defined but without the claims being commensurable with the amount to divide. For instance, c_i could be interpreted as the "contribution", the "need", or the "merit", of agent i. In such context, comparing $\sum c_i$ to E may not be meaningful. Conditions such as *claims boundedness* may not be meaningful either (see our discussion above of *additivity*).

Dagan (1994) considers the taxation problem interpretation of the model and proposes a richer formulation that includes constraints on transfers across agents. He characterizes a class of equal-sacrifice rules, mainly on the basis of *consistency* considerations.

6.2 Generalization to the non-transferable utility case

Chun and Thomson (1988) formulate and analyze a class of bankruptcy problems in which utility functions are not restricted to be linear. The image in utility space of such a "non-tranferable utility bankruptcy problem" can also be seen as a bargaining problem enriched by the addition of a claims point outside of the feasible set. Such problems are "bargaining problems with claims". Chun and Thomson offer several characterizations of the proportional rule: in this setting, this is the solution that selects the maximal feasible point on the line connecting the origin to the claims point (Figure 5).

Bossert (1992), Herrero (1993), and Marco (1994, 1995a,b) have also studied this situation and defined other solutions for it. The main one is the "extended claim egalitarian solution" which selects for each problem the payoff vector at which the utility losses from the claims point are equal across agents, subject to the requirement that no agent receives less than zero. This point can be obtained in either one of the following alternative ways: (i) first, select the maximal feasible point on the path defined by moving down from the claims point in such a way that all agents whose utility is still positive experience equal losses and all other agents receive zero (in Figures 5a, b, this is the path $c, a, 0$, and the point of equal losses is x.) (ii) The other definition is in two steps. Find the maximal feasible point of equal losses in the "comprehensive" hull of the individually rational part of the problem (this is the set of points in $\mathbb{R}^N$ that are dominated by some point of the problem that lies above the origin; in Figures 5a, b, this is the point b). Then set equal to zero the utility of every agent whose individuality rationality constraint is violated at the maximal point of equal losses (this leads us back to x). Further adjustments in both the proportional and the extended claim-egalitarian solution are needed to get Pareto-optimal outcomes. For that purpose, Marco



(a) (b)

Figure 5: ***Non-transferable utility bankruptcy problems.*** In a non-transferable utility bankruptcy problem the boundary of the feasible set is not a straight line of slope -1. Two solutions for such problems are illustrated here for $n = 2$ in panel (a), and $n = 3$ in panel (b). For the proportional solution, P, utilities are proportional to claims. For the extended claim egalitarian solution, ECE, utilities are obtained by imposing equal losses from the claims point, subject to the constraint that no agent end up with a negative utility.

(1995a) uses a lexicographic operation, as in Chun and Peters (1991).

7 Conclusion

Although bankruptcy problems are among the simplest that one may encounter, we have discovered that the model of bankruptcy is surprisingly rich. Axiomatic analysis is of great help in providing support for the bankruptcy rule that is the most commonly used in practice, namely the proportional rule, but that it is invaluable in justifying the other rules that have played a role in practice and theory, as well as in uncovering new rules. Together with the recent studies of bankruptcy problems as non-cooperative games, we now have an incomparably better understanding of the problem than just a few years ago.

8 References

Aggarwal, R., "Efficient and equitable allocation of value in Chapter 11 bankuptcy", Harvard University mimeo, 1992.

Aumann, R. and M. Maschler, "Game theoretic analysis of a bankruptcy problem from the Talmud," **Journal of Economic Theory** 36 (1985), 195-213.

—— and B. Peleg, "A note on Gale's example", **Journal of Mathematical Economics** 1 (1974), 209-211.

Banker, R., "Equity considerations in traditional full cost allocation practices: an axiomatic perspective", in **Joint Cost Allocations** (S. Moriarity, ed), University of Oklahoma Press, 1981, 110-130.

Bernheim, B., B. Peleg, and M. Whinston, "Coalition-proof Nash equilibria I: concepts", **Journal of Economic Theory** 42 (1985), 1-12.

Bossert, W., "Monotonic solutions for bargaining problems with claims", **Economics Letters** 39 (1992), 395-399.

——, "An alternative solution to bargaining problems with claims", **Mathematical Social Sciences** 25 (1993), 205-220.

Chae, S, and J.-A. Yang, "The unique perfect equilibrium of an n-person bargaining game", **Economics Letters 28** (1988), 221-223.

Chun, Y.,"The solidarity axiom for quasi-linear social choice problems", **Social Choice and Welfare** 3 (1986), 297-310.

——, "The proportional solution for rights problem", **Mathematical Social Sciences** 15 (1988), 231-246.

——, "The equal-loss principle for bargaining problems", **Economics Letters** 26 (1988), 103-106.

——, "A non-cooperative justification for egalitarian surplus sharing", **Mathematical Social Sciences** 17 (1989), 245-261.

—— and H. Peters, "The lexicographic equal-loss solution", **Mathematical Social Sciences** 22 (1991), 151-161.

—— and W. Thomson, "A new solution to bankruptcy problems", University of Rochester mimeo, 1990.

—— and ——, "Bargaining problems with claims", **Mathematical Social Sciences** 24 (1992), 19-33.

Corchón. L. and C. Herrero, "A decent proposal", University of Alicante mimeo, August 1995.

Curiel, I. M. Maschler and S.H. Tijs, "Bankruptcy games", **Zeitschrift für Operations Research** 31 (1988), A143-A159.

Dagan, N., "On the least sacrifice principle in taxation", Hebrew University of Jerusalem mimeo, June 1994.

—— , R. Serrano and O. Volij, "A non-cooperative view of consistent bankruptcy rules", Hebrew University of Jerusalem mimeo, November 1993.

——, ——, and ——, "Feasible implementation of taxation methods", Hebrew University of Jerusalem mimeo, September 1994.

——, and O. Volij, "The bankruptcy problem: a cooperative bargaining approach", **Mathematical Social Sciences** 26 (1993), 287-297.

—— and ——, "Bilateral comparisons and consistent fair division rules in the context of bankruptcy problems", Hebrew University of Jerusalem mimeo, February 1994.

van Damme, E., "The Nash bargaining solution is optimal", **Journal of Economic Theory** 38 (1986), 78-100.

Davis, and M. Maschler, "The kernel of a cooperative game", **Naval Reseach Logistic Quaterly** 12 (1965), 223-259.

Dutta, B., and D. Ray, "A concept of egalitarianism under participation constraints", **Econometrica** 57 (1989), 615-635.

de Frutos, M. A. "Coalitional manipulation in a bankruptcy problem", Universitat Autònoma de Barcelona, mimeo, March 1994.

Gale, D., "Exchange equilibrium and coalitions: an example", **Journal of Mathematical Economics** 1 (1974), 63-66.

Herrero, C., "Endogenous reference points and the adjusted proportional solution for bargaining problems with claims", University of Alicante working paper (1993).

—— and M.C. Marco, "A note on the equal-loss principle for bargaining problems", University of Alicante working paper 92-02, 1992.

—— and ——, "Rational equal loss solutions for bargaining problems", **Mathematical Social Sciences** 26 (1993), 273-286.

Imai, H.,"Individual monotonicity and lexicographic maximin solution", **Econometrica** 51 (1983), 389-401.

Kalai, E., "Proportional solutions to bargaining situations: interpersonal comparisons of utility", **Econometrica** 45 (1977), 1623-1630.

—— and M. Smorodinsky, "Other solutions to Nash's bargaining problem", **Econometrica** 43 (1975), 513-518.

Landsburg, S., "Incentive-compatibility and a problem from the Talmud", University of Rochester mimeo, 1993.

Lee, N-C., "A simple generalization of the constrained equal award rule and its characterization", Keio University mimeo, 1994

Lee, R., "Quick proof of a theorem concerning a Talmudic bankruptcy problem", Harvard University mimeo, 1992.

Marco, M.C., "An alternative characterization of the extented claim-egalitarian solution", **Economics Letters** 45 (1994), 41-46.

——, "Efficient solutions for bargaining problems with claims", **Mathematical Social Sciences** 30 (1995a), 57-69.

——, "The extended claim-egalitarian solution across cardinalities", University of Alicante mimeo, 1995b.

Maschler, M., and G. Owen, "The consistent Shapley value for hyperplane games", **International Journal of Game Theory** 18 (1989), 389-407.

Moulin, H., "Egalitarianism and utilitarianism in quasi-linear bargaining", **Econometrica** 53 (1985a), 49-67.

——, "The separability axiom and equal-sharing methods", **Journal of Economic Theory** 36 (1985b), 120-148.

Nash, J.F., "The bargaining problem", **Econometrica** 18 (1950), 155-162.

O'Neill, B., "A problem of rights arbitration from the Talmud," **Mathematical Social Sciences** 2 (1982), 345-371.

Pfingsten, A., "Surplus-sharing methods," **Mathematical Social Sciences** 21 (1991), 287-301.

Pineles, H.M., **Darkah shel Torah**, 1961, Forester, Vienna.

Rubinstein, A., "Perfect equilibrium in a bargaining model", **Econometrica** 50 (1982), 97-109.

Schmeidler, D., "The nucleolus of a characteristic function form game", **SIAM Journal of Applied Mathematics** 17 (1969), 1163-1170.

Serrano, R., "Strategic bargaining, surplus-sharing problems and the nucleolus", **Journal of Mathematical Economics** 24 (1995), 319-329.

Sertel, M., "The Nash bargaining solution manipulated by predonations is Talmudic", **Economics Letters** 40 (1992), 45-55.

Shapley, L., "A value for n-person games," in **Contributions to the Theory of Games II** (Annals of Mathematics Studies 28), ed. by H.W. Kuhn and A.W. Tucker, Princeton: Princeton University Press, 1953, 307-317.

Sonn, S. “Sequential bargaining for bankruptcy problems”, University of Rochester mimeo, 1992

Sprumont. Y, “The division problem with single-peaked preferences: a characterization of the uniform allocation rule”, **Econometrica 49** (1991), 509-519.

Thomson, W., “Population-monotonic allocation rules”, in **Social Choice, Welfare, and Ethics**, (W.A. Barnett, H. Moulin, M. Salles, and N. Schofield, eds), Cambridge University Press, 1995a, 79-124.

——, “Consistent allocation rules”, University of Rochester mimeo, 1995b.

Tijs, S., “Bounds for the core and the τ-value”, in **Game Theory and Mathematical Economics**, (O. Moeschlin and D. Pallaschke, eds), North-Holland, 1981, 123-132.

Young, P., “Taxation and bankruptcy,” University of Maryland Mimeo, July 1986.

——, “When is a tax increase fairly distributed?,” University of Maryland mimeo, 1987a.

——, “On dividing an amount according to individual claims or liabilities,” **Mathematics of Operations Research** 12 (1987b), 398-414.

——, “Distributive justice in taxation,” **Journal of Economic Theory** 48 (1988), 321-335.