Consistent Allocation Rules

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Abstract

The objective of this work is to present a principle that has recently played a fundamental role in axiomatic analysis, and a converse of this principle, which has also played an important role. They are now most commonly known under the names of ***consistency*** and ***converse consistency***. We survey the applications of the principles to a variety of problems in game theory, economics, and political science. They are bargaining problems, coalitional form games, both with and without transferable utility, and strategic games; bankruptcy and taxation problems, quasi-linear cost allocation problems, and pricing problems; fair division in classical economies, in economies with indivisible goods, and in economies with single-peaked preferences; apportionment problems, and finally matching problems. We also present related issues and sketch directions for further research.

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Part 1

CONSISTENCY AND ITS CONVERSE

1.1 INTRODUCTION.

The primary objective of this monograph is to describe the role played by a fundamental principle in the comparative study of solutions to decision problems; we will refer to it as the ***consistency principle.*** This principle unifies important developments that have taken place recently in diverse areas, ranging from abstract models of game theory to concrete models of fair allocation, taxation, apportionment, and others. A secondary objective is to present a converse of this principle, which we will simply name the ***converse consistency principle.***

Although most of the literature reviewed here appeared in the last few years, the consistency principle itself is very old. In fact, it is likely at the root of a 2000 year-old solution to the problem of adjudicating conflicting claims.

A ***decision problem*** is given by a ***list of agents*** together with a ***set of alternatives*** available to them, and their ***preferences*** defined over this set. These preferences conflict. A ***solution*** associates with every decision problem D in some ***admissible domain*** $\mathcal{D}$ an outcome x in the feasible set of that problem; x is called the ***solution outcome of*** D. Depending upon the context, the solution may be seen normatively, as providing the recommendation that an impartial arbitrator could make on how the problem should be solved, or it may be meant to describe or predict how the problem would be solved by the agents on their own. ***Bargaining problems*** and ***bargaining solutions*** on the one hand, ***resource allocation problems*** and ***allocation rules*** on the other, are canonical examples illustrating these general notions.

Two main methodologies have been adopted in the study of solutions. One is ***axiomatic***. Appealing properties of solutions are formulated as ***axioms***, and the existence of solutions satisfying the axioms is investigated. Such studies often result in "characterization" theorems, that is, theorems identifying a particular solution, or a class of solutions, as being the only solution, or the only class, to satisfy a given list of axioms. According to this methodology, the primary concepts are the properties of solutions; properties are used as building blocks in the construction of solutions. The literature on bargaining that makes use of this methodology is extensive.

The other methodology proceeds in the opposite direction. It starts with solutions, which are chosen on the basis of the intuitive appeal of their definitions or as formal representations of schemes encountered in actual practice. Of course, it is usually asked whether each particular solution satisfies basic properties of interest, but the focus is not on properties. As a whole, the study of resource allocation in economic environments has been conducted in this manner.

However, the last few years have seen a considerable expansion of the uses of the axiomatic method. Significant progress has been made in the study of models to which it had been traditionally applied, and a variety of new classes of models for which it was discovered to be equally powerful have been identified, and techniques for their analysis developed. Our aim is to review the role played in these developments by the consistency principle and its converse. Therefore, contrary to most axiomatic studies, in which a specific class of problems is first chosen and the restrictions forced on solutions by various combinations of axioms are identified, we examine here a wide range of models, and we present results unified by the application of these two principles.[1] Naturally, other axioms will be involved, but consistency and its

[1] In most cases, the two principles appear under other names than the ones we use, names that we will indicate as we proceed. It is only recently that the terminology seems to have generally settled on the terms "consistency" and "converse consistency", but other expressions still appear on occasions.

converse will always be the central ones.

This survey is organized as follows. In Part 1, we give a general statement of the consistency principle and of its converse, and we discuss several natural variants.

In Part 2, we turn to the examination of specific models. They exhibit a wide variety. Four are from game theory: bargaining problems, games in coalitional form with transferable utility, and then without transferable utility, and non-cooperative games. Four are from public economics: bankruptcy and taxation problems, quasi-linear cost allocation problems, general cost allocation problems, and finally pricing problems. Three are from welfare economics: resource allocation problems in private good economies with classical preferences, with single-peaked preferences in the one-commodity case, and with indivisible goods. Of the remaining models, one is from political science — it concerns apportionment — and the final model pertains to matching. We show how the consistency principle and its converse have been adapted to each of these models, how in some cases the principles have yielded characterizations of known solutions, and how in others they have led to the discovery and the characterization of new solutions.

In Part 3 we discuss a number of issues related to consistency and its converse, with an emphasis on what we perceive to be promising directions for further research. We first examine models with a large number of agents modelled as a continuum; such models have so far been the object of little attention with respect to our two principles. We formulate the notion of the minimal consistent extension of a solution, and that of its maximal consistent subsolution. We explore possible formulations of consistency for private good economies with individual endowments, and for economies with public goods. Finally, we discuss the computational implications of converse consistency, and the problem of intertemporal allocation.

An earlier survey of this literature is Thomson (1990a). The applications of the consistency principle and of its converse have multiplied to such an extent that updating this relatively recent work already seemed useful. We hope that this revised survey will be out of date as quickly as the first one! Another survey, limited however to coalitional form games, is Driessen (1991).

1.2 GENERAL CONCEPTS

We start with a very general statement of the two principles, and we discuss several useful variants.

Let $\mathcal{I}$ be an infinite set of ***"potential" agents***. Unless otherwise indicated, we take $\mathcal{I}$ to be the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the collection of all finite subsets of $\mathcal{I}$, with generic elements N, N', and so on. For each group $N \in \mathcal{N}$, there is a ***space of alternatives***, X^N, from which the alternatives made available to the group N are taken.[2] Let $\mathcal{D}^N$ be the domain of ***problems*** that the members of N could conceivably face. Each element of $\mathcal{D}^N$ is given by a ***feasible set***, a certain subset of X^N satisfying some regularity conditions, together with data concerning the environment, typically including the preferences of the members of N over this feasible set. How rich is the domain of admissible problems is a modelling choice whose importance should be quite clear by the end of this review.

Given a group $N \in \mathcal{N}$ and a problem $D \in \mathcal{D}^N$, we would like to identify which feasible alternative of D the agents in N will select as a compromise, or depending upon the interpretation, which feasible alternative of D an impartial arbitrator will recommend to them. However, instead of considering each problem separately, we will be more ambitious and look for a general rule that could be applied to any admissible problem that any admissible group could face. Therefore, let $\mathcal{D} = \bigcup_{N \in \mathcal{N}} \mathcal{D}^N$ and $X = \bigcup_{N \in \mathcal{N}} X^N$.

We are now in a position to define our first main concept.

Definition. A ***solution on $\mathcal{D}$*** is a function $\varphi: \mathcal{D} \to X$ that associates with every $N \in \mathcal{N}$ and every $D \in \mathcal{D}^N$ an alternative in the feasible set of D. This alternative, denoted by $\varphi(D)$, is the ***solution outcome of D***; we also say that it is ***φ-optimal for D***.

1.3 CONSISTENCY: THE FUNDAMENTAL DEFINITION.

In most of social choice theory, game theory, and economic theory, the number of agents is assumed to be some fixed number. This number is usually

[2]Later on, the exponent N will denote a cartesian product. It should be clear from the context which is intended.

arbitrary, but it is not allowed to vary. Solutions are rarely subjected to the test of a variable population.[3] Here, we explicitly require solutions to be defined over a domain of problems involving groups of arbitrary cardinalities. When a solution is so defined, the need arises to relate its components relative to different groups, and in particular groups of different sizes. To see this, let φ^1 and φ^2 be two such solutions. The solutions can be used to construct a third one, φ^3, as follows: let $\varphi^3(D) = \varphi^1(D)$ if D involves an odd number of agents and $\varphi^3(D) = \varphi^2(D)$ if D involves an even number of agents. Nothing *a priori* prevents such constructions. But the two solutions φ^1 and φ^2 may be motivated by quite different considerations and as a result the solution φ^3 might appear strange. So we ask: how should the components of solutions be linked across cardinalities? *Consistency* will help us in answering this question, as it will typically disqualify hybrid solutions such as φ^3. But it will actually say much more, since choosing the recommendations made by φ^1 for all cardinalities, or those made by φ^2 for all cardinalities, will not necessarily produce a *consistent* solution.[4]

We start with an informal description of the principle: a solution is ***consistent*** if for any admissible problem faced by some group N, whenever it recommends some outcome x as its solution outcome, then for any subgroup N' of N, it recommends the restriction of x to N' as the solution outcome of the "reduced problem" faced by N': this is the problem derived from the original one by attributing to the members of the complementary subgroup "their components of x." Note that for this operation to be meaningful, outcomes should be "decomposable" in a way that indeed makes it possible to speak of an agent's "component" of an outcome. This issue will be discussed in detail below but at this point we prefer being somewhat vague.

Once the problem $D \in \mathcal{D}^N$ has been solved by applying the solution φ, how does the situation appear to the members of N'? What are their opportunities? Let x denote the alternative selected by φ for D. In answer to these questions, we assume that it is understood that the members of the complementary subgroup $N \backslash N'$ should indeed receive the payoffs $x_{N \backslash N'}$ assigned to them by φ. From their viewpoint, all alternatives in D yielding

[3] In some models, bargaining being an example, investigators have often limited themselves to the case of two agents.

[4] The test of *consistency* is not the only variable population test that one could consider. Certain invariance conditions with respect to replications of problems, and tests of population-monotonicity have also been formulated and investigated in the literature.

these payoffs are equivalent. Therefore, from the perspective of the members of N', this set of alternatives, if it constitutes a well-defined problem, is now the problem that has to be solved. Does solving it produce the payoffs $x_{N'}$ initially assigned to them? The solution is consistent if the answer is always yes. This sort of internal coherence will greatly help in ensuring that the initial decision will not be questionned.

For a formal statement of the principle, we first need to define the notion of a reduced problem:

Definition. Given two groups N, $N' \in \mathcal{N}$ with $N' \subset N$, a problem $D \in \mathcal{D}^N$, and finally an alternative x in the feasible set of D, the ***reduced problem of*** $\boldsymbol{D}$ ***relative to*** $\boldsymbol{N'}$ ***and*** $\boldsymbol{x}$ is the problem comprising all the alternatives of D at which the members of the complementary subgroup $N \backslash N'$ receive their components of x. We denote it by $\boldsymbol{r^x_{N'}(D)}$.

Note that $r^x_{N'}(D)$ may or may not satisfy all of the properties that are required of the members of $\mathcal{D}^{N'}$, although in many of the applications that we will consider, these properties are automatically satisfied: then we say that the "domain is closed under the reduction operation". In some situations the reduced problem belongs to the domain for any x chosen by a particular solution, which is in fact all that we need to discuss the consistency of that solution (See Subsection 1.4.6 for a discussion): the domain is then "closed under the reduction operation ***for the solution***". We are now ready to formally define consistency.

THE FUNDAMENTAL DEFINITION. The solution $\varphi\colon \mathcal{D} \to X$ satisfies ***consistency*** if for all groups N, $N' \in \mathcal{N}$ with $N' \subset N$ and all problems $D \in \mathcal{D}^N$, if x denotes the solution outcome of D, then the restriction of x to N' is the solution outcome of the reduced problem of D relative to N' and x, provided this reduced problem belongs to $\mathcal{D}^{N'}$: for all $N, N' \in \mathcal{N}$ with $N' \subset N$, all $D \in \mathcal{D}^N$, and all $x \in D$, if $x = \varphi(D)$ and $r^x_{N'}(D) \in \mathcal{D}^{N'}$, then $x_{N'} = \varphi(r^x_{N'}(D))$.

The agreement on certain payoffs being paid to the departing agents can be understood in two ways. The most general interpretation is simply that agents are promised specific welfare levels, without the alternative through which these welfare levels will be reached being given. The other interpretation, which makes sense only in situations where the space of alternatives is appropriately decomposable in the sense alluded to earlier, is that the agents who leave do so with their components of the chosen alternative. In some cases, both interpretations are possible. For instance, in exchange economies, agents may leave the scene either with bundles of goods, or with the promise that whatever allocation is finally chosen, they will end up on a certain indifference curve (see Section 1.4.8 for further discussion).

Consider the not uncommon situation in which the alternatives among which a selection has to be performed result from contributions made by agents in a particular temporal sequence. Then, the agents who are done first might want to receive their payoffs and depart. For a *consistent* solution this departure does not create the need to reevaluate the alternative initially selected.[5]

Note that in the above discussion, we considered groups of "named" individuals. This is because in general, two groups may be composed of the same number of individuals with the same characteristics, and yet be dealt with differently. The assumption is made in most of the literature reviewed here that all agents are fundamentally "equal" and accordingly, solutions are minimally required to treat identical agents identically: in fact they are often specified so as to be invariant under exchanges of the "names" of agents (this is the familiar condition of *anonymity*). However our formalism is chosen so as to accommodate the possibility of treating agents differently even though they have identical characteristics. This will provide us with useful flexibility. For example, in some voting bodies, certain voters have more power than others, (*e.g.*, the permanent members of the Security Council of the United Nations). Similarly, in bankruptcy court, certain classes of claims have higher priority than others.

1.4 VARIANTS OF THE FUNDAMENTAL DEFINITION.

At various points in the preceding definition of the reduced game and in the statement of the Fundamental Definition, we could have made other choices.

[5]This motivation is due to Lensberg (1985). If the ordering is really fixed however, the reformulation of *consistency* obtained by considering only subgroups of consecutive agents, starting with the first agent, is natural. Such a formulation is discussed below (Subsection 1.4.3).

We discuss next the nature of these choices. They are illustrated by means of examples of domains and solutions with which most readers will be familiar. Therefore, formal definitions are relegated to the section on applications.

1.4.1 Permitting *multi-valued* solutions.

We required solutions to associate with every admissible problem a ***unique*** outcome ("***the*** solution outcome of $D\ldots$"). Whether a solution is meant to offer predictions or recommendations, uniqueness of the solution outcome is of course greatly desirable. Fortunately, there are interesting domains of problems for which a large number of appealing *single-valued* solutions can be defined; then, it is natural to limit our search to such solutions. The axiomatic theory of bargaining was developed under the almost universal requirement of *single-valuedness* for that reason. However, in many branches of economics and game theory, *single-valuedness* is virtually impossible to obtain or comes at a very high price. For instance, most of the solutions discussed in economic models of exchange and production are *multi-valued*; this is certainly the case for the two central ones, the Walrasian solution and the core. Domain restrictions occasionally exist that guarantee *single-valuedness* (gross substitutability guarantees *single-valuedness* of the Walrasian solution), but they are often too strong to be of much use.

To permit *multi-valued* solutions, rewrite the Fundamental Definition as follows: "For all $N, N' \in \mathcal{N}$ with $N' \subset N$, all $D \in \mathcal{D}^N$, and all $x \in \varphi(D)$, if $r^x_{N'}(D) \in \mathcal{D}^{N'}$, then $x_{N'} \in \varphi(r^x_{N'}(D))$".

Note our choice of quantifiers; we require that starting from ***any*** outcome x among the ones recommended for the initial problem D involving the group N, and given ***any*** subgroup N' of N, the restriction $x_{N'}$ of x to N' be a solution outcome of the associated reduced problem $r^x_{N'}(D)$. We could consider the weaker requirement that there should be ***at least one*** outcome recommended for the initial problem that passes the test for each of the subgroups. A weaker requirement still would be obtained by permitting this outcome to depend on the selected subgroup: given any subgroup N' of N, there is at least one outcome x recommended for D whose restriction to the subgroup is recommended for the associated reduced problem. The usefulness of these weaker definitions remains to be explored. In the latter case, we should of course not go so far as requiring that if a further reduction is carried out relative to an even smaller group of agents $N'' \subset N'$ and that restricted outcome $x_{N'}$, the distinguished outcome $y \in \varphi(r^x_{N'}(D))$ for which $y_{N''} \in \varphi(r^y_{N''}(r^x_{N'}(D)))$ be precisely $x_{N'}$. Indeed, this would simply amount to saying that the solution contains a *consistent* (in the sense we first gave to that term) selection.

In the case of *multi-valued* solutions, it is worth noting at this point that an arbitrary intersection of *consistent* solutions, if non-empty, is a *consistent* solution. The same can be said of arbitrary unions of *consistent* solutions. These facts are exploited in Section 3.2 where we formulate procedures to approximate non-*consistent* solutions by solutions satisfying the property.

1.4.2 Imposing restrictions on the number of potential agents.

We have assumed the set of potential agents $\mathcal{I}$ to be countably infinite. In a number of applications, it is more appealing to draw agents from a finite list. Alternatively, modelling the set of potential agents as a continuum, say the real numbers, may have mathematical advantages. For some models, these alternative choices for $\mathcal{I}$ have significant implications for the theorems. Indeed, the proofs of some of the results that we will present require that there be a fair amount of flexibility in the specification of the class of admissible groups; for instance, one may need to have access to groups of arbitrarily large cardinalities. For other models, a limited class of groups suffices; in some cases, it is enough to include all groups of cardinality up to three.

To illustrate these possibilities, instead of writing "$\mathcal{I} = \mathbb{N}$" in the Fundamental Definition, we would write "$\mathcal{I} \subset \mathbb{N}$ with $|\mathcal{I}| < \infty$" or "$\mathcal{I} = \mathbb{R}$".[6]

1.4.3 Imposing restrictions on the subgroups relative to which the reduced game is calculated.

Starting from some group $N \in \mathcal{N}$, and having solved at x some problem D that it faces, the Fundamental Definition asks us to investigate how the reduced problem $r^x_{N'}(D)$ faced by ***each subgroup N' of N*** would be solved. However, in some situations, it may be natural to impose restrictions on the admissible subgroups. We explore several possibilities.

[6] $\mathbb{R}$ is the set of real numbers.

Imposing restrictions on the size of the subgroups

First, we could limit our attention to subgroups of small cardinality. In particular, when the principle is meant to express the robustness of a compromise under "challenges" by subgroups, it makes sense to consider only small subgroups since coordinated action may be difficult for large groups. In fact, *consistency* is sometimes written with the restriction that only subgroups of cardinality two can form. This weaker version is called ***bilateral consistency***. Usually, but not always, excluding subgroups of cardinality greater than two weakens the axiom in a minor way, and characterization proofs still go through, although with some extra work.

In some contexts, it is sensible to exclude degenerate subgroups of cardinality one, and in others to insist that they be permitted. For instance, when *consistency* is applied to bargaining, the property is intended to link compromises across cardinalities. Degenerate problems with only one person are of course not conflict situations at all, and there is no reason to expect that the way they are solved has any bearing on the resolution of actual conflicts. On the other hand, when *consistency* is applied to non-cooperative games, part of the objective may precisely be to connect the way an isolated individual solves one-person decision problems to the way he handles decision problems affected by the presence of other agents similarly attempting to solve their own decision problems. Then, allowing one-person problems may be necessary.

Endowing the set of agents with a graph structure

The size of the subgroups is not the only relevant consideration however. The set of agents may be equipped with additional structure, and which subgroups are admissible should be specified so as to reflect this structure. For instance, if the situation under study is intertemporal allocation, agents are indexed by time; then, considering only subgroups consisting of successive generations is quite natural. When agents live for several periods, requiring subgroups to be such that the union of the life spans of the agents it comprises is an interval, makes intuitive sense (Section 3.6).

More generally, the set of agents may have a graph structure representing some relevant aspect of social organization, such as channels of communication, hierarchies, business or family connections, racial or ethnic memberships, language groups, historical links, or geographical areas. If agents are interpreted as countries, only countries with a common boundary, or countries between which a trade agreement exists, would be directly connected. In any of these cases, only connected components of the graph should be considered admissible subgroups.

Particulary interesting examples of graphs are (i) complete graphs, when each agent is directly connected to each other agent, (ii) circular graphs, when agents can be arranged in a circle in such a way that each agent is directly connected to, and only to, his two neighbors, (iii) linear graphs, when agents can be ordered in such a way that each agent is directly connected to, and only to, the preceding agent, if there is one, and the following agent, if there is one, (iv) hub-and-spoke graphs, when a particular agent is directly connected to all others, no two other agents being directly connected, and (v) trees, when the graph has a tree structure.

In general, let $\alpha: \mathcal{N} \to \mathcal{N}$ be a correspondence associating with every $N \in \mathcal{N}$ a list $\alpha(N)$ of ***admissible*** subgroups of N. Then, adjust the Fundamental Definition by replacing "for all N, $N' \in \mathcal{N}$ with $N' \subset N$" by "for all N, $N' \in \mathcal{N}$ with $N' \in \alpha(N)$."

In principle, the values of the correspondence α for two different sets of agents N and $\tilde{N} \in \mathcal{N}$ need not be related, but given its intended interpretation, it is natural to require "agreement" between overlapping groups in the sense that if $N' \subseteq N \cap \tilde{N}$, then $N' \in \alpha(N)$ if and only if $N' \in \alpha(\tilde{N})$. In that case, there is a graph on the entire set of potential agents whose restriction to each $N \in \mathcal{N}$ gives the graph on N. At this point, we will note however that connectedness of a graph, a property that will be important in our discussion of *converse consistency*, is not preserved under the reduction operation, so that there are disadvantages to deriving the graph on each group from a single graph on the whole set of players.

When we return to the issue, we will denote by $G = (G^N)_{N \in \mathcal{N}}$ a family of graphs, G^N being a graph on N.

Deriving endogenous restrictions on the permissible subgroups

In some situations, the application of a solution to a problem generates a meaningful order on the set of agents, and it may be useful only to consider agents who leave in that order. Formally, for each $N \in \mathcal{N}$, let π^N be a function defined on X^N which specifies for every $x \in X^N$ which member of N will be first, second, and so on. Now, given $D \in \mathcal{D}^N$, we only allow subgroups of N consisting of initial segments of the set of agents as reordered by π^N.

An application of this idea is when x is a point in a Euclidean "payoff" space, and π^N is a function that orders agents by increasing payoffs. Then, the operation would consist in paying agents with low payoffs first.

This sort of restriction on the formation of subgroups was considered by Blackorby, Bossert, and Donaldson (1994) in the context of bargaining (Subsection 2.2.1) and by Moulin and Shenker (1992) and Kolpin (1994) in studies of cost allocation (Subsection 2.3.3).

1.4.4 Allowing the reduced problem to depend on the solution.

Describing how the original problem appears to the subgroup N' after the members of the complementary group $N \backslash N'$ have received their components of the outcome is not always straightforward. In some cases — the domain of games in coalitional form (Subsection 2.3.2) is the most prominent example — several specifications make sense. The crucial point however is the dependence of the reduced problem on the original problem ***and*** the outcome that is being evaluated. The notion of a reduced problem discussed here should therefore be contrasted with often encountered notions of "subproblems" (or "subeconomies", or "subgames") that depend only on the original problem (or economy, or game). For instance, a subgame of a game in coalitional form v relative to a particular subset of the set of players is defined by setting the worth of each coalition equal to what it was in v. This amounts to taking the restriction of the vector defining v to the subspace corresponding to all coalitions that are subsets of the subgroup. By contrast, the specification of a reduced game involves a proposed compromise for v.

There is an important domain however, where the notions of reduced game and subgame coincide. It pertains to the matching of individuals to individuals, also known as the "marriage problem" (Subsection 2.5.2). This is because agents' "payoffs" are their mates and imagining some agents to leave with their payoffs is simply calculating a subgame. But note that ***which subgames are considered*** in expressing the requirement of *consistency* depends on the matching that is being selected by the solution for the initial problem.

The reduced problem could alternatively be made to depend on the solution itself (Hart and Mas-Colell, 1988,1989). This specification has proved useful too, and its implications have been described in detail for coalitional form games (Subsection 2.2.2).

To allow for the dependence of the reduced problem on the solution, in the Fundamental Definition, replace $r^x_{N'}(D)$ by $r^{\varphi}_{N'}(D)$, where the superscript φ has been substituted for the superscript x.

1.4.5 Allowing other relations between the solution outcome of the original problem and of its reduced problems.

The Fundamental Definition requires ***coincidence*** of the restriction to the subgroup of the initial compromise with the solution outcome of the reduced problem faced by this subgroup. More generally, we could request that a certain relation between the two outcomes holds. We present below two possibilities.

(i) Domination of (a) the restriction of the solution outcome of the initial problem to a subgroup, by (b) the solution outcome of the associated reduced problem this subgroup faces

For instance, the weaker requirement of Pareto-domination of one outcome by the other has been convenient in bargaining theory (Subsection 2.2.1).

Formally, in the Fundamental Definition, replace "then $x_{N'} = \varphi(r^x_{N'}(D))$" by "then $x_{N'} \leqq \varphi(r^x_{N'}(D))$."[7]

(ii) Coincidence between (a) the solution outcome of the initial problem and (b) the average of the solution outcomes of the associated reduced problems

Consider a *single-valued* solution and suppose that it is not *consistent*. Then, the recommendation the solution makes for at least one problem — let us call this recommendation x — does not agree with the recommendation it makes

[7]Vector inequalities: $x \geqq y$, $x \geq y$, $x > y$.

for at least one of the reduced problems associated with x: this means that in that reduced problem at least one agent, say agent i, does not get what he should according to that recommendation, namely x_i.[8] It may be true of course that agent i does not get what he should in other reduced problems. Now, suppose that ***on average***, when the reduced problems associated with x and all the subgroups containing agent i are considered, this agent ***does*** get x_i, and suppose that on average ***every*** agent gets his component of x in the reduced problems associated with x and the subgroups to which he belongs. Then, one might be satisfied with x after all. This is the test of ***average consistency*** proposed by Maschler and Owen (1989) in the context of coalitional form games (Subsection 2.2.2), and also considered by Dagan and Volij (1994) for taxation problems (Subsection 2.3.1). Obviously, for the test to be applicable, outcomes should be decomposable into individuals components, and these components should belong to spaces in which averaging operations are meaningful.

Formally, the proposal here is to require in the Fundamental Definition the equality of x_i and $\frac{1}{(|N|-1)!}\sum_{N':N'\subset N, N'\ni i}\varphi_i(r^x_{N'}(D))$ (as in Dagan and Volij). When the reduced game depends on the solution itself, and using the notation of Subsection 1.4.4, we require $x_i = \frac{1}{(|N|-1)!}\sum_{N':N'\subset N, N'\ni i}\varphi_i(r^\varphi_{N'}(D))$ (this is this formulation that Maschler and Owen suggest).

Variants of the property are obtained by limiting the averaging to subgroups of size 2, or to subgroups of size k for some fixed $k < |N|$, or to subgroups of size at most k.

1.4.6 Requiring admissibility of the reduced problem.

The Fundamental Definition applies only if the reduced problem is in the admissible domain. A stronger version results from ***adding the requirement that this membership holds.*** (Recall our discussion of closedness of domains under the reduction operation that preceded the statement of the Fundamental Definition.) In some applications, the reduced problem is automatically admissible; such is the case for example for the problem of fair allocation in classical economies (Section 2.4). In fact, for some models, admissibility holds even if the outcome with respect to which the reduction

[8] Under efficiency, there is at least one agent who gets less than he should and one agent who gets more than he should.

takes place is an arbitrary feasible outcome of the initial problem, instead of being an outcome selected by the solution for the problem. For other models this is far from being the case; examples here include certain domains of coalitional form games (Subsections 2.2.2-3).

To so strengthen the Fundamental Definition, replace "***if*** $x = \varphi(D)$ and $r^x_{N'}(D) \in \mathcal{D}^{N'}$, ***then*** $x_{N'} = \varphi(r^x_{N'}(D))$" by "***if*** $x = \varphi(D)$, ***then*** $r^x_{N'}(D) \in \mathcal{D}^{N'}$ and $x_{N'} = \varphi(r^x_{N'}(D))$."

1.4.7 Reducing preference relations.

In some situations, the departure of some of the agents can be accompanied by a natural "reduction" of the consumption spaces of the remaining agents, and a corresponding "reduction" of their preference relations. For instance, when the problem is the allocation of indivisible goods, one may want to restrict the preferences of the remaining agents to the set of remaining goods (Subsection 2.3.4). For the marriage problem (Subsection 2.5.2), after the departure of some matched pairs, restricting the preferences of the remaining agents to the set of remaining possible partners is quite appealing. This specification strengthens the independence content of the condition. It may be quite appropriate in models in which the requirement of "independence of infeasible alternatives" is imposed: this says that the choice depends only on the restrictions of preferences to the set of feasible alternatives.[9] But of course, one may not want to go that far, and instead allow the solution outcome of any problem to depend on the whole preference relations; in the two examples just mentioned, these would be preferences over the whole set of "potential objects", and preferences over the whole set of "potential partners" respectively. We will omit formal statements of these various formulations, as they would require more notation than is justified for this section.

[9]In exchange economies, this condition is violated by the Walrasian rule, but not by the constrained Walrasian solution (Hurwicz, Maskin, and Postlewaite, 1979).

1.4.8 Consistency in concrete models: requiring coincidence of outcomes or coincidence of welfare levels?

The Fundamental Definition, which we formulated so as to cover situations as general as possible, can be understood as requiring that for the reduced problem, an alternative be chosen that provides each departing agent the welfare level initially assigned for him. In concrete models, a narrower definition of *consistency* is often natural since solutions do not usually prescribe welfare levels, but rather physical alternatives. Moreover, the space of alternatives may be decomposable in a way that makes it meaningful to imagine the departing agents to leave with physical components of the selected outcome. Then the choice for the remaining agents in the reduced problem can be limited to these alternatives whose components assigned to the departing agents are the ones initially chosen. Alternatives that give them the same welfare levels ***through different components*** would not be admissible. For an illustration of the distinction and of its implications for the way in which the domain of definition of solutions should be specified, see our discussion of fair allocation (Section 2.4).

1.4.9 What consistency does not say.

We conclude this section with a warning. It is important to avoid the tempting analogy of *consistency* with the fact that if a function of several variables is maximized over some constraint set at some point, then it is maximized at the same point over the subset of the constraint set obtained by giving some of the variables their optimal values. To illustrate the possible confusion, let $f: \mathbb{R}^n \to \mathbb{R}$ and $S \subseteq \mathbb{R}^n$ be a set over which f is maximized. Let x be a maximizer and let S' be the subset of S of points whose last $n-m$ coordinates are fixed at $x_{m+1}, \ldots, x_n$. Then consider the function $g: \mathbb{R}^m \to \mathbb{R}$ defined by $g(y) = f(y, x_{m+1}, \ldots, x_n)$ for all $y \in \mathbb{R}^m$. This function is obviously maximized over S' at $(x_1, \ldots, x_m)$. This simple fact seems very much like a consistency property as we defined it. To see that it is not, however, note that the function g depends on $(x_{m+1}, \ldots, x_n)$, whereas *consistency* requires that the method of "solving" subsets of $\mathbb{R}^m$ be specified separately. Of course if the function g happens not to depend on $(x_{m+1}, \ldots, x_n)$, then it can serve as the component of the solution pertaining to subsets of $\mathbb{R}^m$ and *consistency* will obtain in the passage from $\mathbb{R}^n$ to $\mathbb{R}^m$. We will see in our discussion of bargaining problems, which are indeed specified as subsets of some Euclidean space, that the implications of *consistency* go much beyond the requirement that the solution be described in terms of a list of functions indexed by the groups of agents, with each component of the solution being obtained by maximizing for each problem the function in the list that pertains to that group: additive separability will be required too.[10]

1.5 CONVERSE CONSISTENCY: THE DEFINITION.

If *consistency* allows us to deduce, from the desirability of an outcome for some problem, the desirability of its restrictions to all subgroups for the associated reduced problems they face, ***converse consistency*** permits a "dual" operation: to deduce the desirability of an outcome for some problem from the desirability of its restrictions to all subgroups of cardinality two for the associated reduced problems these subgroups face.

DEFINITION: The solution $\varphi: \mathcal{D} \to X$ satisfies ***converse consistency*** if for all groups $N \in \mathcal{N}$, all problems $D \in \mathcal{D}^N$, and all feasible outcomes x of D, if the restriction $x_{N'}$ of x to all subgroups N' of cardinality two is the solution outcome of the reduced problem $r^x_{N'}(D)$ obtained from D by assigning to all agents not in N' their components of x, then x is the solution outcome of D: for all $N \in \mathcal{N}$, all $D \in \mathcal{D}^N$, and all x feasible for D, if for all $N' \subset N$ such that $|N'| = 2$, $r^x_{N'}(D) \in \mathcal{D}^{N'}$, and $x_{N'} = \varphi(r^x_{N'}(D))$, then $x = \varphi(D)$.

Converse consistency can be seen as a condition of two-agent "decentralizability". If an alternative is acceptable in all of its associated reduced two-person problems, then it should be acceptable for the whole group.

Consider a motion on the floor of a political convention, and imagine the following process to decide on its adoption. First, delegates gather in committees of size two and each committee determines whether the motion provides a good compromise between the desires of its two members. If the

[10]However, a domain of problems for which the analogy is appropriate is the domain of games in strategic form (Subsection 2.2.3).

motion successfully passes this stage, it is then examined by committees of size three ... The process is repeated until the motion is either rejected at some stage by some committee, or finally accepted by the whole convention in plenary session. Now, if the way in which the acceptability of a compromise is decided upon satisfies *converse consistency*, acceptance by all committees of size two will guarantee acceptance at the plenary session, and the formation of the committees of size greater than two will be unnecessary.

1.6 VARIANTS OF CONVERSE CONSISTENCY

We will not discuss the possible variants of *converse consistency*. Most of the points raised earlier that led us to formulating variants of *consistency* remain valid here, and it is straightforward to see how to accommodate them. However, a number of additional issues specific to *converse consistency* arise. They are discussed next.

1.6.1 Requiring *Pareto-optimality* of the alternative that is evaluated

An important variant of *converse consistency* is obtained by requiring that the alternative that is being evaluated be undominated in the initial problem, namely that it be ***Pareto-optimal.***[11] It is indeed often the case that without this restriction, the condition is unreasonably restrictive. Since in most cases, we are interested in solutions satisfying Pareto-optimality, there is not much loss of generality in so proceeding.

1.6.2 Writing the hypothesis of *converse consistency* for all subgroups of the original group

A weaker condition than *converse consistency* is obtained by strengthening the hypothesis to the requirement that the restriction $x_{N'}$ of x to any proper subgroup N' of N (not just to any subgroup of cardinality two) be the solution outcome of the reduced problem $r^x_{N'}(D)$. In many applications, this

[11]Precise definitions are given below for several models.

weaker condition is in fact equivalent to the one stated above. Indeed, given $N' \in \mathcal{N}$ of cardinality greater than two, and assuming the hypothesis of that condition, we can deduce $x_{N'} = \varphi(r^x_{N'}(D))$ for all $N' \subseteq N$ with $|N'| = 3$ (this is because proper subgroups of N have at most cardinality two). Then we obtain $x_{N'} = \varphi(r^x_{N'}(D))$ for all $N' \subset N$ with $|N'| \leqq 3$, and in turn $x_{N'} = \varphi(r^x_{N'}(D))$ for all $N' \subset N$ with $|N'| \leqq 4 \ldots$

The equivalence of the two formulations of *converse consistency* holds for any model in which the following ***transitivity of the reduction operation*** holds: given N, N', $N'' \in \mathcal{N}$ with $N'' \subset N' \subset N$, given $D \in \mathcal{D}^N$ and x feasible for D, we have $r^{x_{N'}}_{N''}(r^x_{N'}(e)) = r^x_{N''}(e)$. Transitivity holds, for example, for the models of fair division (Section 2.4). It does not hold for coalitional form games (Subsection 2.2.2).

1.6.3 Converse consistency with respect to a graph.

In our main definition of *converse consistency*, the φ-optimality of the alternative under consideration ***for all*** of its associated two-person reduced problems has to hold in order for the alternative to be declared φ-optimal. However, stronger versions of the condition can be obtained by requiring the hypothesis to hold ***only for selected subgroups***, similar to the way in which we suggested earlier to weaken *consistency* by allowing only reductions of initial problems relative to certain subgroups (Subsection 1.4.3). One way to achieve this is to specify a graph on N, for each $N \in \mathcal{N}$, and simply to require that the hypothesis $x_{N'} \in \varphi(r^x_{N'}(D))$ holds for all groups N' of two agents that are directly connected in the graph:

Converse consistency with respect to a graph: Let $G = (G^N)_{N \in \mathcal{N}}$ be a graph. For all $N \in \mathcal{N}$, all $D \in \mathcal{D}^N$, and all $x \in X(D)$, if [for all $N' \subset N$ with $|N'| = 2$ such that the members of N' are directly connected in G^N, $r^x_{N'}(D) \in \mathcal{D}^{N'}$ and $x_{N'} = \varphi(r^x_{N'}(D))$], then $x = \varphi(D)$.

Note that if $G \subset G'$ (by this notation, we mean that for all $N \in \mathcal{N}$, any link in G^N is a link in G'^N), *converse consistency with respect to G* implies *converse consistency with respect to G'*. If a solution is *conversely consistent with respect to G* but not with respect to any proper subgraph of G, we will say that G is ***minimal.***

The property of *converse consistency* is most interesting when formulated with respect to minimal graphs. In most applications, minimality implies that every agent is, directly or indirectly, connected to every other agent.

Peleg (1986) argues that *converse consistency* is an even more desirable property than *consistency* because it allows to identify desired outcomes for problems with a large number of participants if we know how to check the desirability of outcomes for problems with small numbers of participants:

> This fact [that a solution satisfied *consistency*], may be of high theoretical interest ... However, usually it is more difficult to compute solutions of large (but finite) games than to carry out similar computations for smaller games. Thus, from a practical point of view, the [consistency] property is of limited significance.

The practical interest of *converse consistency* is all the greater in situations where the technical tools needed to solve two-person problems are of lesser sophistication than those needed to solve n-person problems.

1.6.4 Computational implications of converse consistency.

The discussion above can be summarized by saying that *converse consistency* simplifies the task of ***checking*** whether a proposed outcome should be the solution outcome of some problem. But what about the more demanding objective of ***finding*** a solution outcome? Does *converse consistency* have useful implications regarding this issue?

In this section, which is based on Thomson (1992a), we will show how the property can be used for the construction of simple algorithms that might provide a positive answer.

Consider a solution φ satisfying *converse consistency with respect to* G. This fact suggests the computational algorithm for finding φ-optimal alternatives defined as follows:

Given $N \in \mathcal{N}$, let $(N^1, \ldots, N^{\bar{k}})$ be a sequence of links in G^N such that each link of G^N appears at least once. We call such a sequence a ***cover of*** $\boldsymbol{G^N}$. We say that a cover in which each link in G^N appears exactly once is a ***minimal cover of*** $\boldsymbol{G^N}$.

Let $(N^1, \ldots, N^{\bar{k}})$ be an arbitrary cover of G^N. Now given $D \in \mathcal{D}^N$, we consider any sequence $\{x^k\}_{k \in \mathbb{N}}$ such that: $x^0 \in D$, $x^1 \in \varphi(r^{x^0}_{N^1}(D))$, $x^2 \in \varphi(r^{x^1}_{N^2}(D)), \ldots$, $x^k \in \varphi(r^{x^{k-1}}_{N^k}), \ldots$, $x^{\bar{k}} \in \varphi(r^{x^{\bar{k}-1}}_{N^{\bar{k}}})$, $x^{\bar{k}+1} \in \varphi(r^{x^{\bar{k}}}_{N^1}(D)), \ldots$, k being computed modulo $\bar{k}$ (this means that all the links in the graph are visited in successive rounds). For the sequence to be well-defined, we need $r^{x^{k-1}}_{N^k}(D) \in \mathcal{D}^{N^k}$ for all $k \in \mathbb{N}$. In the next paragraphs, we assume this to be the case. We also assume that the spaces X^N are endowed with topological structures. Then, we say that the solution is ***decentralizable with respect to G*** if convergence of $\{x^k\}$ always occurs and the limit point is φ—optimal for D.

In this definition, the cover of G^N is arbitrary; so are the initial outcome $x^0 \in D$ and for each $k \geq 1$, the φ-optimal alternative x^k chosen for the reduced problem $r^{x^{k-1}}_{N^k}(D)$. In applications however, it is conceivable that particular restrictions help convergence. For example, when the graph G^N is connected, the cover of G^N can be chosen to be a sequence of connected links. Should it be? On the other hand, it is tempting to use minimal covers of the chosen graph. Should the temptation be resisted? The extent to which these choices matter, individually and together, that is, how they affect convergence, and, when convergence occurs, its speed, surely will depend on the model at hand. In our examples in Part 2, we start from minimal graphs (they are connected graphs), and we consider minimal covers (they are paths).

Decentralizability with respect to a graph: Let $G = (G^N)_{N \in \mathcal{N}}$ be a graph. For all $N \in \mathcal{N}$, all $D \in \mathcal{D}^N$, all covers $\{N^1, \ldots, N^{\bar{k}}\}$ of G^N, all sequences $\{x^k\}$ in D such that $x^0 \in D$, $x^1 \in \varphi(r^{x^0}_{N^1}(D)), \ldots$, $x^k \in \varphi(r^{x^{k-1}}_{N^k}(D)) \ldots$, with $N^k = N^{k'}$ whenever $k' - k = n\bar{k}$ for some $n \in \mathbb{N}$, there exists $x \in \varphi(D)$ such that $x^k \to x$.

Note that the definition is meaningful whether or not φ is *conversely consistent with respect to G*. However, under appropriate continuity assumptions on φ, if φ is *decentralizable with respect to G*, then it is *conversely consistent with respect to G*.

We now return to the possibility that for some $k \in \mathbb{N}$, $r^{x^{k-1}}_{N^k}(D) \notin \mathcal{D}^{N^k}$. If this happens, we set $x^k = x^{k-1}$ and we continue as before. In principle, nothing prevents x^0 to be such that $r^{x^0}_{N^k}(D) \notin \mathcal{D}^{N^k}$ for all $k \in \{1, \ldots, T\}$ for some $T \in \mathbb{N}$, in which case convergence of $\{x^k\}$ trivially occurs, but of course the limit point will typically not be in $\varphi(D)$. For some models, $r^x_{N'}(D) \in \mathcal{D}^{N'}$ for all $x \in D$ and all $N' \subset N$ and this possibility does not arise. An example is the domain of allocation problems (Section 2.4). For some other models, for each $x^0 \in D$, $r^{x^{k-1}}_{N^k}(D) \in \mathcal{D}^{N^k}$ if k is sufficiently large. An example here is the domain of bargaining problems for some commonly used solutions (Subsection 2.2.1). Finally, simple restrictions on x^0 may guarantee that $r^{x^k}_{N^k}(D) \in \mathcal{D}^{N^{k-1}}$ for all $k \geq 1$. Again, consider the domain of bargaining problems. For the Nash solution, by choosing $x^0 > 0$, we obtain a sequence of problems that are always well-defined.

In order to accommodate these differences, the specific form of *decentralizability* that is used should be tailored to whatever model is being considered.

Although other notions of decentralizability have been proposed in the theory of economic planning, the above definitions seem to capture some important features of what is understood by this term in common language: the social optimum is not directly calculated by a central planner, or "center". Instead, calculations are parceled out to subunits, and carried out in successive steps. At each step, each subunit solves a problem that depends on parameters received from the center. Adjustments are carried out by the center in the parameters sent to the subunits from step to step. These adjustments are made on the basis of information received from the subunits at the previous step.

Features that differentiate the notion of decentralizability proposed here from notions discussed in the theory of planning, however, are that (i) in the latter, the subunits are individual agents (firms, consumers), and (ii) at each round computations are carried out simultaneously by all subunits, whereas here the subunits are ***pairs*** of agents (in the applications that we will consider, consumers or players), and pairs compute sequentially. Another difference is that (iii) for the best-known planning procedures, feasibility of the tentative proposal is not guaranteed at each step, whereas here it is. Feasibility is a particularly desirable feature in situations where computations have to be interrupted after finitely many steps, the typical case.

If *converse consistency* of a solution φ allows us to ***verify*** that an outcome is φ-optimal for some problem by solving a finite number of two-person reduced problems associated with it, the *decentralizability* property we formulated above allows us to ***find*** a socially desirable outcome, or at least to approximate one, by solving a sequence of two-person reduced problems, the reduction taking place with respect to groups reappearing in cycles and outcomes that are repeatedly recomputed. Note however that *decentralizability* does not solve the problem of ***finding all*** φ-optimal outcomes. For each given model, it would be interesting to know the range of limit points that result by varying the initial conditions and the choices made at each step.

1.7 SEVERAL TECHNICAL ISSUES OF GENERAL INTEREST

In this section, we list of number of technical points of general validity pertaining to *consistency*, its *converse*, we discuss possible logical relations between them, and we present some useful lemmas.

1.7.1 Flexibility.

In the definition of *consistency*, the departing agents play a "passive" role. Suppose now that when the original group is divided into two subgroups, each of these subgroups is allowed to make adjustments in what its members receive provided these adjustments are obtained by operating the solution: for each subgroup, only an alternative that is φ-optimal for the associated reduced problem it faces is admissible. Then, these separate choices, when put together, may or may not constitute a φ-optimal alternative for the original problem. If they do however, the partial autonomy given to the subgroups does not get in the way of the overall social objective embodied in the solution. We will say that a solution is ***flexible*** if it permits this sort of readjustments by subgroups.[12]

Flexibility: The solution $\varphi: \mathcal{D} \to X$ is ***flexible*** if for all N, $N' \in \mathcal{N}$ with $N' \subseteq N$, all $D \in \mathcal{D}^N$, all $x \in \varphi(D)$, and all $y \in \varphi(r^x_{N'}(D))$, we have $(y, x_{N \setminus N'}) \in \varphi(D)$.

[12]This concept appears in Balinsky and Young (1982) in their study of apportionment problems. It is a component of the condition they call "uniformity". It is also used by Shimomura (1992), from whom we borrow the term of flexibility, and by Blackorby, Bossert, and Donaldson (1994) in bargaining theory.

1.7.2 Two useful lemmas

As we noted earlier, the models for which the implications of *consistency* and *converse consistency* have been investigated differ considerably in their mathematical structures. An unfortunate consequence of this diversity is that most of the results that we will present were obtained in the context of a specific model, and do not have immediate counterparts for other models. However, two lemmas occur frequently, whose elementary proofs can be stated in a "model-free" fashion. They both pertain to solution correspondences.

The first one essentially states that if a *consistent* solution is contained in a *conversely consistent* solution on the subdomain of two-person problems, then the inclusion holds for all cardinalities. Its proof consists in moving from problems involving an arbitrary number of agents to problems involving two agents by means of *consistency*, and moving back up again by means of *converse consistency*. Using the image of a building whose floors would be indexed by the cardinalities of problems, we refer to this lemma as the "Elevator Lemma": *consistency* is the "Down" button and *converse consistency* the "Up" button.

Lemma 1 *(The* "Elevator Lemma"*) Let φ and φ' be two solutions defined on a domain D that is closed under the reduction operation for the solution φ. If $\varphi \subseteq \varphi'$ on the subdomain of two-person problems, φ is* consistent, *and φ' is* conversely consistent, *then $\varphi \subseteq \varphi'$ on the whole domain D.*

Proof: Let $N \in \mathcal{N}$, $D \in \mathcal{D}^N$, and $x \in \varphi(D)$. Since φ is *consistent*, then for all $N' \subset N$ with $|N'| = 2$, $x_{N'} \in \varphi(r^x_{N'}(D))$. Since $\varphi \subseteq \varphi'$ on the subdomain of two-person problems, then for all $N' \subset N$ with $|N'| = 2$, $x_{N'} \in \varphi'(r^x_{N'}(D))$. Since φ' is *conversely consistent*, $x \in \varphi'(D)$. Q.E.D.

This lemma is of interest because several interesting examples exist for which some special relation holds between two solutions only for the two-person case. Incidentally, note that *bilateral consistency* would suffice in Lemma 1.

We now turn to the second lemma. A number of the results that we will present identify a particular solution that any *consistent* solution satisfying certain conditions has to contain. The conditions often include the requirement that the solution should be a subsolution of some basic solution of interest. A corollary of such a result is that the particular solution is the "minimal" solution satisfying the conditions. The second lemma, whose completely straightforward proof we omit, is the critical step in the proofs of all of these results.

Let φ be a *consistent* solution and suppose that it is contained in some solution $\bar{\varphi}$. Given some problem in the domain and some alternative x that is $\bar{\varphi}$-optimal for it, we note that x will in general not be the only $\bar{\varphi}$-optimal one; we may have some degrees of freedom. However, suppose that additional agents can be introduced and the problem extended to the enlarged set of agents, in such a way that (i) only one alternative is $\bar{\varphi}$-optimal in the augmented problem, (ii) the restriction of that alternative to the initial group is precisely x, and (iii) the reduction of the augmented problem with respect to the initial group of agents and that alternative yields the initial problem. Then, since $\varphi \subseteq \bar{\varphi}$, the conclusion will follow that the augmented alternative is the only φ-optimal alternative for the augmented problem, and by *consistency* of φ, that x is φ-optimal for the initial problem.

Here too, a physical metaphor might be useful: a bookshelf lacking a back panel wobbles. To eliminate this instability, we brace it by means of a board nailed diagonally in the back. More generally, and depending upon the complexity of the structure that we may want to stabilize, one, two, or more such "braces" might be needed (these are the additional agents).

Lemma 2 *(The* "Bracing Lemma"*) Let φ be a* consistent *subsolution of some solution $\bar{\varphi}$. Suppose that $\bar{\varphi}$ is such that for all $N \in \mathcal{N}$, all $D \in \mathcal{D}^N$, and all $x \in \bar{\varphi}(D)$, there are $N' \supset N$, $D' \in \mathcal{D}^{N'}$, and x' in the feasible set of D', such that $\{x'\} = \bar{\varphi}(D')$, $x'_N = x$, and $D = r_N^{x'}(D')$. Then $\varphi = \bar{\varphi}$.*

Let φ^* be a subsolution of $\bar{\varphi}$. If the "extension to uniqueness" just described holds only for all $x \in \varphi^*(D)$, then $\varphi \supseteq \varphi^*$. Therefore, if φ^* is consistent*, it is the smallest* consistent *subsolution of φ.*

To take an example to which we will shortly return, and once again with apologies for not formally introducing the definitions needed to make its discussion self-contained to all readers, consider the domain of games in coalition form having a non-empty core, let $\bar{\varphi}$ be the core, and let us require that φ be a subsolution of the core. Now, given a game and a point x in the core of the game, some of the no-blocking constraints may be met as equalities at x and some may be met strictly. Depending upon the specific configuration of binding and non-binding constraints, it may be that some reallocation of payoffs can be carried out between agents without leaving the core. Now, suppose that additional players can be introduced and the game extended to the newly created coalitions — these are the braces — in such a way that the augmented game admits only one core payoff x', whose restriction to the initial set of players is x and such that its reduction with respect to the initial set of players and x' is the initial game. Then the structure has been stabilized. From the requirement on φ that it selects ***only*** core payoffs, this construction lead us to the conclusion that in fact ***all*** core payoffs have to be included in the φ-optimal set.

Now, the issue is whether this augmentation will be possible, and this will depend on the solution and the richness of the domain of problems under consideration. To return to our analogy, we may not have available a board that is long enough to be nailed diagonally to the back of the bookshelf. Here is an economic example that will make this point obvious. Consider exchange economies and the solution that associates with each economy its set of Pareto-optimal allocations. Starting from an arbitrary Pareto-optimal allocation, there is in general no way to introduce new agents and additional resources so that in the augmented economy, there is a unique Pareto-optimal allocation, let alone only one whose restriction to the initial set of agents is the allocation taken as point of departure.

In some cases, the bracing may not be possible for all φ-optimal alternatives but only for some distinguished ones. If these distinguished alternatives exist for all problems, they constitute a well-defined solution — this is the solution φ^* of the Bracing Lemma — and we will be able to conclude that any solution satisfying the required conditions has to contain it.

In several of the economic examples that we will consider, the φ^*-optimal set is often not a singleton. A useful variant of the lemma in such situations involves the requirement that the solution also satisfies *Pareto-indifference*, which says that if x and x' are feasible outcomes of D such that x is chosen for D and x' is Pareto-indifferent to x', then x' should also be chosen for D. This requirement, which seems innocuous enough, is not always met however.[13] Nevertheless, if we impose it on φ, and if φ^* satisfies it, the inclusion $\varphi \supseteq \varphi^*$ is obtained by a slight modification of Lemma 2.

[13]For instance, for the problem of fair division, the so-called no-envy solution violates it (Thomson, 1983).

1.7.3 Logical relation between *consistency* and its *converse*.

Inspection of our two main definitions does not suggest a logical relation between them but the question is often asked whether such relations exist, especially for *single-valued* solutions, and it may be worthwhile addressing it explicitly.

The answer is that indeed no logical relation exists, even in the presence of *single-valuedness*. We refer to our Section on bargaining, where we discuss two *single-valued* solutions, one of which is *consistent* but not *conversely consistent*, whereas the opposite holds for the other.

In general, and returning to the image of the recommendations made by a solution being stacked up as the floors of the building, with low cardinalities at the bottom, *consistency* implies that the solution be more and more "tapered" as cardinalities increase, whereas *converse consistency* implies greater permissiveness for greater cardinalities. If a solution satisfies both properties, we can think of it as having a "cylinder" shape. If the solution is *single-valued* for the 2-person case, then *single-valuedness* will extend to all cardinalities.

Let us consider an economic example, again without introducing any formal notation: we will see that for the problem of fair division and under somewhat stronger assumptions on preferences than are standard (smoothness is required), both the no-envy solution (which selects for each economy its set of allocations such that each agent prefers his assigned bundle to the bundle assigned to anyone else) and the Walrasian solution from equal division are *consistent* and that the latter is a subsolution of the former. Therefore, the solution that coincides with the no-envy solution up to some fixed cardinality k and with the Walrasian solution for all cardinalities greater than k is *consistent*, and the solution that makes the opposite selections is *conversely consistent*. If a subsolution of the no-envy solution satisfies both properties and coincides with the no-envy solution for cardinality 2, then by the "cylinder" analogy, it coincides with the no-envy solution for all cardinalities (technically, it is a consequence of the elevator lemma applied twice). If it coincides with the Walrasian solution for cardinality 2, the cylinder shape is obtained only if it coincides with it for all cardinalities (again, apply the elevator lemma twice).

We add the obvious fact that in the presence of *single-valuedness*, *consistency* and *flexibility* are equivalent. In fact, if a solution satisfies *essential* *single-valuedness* (for any problem, if it selects several outcomes, then all agents are indifferent between them) and *Pareto-indifference* (for any problem, if it selects some outcome, then it selects any other outcome that is Pareto-indifferent to it), then the solution is *consistent* if and only if it is *flexible.*

1.7.4 Transfer of properties across cardinalities

As we will see, *consistency* and its *converse* are almost never studied by themselves. Instead, it is the implications of these requirements together with other properties that we will consider. In most cases, these other properties will be "fixed population" properties, such as efficiency or symmetry properties. Although these properties are usually required for problems of all cardinalities, it is worth noting that *consistency* often has the effect of "transferring" them across cardinalities, and it may suffice to impose them for two-agent problems. It is often just as natural to impose the properties for all cardinalities, but for the sake of generality, we will note here that this may not be needed. In stating the results in Part II, we have not attempted to identify the minimal assumptions that would guarantee their conclusions.

Part 2

APPLICATIONS

2.1 ABSTRACT VERSUS CONCRETE MODELS.

We now turn to applications. Together, the models described below cover a very broad range of problems commonly studied. At one extreme is our first canonical example, the class of bargaining problems; bargaining problems are specified in utility space, no information about the physical features of the alternatives among which a choice has to be made being retained. At the other extreme is our second canonical example, the class of resource allocation problems, which are specified in commodity space.

It is useful to distinguish between models on the basis of their informational content. Indeed, the amount of information available (i) is relevant to the way conflicts are resolved in practice, and (ii) has clear normative significance. The first point is strongly supported by anecdotal evidence and formal surveys: for instance, Yaari and Bar-Hillel (1984) confronted a group of subjects with several problems involving different sets of physical alternatives having a common representation in utility space. They found systematic differences in the way the problems were solved, depending on the interpretation given to these alternatives. The position that only utility information is relevant to the comparison of alternatives is termed "welfarism" by Sen (1979). Welfarism, intended as a descriptive theory, is in clear violation of these survey results.

The second point, that from a normative perspective, welfarism is inadequate as well, is being argued in a literature that is expanding fast. Roemer (1986a) refers to the position that a precise description of the concrete features of the alternatives available is necessary to their evaluation, as "resourcist". When such information is available, it can be used to enrich the class of admissible solutions, as illustrated by exchange economies; there, the set of available physical choices has a special structure (it is a convex, compact subset of a vector space); preferences can be required to satisfy properties that would not be meaningful otherwise (such as monotonicity, smoothness, convexity); finally, allocation rules can be constructed that make use of this special structure and would not be well-defined without it (an example is the Walrasian solution).

Although we recognize the usefulness of resourcist formulations, we would nevertheless like to advocate a flexible position on the issue of the relative merits of concrete and abstract models. Indeed, the advantage of abstract formulations is their wide applicability. To take just one example, the Shapley value, a solution originally defined for the abstract domain of games in coalitional form, has been very successfully applied to a whole gamut of concrete problems, from the computation of power indices in voting bodies, to cost allocation and the distribution of goods.

The great variety in the models that we will consider illustrates the wide relevance of *consistency* and *converse consistency.* These classes fall into the following broad categories.

Game theory
Bargaining
Games in coalitional form with transferable utility
Games in coalitional form without transferable utility
Games in strategic form

Public economics and cost allocation
Bankruptcy and taxation
Quasi-linear cost allocation
General cost allocation
Pricing

Fair allocation
Fair division in classical private good economies
Fair division in economies with single-peaked preferences

Fair allocation in economies with indivisible goods

Other models
Apportionment
Matching

For each of these classes, we restate the definitions of *consistency* and of its converse when applied to the class, and we present central results that have been based on the principles. We chose these results to be as representative as possible of the state of knowledge, without attempting to be uniform in the amount of detail with which we describe them. For each model, we present several results with some detail, specifying in particular the auxiliary conditions that they involve. We only give informal descriptions of other results, and we limit ourselves to references to the relevant literature for the remaining results. Inevitably, these choices of which results to emphasize reflect this author's personal tastes as well as his greater familiarity with some branches of the literature.

In order for our presentation to be as uncluttered as possible, we only give precise statements, with all the quantifications, of *consistency* and *converse consistency.* For the other properties, variables should be understood to be chosen arbitrarily in their respective domains.

We use the following notational conventions throughout. Given a group of agents $N \in \mathcal{N}$, we use notation such as $\mathcal{D}^N, \mathcal{E}^N \ldots$ for domains of problems that they may face. However, we also use N as a superscript to denote cross-products: for instance, $\mathbb{R}^N$ is our notation for the ***cross-product*** of $|N|$ copies of $\mathbb{R}$ indexed by the members of N. Similarly, in exchange economies, if $\ell \in \mathbb{N}$ is the number of goods, $\mathbb{R}^{\ell N}$ designates the cross-product of $|N|$ copies of the commodity space $\mathbb{R}^\ell_+$ indexed by the members of N. We hope that no confusion will result from this dual use of the superscript N. Given a list of objects $(A_i)_{i \in N}$, and $i \in N$, we denote by A_{-i} the list obtained from A by deleting its i^{th} component. Finally, we recall that φ is our generic notation for a solution.

2.2 GAME THEORY.

The first models of game theory that we examine, bargaining problems and coalitional form games, with and without transferable utility (Subsections

2.2.1-3), can be qualified as "abstract" since their specification only requires sets of feasible utility vectors. Later on, we consider the more "concrete" class of games in strategic form (Subsection 2.2.4). The description of these games includes information about how utility vectors result from profiles of individual choices.

2.2.1 Bargaining problems.

A typical bargaining problem, involving a group of three agents, $N = \{1,2,3\}$, is represented in Figure 2.1a: there is a feasible set T, which is a subset of the three-dimensional utility space, and a disagreement point, d, which we normalize to be 0. The points of T represent the choices available to the agents. What compromise will they reach? Nash (1950) suggested the point x maximizing over T the product of their utility gains from d. Now, let us imagine that agent 3 leaves the scene with a promise of a utility level of x_3 and let us now consider the subset of T comprising all points where his utility is x_3, namely the "slice" of T through x parallel to the $\{1,2\}$ coordinate subspace. This set, denoted by S, can be meaningfully understood as the set of options open to agents 1 and 2 if indeed agent 3 is to receive x_3. Now, we note that (x_1, x_2) is the maximizer of the product of their utility gains in S (gains from $(d_1, d_2) = (0,0)$). This equality of the payoffs chosen for agents 1 and 2 in the initial problem and in the associated reduced problem illustrates the fact that the Nash solution is *consistent*.

Another solution was introduced by Kalai and Smorodinsky (1975): it selects the maximal feasible point proportional to the ***ideal point***, the point whose i^{th} coordinate is equal to the maximal feasible utility for agent $i \in N$. For the problem T represented in Figure 2.1b, which is the convex hull of the points $(0,0,0),(1,0,0),(1,1/2,0),(0,1,0),(0,1/2,0)$ and $(0,0,1)$, the outcome this solution selects is the point $x = (1/2,1/2,1/2)$. Imagining agent 3 to leave the scene with his payoff of x_3 produces the two-person problem S involving the group $N' = \{1,2\}$. It is the convex hull of the points $(0,0),(1/2,0),(1/2,1/2)$ and $(0,3/4)$. Since its Kalai-Smorodinsky outcome, $K(S) = (3/7, 9/14)$, does not coincide with $x_{N'} = (1/2,1/2)$, the Kalai-Smorodinsky solution is not *consistent*!

We now turn to the general definitions. A ***bargaining problem*** is a pair $(S,d) \in 2^{\mathbb{R}^N} \times \mathbb{R}^N$: a group of agents $N \in \mathcal{N}$ can attain any of the points

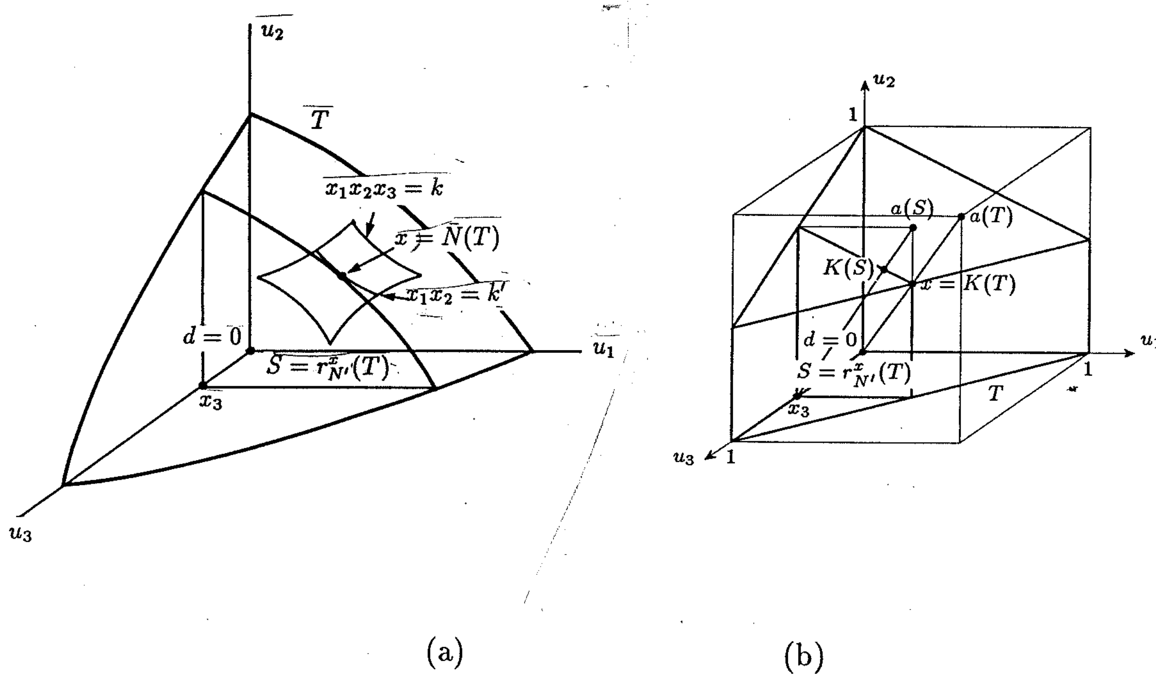


Figure 2.1: *Consistency* in bargaining theory. (a) The Nash solution $\bar{N}$ is *consistent*: $\bar{N}(T)$ is the Nash outcome of the three-person problem T. The Nash outcome of the two-person problem S, which is the slice of T through $\bar{N}(T)$ parallel to the coordinate subspace relative to the group $\{1,2\}$, coincides with the restriction of $\bar{N}(T)$ to that group. (b) The Kalai-Smorodinsky solution K is not *consistent*, since the first two coordinates of $K(T)$ do not give the Kalai-Smorodinsky outcome of the slice of T through $K(T)$ parallel to the coordinate subspace relative to the group $\{1,2\}$.

of S, the ***feasible set***, a subset of their utility space $\mathbb{R}^N$, by unanimously agreeing on it. If they fail to reach an agreement, they end up at d, the ***disagreement point***. We assume that S is convex and compact, and that there exists at least one point of S that strictly dominates d (this is a non-degeneracy assumption). We also require S to be d-comprehensive, (if a point x is feasible, then any point y such that $d \leqq y \leqq x$ is also feasible). This is a natural assmption that guarantees that the solutions that we will want to consider always select outcomes that are at least weakly Pareto-optimal (see below for a formal statement of this property). Finally, and to simplify the exposition, we assume $d = 0$ and we write S instead of $(S, 0)$. Let $\mathcal{B}^N$ be the class of problems satisfying all of the above assumptions, $\mathcal{B} = \bigcup_{N \in \mathcal{N}} \mathcal{B}^N$, and $X_{\mathcal{B}} = \bigcup_{N \in \mathcal{N}} \mathbb{R}^N_+$. A ***solution*** is a function that associates with every $N \in \mathcal{N}$ and every $S \in \mathcal{B}^N$, a unique point of S.[1]

Consistency for bargaining problems: The solution $\varphi: \mathcal{B} \to X_{\mathcal{B}}$ is ***consistent*** if for all N, $N' \in \mathcal{N}$ with $N' \subset N$, all $T \in \mathcal{B}^N$, and all $x \in T$, if $x = \varphi(T)$ and $r^x_{N'}(T) \in \mathcal{B}^{N'}$, where $r^x_{N'}(T) \equiv \{x' \in \mathbb{R}^{N'}: \exists y \in T$ with $y_{N \setminus N'} = x_{N \setminus N'}$ and $y_{N'} = x'\}$, then $x_{N'} = \varphi(r^x_{N'}(T))$.

Geometrically, $r^x_{N'}(T)$ is simply the slice of T through x parallel to the coordinate subspace $\mathbb{R}^{N'}$.

Among the bargaining solutions commonly studied, only two are *consistent.* They are the ***Nash solution***, which, given $N \in \mathcal{N}$ and $S \in \mathcal{B}^N$, selects the maximizer of the product $\prod_N x_i$ for $x \in S$, and the ***lexicographic egalitarian solution***, which selects the point of S that is maximal in the lexicographic order.[2] Now, consider a list of functions $f = (f_i)_{i \in \mathcal{I}}$, where $f_i: \mathbb{R}_+ \to \mathbb{R}$ is strictly monotone increasing, continuous, and such that for all $N \in \mathcal{N}$, the function $f^N: \mathbb{R}^N_+ \to \mathbb{R}$ defined by $f^N(x) = \sum_N f_i(x_i)$ for all $x \in \mathbb{R}^N_+$ is strictly quasi-concave. Then, given $N \in \mathcal{N}$ and $S \in \mathcal{B}^N$, let $\varphi^f(S) = argmax\{f^N(x): x \in S\}$. The ***separable additive*** solution φ^f so defined (Lensberg, 1987) also satisfies *consistency*.[3] The ***egalitarian***

[1] For a survey of the literature on the bargaining problem, see Thomson (1990b).

[2] Given x and $x' \in \mathbb{R}^n$, let $\tilde{x}$ and $\tilde{x}'$ be obtained from x and x' by rewriting their coordinates in increasing order. We say that ***x is lexicographically greater than x'*** if either $\tilde{x}_1 > \tilde{x}'_1$, or $[\tilde{x}_1 = \tilde{x}'_1$ and $\tilde{x}_2 > \tilde{x}'_2] \ldots$, or for some $k \in \{1, \ldots, n-1\}$, $[\tilde{x}_i = \tilde{x}'_i$ for all $i \leq k$ and $\tilde{x}_{k+1} > \tilde{x}'_{k+1}]$.

[3] The Nash solution is the member of this family obtained by choosing $f_i = \log$ for all $i \in \mathcal{I}$.

solution (Kalai, 1977), which selects the maximal point of S of equal coordinates, is not *consistent* but it satisfies the slightly weaker condition obtained by requiring the inequality "$x_{N'} \leqq \varphi(r^x_{N'}(T))$" instead of "$x_{N'} = \varphi(r^x_{N'}(T))$." We will call this condition ***weak consistency***. On domains of problems for which the egalitarian outcome is always Pareto-optimal (see below), the solution does satisfy *consistency*.

The egalitarian solution and its lexicographic extension are *conversely consistent*. The Nash solution and the separable additive solutions are not, but if the domain is restricted to smooth problems, they are. The Kalai-Smorodinsky solution is not.[4]

In bargaining theory, *consistency* was first used by Harsanyi (1959).[5] Harsanyi felt that the Nash solution was the appropriate solution for two-person problems and he asked whether we could deduce in some natural way how n-person problems should be solved. He showed that if a solution is *consistent* and coincides with the Nash solution for two-person problems, then it coincides with the Nash solution for all cardinalities. Lensberg (1985), who rediscovered the condition, is the author of the most general theorems involving it. In particular, he showed that Harsanyi's restrictive hypothesis for two-person problems could be replaced by elementary axioms, as now detailed. To present the results, we need to formulate a few other properties of solutions. The first one is that all gains from cooperation should be exhausted:

Pareto-optimality: If $x \geq \varphi(S)$, then $x \notin S$.

A slightly weaker condition is that there should not be a feasible outcome that all agents prefer to the solution outcome:

Weak Pareto-optimality: If $x > \varphi(S)$, then $x \notin S$.

The solution should be invariant under exchanges of the names of the agents. Given two groups N, $\tilde{N} \in \mathcal{N}$ of equal cardinalities, let π be a

[4]These examples provide an answer to a question raised in Section 1.7.3 and that is often asked: in the presence of *single-valuedness*, is there a logical relation between *consistency* and *converse consistency*? The answer is no since on the domain $\mathcal{B}$ the Nash solution is *consistent* but not *conversely consistent*, whereas the egalitarian solution is *conversely consistent* but not *consistent*. Both are *single-valued*.

[5]Under the name of the "bilateral equilibrium condition".

bijection from N to $\tilde{N}$. Given $x \in \mathbb{R}^N$, we slightly abuse notation and write $\pi(x)$ to denote the point $(x_{\pi(1)}, \ldots, x_{\pi(n)}) \in \mathbb{R}^{\tilde{N}}$, and we write $\pi(S)$ to denote the set $\{\tilde{x} \in \mathbb{R}^{\tilde{N}}: \tilde{x} = \pi(x) \text{ for some } x \in S\}$.

Anonymity: If $|N| = |\tilde{N}|$ and $\pi: N \to \tilde{N}$ is a bijection, then $\varphi(\pi(S)) = \pi(\varphi(S))$.

A positive linear rescaling, defined independently agent by agent, of their utilities, should be accompanied by a similar rescaling of the solution outcome. The function $\lambda: \mathbb{R}^N \to \mathbb{R}^N$ is such a rescaling if there exists a list $(a_i)_{i \in N} \in \mathbb{R}^N_{++}$ such that for all $x \in \mathbb{R}^N$, $\lambda(x) = (a_i x_i)_{i \in N}$.

Scale invariance: $\varphi(\lambda(S)) = \lambda(\varphi(S))$, where $\lambda: \mathbb{R}^N \to \mathbb{R}^N$ is any positive linear rescaling, defined independently agent by agent, of their utilities.

An expansion of the feasible set that is favorable to agent i, in the sense that the range of feasible utility vectors attainable by the others is unaffected but for each such vector the maximal utility attainable by agent i increases, should benefit him (notation: given $S \in \mathcal{B}^N$ and $N' \subset N$, $S_{N'}$ is the projection of S on $\mathbb{R}^{N'}$):

Individual monotonicity: If $S' \supseteq S$ and $S'_{N \setminus \{i\}} = S_{N \setminus \{i\}}$, then $\varphi_i(S') \geqq \varphi_i(S)$.

Small changes in problems (evaluated in the Hausdorff topology) should not produce large changes in solution outcomes:

Continuity: If $\{S^\nu\}$ is a sequence of elements of $\mathcal{B}^N$ such that $S^\nu \to S \in \mathcal{B}^N$, then $\varphi(S^\nu) \to \varphi(S)$.

The burden of supporting additional agents, when their arrival is not accompanied by an expansion of opportunities, should be borne by all agents initially present:

Population-monotonicity: If $N' \subset N$, $T \in \mathcal{B}^N$, $S \in \mathcal{B}^{N'}$, and $T_{N'} = S$, then $\varphi_{N'}(T) \leqq \varphi(S)$.

We are now ready to state the results. They are characterizations of the solutions that have played the major role in the literature:

Theorem 1 (Lensberg, 1988) The Nash solution is the only solution satisfying *Pareto-optimality, anonymity, scale invariance*, and *consistency.*

The following refinement of Theorem 1 is due to Lensberg and Thomson (1988): it says that *Pareto-optimality* can "almost" be dispensed with. Indeed, if a solution satisfies the remaining axioms, then either (i) or (ii) below hold:

(i) it is the Nash solution

(ii) there exists $\lambda \in [0,1[$ such that for all $N \in \mathcal{N}$, and all $S \in \mathcal{B}^N$, the solution outcome x of S has the property that for all $i \in N$, the ratio of x_i to the i^{th} coordinate of the maximal point of S, y, such that $y_{-i} = x_{-i}$, is equal to λ.[6]

There is always at least one point with these properties but there may be several unless $\lambda = 0$.[7] Also, and except in that special case, no *consistent* selection can be made from the correspondence N^λ they define. Consequently, case (ii) is in fact possible only if $\lambda = 0$, in which case the disagreement solution — this is the solution that always selects the disagreement point — results. By excluding this degenerate solution, a characterization of the Nash solution obtains.

Another refinement of Theorem 1 is offered by Lensberg (1987), who shows that its conclusion can be reached in a model in which the number of agents is bounded above (of course there should be at least three agents), provided *continuity* is added to the list of requirements imposed on the solution. Thomson (1985) argues against the simultaneous use of *consistency* and *continuity*, since the latter ignores slices (a sequence of problems may converge in the Hausdorff topology, without slices parallel to a given coordinate subspace through a given point converging), whereas slices are essential in the former. However, the characterization extends when *continuity* is weakened to a condition that recognizes slices in defining convergence.[8]

Next we have:

[6]Recall that by the notation x_{-i}, we mean the vector x from which the i^{th} coordinate has been deleted.

[7]Uniqueness is guaranteed however for bargaining problems obtained as the image in utility space of one-commodity resource allocation problems.

[8]Convergence requires that slices through points that are converging to each other also converge to each other.

Theorem 2 (Lensberg, 1985) The lexicographic egalitarian solution is the only solution satisfying *Pareto-optimality, anonymity, individual monotonicity*, and *consistency.*

Theorem 3 (Thomson, 1984) The egalitarian solution is the only solution satisfying *weak Pareto-optimality, anonymity,*[9] *continuity, population-monotonicity*, and *weak consistency.*

Theorem 4 (Lensberg, 1987) The separable additive solutions are the only solutions satisfying *Pareto-optimality*, *continuity*, and *consistency.*

A result related to Theorem 4 appears in Young (1988b). He considers a model in which agents can be replicated and obtains a characterization in a more direct way.

> **Bibliographic note.** Most of the results just quoted are presented in detail in Thomson and Lensberg (1989).

A characterization of the Kalai-Smorodinsky solution is obtained by Peters, Tijs, and Zarzuelo (1994) on the basis of the following form of *consistency*: the reduced problem of $T \in \mathcal{B}^N$ with respect to $N' \subset N$ and $x \in T$ is defined to be $\lambda T_{N'}$, where $T_{N'}$ is the intersection of T with the coordinate subspace pertaining to N', and $\lambda \in \mathbb{R}_+$ is chosen so that $x_{N'}$ is weakly Pareto-optimal for $\lambda T_{N'}$; the requirement is that if $x = \varphi(T)$ and $\lambda T_{N'}$ is non-degenerate, then $x_{N'} = \varphi(\lambda T_{N'})$.[10] A similar argument leads to a family of solutions (also found in Thomson 1990b) that includes the Kalai-Smorodinsky and egalitarian solutions. Lahiri (1994) proposes a characterization of the egalitarian solution along the same lines.

Although the study of bargaining has been conducted almost entirely under the assumption of *single-valuedness* of solutions, it is natural to wonder how much a relaxation of this requirement would enlarge the class of

[9]The requirement of *symmetry* would suffice: if a problem is invariant under ***all*** exchanges of the agents, then the solution selects a point of equal coordinates.

[10]The Kalai-Smorodinsky obviously also satisfies the weakening of *consistency* obtained by adding the hypothesis that the ideal point of the reduced problem (as we originally defined it) be proportional to the projection of the ideal point of the original problem onto the subspace pertaining to the remaining agents.

admissible solutions. Blackorby, Bossert, and Donaldson (1994) address the consistency question within this more general framework, and identify a family of solutions as being the only ones to satisfy the following conditions: *Pareto-optimality*, *anonymity*, ***homogeneity*** (the weakening of *scale invariance* obtained by making all the multiplicative coefficients equal), versions of *continuity* and ***contraction independence*** formulated for *multi-valued* solutions (*contraction independence* says that if x is the solution outcome of some problem, and x remains feasible when the problem contracts, then it should still be the solution outcome), ***connectedness*** (the requirement that the set of recommendations be a connected set), a weak ***additivity*** condition, the special form of *consistency* obtained by requiring that only the agents with the smallest payoffs be permitted to leave (Subsection 1.4.3), and finally *flexibility* (Subsection 1.7.1).[11]

The solutions they characterize are the ***single-series Gini solutions*** (Weymark, 1981), defined as follows: given a sequence of increasing weights $(a_i)_{i\in\mathcal{I}}$ in $\mathbb{R}_+$, $N \in \mathcal{N}$, and $S \in \mathcal{B}^N$, select any point x of S such that the vector $\tilde{x}$ obtained by rewriting its coordinates in decreasing order maximizes the expression $\sum_N a_i\tilde{x}_i'$ over S. By adding a requirement of ***replication-invariance*** (the requirement that the solution outcome of a replicated problem be obtained by replicating the solution outcome of the problem that is replicated), the ***single-parameter Gini solutions***, which are such that for all $i \in \mathcal{I}$, $a_i = i^\delta - (i-1)^\delta$ for $\delta \in [1, \infty[$ (Donaldson and Weymark, 1980; Bossert, 1990) become the only acceptable ones.

2.2.2 Games in coalitional form with transferable utility

Two domains of coalitional form games are traditionally considered and we discuss them in turn. First, we assume that utility is transferable; then, we dispense with the assumption. For both domains, a number of alternative notions of *consistency* have been proposed. Indeed, the reduced games can be given a variety of forms, and to each of them corresponds such a condition.

Our main results are characterizations of four solutions, the core, the nucleolus, the prekernel, and the Shapley value, as well as characterizations

[11]They also use *individual rationality* (the condition that the solution outcome dominate the disagreement point), a condition that is automatically satisfied on our domain.

of extensions and variants of these solutions. The prenucleolus, introduced in 1969, had never been axiomatized before. Similarly, the characterizations of the core and of the prekernel presented here were the first ones for these solutions. As noted by the author of these latter results, the definition of the core is so intuitive that the need for a characterization may not have been pressing. However, by comparing the characterizations of these various solutions, we gain considerable insights into what really distinguishes all of them.

Consider a group N of differently skilled agents. The productivities of the group and each subgroup $S \subset N$ depend on the complementarities between the skills of the agents composing it. They are given as the numbers denoted by $v(N)$, and $v(S)$ for $S \subset N$. This may be the output that they can jointly produce, or the value of this output at some given prices. We would like to reward agents as a function of what they can contribute to the whole group and to the various subgroups.

(i) Complement consistency. Let v be the list of all these numbers given in Table 2.1 for an example where $N = \{1,2,3\}$. A well-known method of calculating rewards is the ***core***: pick a payoff vector $x = (x_1, x_2, x_3)$ satisfying $\sum_N x_i = v(N)$ that cannot be "improved upon" by any subgroup, that is, such that for all $S \subseteq \{1,2,3\}$, $v(S) \leqq \sum_S x_i$. The vector $x = (10,10,30)$ is in the core. Now, assuming agent 3 to have accepted $x_3 = 30$, how does the situation appear to the group $N' = \{1,2\}$ of remaining agents. We will assume that the right of a coalition to break-off in the reduced game is conditional upon its involving the departing agents and giving them their promised payoffs. Therefore, if agent 1, say, were to break-off, he would get together with agent 3 and pay him his payoff of 30; this would leave him a surplus of $v(\{1,3\}) - x_3 = 30 - 30 = 0$. Similary, agent 2 secures the cooperation of agent 3 by paying him 30, for a surplus of $v(\{2,3\}) - x_3 = 40 - 30 = 10$. Finally, agents 1 and 2 together can divide among themselves the amount $v(N) - x_3 = 50 - 30 = 20$. Now, it is easy to see that the vector $(x_1, x_2) = (10, 10)$ belongs to the core of the reduced game so defined.

On the other hand, consider the solution due to Shapley (1953),[12] for which the computations are also given in Table 1. Shapley recommends

[12]The solution is defined below. It does satisfy *consistency* properties for other definitions of the reduced game, in particular for a definition involving the solution itself. This will be explained in Section 2.2.(iv).

the payoff vector $y = (70/6,\ 100/6,\ 130/6)$ for v, and $(55/6,\ 115/6)$ for the reduced game associated with the subgroup $N' = \{2, 3\}$ and y: it calls for adjustments in players 1 and 2's payoffs after the departure of agent 3.

We now state the general definitions. A ***transferable utility***, or ***TU, game in coalitional form*** is a vector $v \in \mathbb{R}^{2^{|N|}-1}$: there is a group $N \in \mathcal{N}$ of agents whose members can gather in ***coalitions***.[13] What each coalition can achieve on its own, its ***worth***, is given as one of the coordinates of v. Restrictions may be imposed on v making the game (for instance) monotonic (if $S \supset T$, then $v(S) \geqq v(T)$), or super-additive ($v(S) \geq \sum_{k \in K} v(S_k)$ for any partition $(S_k)_{k \in K}$ of S). Let $\mathcal{G}^N$ be a domain of admissible games for the group N, $\mathcal{G} = \bigcup_{N \in \mathcal{N}} \mathcal{G}^N$, and $X_{\mathcal{G}} = \bigcup_{N \in \mathcal{N}} \mathbb{R}^N$. A ***solution*** is a correspondence that associates with every $N \in \mathcal{N}$ and every $v \in \mathcal{G}^N$ a non-empty set of vectors $x \in \mathbb{R}^N$ such that $\sum_N x_i \leqq v(N)$. The i^{th} coordinate of such a vector represents one of the possible payments to agent i for being involved in the game, or an amount that he can "expect" from the game. Note that already in this model, we allow for *multi-valuedness.* Although some interesting *single-valued* solutions exist, many others are *multi-valued*, and it is desirable that they not be eliminated from consideration outright.

For all $N \in \mathcal{N}$ and all $v \in \mathcal{G}^N$, the ***core*** selects all the payoff vectors $x \in \mathbb{R}^N$ satisfying the Pareto-optimality condition $\sum_N x_i = v(N)$ and such that for all $S \subset N$, $\sum_S x_i \geq v(S)$. The ***prenucleolus*** (Schmeidler, 1969) picks the efficient vector $x \in \mathbb{R}^N$ whose associated vector of "excesses" $e(x) \in \mathbb{R}^{2^{|N|}-1}$, where for all $S \subset N$, $e_S(x) = v(S) - \sum_S x_i$, is lexicographically minimal among all excess vectors associated with Pareto-optimal payoff vectors (see Subsection 2.2.1 for a definition of lexicographic orderings). On the domain of games admitting payoff vectors that are individually rational, a property defined below, the nucleolus is defined in a similar way but so as to achieve this property. For the ***Shapley value*** (Shapley, 1953), $x_i = \sum_{S:S \subset N, S \ni i} k_S[v(S) - v(S \setminus \{i\})]$, where $k_S = [(|S|-1)!(|N|-|S|)!]/|N|!$. The vector $x \in \mathbb{R}^N$ belongs to the ***prekernel*** (Davis and Maschler, 1965), if for all i, $j \in N$, $\max_{S:S \subset N, S \ni i, j \notin S}\{e_S(x)\} = \max_{S:S \subset N, S \ni j, i \notin S}\{e_S(x)\}$. Finally, let $c_i(v) = v(N) - v(N \setminus \{i\})$ be player i's "principal contribution in v". Then, let $m_i(v) = c_i(v) + (1/|N|)[v(N) - \sum_N c_j(v)]$. This solution is known as the ***equal allocation of non-separable benefits solution.***[14]

[13] A coalition is a ***non-empty*** subset of N.

[14] See Peleg (1991) for a comprehensive treatment of the modern theory of games in

Game	The core of v contains $x =$	Reduced game of v with respect to $\{1,2\}$ and x	The core of the reduced game contains	The Shapley value of v is $y =$	Reduced game of v with respect to $\{1,2\}$ and y	The Shapley value of the reduced game
$v(\{1\})=0$	10	0	10	70/6	50/6	55/6
$v(\{2\})=0$	10	10	10	100/6	110/6	115/6
$v(\{3\})=0$	30			130/6		
$v(\{1,2\})=20$		20			170/6	
$v(\{1,3\})=30$						
$v(\{2,3\})=40$						
$v(\{1,2,3\})=50$						

Table 2.1: ***Complement consistency*** **for coalitional form games.** Example of a three-person game illustrating the fact that the core is *complement consistent* and showing that the Shapley value is not.

For our first definition of a reduced game, given $N \in \mathcal{N}$, $v \in \mathcal{G}^N$, $x \in \varphi(v)$, and finally $N' \subset N$, we imagine that each coalition $S \subset N'$ is ***required to involve all the members of $N \backslash N'$ and to pay them according to x***. Then, what is left for S is the amount $v(S \cup (N \backslash N')) - \sum_{N \backslash N'} x_i$. We define this difference to be its revised worth. The resulting reduced game and the corresponding *consistency* condition appear in Moulin (1985a). This notion corresponds quite closely to the idea expressed in the Fundamental Definition and to the conditions that have been analyzed on other domains, namely that ***all*** departing agents should receive their agreed upon payoffs. We will name this property "complement" consistency in order to distinguish it from the definitions presented below,[15] and to help ourselves remember that in evaluating the worth of a coalition, the whole complement of the remaining players has to be given satisfaction.[16]

We should note however that the definition that has been the object of the most attention is the one due to Davis and Maschler, to which we will turn next. For a comparative evaluation of these formulations as well as the numerous others that have been offered for coalition form games, and which we will also list, we should keep in mind the possible applications. For some, a particular reduced game might be appropriate but for others, it is a different reduced game that might be more natural. To be most convincing, the choice of the reduced game should be made with the context in mind. Here, we will limit ourselves to abstract models, and admittedly, at this level of generality, it is not easy to decide on the best formulation.

Complement consistency for TU coalitional form games: The solution $\varphi: \mathcal{G} \to X_{\mathcal{G}}$ is ***complement consistent*** if for all N, $N' \in \mathcal{N}$ with $N' \subset N$,

coalitional form.

[15]For readers already familiar with the existing terminology, our decision to introduce new terms may be confusing. Nevertheless, given that one of our objectives here is to help others discover a new literature, and in the face of the multitude of definitions that have been proposed for coalitional form games, we thought that terms suggestive of the operations that are performed in calculating the reduced game might be more useful.

[16]We may feel uneasy about the fact that in order to evaluate the worth of a coalition, we imagine it to cooperate with $N \backslash N'$, and that we will calculate the worths of all other (competing) coalitions under the very same assumption. It may be helpful to think of objecting coalitions as having to take the initiative. In order to upset a proposed payoff vector, the onus is on a coalition to ensure that the agents that are leaving are paid the amounts they were promised.

all $v \in \mathcal{G}^N$, and all $x \in \varphi(v)$, we have $r^x_{N'}(v) \in \mathcal{G}^{N'}$ and $x_{N'} \in \varphi(r^x_{N'}(v))$, where $r^x_{N'}(v)$, the "complement reduced game of v relative to N' and x," is defined[17] by

$$r^x_{N'}(v)(S) = v(S \cup (N \backslash N')) - \sum_{N \backslash N'} x_i \text{ for all } S \subseteq N'$$

To present the results, we will need the following additional properties of solutions. First, payoffs should add up to the worth of the grand coalition:

Pareto-optimality: $\sum_N \varphi_i(v) = v(N)$.

Each agent should be awarded at least what he can achieve on his own:

Individual rationality: $\varphi_i(v) \geqq v(\{i\})$.

Obviously, the core selects Pareto-optimal and individually rational payoff vectors. It also satisfies *complement consistency.* The characterization stated next is mainly based on this property.

Theorem 5 (Tadenuma, 1992) On the domain of games whose core is nonempty, the core is the only solution satisfying *individual rationality* and *complement consistency.*[18]

It is worth noting that the core does not satisfy the natural converse of *complement consistency* (Tadenuma, 1992), but that if (i) the payoff vector that is being evaluated is Pareto-optimal and individually rational for the original game, and (ii) its restrictions to ***all*** proper subsets of the players

[17]Moulin actually defines $r^x_{N'}(v)(N') = \sum_{N'} x_i$. In the presence of *Pareto-optimality,* stated below, this is equivalent to our definition.

[18]It may appear strange at first that the theorem makes no use of *Pareto-optimality.* This is because *individual rationality* can be seen as a "one-person" optimality condition. This property is transferred by the solution from one-person games (which are in the domain), to games of higher cardinalities by means of *complement consistency.* A similar phenomenon occurs with some of the alternative definitions of *consistency* discussed later. This transfer is an example of the phenomenon described in Section 1.7.3. Other properties of solutions are often transferred from low to high cardinalities by means of *consistency.* Then, theorems can be stated with certain properties being required only for low cardinalities (typically one-person or two-person problems).

are in the cores of the associated reduced games, then indeed it is in the core of the original game. The fact that the core satisfies this weakening of *complement converse consistency* plays an important role in the proof of Theorem 5.

To present the next result, we need to introduce two more conditions. The first one says that if two agents contribute equal amounts to all coalitions, they should be awarded equal payoffs:

Symmetry: If $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i,j\}$, then $\varphi_i(v) = \varphi_j(v)$.

A strengthening of this condition says that the solution should be invariant under exchanges of the names of the agents: let v and $\tilde{v}$ be two games involving the groups N and $\tilde{N}$ respectively.

Anonymity: If $|N| = |\tilde{N}|$ and $\pi: N \to \tilde{N}$ is a one-to-one function such that $v(S) = \tilde{v}(\{\pi(i): i \in S\})$ for all $S \subset N$, then $\varphi_i(v) = \varphi_{\pi(i)}(\tilde{v})$ for all $i \in N$.

We will also use the axiom of independence of the choice of origin for the utilities:

Zero independence: If v_1 and $v_2 \in \mathcal{G}^N$ are such that for some $b \in \mathbb{R}^N$, we have that for all $S \subseteq N$, $v_1(S) = v_2(S) + \sum_{N'} b_i$, then $\varphi(v_1) = \varphi(v_2) + b$.

Theorem 6 (Moulin, 1985a) The equal allocation of non-separable benefits solution is the only solution satisfying *Pareto-optimality*, *anonymity*, *zero-independence*, and *complement consistency.*

(ii) Max consistency. We now turn to a different definition of the reduced game, proposed by Davis and Maschler (1965). As we noted earlier, it is this definition that has been the object of the greatest attention. In the reduction of a game $v \in \mathcal{G}^N$ relative to $N' \subset N$ and $x \in \mathbb{R}^N$, the worth of a coalition $S \subset N'$ is calculated under the assumption that ***S can choose the best group of partners in N\N′*** (instead of being forced to get together with all the members of $N \backslash N'$ and pay them according to x). In the numerical example discussed above, agent 1 on his own can obtain 0; by cooperating with agent 3 and paying him x_3, he can obtain $v(\{1,3\}) - x_3 = 30 - 30 = 0$. In either case, he obtains 0. A similar computation for agent 2 involves comparing 0 and 10 (this is the difference $v(\{2,3\}) - x_3 = 40 - 30$), for a maximum of 10. Finally, agents 1 and 2 together can obtain $v(\{1,2,3\}) - x_3 = 50 - 30 = 20$. Is (x_1, x_2) in the core of the 2-person game so defined? The answer is yes. We will name this property *max consistency*, in reference to the fact that it is based on a maximization exercise.

The Shapley value selects the payoff vector $y = (70/6, 100/6, 130/6)$ for v, and $(55/6, 115/6)$ for the reduced game relative to the subgroup $N' = \{2,3\}$ associated with y in a similar way, by solving the corresponding elementary maximization exercises. Therefore, the solution is not *max consistent.*[19]

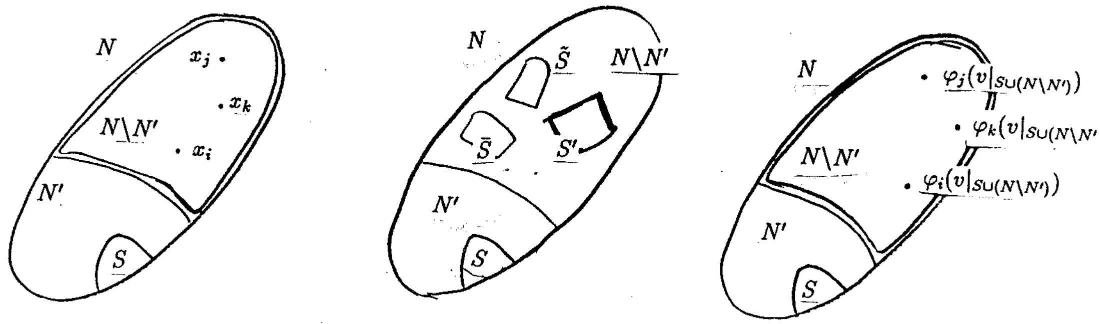
The general definition of the Davis-Maschler reduced game is illustrated in Figure 2. Given a proposed payoff vector $x \in \mathbb{R}^N$, the worth of the coalition S in ***the reduced game of v relative to $N' \subset N$ and x*** is computed under the assumption that S can secure the cooperation of any subgroup S' of N not overlapping with N', provided each member of S' receives his component of x. After these payments are made, what remains for S is the difference $v(S \cup S') - \sum_{S'} x_i$. Maximizing behavior on the part of S involves finding $S' \subseteq N \backslash N'$ for which this difference is maximal. Note that under this scenario, the worths of two distinct coalitions S_1 and S_2 may be achieved by means of cooperation with two overlapping subgroups S_1' and S_2' of $N \backslash N'$.[20]

Max consistency for TU coalitional form games: The solution $\varphi: \mathcal{G} \to X_{\mathcal{G}}$ is ***max consistent*** if for all N, $N' \in \mathcal{N}$ with $N' \subset N$, all $v \in \mathcal{G}^N$, and all $x \in \varphi(v)$, we have $r^x_{N'}(v) \in \mathcal{G}^{N'}$ and $x_{N'} \in \varphi(r^x_{N'}(v))$, where $r^x_{N'}(v)$, the "max reduced game of v relative to N' and x," is defined by

$$\begin{aligned} r^x_{N'}(v)(N') &= v(N) - \sum_{N \backslash N'} x_i \\ r^x_{N'}(v)(S) &= \max\{v(S \cup S') - \sum_{S'} x_i : S' \subseteq N \backslash N'\} \text{ for all } S \subset N' \end{aligned}$$

[19]For our particular game, these are actually the same numbers as for *complement consistency.*

[20]This may create a difficulty of interpretation, but the standard way of deriving a coalitional form game from a strategic form game is subject to the same difficulty. At least, the worth of each coalition in the reduced game is overestimated in a "uniform" way.



(a) (b) (c)

Figure 2.2: ***Reductions of a coalitional form game.*** We start from the game involving the group N, $v \in \mathcal{G}^N$. (a) **Complement consistency:** to define the worth of a coalition S in the ***reduced game of v relative to $N' \subset N$ and $x \in \mathbb{R}^N$***, we require S to get together with all the members of $N \backslash N'$ and pay them their coordinates of x. This leaves a surplus of $v(S \cup (N \backslash N')) - \sum_{N \backslash N'} x_i$. (b) **Max consistency:** alternatively, we let S cooperate with any subgroup S' of $N \backslash N'$, provided that once again it pays the members of S' their coordinates of x. This leaves the surplus $v(S \cup S') - \sum_{S'} x_i$. The coalition S is allowed to look for the best coalition S' with which to cooperate: this is the one for which the surplus is the greatest. (c) **Self-consistency:** a third formulation is to require S to get together with the whole group $N \backslash N'$, but this time to pay each of its members what ***the solution*** would recommend for them in the ***sub***problem faced by the group $S \cup (N \backslash N')$. Here, the surplus left for S is $v(S \cup (N \backslash N')) - \sum_{N \backslash N'} \varphi_i(v_{|S \cup (N \backslash N')})$.

We also have the following natural converse of *max consistency.*

Max converse consistency for TU coalitional form games: The solution $\varphi: \mathcal{G} \rightarrow X_{\mathcal{G}}$ is ***max conversely consistent*** if for all $N \in \mathcal{N}$, all $v \in \mathcal{G}^N$, and all $x \in \mathbb{R}^N$ such that $\sum_N x_i = v(N)$, and [for all $N' \subset N$ with $|N'| = 2$, $r^x_{N'}(v) \in \mathcal{G}^{N'}$ and $x_{N'} \in \varphi(r^x_{N'}(v))$], we have $x \in \varphi(v)$.

The *max consistency* of the pseudokernel is established by Davis and Maschler (1965). Other early studies of the property are due to Maschler and Peleg (1967), Maschler, Peleg, and Shapley (1972), Aumann and Drèze (1974), and Sobolev (1975).[21] It is satisfied by the core, the prenucleolus, and the prekernel.[22] The *max converse consistency* of the pseudokernel is established by Davis and Maschler (1965). The property is further studied by Peleg (1985, 1986, 1989),[23] who shows that it is satisfied by the core and the prekernel.[24]

(a) The prenucleolus. We will start by presenting a characterization of the prenucleolus. It involves one auxiliary condition not introduced yet. Consider situations where the worth of a coalition is obtained by adding up the utilities of its members at one of its available undominated alternatives. The condition says that the choice of utility scales within a certain class should be irrelevant. Specifically, the multiplication by a common positive

[21]These authors use the phrase "reduced game property".

[22]These facts are established by Aumann and Drèze (1974). Other solutions, which we will not define, are *max consistent*: the pseudokernel, the pseudonucleolus, the pseudobargaining set. On the other hand, the von-Neumann-Morgenstern solution is not *max consistent*, as shown by Chang (1988) and contrary to a claim made by Aumann and Drèze (1974). Chang and Kan (1992) provides additional results on the issue. Dutta, Ray, Sengupta, and Vohra (1989) propose a definition of the bargaining set under the name of "consistent" bargaining set, but they use the term in a different sense. As they show, their solution does not satisfy the *consistency* property studied here. Nor does it satisfy any of the other notions defined below.

[23]Under the name of "converse reduced game property."

[24]It is useful to note that the worth of a coalition in the complement reduced game is never greater than its worth in the max reduced game. Therefore, if a payoff vector meets the no-blocking conditions in the complement reduced game, it automatically meets the no-blocking conditions in the max reduced game. Conversely of course, the hypotheses of *max converse consistency* are harder to satisfy than the hypotheses of *complement converse consistency.*

constant of all the utilities and the addition of arbitrary constants to each of the utilities affect the payoffs in the same way:[25]

Homogeneity: If there exist $\alpha \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}^N$ such that for all $S \subset N$, $w(S) = \alpha v(S) + \sum_S \beta_i$, then $\varphi(w) = \alpha\varphi(v) + \beta$.

Theorem 7 (Sobolev, 1975) The prenucleolus is the only *single-valued* solution satisfying *Pareto-optimality, anonymity, homogeneity*, and *max consistency.*[26]

The follow-up literature has been considerable and we limit ourselves to a short summary. First, we note that if an upper bound greater than 3 is placed on the number of potential agents, solutions other than the nucleolus satisfy the properties of Theorem 7, as demonstrated by Gurvich, Menshikov, and Menshikova (1992). On the other hand, by working with a special class of games, and imposing the requirement that the solution be a subsolution of the solution known as the "least core", Maschler, Potters, and Tijs (1992) obtain a characterization of the nucleolus in a considerably more direct way than Sobolev.

A characterization of the nucleolus itself on the basis of a weakening of *max consistency* is provided by Potters (1991) on the domain of games for which there are efficient vectors meeting the individual rationality constraints. He applies the condition only to a pair of a payoff vector and a subgroup of players such that the restriction of that payoff vector to that subgroup is individually rational in the associated max reduced game. He proves that on the domain he considers, if a solution satisfies this ***weak*** *max consistency*, in addition to *Pareto-optimality*, *individual rationality*, *single-valuedness*, *anonymity*, *homogeneity*, and a certain "limit" condition which is somewhat too technical to describe here, then it coincides with the nucleolus on the subclass of games for which the prenucleolus happens to be individually rational. This subclass contains all games with a non-empty core and

[25]This condition is often referred to as "strategic equivalence" or "relative invariance under strategic equivalence" (RISE). Hart and Mas-Colell (1989) call it "TU-equivalence". We have chosen the term that is used in the theory of bargaining.

[26]Sobolev does not have an optimality condition as he uses a slightly different definition of the reduced game.

all games whose 0-normalization is weakly monotonic.[27]

The alternative weakening of *max consistency* obtained by restricting its application to subgroups consisting of agents for whom the payoff vector under evaluation meets strictly the individual rationality constraints of the original game, is also satisfied by the nucleolus.

Another characterization of the nucleolus, based on a slight redefinition of the reduced game, is offered by Snijders (1991): indeed, *individual rationality* can be recovered simply by changing, in the Davis-Maschler definition of the reduced game relative to a payoff vector x, the worth of each one-player "coalition" to the minimum of two numbers, (i) the payoff to the player at x and (ii) what this worth would be according to the original definition. This weakens the individual rationality constraints in the reduced game in such a way that given any individual rational and Pareto-optimal payoff vector, the set of payoff vectors having these properties in any reduced game associated with it is non-empty (using the language of Section 1.3, we can say that the domain of games is closed under Snijders' reduction operation for the individual rational and Pareto solution, whereas the domain is not closed under the Davis-Maschler reduction operation for the nucleolus). The other properties are as in Sobolev's paper (Theorem 7).

Sobolev (1995) considers a more general domain of problems consisting of a pair of a game and a set of individual rationality constraints. He imposes axioms on solutions defined on such extended games. In addition to the standard requirements of *anonymity*, *homogeneity*, and a *max consistency*-type condition, he imposes a ***boundedness condition*** relative to the individual constraints, which says that the solution outcome should be bounded below by the vector of individual constraints, and bounded above by the vector whose i^{th} coordinate is the difference between the worth of the grand coalition and the sum of the other agents' individual constraints; an ***independence condition with respect to tightening of the individual constraints***, which says that if the individual constraints increase but remain below the solution outcome of some initial game, then the solution outcome should be unchanged; and an ***independence condition with respect to a maximum operation*** performed on games and the individual constraints. He

[27]A 0-normalized game is one for which the worths of one-player coalitions are equal to 0. A weakly monotonic game is a game v such that for any pair of coalitions S, T with $S \subset T$, $v(S) + \sum_{T \setminus S} v(i) \leqq v(T)$.

establishes the uniqueness of a solution satisfying all of these conditions. This solution coincides with the prenucleolus when the individual constraints are set equal to $-\infty$ and with the nucleolus when they are set equal to the worths of the one-person coalitions.

Sudhölter (1993) proposes an alternative definition of the nucleolus, under the name of "modified nucleolus". This definition is based on a lexicographic minimization of the vector of differences, for pairs of coalitions, of their excesses. He then formulates a corresponding notion of *max consistency* that is satisfied by the solution, and bases a characterization of it on this definition.

(b) The core. Another important theorem involving *max consistency* is stated next. It involves the requirement that the set of recommended payoff vectors of the sum of two games should contain all the sums of recommended payoff vectors of each of the games:

Super-additivity: $\varphi(v_1 + v_2) \supseteq \varphi(v_1) + \varphi(v_2)$.

Theorem 8 (Peleg, 1986a) On the domain of games whose core is non-empty, the core is the only solution satisfying *individual rationality, super-additivity,* and *max consistency.*

This result should be compared to that stated as Theorem 5. In particular, note that *super-additivity* is not used there.[28]

A characterization of the core of "market games" along the lines of Theorem 8 appears in Peleg (1989, 1992). A "market game" is a game whose core is non-empty and all of whose subgames also have non-empty cores (Shapley and Shubik, 1969). The phrase is motivated by the fact that games associated with exchange economies in a natural way have that property. The games are also known as "totally balanced" games. It turns out that the domain of market games is not closed under the reduction operation, and as a result, the core is not *max consistent* on that domain. However, when the reduction is to two-person groups, total balancedness amounts to balancedness and the difficulty disappears. Let us call *unilateral and bilateral max consistency* the weakening of *max consistency* obtained by requiring the subgroups to have either one member or two members. This property, which is satisfied by the core on the domain of market games, retains enough strength to allow

[28]Moreover, it holds as soon as the number of potential agents is 3 or more.

another characterization of the solution, provided two additional properties are brought in: the first one is ***weak symmetry***, which says that for a two-person game in which the worths of the one-person coalitions are equal, the set of solution outcomes should be globally symmetric, and the second one is *max converse consistency*:

Theorem 9 (Peleg, 1989, 1992) On the domain of market games, the core is the only solution satisfying *individual rationality*, *weak symmetry*, *super-additivity*, *unilateral and bilateral max consistency*, and *max converse consistency.*[29]

Peleg shows that in Theorem 9, *super-additivity* can be replaced by the requirement that the solution coincides with the core for the two-person case.

> **Bibliographic note.** Results related to Theorem 5 appear in Tadenuma (1989), and results related to Theorem 7 in Suematsu (1988). Peleg (1986a) proves a counterpart of Theorem 8 for games with a coalition structure. Shimomura (1994) offers a characterization of the ϵ-core (obtained by requiring that the no-blocking conditions defining the core be met up to an ϵ). The characterization is based on a version of *max consistency* relative only to two-person subgroups and a converse relative to connected graphs (Subsection 1.6.3). Aumann and Drèze (1974) prove that the core is *flexible* with respect to the max reduced game (Subsection 1.7.1): for all $v \in \mathcal{G}^N$, all $x \in C(v)$, all $N' \subset N$, and all $y \in C(r^x_{N'}(v))$, we have $(y, x_{N \setminus N'}) \in C(v)$.

(c) The prekernel. The following result is the only existing characterization of the prekernel:

Theorem 10 (Peleg, 1986a) The prekernel is the only solution satisfying *Pareto-optimality, homogeneity, symmetry, max consistency,* and *max converse consistency.*

[29] The original theorem in Peleg (1989) does not have *weak symmetry* and it is not known whether this axiom is independent of the others.

Does the kernel itself satisfy an interesting version of *consistency*? Here, the problem is far from being completely understood but Potters (1991) establishes the following facts concerning the issue: on the domain of games whose 0-normalization is monotonic, the kernel satisfies *Pareto-optimality, individual rationality, symmetry, max converse consistency*, the two variants of *max consistency* presented earlier in connection with his characterization of the nucleolus (see the paragraphs following Theorem 7) and the *limit* property that he also uses in that characterization; these properties might eventually provide the basis for a complete characterization. We also note that the kernel satisfies Snijders (1991)'s version of *max consistency.*

Peleg (1989) gives an axiomatization of the intersection of the core and the prekernel along the lines of Theorem 9 by strengthening *weak symmetry* to *symmetry*, and imposing *homogeneity* instead of *super-additivity.*

Sudhölter (1993) offers a characterization of a notion of "modified kernel", analogous to his "modified nucleolus". This characterization uses his notion of *max consistency* as well as a similarly defined condition of *converse consistency* (again, see the discussion following Theorem 7). The axioms he uses are otherwise as in Theorem 10.

(iii) Other reduced games that do not depend on the solution. Note that in each of the definitions of the reduced games examined up to now, the solution itself does not appear. Before turning to a proposal for which the solution does appear, we list the remaining definitions that are independent of the solution.

In the first definition, proposed by Ruiz, Valenciano, and Zarzuelo (1996), the worth of a coalition S in the reduced game of v relative to x and $N' \subset N$ is the ***simple average*** (instead of the maximum) of the surpluses $v(S \cup S') - \sum_{S'} x_i$ when S' ranges over the subsets of $N \backslash N'$. They show that there exists a unique *single-valued* solution satisfying *Pareto-optimality, symmetry, homogeneity*, and this condition. It is the solution that selects the minimizer over the set of efficient vectors of the variance of excesses.[30] This solution is also obtained as follows. First, define a "power index" to be a function that associates with every game $v \in \mathcal{V}^N$ a vector in $\mathbb{R}^N$ whose i^{th} coordinate is interpreted as the "power" of player i: for the ***Banzhaf index***

[30]They refer to it as the least-square prenucleolus. Note that it is not based on a lexicographic operation as the prenucleolus is.

(Banzhaf, 1965; Owen, 1982), the power of a player is the simple average of his contributions to all coalitions not containing him.[31] The coordinates of the Banzhaf vector need not add to the worth of the grand coalition. To obtain *Pareto-optimality*, and therefore a well-defined solution as we understand this term, Hammer and Holzman (1987) suggest adding a common constant to each agent's coordinate of the Banzhaf index. This is the solution to which Ruiz, Valenciano, and Zarzuelo arrive. These authors (1995) also define a family of solutions, under the name of "least square values", generalizing their earlier proposal by means of weights assigned to the various coalitions. They formulate a notion of *consistency* involving weighted averages of contributions, and derive a characterization of a subfamily (a certain "consistency" property of the weights has to hold) in the spirit of their earlier theorem. Dragan (1996) proposes another characterization in which the solution that is applied to the reduced game may differ from the solution used for the initial game. Then *consistency* is a property of the pair.

In the second contribution, due to Nagahisa and Yamato (1992), the options open to the remaining agents are even more limited than they were according to the earlier definitions. When the members of $N \backslash N'$ leave, no cooperation with them is possible anymore but the commitment to their payoffs has to be honored by the grand coalition N' in the reduced game: its worth is changed to the maximum of its original value $v(N')$ and the difference of the worth of the grand coalition in the original game and the sum of the payoffs promised to the agents who left, namely $v(N) - \sum_{N \backslash N'} x_i$; the worth of each proper subcoalition is what it was in the original game. Because the reduced game is almost a subgame, we will designate it by the perhaps awkward phrase of "projected" reduced game, and to the condition as *projection consistency*. This definition is analyzed by Nagahisa and Yamato, who base on it yet another characterization of the core: on the domain of games whose core is non-empty, the core is the only solution satisfying *Pareto-optimality* and *projection consistency*.

If the set of potential players is finite, a characterization of the core is obtained by adding either one of the following two conditions. One of them is a converse of the *projection consistency*, which we will name ***weak projection converse consistency***: if a payoff vector is Pareto-optimal for some game and its restriction to each proper subset of the players (not just

[31]Recall that the Shapley value is obtained as a ***weighted*** average of contributions.

subsets of size two) is a solution outcome of the associated projected reduced game, then it is one of the payoff vectors recommended by the solution for the original game. The other is ***anti-monotonicity*** (Keiding, 1986): if the worth of every coalition, except that of the grand coalition, increases, the set of solution payoff vectors should not enlarge. The results are that on the domain of games whose core is non-empty, the core is the only solution satisfying *Pareto-optimality*, *projection consistency*, and *weak projection converse consistency*. Also, it is the only solution defined on that domain satisfying *Pareto-optimality*, *projection consistency*, and *anti-monotonicity*. Nagahisa and Yamato also provide a detailed study of the logical relations between their conditions and the conditions used by Peleg in his characterization of the core based on *max consistency* (Theorem 8).

Projection consistency can be understood as a particular case of a general notion formulated by Maschler, Potters and Tijs (1992). These authors propose a definition in which different coalitions may be made to play different roles and base on it a characterization of the nucleolus. One of their main axioms is the important restriction that the solution be a subsolution of the least core.

In the light of the various characterizations of the core based on some notion of *consistency*, it is natural to ask whether they can all be obtained as special cases of a general theorem. Funaki and Yamato (1994) provide a positive answer. They propose a class of reduced games, show that the core satisfies each of the corresponding *consistency* notions, and establish a number of uniqueness results based on their general definition and standard auxiliary conditions. Theorems 5 and 8 are special cases of their general characterization.

Driessen and Funaki (1993) consider a family of solutions obtained by first defining the notion of "an agent's individual contribution", and then distributing evenly among all players the surplus over the sum of their individual contributions. For each of four possible specifications of individual contributions, they state a form of the reduced game such that the implications of the resulting *consistency* condition can be completely described, when imposed together with standard conditions.

Sobolev (1977) and Driessen (1991) formulate definitions of the reduced game with respect to which the Shapley value is *consistent*.[32] Sobolev shows that the Shapley value is the only solution satisfying his *consistency* condition together with *Pareto-optimality*, *symmetry*, and *invariance*. R. Lee (1992b) characterizes the Shapley value on the basis of Driessen's proposal: it is the only solution to satisfy it together with *symmetry* and *homogeneity*.

Driessen (1992) characterizes a value proposed by Tijs (1981) by means of yet a different *consistency* condition.

Funaki (1995) characterizes solutions to dual games with the help of duals of the standard axioms and of *consistency* conditions.

> **Bibliographic note.** Peleg (1990, 1992) reviews the characterizations of the core developed up to 1989.

(iv) Self-consistency. A notion of *consistency* quite different from any of the ones examined until now was introduced by Hart and Mas-Colell (1988, 1989). The scenario underlying their definition of the reduced game is as follows. Let φ be a *single-valued* solution and $v \in \mathcal{G}^N$ be given. In order to compute the worth of a coalition $S \subseteq N'$ in the reduced game of v relative to $N' \subset N$ ***and the solution***, we require S to cooperate with the complement of N'. This cooperation is assumed to be feasible provided the members of $N \backslash N'$ are paid what φ would recommend for them in the ***subgame*** of v faced by the group $S \cup (N \backslash N')$.[33] That is, we imagine the disappearance pure and simple of the group $N' \backslash S$ and we ask: what should S receive in total in the subgame $v_{S \cup (N \backslash N')}$? The suggested answer is the difference $v(S \cup (N \backslash N')) - \sum_{N \backslash N'} \varphi_i(v|_{S \cup (N \backslash N')})$. Note that for this calculation to be meaningful, we have limited ourselves to *single-valued* solutions. *Multi-valued* solutions could

[32]Only reduced games obtained by imagining the departure of a single agent are considered: given $N \in \mathcal{N}$, $v \in \mathcal{G}^N$, $i \in N$, and $x \in \mathbb{R}^N$, let $N' = N \backslash \{i\}$. Sobolev defines the reduced game of v relative to N' and x by setting the worth of a coalition $S \subset N'$ equal to $(|N|-1)^{-1}[|S|(v(S \cup \{i\}) - x_i) + (|N| - |S| - 1) - 1)v(S)]$, whereas Driessen sets the worth of the grand coalition N' equal to $v(N) - x_i$, and the worth of any other coalition $S \subset N'$ equal to $(|N|-1)^{-1}[(|N|-|S|-1)(v(S) - v(N \backslash S)) - |S|x_i]$. Note that, in contrast with the Hart-Mas-Colell definition, neither one of the reduced games involves the solution itself.

[33]Note the conceptual difference between a subgame and a reduced game. A subgame of a game v relative to a subset N' of N is simply the restriction of the vector v to all of the subsets of N'.

be accommodated, but at the price of some arbitrariness: among the possible values of the sum $\sum_{N\backslash N'} x_i$ when x is chosen in $\varphi(v|_{S\cup(N\backslash N')})$, which one should we take? (Dutta (1990) discusses this issue.) Also, we emphasize that φ itself is used in specifying the reduced game. This dependence is what motivates our term of "self"-consistency to designate the property[34] and our notation $r^{\varphi}_{N'}(v)$ for the reduced game (Figure 2.2).

Once the reduced game is defined, the requirement of *consistency* is as before, namely $\varphi(r^{\varphi}_{N'}(v)) = \varphi_{N'}(v)$: the members of the subgroup N' receive the same payoffs in the original game as they do in the reduced game.

Self-consistency for TU coalitional form games: The *single-valued* solution $\varphi: \mathcal{G} \rightarrow X_{\mathcal{G}}$ is ***self-consistent*** if for all N, $N' \in \mathcal{N}$ with $N' \subset N$, and all $v \in \mathcal{G}^N$, we have $x_{N'} = \varphi(r^{\varphi}_{N'}(v))$, where $r^{\varphi}_{N'}(v)$, the "self-reduced game of v relative to N' and φ," is defined by

$$r^{\varphi}_{N'}(v)(S) = v(S \cup (N\backslash N')) - \sum_{N\backslash N'} \varphi_i(v|_{S\cup(N\backslash N')}) \text{ for all } S \subseteq N'.$$

In conjunction with several elementary conditions already stated, *self-consistency* leads to the Shapley value and to nothing else:

Theorem 11 (Hart and Mas-Colell, 1989) The Shapley value is the only solution satisfying *Pareto-optimality*, *symmetry*, *homogeneity*, and *self-consistency*.[35]

We emphasize that in defining the worth of a coalition $S \subseteq N'$ in a reduced game, ***all*** members of $N\backslash N'$ are involved.

The members of S do not have the option of looking for the best subset of $N\backslash N'$ to cooperate with (this is in contrast with the Davis-Maschler reduced game). Hart and Mas-Colell also consider a formulation of the reduced game

[34]Aumann and Maschler (1985) use this expression in the sense of what we call *consistency* in this survey.

[35]It is sufficient to impose the first three properties for two-person games. The result actually holds for a solution defined on a domain consisting of a single game and all of its subgames. By dropping *symmetry*, and imposing instead the requirement that the solution be monotonic with respect to the worth of the grand coalition (see below our formulation for the non- transferable utility case), Hart and Mas-Colell also characterize weighted generalizations of the Shapley value.

in which each group $S \subset N'$ is allowed to search for a best group of partners S', the members of S' being paid the amounts they would get by applying the solution φ to the subgame relative to $S \cup S'$. This alternative reduced game w is defined by:

$$w(S) = \begin{cases} v(N) - \sum_{N \setminus N'} \varphi_i(v) & \text{if } S = N' \\ \max_{S' \subset (N \setminus N')} [v(S \cup S') - \sum_{S'} \varphi_i(v|_{S \cup S'})] & \text{if } S \subset N' \end{cases}$$

However, as they show, there is no solution satisfying *consistency* with respect to this reduced game together with *Pareto-optimality*, *symmetry*, and *homogeneity.*

Bibliographic note. Driessen (1991) compares a number of the results based on the properties of *consistency* for coalitional form games just discussed (see also Maschler , 1990).

The two contributions described next concern models in which the role of coalitions is described in richer detail. Winter (1992) considers games with a coalition structure, that is, games where a partition of the players is given a priori. The two principal solutions for this class of games are a generalization of the Shapley value due to Aumann and Drèze (1974), and a solution introduced by Owen (1977). The main difference between them has to do with the way *Pareto-optimality* is specified: in the first case, it says that for each coalition in the partition, the members of that coalition together receive its worth; in the second case, it says that the worth of the grand coalition is divided among all the players. Winter defines two self-reduced games relative to the partition and shows that the resulting *self-consistency* conditions can help distinguish between the two solutions. He patterns his characterization of the ***Aumann-Drèze value*** after the Hart-Mas-Colell characterization of the Shapley value (Theorem 11). The characterization of the ***Owen value*** that he offers also relies on an axiom that essentially says that the computation of payoffs can be carried out in two stages: in the first stage, coalitions in the partition behave as players, and in a second stage the members of each of these coalitions divide what the coalition has received.

In the second contribution, due to Derks and Peters (1993), certain coalitions are made to play a special role, which is reflected in the formulation of the axioms: a coalition ***S is effective for a solution*** φ if given any two games that differ only in the worth of S, the solution chooses two different payoff vectors. The coalition is ***ineffective for*** $\boldsymbol{\varphi}$ if under the same assumption the resulting payoff vectors are always the same. They impose two conditions on the class of effective coalitions; that each coalition be either effective or ineffective, and that the class be closed under unions. The optimality requirement takes the form that the sum of the payoffs be equal to the worth of a maximally effective coalition. Finally, they formulate counterparts for elementary games of the *Pareto-optimality* and *symmetry* requirements that take into account the effectiveness structure. They search for solutions that satisfy the weakening of *self-consistency* obtained by applying it only to reduced games relative to effective coalitions. Their main result is a characterization of a family of extensions of the Shapley value that differ from it in that the contributions of the players are calculated in terms of maximal effective coalitions.

Joosten, Peters and Thuijsman (1994) introduce a family of solutions designed to provide certain payoff lower and upper bounds that are to be respected independently of their actual contributions,[36] and they define a version of *self-consistency* that their solutions satisfy. The solutions are convex combinations of the Shapley value and the solution that divides the worth of the grand coalition equally among all players. They also offer a charaterization of the family on the basis of this condition.

Dragan (1995) obtains a characterization of the Banzhaf index (see above) by means of a *consistency* property relative to a self-type reduced game. The other axioms are the ones used by Hart and Mas-Colell. In the version of *self-consistency* formulated by Ruiz (1995), the worth of each coalition in the reduced game is calculated as if it were a single player. He too obtains a characterization of the Banzhaf index along the lines of Theorem 11.

Finally, Evans (1996) characterizes the Shapley value by means of a *consistency* notion based on applying the standard solution (see below) to all two-person subgames in which the two "players" are (i) an arbitrary coalition and (ii) its complement.

> **Bibliographic note.** McLean and Sharkey (1993) use *self-consistency* in a study of pricing mechanisms. We will come back

[36]The solutions do not satisfy the "dummy condition", which says that any agent whose contributions to all coalitions is zero receives a zero payoff.

to their contribution later as it pertains to a different model (see Subsection 2.3.4.)

(v) Compatibility of *max consistency* and *self-consistency*. A natural question concerns the compatibility of *max-consistency* and *self-consistency*. Are there solutions satisfying both properties? An answer is given by Dutta (1990). On the domain of "convex games", (that is, games for which a player contributes more to each coalition S than to any subcoalition that S contains), he identifies a solution that has both properties. It is the solution, introduced by Dutta and Ray (1989), that selects the Lorenz maximal element in the core. For the two-person case, it picks, among the Pareto-optimal points meeting the individual rationality constraints, the one whose coordinates are the closest to being equal, the ***constrained egalitarian solution*** outcome. Note that the solution does not satisfy *homogeneity* (see the definition preceding Theorem 7) although it satisfies the weaker requirement obtained by requiring equality of all the "translation" coefficients β_i appearing in this definition. The solution is the only one to always select that point in the two-person case and to satisfy either one of the *consistency* conditions. It is also *max conversely consistent*, but this property is not imposed as one of the axioms in either characterization.

Theorem 12 (Dutta, 1990) On the domain of convex games, the solution defined by selecting for every game the Lorenz-maximal element of the core is the only solution to coincide with the constrained egalitarian solution in the two-person case, and to satisfy *max consistency*. The same conclusion holds if *max consistency* is replaced by *self-consistency*.

Supppose now that for the two-person case we require that the solution coincides instead with the ***standard solution***: this is the two-person solution defined by dividing equally among both agents the surplus above the individual rationality levels. Then, even on the domain of convex games, the two *consistency* conditions cannot be met jointly. In fact, on that domain, no solution satisfies ***strictly monotonicity with respect to the individual rationality levels*** in the two-person case, *weak homogeneity*, and the two *consistency* notions.

2.2.3 Games in coalitional form without transferable utility

We now turn to a richer model in which what each coalition S can achieve is given as a ***subset*** $V(S)$ of its utility space $\mathbb{R}^S$, $V(S)$ being required to satisfy certain regularity properties. These games are called ***NTU*** (non-transferable utility) games, as opposed to the TU (transferable utility) games described earlier. Let $\mathcal{H}^N$ be a class of admissible NTU games involving the group N, $\mathcal{H} = \cup_{N \in \mathcal{N}} \mathcal{H}^N$, and $X_{\mathcal{H}} = \cup_{N \in \mathcal{N}} \mathbb{R}^N$. A ***solution*** on $\mathcal{H}$ associates with every $N \in \mathcal{N}$ and every $V \in \mathcal{H}^N$ a non-empty subset of $V(N)$.

Most of the definitions of reduced games and the associated *consistency* conditions presented in the previous section have been extended to the NTU case.

(i) Complement consistency. We start with the extension of the definition based on requiring that when a group of players leave, and in evaluating the worth of a coalition in the resulting reduced game, the coalition be obliged to get together with the coalition consisting of all of the departing players, and to pay each of them his promised payoff:

Complement consistency for NTU coalitional form games: The solution $\varphi: \mathcal{H} \to X_{\mathcal{H}}$ is ***complement consistent*** if for all N, $N' \in \mathcal{N}$ with $N' \subset N$, all $V \in \mathcal{H}^N$, and all $x \in \varphi(V)$, we have $r^x_{N'}(V) \in \mathcal{H}^{N'}$ and $x_{N'} \in \varphi(r^x_{N'}(V))$, where $r^x_{N'}(V)$ is the game defined by

$$r^x_{N'}(V)(S) = \{y \in \mathbb{R}^S : (y, x_{N \setminus N'}) \in V(S \cup (N \setminus N'))\} \text{ for all } S \subseteq N'$$

The following result is an exact counterpart of Theorem 5. It extends to the class of NTU games with a non-empty core the characterization of the core given as Theorem 5 for the TU case.

Theorem 13 (Tadenuma, 1992) On the domain of NTU games with a non-empty core, the core is the only solution satisfying *individual rationality* and *complement consistency.*

Extensions of Theorem 13 to games with coalition structures appear in Tadenuma (1989). A class of games "dual" to the class of games with a non-empty core is the class of games for which there exists an efficient payoff vector at which each coalition gets no more than its worth. The set of payoff vectors with that property is the ***anticore*** of the game. On the class of games with a non-empty anti-core, Tadenuma (1992) proposes a characterization of the anticore dual to his characterization of the core.

(ii) Max consistency. The next definition extends the Davis-Maschler proposal for the TU case based on a maximization operation. The axiom was first used by Greenberg (1985).

Max consistency for NTU coalitional form games: The solution $\varphi : \mathcal{H} \to X_{\mathcal{H}}$ is ***max consistent*** if for all N, $N' \in \mathcal{N}$ with $N' \subset N$, all $V \in \mathcal{H}^N$, and all $x \in \varphi(V)$, we have $r^x_{N'}(V) \in \mathcal{H}^{N'}$ and $x_{N'} \in \varphi(r^x_{N'}(V))$, where $r^x_{N'}(V)$ is the game defined by

$$r^x_{N'}(V)(N') = \{y \in \mathbb{R}^{N'}: (y, x_{N\setminus N'}) \in V(N)\}$$

$$r^x_{N'}(V)(S) = \cup_{S' \subset N\setminus N'}\{y \in \mathbb{R}^S: (y, x_{S'}) \in V(S \cup S')\} \text{ for all } S \subset N'$$

We have the following characterization of the core. Note that, in contrast with the TU case, the auxiliary axiom it involves is the same as in the characterization based on *complement consistency* (Theorem 13).

Theorem 14 (Peleg, 1985) On the domain of strictly comprehensive NTU games with a non-empty core, the core is the only solution satisfying *individual rationality* and *max consistency.*[37]

> **Bibliographic note.** Moldovanu (1989) considers a *flexibility* notion (Subsection 1.7.1) based on the max reduced game. He shows that on the domain of NTU games whose core is non-empty, the core is not what can therefore be called *max flexible* (in contrast to the TU case). However, he identifies a weakening of the property that is satisfied by the core.

Moldovanu and Winter (1990) offer a slightly different definition of *max consistency* and use it to characterize a solution[38] introduced by Albers

[37]On the class of games involving at most k players, where $k \geq 3$, a characterization of the core is obtained if *max converse consistency* is added.

[38]The term is used in a different sense here, as feasibility is not required of the payoff vectors.

(1979), Bennett (1983), and Bennett and Zame (1988), and which also appears in a non-cooperative model studied by Selten (1981). The "stable demand correspondence" is defined as follows. Given $N \in \mathcal{N}$, and a game $V \in \mathcal{H}^N$, the vector $x \in \mathbb{R}^N$ is ***semi-stable*** if (i) for every player $i \in N$, there is a coalition S to which agent i belongs and such that "S can afford x," in the sense that the restriction of x to $\mathbb{R}^S$, x_S, belongs to $V(S)$, and (ii) for no coalition $S \subseteq N$, x_S belongs to the interior of $V(S)$. The number x_i is interpreted as a demand made by player i for his participation. The first condition says that each player is part of a coalition that can meet the demands of all of its members, and the second condition states that demands are maximal. The payoff vector is ***stable*** if it is semi-stable and in addition, for any pair of players $i, j \in N$, it is not the case that the set of coalitions containing i that can afford x is a proper subset of the set of coalitions containing j that can afford x. The solutions associating with every game (i) its set of stable demand vectors on the one hand, and (ii) its set of semi-stable demand vectors on the other, satisfy both definitions of *consistency* and *converse consistency* formulated by Moldovanu and Winter. Their characterization of the stable demand solution involves in addition the requirement that in the two-player case, the solution coincides with the core if the core is not-empty, and the vector of individual rational levels otherwise. For the semi-stable solution, this last requirement is changed to the requirement that the solution coincides with the interior of the core if this set is non-empty. These results extend earlier work by Winter (1989) for the TU case.

(iii) Other reduced games that do not depend on the solution. Nagahisa and Yamato (1992) extend their notion of *consistency* to the class of NTU games and prove exact counterparts of their results for the TU case: in particular, on the domain of NTU games with a non-empty core, the core is the only *Pareto-optimal* and *projection consistent* solution.[39]

(iv) Self-consistency. The notion of a reduced game proposed by Hart and Mas-Colell can also be extended to the NTU case:[40] given N, $N' \in \mathcal{N}$ with

[39] Also, if the number of potential players is finite, the core is the only solution satisfying either *Pareto-optimality, projection consistency*, and *weak projection converse consistency*, or *Pareto-optimality, projection consistency*, and *anti-monotonicity*, these conditions being the straightforward extensions of the conditions Nagahisa and Yamato use for the TU case.

[40] Hart and Mas-Colell credit Owen and Maschler for the definition.

$N' \subset N$, and $V \in \mathcal{H}^N$, we define the "self-reduced game of v relative to N' and φ" by setting for every $S \subseteq N'$,

$$W(S) = \{x \in \mathbb{R}^S : (x, (\varphi_i(V|_{S\cup(N\backslash N')}))_{i\in N\backslash N'}) \in V(S \cup (N\backslash N'))\}$$

To present the next result, we need to introduce an additional solution. Given a list $(w_i)_{i\in\mathcal{I}}$ of positive weights, the ***Kalai-Samet w-egalitarian solution*** is defined by successive distributions of dividends as follows: given $N \in \mathcal{N}$ and $V \in \mathcal{H}^N$, each agent $i \in N$ starts with an initial dividend equal to $\max\{x_i : x_i \in V(\{i\})\}$. Given a coalition $S \subseteq N$ and $i \in S$, the dividend d_{iS} received by agent i from his membership in the coalition S is obtained by distributing among all members of S the surplus over and above the vector of accumulated dividends of all subcoalitions, the distribution being effected proportionally to their weights $(w_i)_{i\in S}$. Precisely, $d_{iS} = w_i t$, where $t = max\{t' : (\sum_{S':S'\supset S, S'\ni i} d_{iS'})_{i\in S} + t'w \in V(S)\}$.

We will also need one more property of solutions. It says that an enlargement of the feasible set relative to the grand coalition, the other components of the game being kept fixed, benefits everyone:

Grand coalition monotonicity: (Meggido, 1974) If $W(N) \supseteq V(N)$ and $W(S) = V(S)$ for all $S \neq N$, then $\varphi(W) \geq \varphi(V)$.

The next result pertains to the class of "strictly comprehensive games" (games V such that for all coalitions S and for all x, $y \in V(S)$, if $x \geq y$, then there exists $z \in V(S)$ such that $z > y$).

Theorem 15 (Hart and Mas-Colell, 1989) On the domain of strictly comprehensive NTU games, a solution satisfies *Pareto-optimality*, *homogeneity*, *grand coalition monotonicity*, and *self-consistency* if and only if there are positive weights $(w_i)_{i\in\mathcal{I}}$ such that it is the w-egalitarian solution.[41]

Maschler and Owen (1989) note — it is a consequence of their main results discussed below in (v) — that the axioms shown by Sobolev (Theorem 7) to characterize the nucleolus on the class of TU games, with *max consistency*

[41]The uniqueness part of this theorem still holds if the first three axioms are only imposed for two-person games.

replaced by *self-consistency* and *homogeneity* replaced by *scale invariance*,[42] are incompatible on the class of NTU games.

(v) Average self-consistency. Here, we consider an application of the idea of *average consistency* informally discussed in Subsection 1.4.5. A solution is only required to be such that for each game $v \in \mathcal{H}^N$ and for each player $i \in N$, his payoff x_i be equal to the average of his payoffs in all of the reduced games associated with x and all of the subgroups of the initial set of players containing him. Note that such a requirement is most meaningful for *single-valued* solutions. A special case of it is obtained by limiting the averaging operation to groups of two agents.

Maschler and Owen (1989) focus on the class $\mathcal{H}_{hyper} \subset \mathcal{H}$ of "hyperplane games", that is, games in which for each S, $V(S)$ is a half-space.[43] Note that their formulation is based on the self-reduced games (as in Hart and Mas-Colell, 1989).

Average self-consistency: The *single-valued* solution $\varphi: \mathcal{H}_{hyper} \to X_{\mathcal{H}}$ is ***average self-consistent***[44] if for all $N \in \mathcal{N}$, all $V \in \mathcal{H}^N_{hyper}$, and all $i \in N$, we have $x_i = \frac{1}{2^{|N|-1}-1} \sum_{N': N' \subset N, N' \ni i} \varphi_i(r^{\varphi}_{N'}(V))$, where $r^{\varphi}_{N'}(V)$ is the self-reduced game of V relative to N' and φ (Section (iv)).[45]

Maschler and Owen propose a solution for hyperplane games, and show that this solution is *average self-consistent.* It is defined by an averaging of marginal contributions similar to that defining the Shapley value (and indeed in the TU case, it just gives the Shapley value payoffs): specifically, given an ordering of the players, pay the first player the most that he could get on his own; pay the second player the most that he could get in the coalition consisting of the first two players subject to the condition that the first player receives his payoff as just calculated ... Proceed in this way until the last player. Finally, define the ***Maschler-Owen*** payoff of a player to be the

[42]This is the requirement, already encountered in our discussion of bargaining problems, that the solution be invariant with respect to linear transformations, independent agent by agent, of their utilities.

[43]A TU game can be represented as a hyperplane game in which all the hyperplanes are normal to vectors of ones: given $v \in \mathcal{G}^N$ and $S \subseteq N$, set $V(S) = \{x \in \mathbb{R}^N: \sum_N x_i = v(S)\}$.

[44]Maschler and Owen use the term *consistency* and refer to the property that we have designated by that name as *strong consistency.*

[45]Note that the self- reduced games are hyperplane games.

average of his payoffs so calculated when all orders are equally likely. The average of these vectors of marginal contributions is Pareto-optimal because the feasible set for the grand coalition is a hyperplane.

In fact, the Maschler-Owen solution satisfies the stronger version of *average self-consistency* obtained by averaging over all coalitions of a given size, no matter what that size is. Given $k \in \mathbb{N}$, let us call ***k-average self-consistency*** the version of the property obtained by averaging only over coalitions of size k (when meaningful, that is, for games with at least k players): using this terminology, the Maschler-Owen solution is *k-average self-consistent* for any k. We also have the following characterization:

Theorem 16 (Maschler and Owen, 1989) On the domain of hyperplane games, the Maschler-Owen solution is the only *single-valued* solution satisfying *Pareto-optimality*, *symmetry*, *scale invariance*, and *2-average self-consistency.*[46]

One may wonder whether the use of *average self-consistency* itself would allow other solutions. The answer is no. Indeed, Orshan (1992) shows that the axioms of Theorem 16 with *average self-consistency* substituted for *2-average self-consistency* still characterize the Maschler-Owen solution.

In a second step, Maschler and Owen extend their solution to general NTU games.[47] However, this extension is not *2-average self-consistent* (Owen, 1992).

For each problem, starting from an arbitrary Pareto-optimal payoff vector, adjust its coordinates in the direction of the 2-averages given in the definition of *2-average consistency.* The Maschler-Owen outcome can be understood as the rest point of this dynamic adjustment process, and it is natural to wonder under what conditions such a process converges. Maschler

[46]Owen (1992) states that the result also holds with *2-average self-consistency* replaced by *k-average self-consistency* for any k.

[47]We will denote it by $\boldsymbol{MO^*}$: given $N \in \mathcal{N}$ and $V \in \mathcal{H}^N$, let $x = (x_S)_{S \subseteq N}$ be a ***payoff configuration for v***, that is, x_S is a point in $V(S)$ for each $S \subseteq N$. For each $S \subseteq N$, let H_S be a hyperplane of support of $V(S)$ at x_S. Then, consider the hyperplane game $H = (H_S)_{S \subseteq N}$, and calculate $(MO(H_S))_{S \subseteq N}$. If $x_S = MO(H_S)$ for all $S \subseteq N$, then $x \in MO^*(V)$. The non-emptiness of the set $MO^*(V)$ is established for a large class of games by means of a fixed point theorem. Owen (1992) refers to it as the "inductive" value, and Maschler and Owen as the "consistent" value, a name that we prefer avoiding, given Owen (1992).

and Owen identify an adjustment factor such that geometric convergence takes place towards the payoff vector chosen by their solution, starting from any Pareto-optimal payoff vector. Orshan (1992) shows that convergence also takes place, of course to the same outcome, when the adjustments are made on the basis of the averages over coalitions of all sizes, as in the definition of *average self-consistency*.

2.2.4 Games in strategic form

The consistency principle has mainly been investigated in what can be called "cooperative" models, as opposed to "strategic" models, these terms being used to differentiate between (i) models in which only the opportunities collectively available to groups of agents are specified (the cooperative models), and (ii) models in which a set of possible actions is specified for each agent, together with a payoff function mapping from profiles of actions into payoff space; the issue there is to identify what is the best action, or "strategy", an agent should take (the strategic models). An interesting question concerns the extent to which the two models, when applied to the same basic situation, yield the same results. Since in the formulation of *consistency*, we imagine agents leaving the scene with their payoffs, it is natural to attempt to establish a link by considering strategic models in which it is an agent's option actually to also leave the scene. Here, agents leave the scene after playing their components of the strategy profile under consideration.

Consider the following three-player game below. Player 1 is the row player, player 2 is the column player and player 3 is the matrix player. We allow mixed strategies. The ***Nash equilibrium solution*** chooses the strategy profiles such that each player's strategy is a best response to the strategies chosen by the other two players.

	Left	Right
Up	1,1,1	1,0,1
Down	1,1,1	0,0,1

L

	Left	Right
Up	0,1,0	0,0,0
Down	1,1,0	0,0,0

R

Consider such a strategy profile and imagine some of the players leaving with a commitment to play their components of the profile. In the game faced by the remaining players, their strategy spaces are unchanged but the payoff function is derived from the original one by fixing at the chosen values the strategies of the players who left. Now, we ask whether the players who stay will still play their own components of the profile. It is trivial to check that the Nash equilibrium solution satisfies this definition.

The ***perfect equilibrium solution*** chooses the strategy profiles satisfying the following condition: given $\epsilon > 0$, say that a strategy profile is an ***ϵ-equilibrium*** if it assigns positive weights to all pure strategies, but is assigns a weight smaller than ϵ to any pure strategy that is not a best response to the strategies chosen by the other players. A ***perfect equilibrium*** is a limit of such ϵ-equilibria as ϵ goes to 0. Note that for the example, $s = (D, L, L)$ is a perfect equilibrium but (D, L) is not a perfect equilibrium of the game derived from it by fixing player 3's strategy at L: therefore, the perfect equilibrium solution is not *consistent.*

We now turn to the formal definitions. A ***strategic form game*** is a pair (S, h) where $N \in \mathcal{N}$ is a set of players, $S = \Pi_N S_i$ is the cartesian product of their ***strategy spaces***, and $h: S \to \mathbb{R}^N$ is the ***payoff function***: given $s \in S$ and $i \in N$, the coordinate $h_i(s)$ of $h(s)$ is interpreted as the payoff received by player i when the profile of strategies is s. Let $\mathcal{S}^N$ be the domain of games in which the group N may be involved, $\mathcal{S} = \bigcup_{N \in \mathcal{N}} \mathcal{S}^N$, and $X_{\mathcal{S}} = \bigcup_{N \in \mathcal{N}} \mathbb{R}^N$. A ***solution*** is a correspondence that associates with every $N \in \mathcal{N}$ and every game $(S, h) \in \mathcal{S}^N$ a (possibly empty)[48] subset of S. The Nash equilibrium solution and the perfect equilibrium solution are examples of solutions. The ***strong Nash equilibrium*** solution (Aumann, 1959) is another: it selects the strategy profiles such that there is no group whose members can all strictly benefit by jointly switching to other strategies, assuming that the members of the complementary group do not switch.

Most of the *consistency* and *converse consistency* concepts of this section, and the results, are due to Peleg and Tijs (1996).[49]

We start with a formal definition of *consistency.* Let $N \in \mathcal{N}$ and $(S, h) \in \mathcal{S}^N$ be a game. Given $N' \subset N$ and $s \in S$, the ***reduced game of (S, h) relative to N' and s*** is the game $(S_{N'}, h^s_{N'})$ where $h^s_{N'} = (h^s_i)_{i \in N'}$ and $h^s_i: S_{N'} \to \mathbb{R}^{N'}$ is defined by $h^s_i(s') = h_i(s', s_{N \setminus N'})$ for all $s' \in S_{N'}$. Let

[48] This is in contrast with the other models reviewed in this survey, but as we will see, for some of the solutions that have been characterized, very little is known about existence.

[49] A very brief discussion of the issue appears in Aumann (1986), and ideas of *consistency* are an important ingredient of the notion of a coalition-proof equilibrium (Bernheim, Peleg and Whinston, 1987).

$r^s_{N'}(S, h) = (S_{N'}, h^s_{N'})$.

Consistency for games in strategic form: The solution $\varphi: \mathcal{S} \to X_\mathcal{S}$ is ***consistent*** if for all N, $N' \in \mathcal{N}$ with $N' \subset N$, all $(S, h) \in \mathcal{S}^N$, and all $s \in \varphi(S, h)$, we have $s_{N'} \in \varphi(r^s_{N'}(S, h))$, where $r^s_{N'}(S, h) = (S_{N'}, h^s_{N'})$.

As far as *converse consistency* is concerned, several interesting alternatives are possible. We start with the formulation that is the closest to the one used in other models.

Converse consistency for games in strategic form: The solution $\varphi: \mathcal{S} \to X_\mathcal{S}$ is ***conversely consistent*** if for all $N \in \mathcal{N}$, all $(S, h) \in \mathcal{S}^N$, and all $s \in S$, if [for all $N' \subset N$, $s_{N'} \in \varphi(r^s_{N'}(S, h))$], then $s \in \varphi(S, h)$.

As noted above, the Nash equilibrium solution is *consistent*. The example used above to show that the perfect equilibrium solution is not *consistent* also shows that the proper equilibrium solution and the stable equilibrium solution are not either. The Nash equilibrium solution is *conversely consistent*. In fact, our first result is that it is the only solution to satisfy both conditions as well as the following requirement, which pertains to one-person decision problems.

One-player payoff maximization: For a one-player game, the solution should select the set of strategies maximizing the player's payoff function.

Theorem 17 (Peleg and Tijs, 1996) The Nash equilibrium solution is the only solution satisfying *one-player payoff maximization*, *consistency*, and *converse consistency*.

Consider the following additional properties, whose interpretation is straightforward:

Independence of irrelevant strategies: If a strategy profile is selected by the solution for some game and it remains feasible in the subgame obtained by deleting some of the strategies of each player, then it is selected by the solution in the subgame.[50]

[50]This condition bears a certain resemblance to Nash's (1950) condition of *contraction independence* (Section 2.2.1).

Dummy property: If a player has only one strategy, he can be deleted from the game without affecting the strategy profiles selected by the solution for the other players.

Given that *independence of irrelevant strategies* and the *dummy property* together imply *consistency*, a corollary of Theorem 17 is that the Nash equilibrium solution is the only solution satisfying *one-player payoff maximization*, *converse consistency*, *independence of irrelevant strategies*, and the *dummy property.*

Here is an alternative formulation of *converse consistency.* We qualify it of "conditional" because we impose on the strategy profile that is being considered the requirement that it lead to a payoff vector that is weakly Pareto-optimal among all payoff vectors resulting from some strategy profile: given a game (S, h), let $WPO(S, h)$ be the subset of S of strategy profiles whose associated payoff profiles are not strongly Pareto-dominated by any payoff profile. The set $PO(S, h)$ is defined in a similar way, replacing strong Pareto-domination by Pareto-domination.

WPO-conditional converse consistency for games in strategic form: The solution $\varphi: \mathcal{S} \rightarrow X_{\mathcal{S}}$ is ***WPO-conditional conversely consistent*** if for all $N \in \mathcal{N}$, all $(S, h) \in \mathcal{S}^N$, and all $s \in S$, if $s \in WPO(S, h)$ and [for all $N' \subset N$, $s_{N'} \in \varphi(r^s_{N'}(S, h))$], then $x \in \varphi(S, h)$.

Theorem 18 (Peleg and Tijs, 1996) The strong Nash equilibrium solution is the only solution satisfying *one-person payoff maximization*, *weak Pareto-optimality*, *consistency*, and *WPO-conditional converse consistency.*

The subsolution of the strong Nash equilibrium solution defined by requiring that no group of players be able to make all of its members better-off and at least one of them strictly better-off by jointly switching to other strategies, can be characterized along the lines of Theorem 18 by simply changing the conditional statement in *WPO-conditional converse consistency* from $s \in WPO(S, h)$ to $s \in PO(S, h)$. It could be referred to as ***PO-conditional** converse consistency.*

The next result is a characterization of a relatively recent solution. To define it, and given $N \in \mathcal{N}$, $(S, h) \in \mathcal{S}^N$, $s \in S$, and $N' \subset N$, $s' \in S_{N'}$ is an ***internally coherent***[51] ***improvement of N' over s*** if the following holds: if $|N'| = 1$, that is, if $N' = \{i\}$ for some $i \in N$, $h_i(s'_i, s_{N\setminus\{i\}}) > h_i(s)$; if $|N'| > 1$, $h_i(s'_i, s_{N\setminus N'}) > h_i(s)$ for all $i \in N'$, and no $N'' \subset N'$ has an internally coherent improvement over $(s_{N'}, s_{N\setminus N'})$. Finally, s ***is a coalition-proof equilibrium*** if no $N' \subseteq N$ has an internally coherent improvement over s (Berheim, Peleg, and Whinston, 1987).

The characterization of the solution stated next involves the requirement of **weak two-person Pareto-optimality**, which says that for any game $(S, h) \in \mathcal{S}^N$ with $|S| = 2$, if s is chosen by the solution, then there is no $t \in S$ such that for all $N' \subset N$, $t_{N'} \in \varphi(r^t_{N'}(S, h)$, $h_i(t) > h_i(x)$ for all $i \in N$. It also involves the following weakening of *converse consistency.*

Weak converse consistency for games in strategic form: The solution $\varphi: \mathcal{S} \to X_{\mathcal{S}}$ is ***weakly conversely consistent*** if for all $N \in \mathcal{N}$, all $(S, h) \in \mathcal{S}^N$, and all $s \in S$ such that [for all $N' \subset N$, $s_{N'} \in \varphi(S_{N'}, h^s_{N'})$], and there is no $t \in S$ such that [for all $N' \subset N$, $t_{N'} \in \varphi(r^t_{N'}(S, h))$] and $h_i(t) > h_i(s)$ for all $i \in N$, then $s \in \varphi(S, h)$.

Theorem 19 (Peleg and Tijs, 1996) The coalition-proof Nash equilibrium solution is the only solution satisfying *one-player payoff maximization*, *weak two-person Pareto-optimality*, *restricted Pareto-optimality*, *consistency*, and *weak converse consistency.*

Other results in the spirit of Theorems 17, 18, and 19 are offered by Peleg and Tijs. In particular, they characterize the correspondence associating with each game its set of profiles of dominant strategies. They also consider applications to Bayesian games and to sequential games, and offer a characterization of the Bayesian Nash solution. This latter theorem is an exact counterpart of their characterization of the Nash solution (Theorem 17). Van Heumen, Peleg, Tijs, and Borm (1996) provide a detailed study of Bayesian games, and offer characterizations of the Bayesian, strong Bayesian, and coalitional-proof Bayesian equilibrium solutions. For extensive form games, Peleg and Tijs (1996) derive an almost identical result to their Theorem 17; the only difference is that *one-player payoff maximization* is

[51] We use the term "coherent" instead of the term "consistent" which is commonly used, in order to avoid a possible confusion.

strengthened to ***"perfect" one-player payoff maximization***, a condition which is equivalent to the principle of backward induction for one-person decision problems.

Norde, Potters, Reijnierse, and Vermeulen (1993) focus on two classes of games, (i) the class of mixed extensions of finite games, and (ii) the class of games with compact and convex strategy spaces and continuous and concave payoff functions. They show that in either case the Nash equilibrium solution is the only solution to satisfy *non-emptiness*, *weak one-player payoff maximization*, obtained from *one-player payoff maximization* by requiring that for a one-person game, the solution selects a subset of the set of strategies maximization the player's payoff function, and *consistency*. Peleg, Potters, and Tijs (1993) identify general graph-theoretic conditions for a class of games to be such that on the class, the Nash solution is the only well-defined solution to satisfy these axioms. They show that the conditions are met by the class of finite games having at least one Nash equilibrium, but not by several interesting classes of potential games.

Peleg and Sudölter (1994) consider abstract economies in the sense of Debreu (1952). Such an economy consists of a list of agents, and for each agent, a strategy space, a payoff function defined on the cross-product of the strategy spaces, and finally a feasibility correspondence: agent i's feasibility correspondence specifies, for each list of strategies chosen by the other agents, a subset of his strategy space from which he can actually choose. The assumptions are that for each agent, (i) his strategy space is a non-empty convex, and compact subset of a Euclidean space, (ii) his feasibility correspondence is continuous and takes non-empty, closed, and convex values, (iii) his payoff function is continuous on the graph of his feasibility correspondence, and (iv) for each list of strategies chosen by the other agents, his payoff function is quasi-concave with respect to his strategy over the set of allowable strategies for him, as specified by his feasibility correspondence. The theorem is that the Nash equilibrium solution is the only solution satisfying *one-player payoff maximization* and *consistency* appropriately rewritten for this class of situations (in a reduction, the payoff functions ***and*** the feasibility correspondences have to be respecified so as to take into account the strategy choices made by the departing agents).

A counterpart of the Peleg-Sudhölter characterization for the class of games with (i) convex and weakly compact strategy spaces in infinite dimensional Banach spaces, (ii) weakly continuous and convex-valued feasibility correspondences, and (iii) weakly continuous and quasi-concave payoff functions, has been proved by Shinotsuka (1994b). This formulation covers in particular the class of games in which strategy spaces are probability distributions over compact intervals.

We close this section by noting that the idea of *consistency* has greatly helped in the understanding of a number of non-cooperative models of bargaining. Contributions in point are due to Krishna and Serrano (1996), who consider a model of sequential bargaining, and Sonn (1992, 1993) who studies a class of bankruptcy problems (Subsection 2.3.2) as well as an NTU formulation of the Rubinstein's (1982) game of alternative offers. In each of these papers, the Nash solution outcome emerges as the limit of equilibrium outcomes when a critical parameter (discount rate in most models) tends to its extreme value. It turns out that the system of equations satisfied by the equilibria is the same as the system of equations defining the one-parameter family of Nash-like solutions of Lensberg and Thomson (1988) (See the discussion following Theorem 1). These solutions had been obtained entirely from axiomatic considerations, with *consistency* playing the main role.

In a series of contributions, Serrano has shown that certain strategic form games could be analyzed by drawing on the *consistency* and *converse consistency* properties of their equilibrium correspondences.

Serrano (1993a) suggests associating with each three-person superadditive TU game in coalitional form v an infinite alternating-offer bargaining game with exit and outside options, and proves that if the order in which players are allowed to make offers corresponds to an intuitive notion of "power", as reflected by the coalition form, the non-cooperative game has a unique Markov perfect equilibrium outcome, which is the nucleolus of v.[52] Serrano (1994) defines, for every convex TU game, a sequential game whose set of subgame perfect equilibrium outcomes coincides with the core of the game. Serrano (1995a) associates with each bankruptcy problem (see Subsection 2.3.1 for a formal definition) a sequential game and shows that there is a unique subgame perfect equilibrium with payoff vector equal to the nucleolus of the TU coalition form associated with the bankruptcy problem, and he obtains a similar result for surplus-sharing problems. Finally, Serrano (1995b) derives the kernel of a class of TU games in coalitional form as well as

[52]As he notes, the result does not extend to the case of more than 3 players.

the Nash bargaining solution of bargaining problems and an extension of the kernel to a class of NTU games in coalitional form, as equilibria of a certain sequential game. These results rely crucially on the *converse consistency* of these solutions on the domains under consideration.

Moldovanu (1990b) invokes the idea of *consistency* in his analysis of a strategic market game in economies with indivisibilities.

The investigation of the implications of *consistency* for other strategic models appears to be one of the most promising directions for future research.

2.3 PUBLIC FINANCE

In this section, we present results pertaining to several models of public finance. We begin with bankruptcy and taxation models, continue with two models of cost allocation, and we close with pricing problems.

2.3.1 Bankruptcy and taxation problems

We start with two problems discussed in the Talmud.

The ***contested garment problem***: two men disagree over the ownership of a garment, worth 100. The first man claims half of it (50) and the other claims it all (100). Assuming both claims to be made in good faith, how should the worth of the garment be divided among the two men? The Talmud recommends 25 for the first one and 75 for the second (Baba Metzia, Babylonian Talmud).

The ***estate division problem***: a man has three wives whose marriage contracts specify that in case of his death they should receive 100, 200, and 300 respectively. The man dies and his estate is found to be worth only 100. How should this amount be divided among the wives? The Talmud recommends equal division. If the estate is worth 300, the Talmud recommends proportional division, but if it is worth 200, it recommends (50, 75, 75)! (Ketuboth 93a, Babylonian Talmud)

To clarify the mystery posed by the numbers given as resolutions of these problems, we should first of all find a general and natural formula that generates them. Consider the following method to divide the value of the estate among n claimants. The method, proposed by Aumann and Maschler (1985), is illustrated in Figure 2.3 for the two problems in the Talmud: the first units of the estate are divided equally until each claimant has received an amount equal to half of the smallest claim; then the claimant with the smallest claim does not receive anything for a while; instead, any additional amount is divided equally among all others until each of them has received an amount equal to half of the second smallest claim; then the two claimants with the smallest claims do not receive anything for a while...; the algorithm proceeds in this way until a value of the estate equal to $\sum c_i/2$; at that point, each claimant has received half of her claim; for values of the estate greater than $\sum c_i/2$, awards are computed in a symmetric way, by successively equating incremental losses instead of incremental gains, and starting from a value of the estate equal to the sum of the claims, and for which each claimant is fully compensated. It is a simple matter to check that when this method is applied to the two Talmudic problems, it yields the numbers given by the Talmud. Henceforth, we will call it the ***Talmudic solution.***

Now, for an estate of 200 in the 3-person case, the amounts awarded to claimants 1 and 2 are 50 and 75 respectively, for a total of 125. Applying the Talmudic solution to divide an estate of 125 between the first two claimants returns the same numbers 50 and 75! In fact, given any value of the estate, if x denotes the solution outcome of the 3-person problem, applying the solution to the division of an estate of $x_i + x_j$ between any pair $\{i, j\}$ yields the settlement (x_i, x_j). This coincidence always occurs. It is because the Talmudic solution is *consistent*!

Here are the general definitions: a ***bankruptcy problem*** is a pair $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$ with $\sum_N c_i \geq E$: $N \in \mathcal{N}$ is a group of claimants on the ***net worth*** E of a bankrupt firm, c_i being the ***claim*** of claimant $i \in N$.[53]

A different interpretation of pairs in $\mathbb{R}_+^N \times \mathbb{R}_+$ gives us the class of tax collection problems, and in what follows we will mainly focus on that interpretation: a ***tax collection problem*** is a pair $(w, T) \in \mathbb{R}_+^N \times \mathbb{R}_+$ with $\sum_N w_i \geq T$; $N \in \mathcal{N}$ is a group of ***taxpayers*** with ***incomes*** given by the coordinates of w, and who among themselves must cover the ***cost*** T of a project. Let $\mathcal{T}^N$ be the class of these problems, $\mathcal{T} = \bigcup_{N \in \mathcal{N}} \mathcal{T}^N$, and $X_{\mathcal{T}} = \bigcup_{N \in \mathcal{N}} \mathbb{R}^N$. A ***solution*** is a function associating with every $N \in \mathcal{N}$

[53]Bankruptcy problems have been considered by O'Neill (1982), Aumann and Maschler (1985), Chun (1988), Chun and Thomson (1990), Serrano (1993), Landsburg (1993), Dagan and Volij (1993), Dagan, Serrano and Volij (1993), R. Lee (1992a) and and N-C. Lee (1994). See Thomson (1995) for a survey.

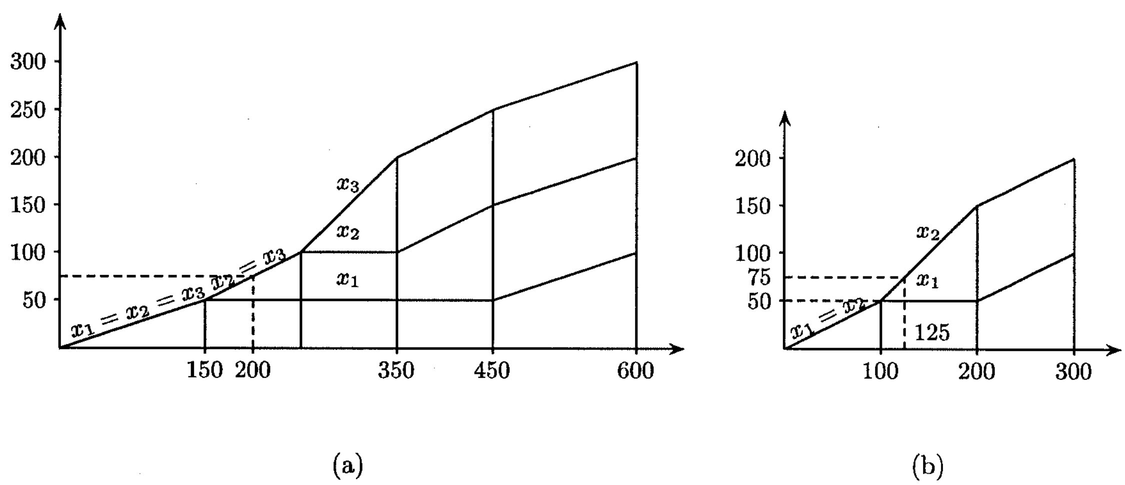


Figure 2.3: The *consistency* of the Talmudic solution to the estate division problem. The value of the estate is measured horizontally. The payments to the claimants are measured vertically. (a) Claims are $(c_1, c_2, c_3) = (100, 200, 300)$. If the estate is worth 200, the Talmudic solution recommends (50,75,75). (b) Claimant 3 has received 75, which leaves $200 - 75 = 125$ for the other two claimants. Their claims are $(c_1, c_2) = (100, 200)$. The Talmudic solution recommends (50,75), which is exactly what they had initially received.

Taxpayers' incomes	Proportional taxation applied to		Rank taxation applied to	
w	$(w, 30)$	$(w_1, w_2, 15)$	$(w, 30)$	$(w_1, w_3, 20)$
$w_1 = 10$	5	5	5	20/3
$w_2 = 20$	10	10	10	
$w_3 = 30$	15		15	40/3

Table 2.2: *Consistency* of taxation methods. Proportional taxation is *consistent.* "Rank taxation", which assesses taxes proportionally to the relative ranks agents hold when they are ordered by incomes, is not *consistent.* Indeed, relative ranks are disturbed when some of the agents leave. In the example, $N = \{1, 2, 3\}$, $w = (10, 20, 30)$, and $T = 30$.

and every $(w, T) \in \mathcal{T}^N$, a vector in $\mathbb{R}^N$ whose coordinates add up to T. Taxation problems have been extensively investigated by Young (1986, 1987a,b, 1988a).

Interesting examples of solutions are: the ***proportional solution***, which gives the vector of taxes x as λw, λ being adjusted, as in the next three examples, so that $\sum x_i = T$; the ***leveling tax***, where $x_i = \max\{w_i - 1/\lambda, 0\}$; ***Stuart's solution***, where $x_i = \max\{0, w_i - w_i^{1-\lambda}\}$; finally, ***Cassel's solution***, where $x_i = w_i^2/(w_i + 1/\lambda)$.

Table 2 illustrates the fact that proportional taxation is *consistent*[54] whereas the following ***rank taxation*** method[55] is not: order the taxpayers by increasing incomes. Then, assess them proportionally to their ranks (for example, agent of rank 5 is assessed 5/3 times what agent of rank 3 is assessed).

Consider now the following class of solutions. Let $f: \mathbb{R}_+ \times [a, b] \to \mathbb{R}_+$, where $[a, b] \subset [-\infty, +\infty]$, be continuous, weakly monotone increasing in its second argument and such that $f(w_i, a) = 0$ and $f(w_i, b) = w_i$ for all $w_i \in \mathbb{R}_+$. Then, given $N \in \mathcal{N}$ and $(w, T) \in \mathcal{T}^N$, let $x = \varphi(w, T)$ if $\sum_N x_i = T$ and for some $\lambda \in [a, b]$, $x_i = f(w_i, \lambda)$ for all $i \in N$. Young (1987) calls any

[54]This is true for this model as it is for any model where proportionality is well-defined.
[55]I owe this example to P. Young.

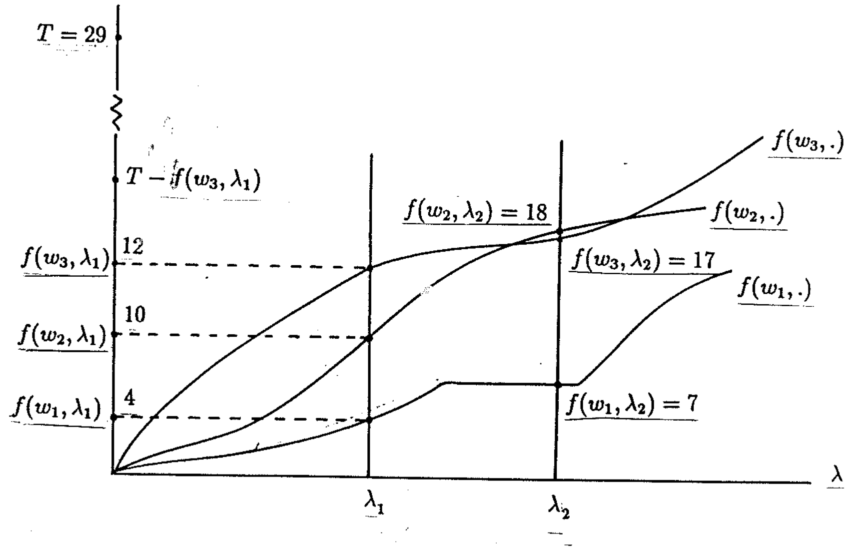


Figure 2.4: Parametric solutions are *consistent*. Given the three incomes (w_1, w_2, w_3), the parameter λ is chosen so that the amounts $f(w_1, \lambda)$, $f(w_2, \lambda)$, and $f(w_3, \lambda)$ add up to what has to be collected, T. Now, if the amount $T' = T - f(w_3, \lambda)$ is to be collected from taxpayers 1 and 2, the value λ' for which the amounts $f(w_1, \lambda')$ and $f(w_2, \lambda')$ add up to T' is of course $\lambda' = \lambda$. After taxpayer 3 has paid $f(w_3, \lambda)$, taxpayers 1 and 2 are still assessed the same amounts.

solution so defined ***parametric***. It is straightforward to check that they are all *consistent*.

Figure 2.4 depicts the graphs of such an f for three possible values of the first argument. The choice of $\lambda = \lambda_1$ allows $4 + 10 + 12$ to be collected, and the choice of $\lambda = \lambda_2$ allows $7 + 18 + 17$ to be collected. Note that the graph corresponding to w_1 is not strictly monotone increasing whereas the other graphs have that property, and that the graph corresponding to w_3 does not lie entirely above that corresponding to $w_2 < w_3$. At this stage these are indeed possibilities. It may be judged desirable that when the burden imposed on some agent increases, then so does the burden imposed on any other agent. To achieve this, require that the functions never be constant.[56] When an agent's income is greater than some other agent's income, we may find it appropriate for him never to pay less taxes. This is accomplished by requiring that the functions be ordered by incomes. The proportional and Talmudic solutions are parametric and Figures 2.5a and 2.5b give parametric representations for them, the latter in the case in which an upper bound on

[56]They could of course be constant over the same intervals.

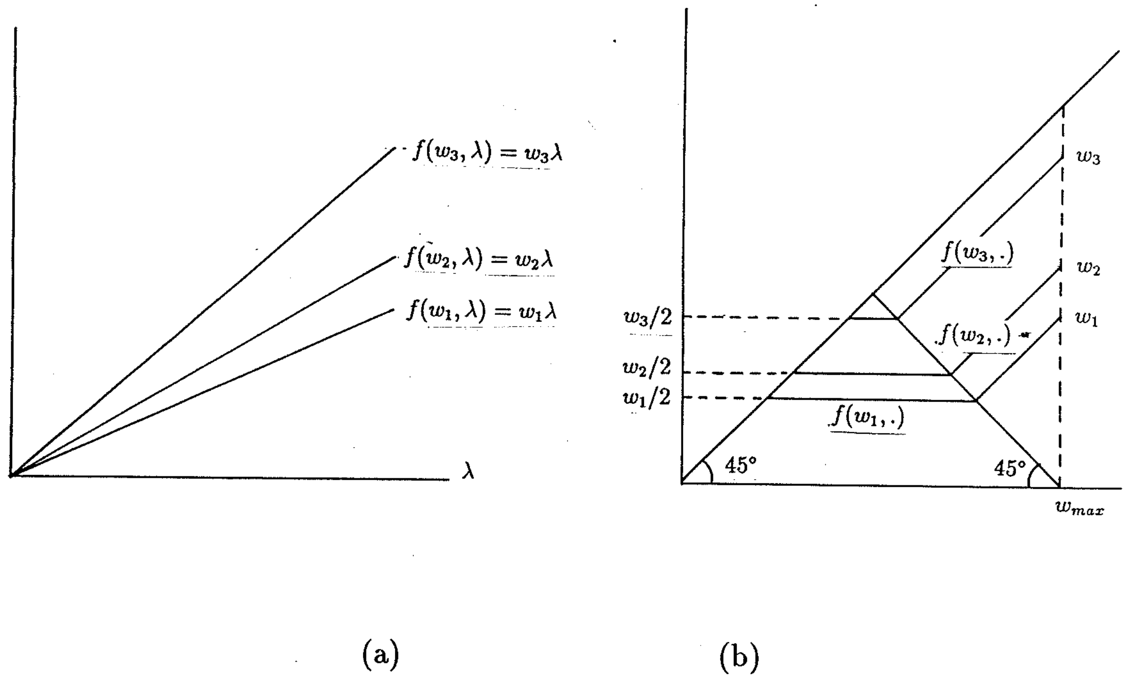


Figure 2.5: Parametric representations of two solutions. (a) Proportional solution: the schedules are straight lines through the origin, of slopes equal to incomes. (b) Talmudic solution: The schedule relative to income w_1 follows the 45° line up to the point $(w_1/2, w_1/2)$, continues horizontally until it meets the line of slope -1 emanating from $(w_{\mathbf{max}}, 0)$, then again follows a line of slope 1, until it reaches the point (w_{max}, w_1).

incomes exists, $w_{\max}$.[57]

In the formulation of *consistency*, we go back to the requirement of *single-valuedness* since all taxation methods discussed in the literature satisfy it.

Consistency for taxation problems: The solution $\varphi: \mathcal{T} \to X_{\mathcal{T}}$ is ***consistent*** if for all N, $N' \in \mathcal{N}$ with $N' \subset N$, all $(w,T) \in \mathcal{T}^N$, and all $x \in \mathbb{R}^N$, if $x = \varphi(w,T)$, then $x_{N'} = \varphi(r^x_{N'}(w,T))$, where $r^x_{N'}(w,T) = (w_{N'}, \sum_{N'} x_i)$.

The first two of the requirements that we will impose in conjunction with *consistency* are straightforward and they have wide appeal. The next three, used in the subsequent theorem, have also been extensively discussed in the literature, although they are perhaps slightly less compelling.

Taxpayers with identical incomes should be assessed identical taxes:

Equal treatment of equals: If $w_i = w_j$, then $\varphi_i(w,T) = \varphi_j(w,T)$.

Small changes in the parameters of the problem should not produce large changes in taxes. Let $\{(w^\nu, T^\nu)\}$ denote an arbitrary sequence of taxation problems in $\mathcal{T}^N$:

Continuity: If $(w^\nu, T^\nu) \to (w,T)$, then $\varphi(w^\nu, T^\nu) \to \varphi(w,T)$.

We are now ready for the main theorem:

Theorem 20 (Young, 1987) The continuous parametric solutions are the only solutions satisfying *equal treatment of equals, continuity,* and *consistency.*[58]

Subfamilies of the parametric family can be identified by imposing the requirement that no group of agents should be able to benefit by consolidating their incomes and being treated as a single taxpayer:

[57] This assumption restricts somewhat the scope of the solution but it permits a very simple (piecewise linear) representation (Chun and Thomson, 1990). See Young (1987) for a representation without the upper bound.

[58] Any such solution is also obtained by maximization of a symmetric, continuous, and separable additive objective function $\sum H(w_i, t)$ over the constraint set $\{t \in \mathbb{R}^N : \sum t_i = T, 0 \le t_i \le x_i\}$, where $H(.,.)$ is a strictly convex function of its second argument.

Non-manipulability by merging: If $i \in N' \subset N$, then $\sum_{N'} \varphi_j(w,T) \leq \varphi_i(\sum_{N'} w_j, w_{N \setminus N'}, T)$.

Alternatively, ***non-manipulability by splitting*** says that no taxpayer should be able to benefit by representing several taxpayers whose incomes add up to his (this requirement is obtained from the one just stated by reversing the inequality).

As a corollary of Theorem 20, we obtain that if a solution satisfies *continuity*, *equal treatment of equals*, and *consistency*, and in addition is *non-manipulable by merging*, then it is a parametric solution relative to a function that is concave with respect to its first argument. If instead, it is *non-manipulable by splitting*, then the function is convex in its first argument (De Frutos, 1994).

Aumann and Maschler (1985) show that the Talmudic solution to the bankruptcy problem is the only *bilaterally consistent* solution to coincide in the two-person case with the solution to the contested garment problem. They also show that, interestingly, this solution picks for every bankruptcy problem the nucleolus (Subsection 2.2.2) of the TU coalitional form game associated with the problem in the following way (O'Neill, 1982): define the worth of a coalition to be the amount that is left over after all the members of the complementary coalition have been given full satisfaction if that left-over is non-negative, and 0 otherwise. R. Lee (1992a) gives a short proof of this result.

Bibliographic note. Chun and Thomson (1990) define a particular member of the parametric family. It is inspired by a solution to the problem of fair division when preferences are single-peaked known as the uniform rule (Subsection 2.4.3.) N-C. Lee (1994) proposes a weighted generalization of the "constrained equal award" solution to bankruptcy problems (the counterpart of the leveling tax), and develops a characterization of it based on *consistency*. This characterization exploits duality relations between cores, anticores, and their reductions. We noted above (Section 2.2.3) Sonn (1992, 1993)'s derivation of the equilibria of games of alternating offers associated with bankruptcy problems;

this derivation relies in an important way on *consistency* arguments. Finally, as also mentioned there, we recall that Serrano (1993) makes use of the *consistency* of the nucleolus, when applied to the TU game associated with a bankruptcy problem, in order to construct a sequential game whose equilibrium outcome is the payoff vector the nucleolus would select.

Within the class of parametric solutions, a narrow subclass of great interest can be identified by imposing the following additional properties:

If incomes and cost are multiplied by the same positive number, so should all taxes:

Homogeneity: $\varphi(\alpha w, \alpha T) = \alpha \varphi(w, T)$ for all $\alpha > 0$.

Taxpayers with greater incomes should pay relatively more:

Progressivity: If $w_i > w_j > 0$, then $\frac{\varphi_i(w,T)}{w_i} > \frac{\varphi_j(w,T)}{w_j}$.

Taxes can be assessed indifferently at one time or in installments:

Decomposability: $\varphi(w, T + T') = \varphi(w, T) + \varphi(w - \varphi(w, T), T')$.

Theorem 21 (Young, 1986) A parametric solution satisfies *progressivity, homogeneity*, and *decomposability* if and only if it can be represented in one of the following ways:

$$
\begin{array}{lll}
f(w_i, \lambda) & = \lambda w_i & 0 \leq \lambda \leq 1 \\
f_p(w_i, \lambda) & = w_i - w_i/(1 + \lambda w_i^p)^{1/p} & 0 \leq \lambda \leq \infty \quad p > 0 \\
f_\infty(w_i, \lambda) & = \max\{0, w_i - 1/\lambda\} & 0 \leq \lambda \leq \infty
\end{array}
$$

Other interesting subclasses of the class of *consistent* solutions are identified by Young (1987, 1988a). Consider for instance the following family:

Equal-sacrifice solutions : Let $u: \mathbb{R}_{++} \to \mathbb{R}$ be a continuous and strictly increasing function. Then, given $N \in \mathcal{N}$ and $(w, T) \in \mathcal{T}^N$ with $c > 0$, the ***equal-sacrifice solution relative to u*** selects the point $x \in \mathbb{R}^N$ such that for some $\lambda \geq 0$, and for all $i \in N$, we have $u(c_i) - u(c_i - x_i) = \lambda$.

The next result involves the requirement of ***strict tax bill monotonicity***, which says that all assessments should strictly increase if the tax bill increases, and ***strict order preservation***, which says that if agent i's income is greater than agent j's, he should be assessed a strictly greater amount.

Theorem 22 (Young, 1988) On the domain of problems for which incomes are all positive, the equal-sacrifice solutions are the only solutions satisfying *continuity*, *equal treatment of equals*, *strict tax bill monotonicity*, *strict order preservation*, *decomposability*, and *consistency*. If in addition, *homogeneity* is imposed, then the solution is an equal-sacrifice solution relative to u such that either $u(x) = ln(x)$ or $u(x) = -x^p$ for $p < 0$. In the first case, the solution is the flat tax; in the second case, it is a parametric method of representation $f(w_i, \lambda) = w_i - [w_i + \lambda^p]^{1/p}$.

> **Bibliographic note.** Pan (1996) establishes a variant of Theorem 22 in a model in which the labor-leisure trade-off is explicitly described, but is the same for all agents.

A requirement related to *consistency* can be formulated for situations in which one of the taxpayers has an income equal to zero: then, (i) he pays nothing, and (ii) deleting him does not change the taxes assessed the others. Part (i) corresponds to the condition known in the theory of coalitional form games as the "dummy condition". (It is used in that form in de Frutos, 1994). It is part (ii) that corresponds to *consistency*. Since its coverage is not as wide as that of the condition that we used under that name, we will refer to it as ***limited consistency***. The condition amalgamating the two parts appears in O'Neill (1982) and Chun (1988).

Dummy: If $N' \subset N$ and $w_{N'} = 0$, then $\varphi_{N'}(w, T) = 0$.

Limited consistency for taxation problems: Under the hypotheses of *dummy*, if $x = \varphi(w, T)$, then $x_{N \setminus N'} = \varphi(w_{N \setminus N'}, \sum_{N \setminus N'} x_i)$.

The next characterization also involves the requirement that taxpayers not be able to benefit by redistributing incomes among subgroups. This condition is particularly meaningful in the context of bankruptcy.

No-advantageous reallocations: If $N' \subset N$ and $\sum_{N'} w_i = \sum_{N'} w'_i$, then $\sum_{N'} \varphi_i(w', T) = \sum_{N'} \varphi_i(w, T)$.

Theorem 23 (Chun, 1988) The proportional solution is the only solution satisfying *anonymity*, *continuity*, *no-advantageous reallocation*, *dummy*, and *limited consistency.*

Next, we present an alternative weakening of *consistency*, which is based on the averaging operation discussed in Subsection 1.4.5. Consider a solution that is not *consistent.* Then, for at least one problem — let the recommendation made for it by the solution be denoted by x — there is at least one subgroup and one taxpayer that belongs to it, say taxpayer i, such that in the reduced problem relative to that subgroup and x, he is assessed an amount that is different from what he was initially assessed, x_i. It may be the case however, that ***on average***, when all the reduced problems associated with x relative to groups to which he belongs are considered, he is still assessed x_i. If this is always true, we may be satisfied with x after all. To the extent that the formation of subgroups is a thought experiment anyway, this weaker notion may be quite acceptable.

Average consistency: For all $N \in \mathcal{N}$, all $(w, T) \in \mathcal{T}^N$, and all $i \in N$, $x_i = \frac{1}{2^{|N|-1}} \cdot \sum_{N':N' \subset N, N' \ni i} \varphi_i(w_{N'}, \sum_{N'} x_j)$.

This form of *consistency* is studied by Dagan and Volij (1994) who suggest that the averaging be limited to coalitions of size two. They have in mind situations in which a solution for the two-claimant case has been chosen. Then the idea of *average consistency* can be exploited to provide an extension of the solution to all cardinalities, as follows: given a problem $(w, T) \in \mathcal{T}^N$, select $x \in \mathbb{R}^N_+$ such that $\sum_N x_i = T$ and for all $i \in N$, $x_i = \frac{1}{|N|-1} \sum_{j \in N \setminus \{i\}} \varphi_i(w_i, w_j, x_i + x_j)$. Two questions are whether such an x exists, and if it does, whether it is unique. The following theorem states that provided the two-person solution satisfies ***boundedness***, which says that no agent's assessment should exceed his income, ***anonymity***, which says that the rule should be invariant with respect to permutations of the names of agents, and ***tax bill monotonicity***, which says that no assessment should decrease if the tax bill increases, both questions have positive answers:

Theorem 24 (Dagan and Volij, 1994) For all two-person solution satisfying *boundedness*, *anonymity*, and *tax bill monotonicity*, φ, all $N \in \mathcal{N}$, and all $(w, T) \in \mathcal{T}^N$, there is a unique $x \in \mathbb{R}_+^N$ such that (i) $\sum x_i = T$ and (ii) for all $i \in N$, $x_i = \frac{1}{|N|-1} \sum_{j \in N \setminus \{i\}} \varphi_i(w_i, w_j, x_i + x_j)$.

Problems closely related to taxation problems are surplus-sharing problems. Such a problem is a pair $(w, S) \in \mathbb{R}_+^N \times \mathbb{R}_+$, where w_i is the investment in a joint venture made by agent $i \in N$, and $S > 0$ is the surplus generated by this venture. Moulin (1985a)[59] uses *consistency* together with some other natural conditions and characterizes one-parameter families of surplus-sharing methods that contain as particular cases both equal sharing and proportional sharing. One of his auxiliary axioms is *homogeneity* (see above). Pfingsten (1991) describes how the class of admissible solutions enlarges when *homogeneity* is dropped. Herrero, Maschler, and Villar (1995) study the implications of *consistency* in a model that includes both bankruptcy and surplus-sharing as special cases.

2.3.2 Quasi-linear cost allocation problems

Three agents have the choice between two projects, a_1 and a_2, costing $(c_1, c_2) = (20, 30)$. The benefits they derive from these projects are $u_1 = (70, 50)$, $u_2 = (10, 50)$, and $u_3 = (30, 10)$. Monetary transfers can be effected among them. Which project should they select and how should its cost be allocated? Consider the method consisting in first selecting the project generating the highest surplus and then calculating contributions so that all agents receive an equal share of this surplus. For the example, the best project is a_1 since $u_{11} + u_{21} + u_{31} - c_1 = 70 + 10 + 30 - 20 = 90 > u_{12} + u_{22} + u_{32} - c_2 = 50 + 50 + 10 - 30 = 80$, and the resulting utility levels are $(90/3, 90/3, 90/3) = (30, 30, 30)$. To check whether the method is *consistent*, we note that in order to guarantee agent 3 a utility of 30, agents 1 and 2 should pay him $30 - 30 = 0$ if they choose a_1, and $30 - 10 = 20$ if they choose a_2, leading to an "adjusted cost vector" $c' = (20 + 0, 30 + 20) = (20, 50)$. The project that produces the highest surplus in the two-person economy that results from this adjustment is of course

[59]Interestingly, Moulin uses an axiom of *separability* which pertains to the fixed population case, and shows that together with his auxiliary axioms, the *separability* of a solution for each cardinality implies its *consistency*.

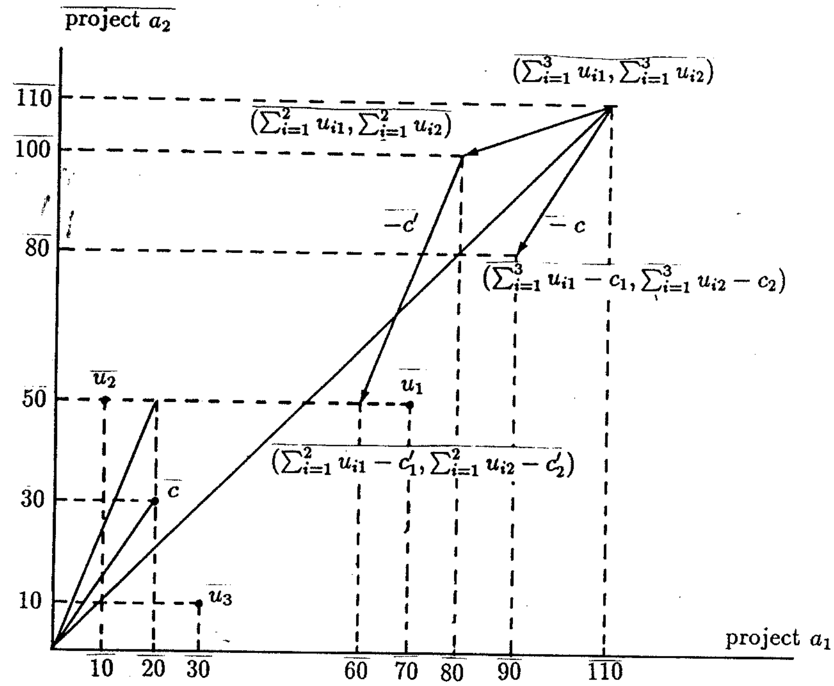


Figure 2.6: *Consistency* for quasi-linear cost-allocation problems. This figure illustrates the *consistency* of the "equal surplus sharing" solution for a two-project example. When agents 1, 2, and 3 are present, project a_1 is chosen and they achieve the final utilities (30,30,30). If agent 3 leaves with his promised utility of 30, then agents 1 and 2 can still choose either one of the two projects but they have to transfer money to agent 3 as a function of which project they select so that his utility be indeed 30. This leads to a revised cost vector for the reduced problem they face. When this reduced problem is solved, agents 1 and 2 end up with the same utilities.

a_1 since $u_{11}+u_{21}-c'_1 = 70+10-20 = 60 > u_{12}+u_{22}-c'_2 = 50+50-50 = 50$. Equal sharing of that surplus of 60 yields the utilities $(60/2, 60/2) = (30, 30)$, as initially determined. This is because the method is *consistent.*

The general definitions are as follows. Given a finite set A of ***public projects***, a ***quasi-linear cost allocation problem*** is a pair $(u, C) \in \mathbf{R}^{|A|N} \times \mathbf{R}^{|A|}$. Here, C is interpreted as a ***cost vector***. Each coordinate of C is the cost of the corresponding project. In addition, there is a private good called "money". The preferences of agent $i \in N$, defined over the product $A \times \mathbf{R}$, admit a quasi-linear utility representation: given the project $a \in A$ and given agent i's holdings of money $m_i \in \mathbf{R}$, his utility is $u_{ia} + m_i$. Let $\mathcal{M}^N$ be the class of these problems, $\mathcal{M} = \cup_{N \in \mathcal{N}} \mathcal{M}^N$, and $X_{\mathcal{M}} = \cup_{N \in \mathcal{N}} \mathbf{R}^N$. A ***solution*** is a function that associates with every $N \in \mathcal{N}$ and every $(u, C) \in \mathcal{M}^N$ a vector $x \in \mathbf{R}^N$ such that $\sum_N x_i \leqq \max_{a \in A}\{u_{ia} - C_a\}$.

Moulin (1985a, 1985b) carries out an extensive analysis of this class of problems, which generalizes the class of bankruptcy and taxation problems of the previous section, as it is given in a space of higher dimensionality. He defines a rich class of solutions which can be described as "egalitarian" since they are based on equating utility gains. They differ from each other in the specification of the reference point from which utility gains are measured. The formal statement of *consistency* for this model is as follows:

Consistency for quasi-linear cost allocation problems: The solution $\varphi: \mathcal{M} \to X_{\mathcal{M}}$ is ***consistent***[60] if for all N, $N' \in \mathcal{N}$ with $N' \subset N$ and all $(u, C) \in \mathcal{M}^N$, if $x = \varphi(u, C)$, we have $x_{N'} = \varphi(r^x_{N'}(u, C))$, where $r^x_{N'}(u, C) = (u_{N'}, \tilde{C})$ with $\tilde{C}_a = C_a + \sum_{N \setminus N'}(x_i - u_{ia})$ for all $a \in A$.

We will consider solutions satisfying the following requirements. First, the decision should maximize the net aggregate benefit:

Pareto-optimality: $\sum_N \varphi_i(u, C) = \max_{a \in A}\{\sum_N u_{ia} - C_a\}$.

The solution should be invariant under exchanges of the names of agents: let $N, \tilde{N} \in \mathcal{N}$ be such that $|N| = |\tilde{N}|$, and let $(u, C) \in \mathcal{M}^N$ and $(v_i)_{i \in \tilde{N}} \in \mathbb{R}^{|A|\tilde{N}}$.

Anonymity: If $|N| = |\tilde{N}|$, $\pi: N \to \tilde{N}$ is a bijection, and $(v_i)_{i \in \tilde{N}} = (\pi(u))$, then $\varphi((v_i)_{i \in \tilde{N}}, C) = \pi(\varphi(u, C))$.

The solution should be invariant under the addition of an arbitrary constant to an agent' utility vector:

Independence of the zero of the utility functions: If for some $\alpha \in \mathbb{R}$, $v_i = u_i + \alpha(1, \ldots, 1)$, and $v_j = u_j$ for all $j \in N \setminus \{i\}$, then $\varphi_i(v, C) = \varphi_i(u, C) + \alpha$ and $\varphi_j(v, C) = \varphi_j(u, C)$ for all $j \in N \setminus \{i\}$.

An increase in the cost function that is uniform across all alternatives should be distributed evenly among all agents:

Independence of the zero of the cost function: If $C' = C + \alpha(1, \ldots, 1)$, then $\varphi_i(u, C') = \varphi_i(u, C) - \alpha/|N|$ for all $i \in N$.

[60] Moulin uses the term "separable".

Theorem 25 (Moulin, 1985a) A solution $\varphi: \mathcal{M} \to X_{\mathcal{M}}$ satisfies *Pareto-optimality*, *anonymity*, the two *independence* axioms, and *consistency* if and only if there is a function $g: [\mathbb{R}^A]^2 \to \mathbb{R}$ satisfying

(i) $g(x + \alpha(1, \ldots, 1), z) = g(x, z) + \alpha$ for all x, $z \in \mathbb{R}^A$, and all $\alpha \in \mathbb{R}$; $g(0, z) = 0$ for all $z \in \mathbb{R}^A$,

(ii) $g(x, z + \alpha(1, \ldots, 1)) = g(x, z)$ for all x, $z \in \mathbb{R}^A$, and all $\alpha \in \mathbb{R}$,

and such that for all $N \in \mathcal{N}$, all $(u, C) \in \mathcal{M}^N$, and all $i \in N$, $\varphi_i(u, C) = (1/|N|) \max_{a \in A}\{\sum_N u_{ia} - C_a\} + (1/|N|)\{(|N| - 1)g(u_i, \sum_N u_i - C) - \sum_{N\setminus\{i\}} g(u_j, \sum_N u_i - C)\}$.

The class identified in this theorem is quite large. Interesting subclasses result by imposing two further conditions.

The first condition is that an increase in the cost function should be borne by all agents:

Cost monotonicity: If $C' \geqq C$, then $\varphi(u, C') \leqq \varphi(u, C)$.

The second condition is that no agent should be able to benefit by pretending his utility to be smaller than what it really is:

Immunity to manipulation by disposal of utility: If $u_i < v_i$ and $u_j = v_j$ for all $j \in N\setminus\{i\}$, then $\varphi_i(u, C) \leqq \varphi_i(v, C)$.

Theorem 26 (Moulin, 1985a) A solution of the form identified in Theorem 25 satisfies *cost monotonicity* and is *immune to manipulation through disposal of utility* if and only if $g(x, z) = \tilde{g}(x)$ for all $x, z \in \mathbb{R}^A$, for some monotone function $\tilde{g}: \mathbb{R}^A \to \mathbb{R}$.

A number of corollaries follow from Theorem 25, by imposing additional requirements such as various lower or upper bounds on welfares. An example is the self- explanatory ***individual rationality above some reference lottery σ***. Another is ***participation***, which says that an agent should not be better off by withdrawing from the economy. A third is ***no free lunch***, which says that no agent should be better off than he would be by optimally choosing the public decision under the constraint that its cost be divided equally. A final condition is ***no advantageous reallocation***, which says that no coalition of agents should be able to make all of its members better off by "transfers of utility" defined as follows: the problem is changed into one in which the members of the coalition announce different utility vectors but for each decision, the sum of the utilities of its members remains the same. When these additional conditions are used in turn, characterizations of various subfamilies of the family described in Theorem 25 obtained by placing restrictions on the function g are obtained. One of these corollaries singles out utilitarianism.

The final result that we will state for this model is of particular interest as it also pertains to possible changes in populations. It says that an increase in the number of agents should affect all agents initially present in the same direction.

Weak population-monotonicity: If $N' \subset N$, either $\varphi_i(u, C) \geqq \varphi_i(u_{N'}, C)$ for all $i \in N'$, or $\varphi_i(u, C) \leqq \varphi_i(u_{N'}, C)$ for all $i \in N'$.

Theorem 27 (Chun, 1986) A solution of the form identified in Theorem 25 satisfies *weak population-monotonicity* if and only if there is a function $\tilde{g}: \mathbb{R}^A \to \mathbb{R}$ satisfying

(i) $\tilde{g}(x + \alpha(1, 1, \ldots, 1)) = \tilde{g}(x) + \alpha$ for all $x \in \mathbb{R}^A$ and all $\alpha \in \mathbb{R}$

(ii) $\tilde{g}(0) = 0$

and such that for all $N \in \mathcal{N}$, all $(u, C) \in \mathcal{M}^N$, and all $i \in N$,

$\varphi_i(u, C) = (1/|N|) \max_{a \in A}\{\sum_N u_{ia} - C_a\} + (1/|N|)\{(|N| - 1)\tilde{g}(u_i) - \sum_{N \setminus \{i\}} \tilde{g}(u_j)\}$.

A corollary of this result is that if in addition to the axioms of Theorem 27, the solution is required to be *immune to manipulation through disposal of utility*, then the function $\tilde{g}$ identified in the theorem should be such that for all x, $y \in \mathbb{R}^A$ with $x \leqq y$, $\tilde{g}(x) \leqq \tilde{g}(y)$. A final characterization, which is a consequence of this corollary, involves the further requirement that the solution should guarantee to each agent a utility level that depends only on his own utility vector and the per capita cost vector, but not on the size of the society to which he belongs. The family that results is a subfamily of the one just described, obtained by choosing $\tilde{g}$ such that for some $\sigma \in \Delta^{|A|}$, $\tilde{g}(x) = x.\sigma$ for all $x \in \mathbb{R}^A$. All of these additional characterizations are due to Chun (1986).

2.3.3 General cost allocation problems

The models of bargaining theory and coalitional form games as well as the cost allocation models examined earlier can be interpreted as "abstract" production models in which the alternatives to choose from are the images in utility space of production and distribution decisions, and as explained earlier the results obtained for these abstract models can be used to define solutions to concretely specified production problems. Such problems have not been the object of much attention with regards to *consistency*. The exceptions are Moulin and Shenker (1994) who study cost-sharing in a model with one input and one output, and characterize a certain "serial" cost sharing rule, and Kolpin (1994) who considers an extension of the model to the multicommodity case.

In a production model, it is not immediately obvious how *consistency* should be defined. Indeed, the most natural reduction involves translating the technology by the vector of goods allocated to the departing agents but classical domains are not closed under such operations. For that reason, Moulin and Shenker consider more general domains for which closedness is ensured.

The formal model is as follows. A group of consumers $N \in \mathcal{N}$ demand the amounts $d = (d_i)_{i \in N} \in \mathbb{R}^N_+$ of a good, d_i being the ***demand*** of consumer i. The good is produced according to a technology described by a ***cost function*** $C: \mathbb{R}_+ \to \mathbb{R}_+$. This cost function belongs to the domain $\mathcal{F}$ of nondecreasing functions that can be expressed as the difference of two convex functions taking the value 0 at 0.[61] A ***cost allocation problem*** is a pair $(d, C) \in \mathbb{R}^N_+ \times \mathcal{F}$. Let $\mathcal{C}^N$ be the class of all such problems involving the group $N \in \mathcal{N}$, $\mathcal{C} = \bigcup_{N \in \mathcal{N}} \mathcal{C}^N$, and $X_{\mathcal{C}} = \bigcup_{N \in \mathcal{N}} \mathbb{R}^N_+$. A ***solution*** is a function that associates with every $N \in \mathcal{N}$ and every $(d, C) \in \mathcal{C}^N$ a unique vector $x \in \mathbb{R}^N_+$ such that $\sum_N x_i = C(\sum_N d_i)$.

Average cost sharing is one of the best-known methods. Simply divide the cost of satisfying aggregate demand proportionally to individual demands. ***Serial cost sharing*** is defined as follows. Suppose agents are ordered by increasing demands. Then, agent 1 pays $(1/n)$ of the cost of producing nd_1. Agent 2 pays $1/(n-1)$ of the difference between the

[61] This domain contains all non-decreasing and twice differentiable functions taking the value 0 at 0. It is dense in the domain of nondecreasing functions taking the value 0 at 0.

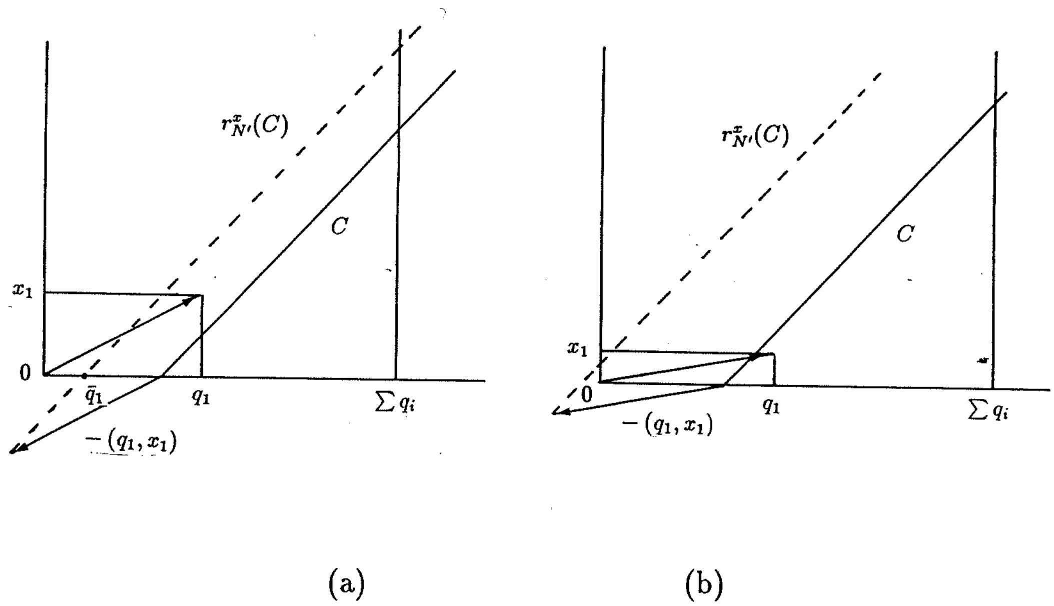


Figure 2.7: *Consistency* for cost allocation. In the two examples depicted here, the cost function is piece-wise linear. In each case, we imagine agent 1 leaving. His demand q_1 has to be satisfied and his payment x_1 is taken into account in the calculation of the costs that are still to be covered by agents 2 and 3. This amounts to translating the cost function by the vector $-(q_1, x_1)$ (the dashed line). (a) In this case, an adjustment is needed so that the reduced cost function only takes non-negative values: set it equal to 0 for every q in the interval $[0, \bar{q}_1]$. (b) Here, no adjustment is needed.

cost of producing $d_1 + (n-1)d_2$ and what agent 1 has paid ... In general, agent k pays $1/(n-k+1)$ of the difference between the cost of producing $d_1 + d_2 + \ldots + (n-k+1)d_k$ and what agents $1, \ldots, k-1$ have paid.

Here is the form of *consistency* for this model:

Consistency for cost allocation problems: The solution $\varphi: \mathcal{C} \to X_{\mathcal{C}}$ is ***consistent*** if for all N, $N' \in \mathcal{N}$ with $N' \subseteq N$ and all $(d, C) \in \mathcal{C}^N$, if $x = \varphi(d, C)$, we have $x_{N'} = \varphi(d_{N'}, r^x_{N'}(C))$, where $r^x_{N'}(C)$ is the cost function defined by $r^x_{N'}(C)(y) = \max\{C(y + \sum_{N \setminus N'} d_i) - \sum_{N \setminus N'} x_i, 0\}$ if $y > 0$, and $r^x_{N'}(C)(0) = 0$.

To present the results, we need the following additional properties of solutions. First, two agents with the same demands should pay the same amounts:

Equal treatment of equals: If $d_i = d_j$, then $\varphi_i(d, C) = \varphi_j(d, C)$.

If the cost function is linear, individual payments should be proportional to individual demands:

Proportionality for linear cost: If for some $\lambda \in \mathbb{R}_+$ and for all $d_0 \in \mathbb{R}_+$, $C(d_0) = \lambda d_0$, then $\varphi_i(d, C) = \lambda d_i$.

The payments associated with a cost function that is the sum of two functions should be the sums of the payments associated with each of the components:

Cost additivity: $\varphi(d, C_1 + C_2) = \varphi(d, C_1) + \varphi(d, C_2)$.[62]

Theorem 28 (Moulin and Shenker, 1994) Average cost sharing is the only solution satisfying *equal treatment of equals, proportionality for linear cost, cost additivity,* and *consistency.*

For the next result, we need two other fairness axioms and we also use a weak form of *consistency.*

If agent i's demand is greater than agent j's demand, he should pay more than agent j.

[62]Note that the domain of cost functions is closed under addition.

Order preservation: If $d_i \geq d_j$, then $\varphi_i(d, C) \geq \varphi_j(d, C)$.

Next, imagine an agent whose demand is such that if every one else had the same demand, the total cost of satisfying them all would be zero. Then, the requirement is that the agent should bear no part of the actual total cost.

Free lunch for agents with small demands: For all N, $N' \in \mathcal{N}$, all $i \in N$, and all $(d, C) \in \mathcal{C}^N$, if $N' = N \backslash \{i\}$ and $C(|N|d_i) = 0$, then $\varphi_i(d, C) = 0$.

The announced weakening of *consistency* is the requirement that in the circumstances described in the previous axiom, the other agents should be charged as they would be in the resulting reduced economy:

Limited consistency for cost allocation problems: Under the hypothesis of *free lunch for agents with small demands*, if $x = \varphi(d, C)$, we have $x_{N'} = \varphi(d_{N'}, r^x_{N'}(C))$.

Theorem 29 (Moulin and Shenker, 1994) Serial cost sharing is the only solution satisfying *order preservation*, *proportionality for linear cost*, *cost additivity*, *free lunch for agents with small demands*, and *limited consistency.*

An investigation of the multi-commodity case was initiated by Kolpin (1994), who offers characterizations of an extension of serial cost sharing defined as follows: to each coalition of consumers is associated an implicit "social burden", which is the production cost that would result if the demands of society at large "mirrored" those of the members of the coalition; these reference burdens are used to calculate rights that coalitions have with respect to protection from costs; rights are prioritized and the coalition with the highest priority allocates in a uniform manner its burden among its members; the residual cost is distributed among the remaining agents by repeated application of the rule. Kolpin's characterization is along the lines of Theorem 29. Kolpin also shows that his solution satisfies a strenghtening of *limited consistency* obtained by first identifying the minimal amount any agent is paying, and imagining the departure of all agents paying this minimal amount (see Subsection 1.4.3). For an alternative approach to the cost allocation problem based on the concept of an expenditure function, and for characterizations of expenditure functions inducing serial cost sharing, see Kolpin (1995).

2.3.4 Pricing problems

Here we turn to the problem of identifying well-behaved pricing formulas. A feature of the *consistency* condition that has been considered in this context is that it is the number of goods that is made to vary. The reduction operation is the counterpart for this class of problems of the operation used by Hart and Mas-Colell, and which they showed led to the Shapley value (Section 2.2.1.(iv). Perhaps not surprisingly then, we will obtain a characterization of the pricing mechanism based on the Shapley value, and known as "Shapley value pricing mechanism". We will also offer a characterization of the so-called Aumann-Shapley pricing formula. This section is based on McLean, Pazgal, and Sharkey (1994).

The formal model is as follows. There is a set $N \in \mathcal{N}$ of ***commodities***. Given $\alpha \in \mathbb{R}^N_+$, to be interpreted as a maximal output vector, there is a ***cost function*** $C: \{x \in \mathbb{R}^N_+ : x \leqq \alpha\} \to \mathbb{R}$ such that $C(0) = 0$. Given x in its domain of definition, the number $C(x)$ is interpreted as the cost of producing the output vector x. Let $\mathcal{P}r^N$ be the class of problems involving the set of commodities N, $\mathcal{P}r = \bigcup_{N \in \mathcal{N}} \mathcal{P}r^N$, and $X_{\mathcal{P}r} = \bigcup_{N \in \mathcal{N}} \mathbb{R}^N$. A ***pricing mechanism*** is a function φ that associates with each $N \in \mathcal{N}$ and each pricing problem $(C, \alpha) \in \mathcal{P}r^N$ a vector $P(C, \alpha) \in \mathbb{R}^N$. Two examples of pricing mechanisms are the following: the ***Shapley value pricing mechanism*** selects for each (C, α) the pricing vector whose i^{th} coordinate is $Sh^*_i(C, \alpha) = Sh_i(v_{(C,\alpha)})/\alpha_i$, where Sh is the Shapley value solution (Subsection 2.2.2), and $v_{(C,\alpha)}$ is the cooperative game whose coordinate relative to coalition S is $C(0_{N \setminus S}, \alpha_S)$. Now, consider the domain of pricing problems for which the cost function is differentiable infinitely many times and such that $\int_0^1 C(t\alpha)dt$ exists. Given a problem (C, α) in that class, the ***Aumann-Shapley pricing mechanism*** selects the vector whose i^{th} coordinate is $\int_0^1 C_i(t\alpha)dt$, where C_i indicates the ith partial derivative of C.[63]

In the definition of a reduced pricing problem, one possibility is to consider problems with fewer commodities.

Self-consistency for pricing problems: The solution $\varphi: \mathcal{P}r \to X_{\mathcal{P}r}$ is ***self-consistent*** if for all $N, N' \in \mathcal{N}$ with $N' \subset N$, all $(C, \alpha) \in \mathcal{P}r^N$, we have $\varphi_i(r^{\varphi}_{N'}(C), \alpha_{N'}) = \varphi_i(C, \alpha)$ for all $i \in N'$ such that $\alpha_i > 0$, where

[63]For a survey of the literature devoted to the analysis of this model, see Tauman (1988).

$r^{\varphi}_{N'}(C)$, the "self-reduced cost function of C relative to N' and φ," is defined by

$$r^{\varphi}_{N'}(C)(z) = \begin{cases} C(z, \alpha_{N\setminus N'}) - \sum_{N\setminus N'} \alpha_j \varphi_j(C, (z, \alpha_{N\setminus N'})) & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Another possibility is to decrease some of the coordinates of z and make the corresponding adjustments in the cost function by pricing those goods whose coordinates have decreased by means of the prices chosen for the original problem. Note that this reduction need not involve a decrease in the dimension of the commodity space.

We will impose the requirement that the solution should produce Shapley prices for the two-commodity case. This requirement is satisfied by many pricing mechanisms, and for that reason we will say that the mechanism is ***standard***. In the second theorem, the term should be understood in a similar sense: for the two-commodity case, the mechanism should coincide with the Aumann-Shapley mechanism.

Theorem 30 (McLean, Pazgal, and Sharkey, 1994) The Shapley value pricing mechanism is the only pricing mechanism to be *standard* and *self-consistent.*

An exact counterpart of this result can be obtained for the domain of infinitely differentiable cost functions.

Theorem 31 (McLean, Pazgal, and Sharkey, 1994) On the subdomain of pricing problems where the cost function is differentiable infinitely many times, the Aumann-Shapley pricing mechanism is the only pricing mechanism to be *standard* and *self-consistent.*

McLean, Pazgal, and Sharkey (1994) also characterize the class of pricing mechanisms that do not satisfy the symmetry condition.

2.4 FAIR ALLOCATION

In this section we turn to economic models of fair allocation. The first one is the standard model: agents are equipped with classical preferences over an ℓ-dimensional commodity space. In the second model, there is only one commodity and agents' preferences are single-peaked. In the third model, indivisible goods are present.

2.4.1 Fair division in classical private good economies

The main model we consider pertains to problems of distribution, but we will also examine a model with production in which agents' abilities may differ. Finally we will turn to a model in which it is the number of goods that is allowed to vary.

(i) Distribution of an unproduced bundle. The fair allocation of a "social endowment" of resources among agents with equal rights on them, say a group of heirs, has to be determined. A solution is applied and each heir is given his share of the social endowment. Now, we focus on a particular subgroup, add up what the members of that subgroup have received, and consider the problem of fairly allocating these resources among them. Should each of them still receive what he had initially received? If the answer is yes and if the answer would be yes independently of preferences, the social endowment, the subgroup, and the allocation initially selected among those chosen by the solution, then the solution is *consistent.*

For instance, suppose that resources are allocated by operating the Walrasian solution from equal division, as illustrated in the two-good three-person example of Figure 2.8a. When the social endowment is Ω, the solution leads to the allocation $z = (z_1, z_2, z_3)$, with associated equilibrium prices p. If the amount $z_1 + z_2$ is to be distributed to the subgroup consisting of the first two heirs, applying the same solution (now, each of them starts out with $\frac{z_1+z_2}{2}$), takes us to the allocation (z_1, z_2), the same prices serving as equilibrium prices: they both end up with the same bundle. This simple example illustrates the fact that the Walrasian solution operated from equal division is *consistent.* Of course, there could be other equilibrium prices in the two-person reduced economy that results by letting agent 3 leave with his assigned bundle, but the initial equilibrium prices remain equilibrium prices.

On the other hand, consider the "Ω-egalitarian-equivalent" method, defined by selecting the allocation(s) at which each agent is indifferent between his assigned consumption and a common multiple of the social endowment Ω. Figure 2.8b shows that neither this solution nor its intersection with the Pareto solution are *consistent.* Indeed, the allocation $z = (z_1, z_2, z_3)$ satisfies both criteria in the three-person economy this figure depicts, but agents 1 and 2's indifference curves through z_1 and z_2 respectively do not intersect on the ray passing through $z_1 + z_2$, the social endowment of the reduced economy associated with the group $\{1, 2\}$ and z.

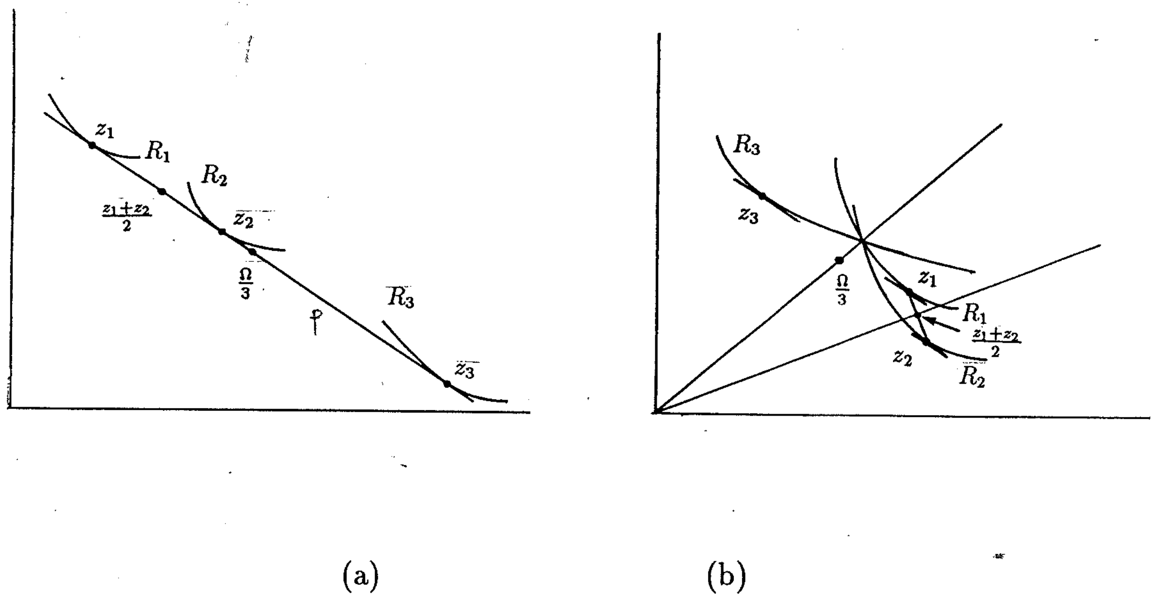


Figure 2.8: ***Consistency*** **for fair division.** (a) The Walrasian solution operated from equal division is *consistent.* (b) The Ω-egalitarian-equivalence and Pareto solution is not *consistent.*

We now give the general definitions. There are ℓ goods and a group $N \in \mathcal{N}$ of agents. For each $i \in N$, R_i is agent i's ***preference relation.*** It is drawn from some class $\mathcal{R}$ of admissible preferences defined over $\mathbb{R}^{\ell}_{+}$. We will often consider the class $\mathcal{R}_{cl}$ of preference relations satisfying the "classical" assumptions of continuity, convexity, and monotonicity. There is also a ***social endowment*** $\Omega \in \mathbb{R}^{\ell}_{+}$. A ***fair division problem*** is a pair $(R, \Omega) \in \mathcal{R}^N \times \mathbb{R}^{\ell}_{+}$. This model is to be distinguished from the usual one in which each agent starts out with a particular share of society's resources, his individual endowment, a formulation that is discussed in Section 3.3. For now, we think of agents as being ***collectively*** entitled to the resources Ω. Let $\mathcal{E}^N$ be a class of admissible problems involving the group N, $\mathcal{E} = \bigcup_{N \in \mathcal{N}} \mathcal{E}^N$, and $X_{\mathcal{E}} = \bigcup_{N \in \mathcal{N}} \mathbb{R}^{\ell N}_{+}$. A ***solution*** is a correspondence that associates with every $N \in \mathcal{N}$ and every $(R, \Omega) \in \mathcal{E}^N$ a non-empty subset of the set of feasible allocations of (R, Ω), namely $\{z \in \mathbb{R}^{\ell N}_{+} : \sum_N z_i \leqq \Omega\}$.

Consistency for fair division problems: The solution $\varphi: \mathcal{E} \rightarrow X_{\mathcal{E}}$ is ***consistent***[64] if for all N, $N' \in \mathcal{N}$ with $N' \subset N$, all $(R, \Omega) \in \mathcal{E}^N$, and all

[64]Thomson (1988) uses the phrase "stability under arbitrary formation of subgroups" and "stability under aggregation" for *converse consistency* defined below.

$z \in \varphi(R,\Omega)$, we have $z_{N'} \in \varphi(r^z_{N'}(R,\Omega))$, where $r^z_{N'}(R,\Omega) = (R_{N'}, \sum_{N'} z_i)$.

It is important to understand how the form we give to *consistency* here relies on the special structure of the feasible set. In the Fundamental Definition, we required that in the reduced problem an alternative be chosen that all departing agents find indifferent to the alternative initially selected. Here, we imagine the departing agents to leave with the ***bundles of goods*** assigned to them by the solution. The opportunities available to the remaining agents are therefore defined as $\{(z'_i)_{i\in N'} \in \mathbb{R}^{\ell N'}_+ : \sum_{N'} z'_i = \Omega - \sum_{N\setminus N'} z_i\}$ instead of $Z' = \{(z'_i)_{i\in N'} \in \mathbb{R}^{\ell N'}_+$: there exists $(z'_i)_{i\in N\setminus N'}$ satisfying $z'_i I_i z_i$ for all $i \in N\setminus N'$ and $\sum_{N'} z'_i = \Omega - \sum_{N\setminus N'} z'_i\}$. The second definition is not unreasonable but note that to apply it, we would actually need to respecify the domain over which solutions are defined.[65]

We stated the condition for solution correspondences. This is the most natural formulation here since resource allocation rules are rarely *single-valued*; we would not want to eliminate from consideration the Walrasian solution operated from equal division, for instance, just because it does not usually select a single allocation.

We noted earlier that this solution is *consistent*. However, other important solutions are too. The ***Pareto*** solution is among them; so are the ***no-envy*** solution (Foley, 1967), which selects the set of feasible allocations z such that for no pair $\{i,j\} \subseteq N$, $z_j P_i z_i$ — at an envy-free allocation no agent would want to exchange bundles with anyone else — and the ***egalitarian-equivalence*** solution (Pazner and Schmeidler, 1978), which selects the set of feasible allocations z such that for some $z_0 \in \mathbb{R}^\ell_+$ and all $i \in N$, $z_0 I_i z_i$. Various solutions designed to evaluate the relative welfares of groups are *consistent* as well; the main example is the solution that selects the allocations for which there is no pair of groups of agents of equal cardinalities such that one group could make all of its members better off if they had access to the resources allocated in total to the other group and could freely redistribute these resources among themselves: this is the ***group no-envy solution***. However, none of the following solutions satisfy the condition: the ***equal division lower bound solution*** — this is the solution that selects the set of feasible allocations that all agents prefer to equal division — its intersection

[65]Indeed, there is in general no $\Omega' \in \mathbb{R}^\ell_+$ such that the feasible set of $(R_{N'}, \Omega')$ is equal to Z', and we initially specified economies to be pairs of a list of preference relations together with a social endowment (see our discussion of closedness of domains in Subsection 1.4.8).

with the Pareto solution, and the ***equal division core*** — this is the solution that selects the set of allocations such that no group of agents could make all of its members better-off if each of them had access to an equal share of the social endowment and they could freely redistribute these resources among themselves. As we noted earlier, the Ω-egalitarian-equivalence solution is not *consistent* either.

Turning now to *converse consistency*, we find that the Pareto solution satisfies the property if appropriate domain restrictions, such as smoothness of preferences, and interiority of the allocation that is being evaluated, are imposed. The issue whether, using modern terminology, the Pareto-optimality of an allocation for some economy can be inferred from the Pareto-optimality of its restrictions to all two-person associated reduced economies (or to all associated reduced economies of cardinality at most t for some $t \in \mathbb{N}$), has been the subject of a number of studies (Rader, 1968; Feldman, 1973; Madden, 1976; Goldman and Starr, 1982; Lainé, 1987.) Figure 2.9 shows that the Walrasian solution operated from equal division is not *conversely consistent* when indifference curves are allowed to have kinks. However, if the class of admissible preferences is restricted so as to guarantee uniqueness of hyperplanes of support to the upper contour sets at every point, *converse consistency* obtains for that solution. No such domain restriction makes the equal division core, or the equal division lower bound solution, or the intersection of the latter with the Pareto solution, *conversely consistent.* On the other hand, the no-envy solution is *conversely consistent* since it is precisely based on pairwise comparisons.

Characterizations of the Walrasian solution operated from equal division are obtained when *consistency* is imposed in conjunction with other related requirements expressing certain forms of invariance of the solution under deletion, or addition, of agents. A representative example is the next result, which involves the requirement that if an allocation is chosen by the solution for some economy, then for all $k \in \mathbb{N}$, its k-times replica should be chosen by the solution for the k-times replica of the economy: in the k-replica of (R, Ω), there are k agents with preferences identical to those of each of the agents in N and the social endowment is $k\Omega$.[66]

[66]Note that it is a direct consequence of the Debreu and Scarf (1963) theorem on the convergence of the core towards the set of Walrasian allocations that under their assumptions, if a subsolution of the equal division core satisfies *replication invariance*, then it is

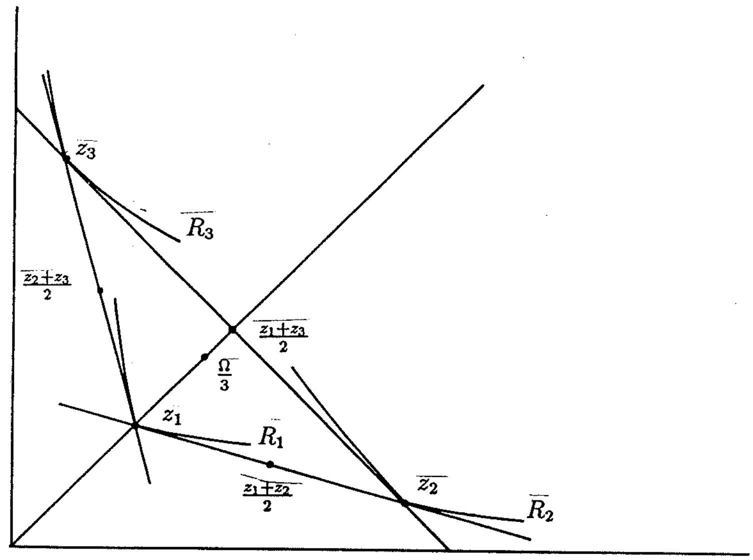


Figure 2.9: ***Converse consistency*** **and the Walrasian solution operated from equal division.** If kinks in indifference curves are permitted, the Walrasian solution operated from equal division is not *conversely consistent.* Indeed, in the example $e = (R, \Omega) \in \mathcal{E}_{cl}^N$ depicted here, where $N = \{1, 2, 3\}$, for each pair of agents $\{i, j\} \subset N$, we have $(z_i, z_j) \in W_{ed}(R_i, R_j, z_i + z_j)$; yet $z \notin W_{ed}(e)$.

Replication invariance: If $z \in \varphi(R, \Omega)$, then $k * z \in \varphi(k * R, k\Omega)$, where $k * z$ is the k-replica of z and $k * R$ is the k-replica of R.

Let $\mathcal{E}_s$ be the subdomain of $\mathcal{E}_{cl}$ of economies where preferences have differentiable numerical representations.

Theorem 32 (Thomson, 1988) On the domain $\mathcal{E}_s$, if a subsolution of the equal division lower bound and Pareto solution satisfies *consistency* and *replication invariance*, then it is a subsolution of the Walrasian solution operated from equal division.

The following result involves an additional condition: any allocation that is Pareto-indifferent to an allocation that the solution selects should also be selected. This is the condition of ***Pareto-indifference***, whose relevance to the understanding of resource allocation problems was discussed by Thomson (1983) and Gevers (1986).

Theorem 33 (Thomson, 1992) On the domain $\mathcal{E}_s$, if a subsolution of the equal division lower bound and Pareto solution satisfies *Pareto-indifference*, *anonymity*, and *converse consistency*, then it is a subsolution of the Walrasian solution operated from equal division for the two-person case. If coincidence actually holds for the two-person case, then the solution contains the Walrasian solution operated from equal division for all cardinalities.

Conditions related to *consistency* and its *converse* can be found in Thomson (1988). For instance, ***separation independence*** says that if for an allocation that is φ-optimal for some economy, agents can be partitioned into two groups such that each group on average receives the average endowment of the whole economy, then the restriction of the allocation to each of the two groups is φ-optimal for the reduced economy relative to the group and the allocation. A converse of this condition, ***juxtaposition independence***, pertains to the juxtaposition of two allocations that are φ-optimal for two disjoint economies with the same endowment per capita. It says that if the resulting allocation is Pareto-optimal for the combined economy, then it is φ-optimal for it. This condition is a strengthening of *replication invariance*.

a subsolution of the Walrasian solution operated from equal division.

In Theorem 32, if *juxtaposition independence* is used instead of *replication invariance*, the distributional requirement can be considerably weakened, to yield another characterization of the Walrasian solution operated from equal division. ***Equal treatment of equals*** says that two agents with the same preferences should receive bundles that lie on a common one of their indifference surfaces.

Theorem 34 (Maniquet, 1996) On the domain $\mathcal{E}_s$, if a subsolution of the Pareto solution satisfies *equal treatment of equals*,[67] *Pareto-indifference*, *juxtaposition independence*, and *consistency*, then it is a subsolution of the Walrasian solution operated from equal division.

If in Theorem 34, instead of *equal treatment of equals* we use the distributional requirement that when equal division is efficient, the rule only selects allocations that are Pareto-indifferent to it, the conclusion this time is that the solution contains the Walrasian solution operated from equal division (even if the solution is not required to be a subsolution of the Pareto solution).

Young (1993) considers a formulation of the problem in which agents may have different "rights", formalized as non negative numbers. In this richer framework, he shows that the Walrasian solution in which incomes are proportional to these numbers is characterized by *consistency* and *replication invariance*, thereby generalizing Theorem 32 to situations where *equal treatment of equals* may not be desirable, as when otherwise identical agents actually represent entities of possibly different sizes, such as households or countries.

Fleurbaey's (1995b) objective is to formalize notions of equal opportunities, and in particular to define methods of compensating agents for non-tranferable characteristics such as handicaps. One of the conditions he imposes is *consistency*. He obtains characterizations of two solutions in the spirit of egalitarian-equivalence.

Finally, we note the logical relations derived by Shimomura (1993) between *consistency*, a condition akin to the weak axiom of revealed preferences, and monotonicity with respect to resources and with respect to population.

[67]The requirement that if all agents have the same preferences the solution selects an allocation consisting of bundles that are all indifferent according to that common preference relation, together with *consistency*, imply *equal treatment of equals*.

(ii) Models with production. Let us now enrich the model by adding production possibilities.[68] First, we consider a model in which each agent is described in term of his preferences over a two-dimensional commodity space and a productivity parameter. Solutions of interest for this model are the following: the ***constant returns-to-scale lower bound solution***, which selects all the allocations that all agents find indifferent to the best they could achieve if they had access to a common constant returns-to-scale technology; the ***work alone lower bound solution***, which selects all the allocations that each agent prefers to the best he could achieve by operating the technology by himself and providing consumptions to the others to which he would not prefer his own. Two *essentially single-valued* solutions are the ***equal-wage equivalent and Pareto solution***, which selects the efficient allocations that each agent finds indifferent to the best he could achieve by maximizing his preferences on a budget set defined by a wage rate that is the same for all agents, and the ***output-egalitarian-equivalence and Pareto solution***, which selects the efficient allocations that all agents find indifferent to a common consumption consisting of only some amount of the output.

We will use the condition of ***equal welfare for equal preferences***, which says that two agents with the same preferences but possibly different productivities should receive consumptions belonging to one of their common indifference curves. Also, ***contraction independence*** says that if an allocation is chosen for some economy and the technology changes in such a way that the set of feasible allocations shrink but the allocation remains feasible, then it should still be chosen.

Theorem 35 (Fleurbaey and Maniquet, 1994) On the general domain of production technologies, the equal-wage equivalent and Pareto solution is the only *essentially single-valued* selection from the constant returns-to-scale lower bound solution satisfying *equal welfare for equal preferences*, *Pareto-indifference*, *contraction independence*, and *consistency.*

Note the difference of domains in the next theorem:

Theorem 36 (Fleurbaey and Maniquet, 1994) On each of the following domain: (i) economies with general production technologies, (ii) economies

[68]The model of Subsection 2.3.3 is of course a production model, but agents are described in terms of a fixed demand parameter, which is irresponsive to the solution.

with decreasing returns-to-scale technologies, (iii) economies with concave technologies, the output-egalitarian-equivalence and Pareto solution is the only *essentially single-valued* selection from the work-alone lower bound solution satisfying *equal welfare for equal preferences*, *Pareto-indifference*, and *consistency.*

Maniquet (1992) considers a class of production economies and formulates a *consistency* condition that is satisfied by the equal-income Walrasian rule and the ***constant returns-to-scale equivalent solution***: this is the solution that selects the efficient allocations that would be obtained by giving each agent access to a common constant returns-to-scale "reference" technology and letting him maximize his preferences. This solution is the only one to satisfy this *consistency* notion together with several other appealing requirements. In a model in which productivities may differ from agent to agent, Maniquet (1996) provides additional characterizations of subsolutions of the Pareto solution based on *replication invariance* and *consistency.* The auxiliary conditions are a lower bound on welfares, a condition of ***skill-solidarity*** which says that all agents should be affected in the same direction by a change in the skill profiles, *Pareto-indifference*, and the implementability condition of *Maskin-monotonicity.*

(iii) Model with a variable number of goods. In all of the models discussed so far, we imagined variations in the number of agents. However, the dimensionality of problems can be affected in other ways. Roemer (1986a,b, 1988, 1990) considers exchange economies and allows for variations in the number of goods instead. He formulates a *consistency* condition with respect to such variations, and bases on it a characterization of egalitarian-type solutions.

In order to state the condition, we need to introduce the number of goods explicitly in the notation. An economy is a list (Q, u, Ω), where $Q \in \mathcal{N}$ is a set of goods, $u = (u_1, \ldots, u_n)$ is a list of n utility functions defined on $\mathbb{R}^Q_+$, and $\Omega \in \mathbb{R}^Q_+$ is a social endowment. Note that we use utility information. We consider the domain of utility functions that are continuous, monotone, concave and such that for all $x \in \mathbb{R}^Q_+$, $\lim_{t \to \infty} t^{-1} u_i(tx) = 0$. Let $\mathcal{D}^Q$ be the domain of economies so defined, $\mathcal{D} = \bigcup_{Q \in \mathcal{N}} \mathcal{D}^Q$, and $X_{\mathcal{D}} = \bigcup_{Q \in \mathcal{N}} \mathbb{R}^{Qn}$. Let a solution defined on $\mathcal{D}$ be given. In the formulation of *consistency*, we apply the solution to a problem in which a vector of goods in $\mathbb{R}^Q_+$ has to be allocated.

Then, given $Q' \subset Q$, we keep at their chosen values all agents' consumptions of the goods in the set $Q \backslash Q'$, and ask whether the solution would distribute in the same way the remaining goods among the same agents when their utility functions are adjusted so as to take into account the fact that they have already received something.[69] A basic solution here is the ***weak Pareto solution***, which selects the feasible allocations such that there is no other feasible allocation that all agents strictly prefer. We will refer to the solution that selects the set of weakly efficient allocations at which all utilities are equal as the ***equal utility and weak Pareto solution***.[70]

In the theorem to be stated below, we impose the requirement of ***equal treatment of equals***, which says that two agents with identical utility functions should receive bundles of equal utilities, and ***continuity***, which says that small changes in the data of the problem should not be accompanied by large changes in the chosen allocation. Apart from the consistency condition, another substantial requirement is ***resource-monotonicity***, which says that all agents should benefit from an increase in the social endowment. Given $z_i \in \mathbb{R}^Q$ and $Q' \subset Q$, we write $z_{i,Q\backslash Q'}$ to denote the projection of z_i onto the subspace $\mathbb{R}_+^{Q\backslash Q'}$.

Consistency across dimensions: The solution $\varphi : \mathcal{D} \to X_{\mathcal{D}}$ is ***consistent across dimensions*** if for all $Q, Q' \in \mathcal{N}$ with $Q' \subset Q$, all $e = (Q, u, \Omega) \in \mathcal{D}^Q$ and $e' = (Q', u', \Omega') \in \mathcal{D}^{Q'}$, and all $z \in \varphi(e)$, if (i) for all $i \in N$, and all $y_i \in \mathbb{R}_+^{Q'}$, $u'_i(y_i) = u_i(z_{i,Q\backslash Q'}, y_i)$, and (ii) $\Omega' = \Omega_{Q'}$, then $z_{Q'} \in \varphi(e')$.

A weaker condition in which each of the goods that is eliminated affects favorably at most one agent (the others' utility functions are constant with respect to it) can be formulated. In that case, an efficient allocation rule allocates the entire amount available to the only agent whose utility function actually depends on the consumption of the good. Let us call this condition ***weak consistency across dimensions***. Note that the Walrasian solution operated from equal division does not satisfy this requirement.

[69]Note that this formalism allows us to treat different goods differently, depending on what kind of goods they are (luxuries or necessities).

[70]Roemer refers to it as "egalitarian".

Theorem 37 (Roemer, 1986) On the domain $\mathcal{D}$, the equal utility and weak Pareto solution is the only subsolution of the weak Pareto solution satisfying *essential single-valuedness*, *Pareto-indifference*, *equal treatment of equals*, *resource-monotonicity*, and *weak consistency across dimensions.*

The proof of this result is based in part on a result of Howe (1987) which states conditions under which two utility functions on $\mathbb{R}^{\ell}_{+}$ can be seen as the restrictions to $\mathbb{R}^{\ell}_{+}$ of a unique utility function defined on the larger space $\mathbb{R}^{\ell+1}_{+}$ by giving the additional argument two different values. Such an extension can be interpreted in terms of the Kolm's (1972) concept of "fundamental preferences". According to that interpretation, if two individuals have different utility functions, it is only because they differ in their consumptions of some "hidden" and non-transferable good. In a series of additional contributions Roemer (1988, 1990) developed characterizations of most of the central solutions of the theory of bargaining. In all of these theorems, *consistency across dimensions* plays a critical role.

> **Bibliographic note.** As usual, in order to recover full optimality, a lexicographic extension of the equal utility solution is natural. A characterization of this extension along the lines of Theorem 37 is obtained by Nieto (1992). Iturbe and Nieto (1996) base another characterization of the solution on a weaker monotonicity requirement, but both *consistency* and *consistency across dimensions.* Donaldson and Roemer (1987) consider the problem of ranking allocations as a function of profiles of individual utility functions. Their central condition is *consistency across dimensions.* Herrero (1995) develops characterizations of solutions to claims problems (Chun and Thomson, 1992) formulated in commodity space, again by using *consistency* and *consistency across dimensions.*

2.4.2 Fair division in economies with single-peaked preferences

A team of three workers is assigned a job that will take Ω hours of work. Each worker is paid a fixed hourly wage and his disutility of labor is a concave function of the time he spends working. As a result, his total utility is

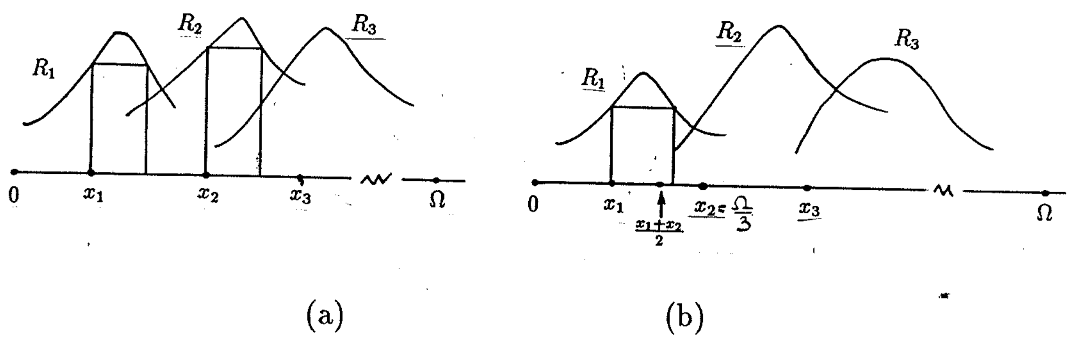


Figure 2.10: *Consistency* in economies with single-peaked preferences. (a) The no-envy solution is *consistent.* (b) The equal division lower bound and Pareto solution is not *consistent.* Indeed, in the three-agent economy depicted here, each agent prefers what he receives to equal division, but if agent 3 leaves with his assigned consumption x_3, agent 1 prefers the new point of equal division $\frac{x_1+x_2}{2}$ to his assigned consumption x_1.

a concave function of the labor he supplies. Figure 2.10 gives numerical representations of the three workers' preferences over labor supplied for two examples. How should the job be divided among them? Fairness, in addition to efficiency, is one of our objectives.

It is easy to see that if Ω is larger than the sum of the preferred supplies, efficiency requires that all three workers work more than they would prefer, and if the opposite holds, that all three workers work less than they would prefer. This condition is sufficient for efficiency. As far as fairness is concerned, we will focus on no-envy and the equal division lower bound, requirements already discussed in the previous section. An envy-free and efficient allocation is illustrated in Figure 2.10a.

As on any domain, the no-envy solution is obviously *consistent*: if x is envy-free, then the restriction of x to the subgroup consisting of agents 1 and 2, say, constitutes an envy-free way of dividing among them the sum $x_1 + x_2$ that they are being assigned. On the other hand, consider the equal division lower bound solution. That it fails *consistency* is illustrated in Figure 2.10b for a three-person example. At $x = (x_1, x_2, x_3)$, each of the three workers prefers what he gets to $\Omega/3$; yet, if we had to divide $x_1 + x_2$ among workers 1 and 2, we would find that (x_1, x_2) does not Pareto-dominate $(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2})$. Note that x is efficient, so that the same example actually shows that the equal division lower bound and Pareto solution is not *consistent.*

For most economies, there are continua of efficient allocations satisfying no-envy or meeting the equal division lower bound, and the question of selection arises. We will look for *consistent* subsolutions of the no-envy and Pareto solution and of the equal division lower bound and Pareto solution. An appealing *single-valued* solution, which is a selection from both, is the ***uniform rule***, introduced in the fix-price literature and recently characterized by Sprumont (1991) on the basis of incentive considerations. It is defined as follows: for all $i \in N$, let $p(R_i)$ be the preferred "consumption" of worker i, that is, his preferred labor supply. If $\sum p(R_i) \geqq \Omega$, let $\lambda \in \mathbb{R}_+$ such that (i) giving his preferred consumption to every worker whose preferred consumption is smaller than λ and, (ii) giving λ to the others, defines a feasible allocation. If $\sum p(R_i) \leqq \Omega$, pick λ such that (i) giving λ to every worker whose preferred consumption is greater than λ and, (ii) giving their preferred consumptions to the others, defines a feasible allocation. The existence and uniqueness of λ is easily established. The allocation that results for that λ is the uniform allocation. The rule, which is illustrated in each of the two cases in Figure 2.11, is clearly *consistent.*

We now turn to the formal definitions. There is a group $N \in \mathcal{N}$ of agents among whom to allocate Ω units of an infinitely divisible commodity. For all $i \in N$, R_i is agent i's preference relation. This preference relation, defined over $\mathbb{R}_+$, is continuous and ***single-peaked***: this means that there is a number $p(R_i) \in \mathbb{R}_+$ such that for all x_i, $x_i' \in \mathbb{R}_+$, if $x_i' < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < x_i'$, then $x_i P_i x_i'$. Let $\mathcal{R}_{sp}$ designate the class of all such preference relations. A ***fair division problem with single-peaked preferences*** is a pair $(R, \Omega) \in \mathcal{R}_{sp}^N \times \mathbb{R}_+$. Let $\mathcal{E}_{sp}^N$ be the domain of such problems, $\mathcal{E}_{sp} = \bigcup_{N \in \mathcal{N}} \mathcal{E}_{sp}^N$, and $X_{sp} = \bigcup_{N \in \mathcal{N}} \mathbb{R}_+^N$. A ***solution*** is a correspondence that associates with every $N \in \mathcal{N}$ and every $(R, \Omega) \in \mathcal{E}_{sp}^N$ a non-empty subset of $\{z \in \mathbb{R}_+^N : \sum_N z_i = \Omega\}$, the set of feasible allocations of (R, Ω). Note that feasibility is defined with an equality sign, indicating that we do not assume

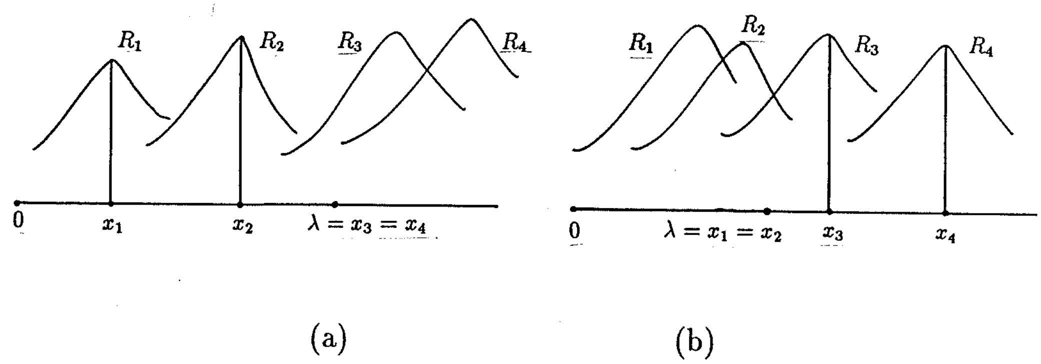


Figure 2.11: The uniform rule. (a) The case $\Omega \leqq \sum p(R_i)$: all agents whose preferred consumptions are smaller than λ get their preferred consumptions; the others get λ. (b) The case $\sum p(R_i) \leqq \Omega$: all agents whose preferred consumptions are greater than λ get their preferred consumptions; the others get λ. In each case, λ is chosen so as to obtain feasibility.

free disposability of the commodity.[71]

The Pareto solution is clearly *consistent.*[72] We have already noted that so are the uniform rule and the no-envy solution, but that the equal division lower bound and Pareto solution is not. A negative result holds for the equal division core as well.[73] This pattern of positive and negative results is repeated for *converse consistency.*

The implications of *consistency*, and in fact, of its bilateral version and of its converse, can be very completely described in this model, when these properties are used in conjunction with no-envy or the equal division lower bound. One of the auxiliary conditions in the theorem below is the following continuity requirement with respect to the amount to be divided. Let ν be our generic notation for the natural numbers, and $\{\Omega^\nu\}$, $\{z^\nu\}$ denote sequences in $\mathbf{R}_+$ and $\mathbf{R}_+^N$ respectively.

[71]As noted earlier, the axiomatic analysis of this model was initiated by Sprumont (1991). Additional studies of this model, not dealing with *consistency*, are due to Ching (1992,1993), Ching and Serizawa (1994), Gensemer, Hong, and Kelly (1992, 1996), Klaus, Peters, and Storken (1995), and Barberà, Jackson, and Neme (1995).

[72]We do not explicitly write out the statements of *consistency* and *converse consistency* for this domain since they are the same as for classical economies.

[73]To obtain non-emptiness of the equal division core, we need to define it as the solution that selects all the allocations such that no subgroup of agents can make all of its members strictly better-off when each of them is given access to equal division.

Continuity: If $z^\nu \in \varphi(R, \Omega^\nu)$ for all $\nu \in \mathbb{N}$, $\Omega^\nu \to \Omega$ as $\nu \to \infty$, and $z^\nu \to z$ as $\nu \to \infty$, then $z \in \varphi(R, \Omega)$.

The main result on the issue is the following:

Theorem 38 (Thomson, 1994a) If a subsolution of the no-envy and Pareto solution satisfies *continuity* and *consistency*, then it contains the uniform rule.

Since the equal division lower bound solution is a subsolution of the no-envy solution in the two-person case and the no-envy solution is *conversely consistent*, we deduce as a corollary of Theorem 38 that if a subsolution of the equal division lower bound and Pareto solution satisfies *consistency* and *continuity*, then it too contains the uniform rule. As other corollaries, we obtain complete descriptions of the classes of subsolutions of the no-envy and Pareto solution, and alternatively of the equal division lower bound and Pareto solution, satisfying *consistency* and *continuity* together, or *bilateral consistency* and *continuity* together. By imposing *single-valuedness* in addition, we derive characterizations of the uniform rule. Dagan (1996) shows that under that requirement this rule can in fact be obtained in a very direct way, without imposing *continuity* in addition, and even if the maximal number of agents is finite.[74]

A different set of axioms, which does not include *continuity* either, also leads to the uniform rule. This result is a simple corollary of a characterization of the class of subsolutions of the equal division lower bound and Pareto solution satisfying *converse consistency* and the requirement that if two agents have identical preferences, exchanging their consumptions in an allocation chosen by the solution gives an allocation that should also be chosen:

Weak equal treatment of equals: If $R_i = R_j$ and $x \in \varphi(R, \Omega)$, then x' defined by $x'_i = x_j$, $x'_j = x_i$ and $x'_k = x_k$ for all $k \notin N \backslash \{i, j\}$, is such that $x' \in \varphi(R, \Omega)$.

Theorem 39 (Thomson, 1994) The uniform rule is the only *single-valued* selection from the equal division lower bound and Pareto solution satisfying *weak equal treatment of equals* and *converse consistency.*

[74]This result is obtained under the assumption that the set of potential agents has cardinality greater than, or equal to, 4.

Finally, we have the following counterpart of Theorem 33:

Theorem 40 (Thomson, 1994) The uniform rule is the only subsolution of the equal division lower bound and Pareto solution satisfying *consistency* and *replication invariance.*

Sönmez (1994) obtains characterizations of the uniform rule without imposing efficiency, but instead combining *consistency* with monotonicity requirements with respect to the amount to be divided on the one hand and the number of agents on the other. He derives parallel results with *converse consistency* replacing *consistency.*

Otten, Peters, and Volij (1996) show that the uniform allocation of an economy can be obtained as the Nash solution outcome (Section 2.2.1) of an associated bargaining problem defined as follows: suppose that each agent has preferences that are symmetric with respect to his preferred consumption. Such preferences can be represented by a piece-wise linear "utility" function having a slope of 1 to the left of the preferred consumption and a slope of -1 to the right. Now, take the image in utility space of the set of feasible allocations when agents are equipped with such utility functions. This image is a truncated simplex. This is the problem to which they apply the Nash solution. The uniform allocation is also the lexicographic egalitarian solution outcome of that problem (again, see Section 2.2.1). Since the uniform rule depends only on preferred consumptions, the uniform allocation can be obtained in this way even if preferences do not have the symmetry postulated above. Interestingly, these authors' proof of the coincidence of these solutions relies on the *consistency* of the Nash and lexicographic egalitarian solutions. These equivalences allow them to characterize the uniform rule by means of lists of axioms related to lists known in the theory of bargaining to characterize the Nash and lexicographic egalitarian solutions.

De Frutos and Massó (1995) suggest a way of associating with each economy a coalitional form game, and they study the correspondences between counterparts of *max-consistency* and *self- consistency* (Section 2.2.2-3) for the class of games so obtained and *consistency* as we defined it earlier. Of particular interest is the equivalence under efficiency of *max-consistency* and *consistency.* They also provide a characterization of the uniform rule exploiting these correspondences.

Klaus, Peters, and Storcken (1996) consider a "dual" version of the model in which agents have "single-troughed" preferences[75] and identify the class of selections from the Pareto solution satisfying *strategy-proofness* and *consistency.*

Shimomura (1993) provides a characterization of the natural extension of the uniform rule to the domain of economies with "single- plateaued" preferences (here, there is a possibly non-degenerate interval of preferred consumptions), along the lines of Theorem 38. He uses *flexibility* instead of *consistency*, exploiting the equivalence of the two properties for economies in which uniqueness of the φ-optimal outcome holds (Section 1.7.3).

Bibliographic note. Sonn (1994) derives the uniform allocation as the limit of solution outcomes of sequential strategic games. This derivation relies importantly on *consistency* arguments.

2.4.3 Fair allocation in economies with indivisible goods

There are 3 jobs that have to be assigned to 3 workers with the same seniorities and qualifications. The jobs are not identical and the workers' preferences over job-salary packages differ. Salaries can be chosen to compensate agents for being assigned jobs that they find less desirable. The sum of the salaries is not to exceed a certain budget. How should jobs be assigned and salaries determined?

Figure 2.12a represents 3 axes, indexed by the three jobs. Along each axis is measured the salary that will be associated with the corresponding job. To keep track of which job-salary combinations an agent finds indifferent to each other, we connect them by an "indifference curve". A few sample indifference curves are indicated for agent 1, and one indifference curve for each of agents 2 and 3.

The notion of an envy-free allocation applies to this situation just as well as to the classical domain. Depending on how the specification of the model is completed, envy-free allocations may or may not exist, but when they do, there often is a continuum of them, and again a natural question is how to make selections from this continuum.

[75]For such a preference relation, there is a worst consumption, and moving away from it, in either direction, makes the agent better off.

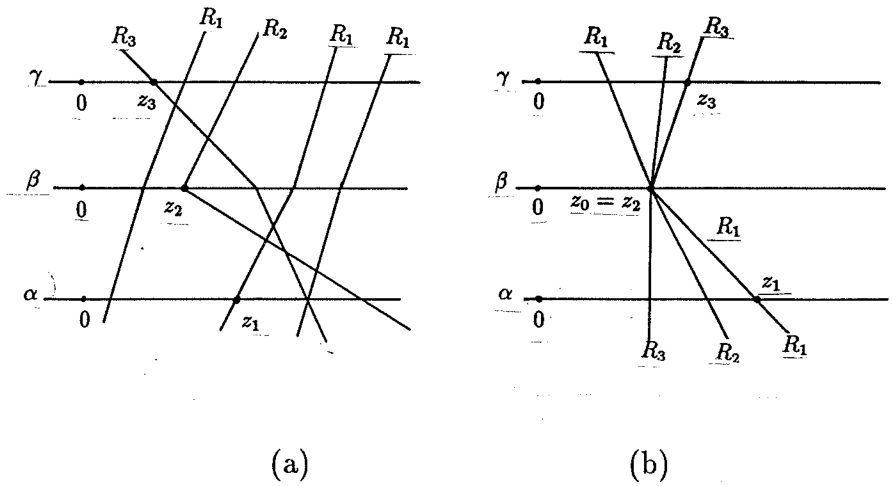


Figure 2.12: *Consistency* in economies with indivisible goods. (a) The no-envy solution is *consistent.* (b) The egalitarian-equivalence and Pareto solution is not *consistent.*

Here too, we will attempt to do this by requiring *consistency* of the selection method. In the present context, we find it most natural in the definition of a reduced economy to restrict preferences to the consumption subspace corresponding to the remaining jobs, that is, to use what we called "reduced preference relations" (see Part 2, section 3g). Some of the results below depend on this reduction having taken place. Consider in particular the solution that picks the efficient allocations for which there is a reference bundle (α_0, m_0) that each agent finds indifferent to his assigned consumption, α_0 being chosen arbitrarily among the objects actually present. This straightforward adaptation of the egalitarian-equivalence and Pareto solution of Pazner and Schmeidler (1978) is not *consistent*, in contrast with the behavior of the solution on the classical domain. For instance, the allocation $z = (z_1, z_2, z_3)$ of Figure 2.12b satisfies this definition with reference bundle z_2. However, if agent 2 leaves the scene, he takes with him the object used in defining the reference bundle making z one of the egalitarian-equivalent allocations in the original economy. On the other hand, it is easy to see that the no-envy solution is *consistent*, whether or not preference relations are reduced.

(i) Multiple-object case. We will consider several versions of the model, and give the formal definitions first in the general case of multiple objects. There is a group $N \in \mathcal{N}$ of agents and a collection A of ***objects*** taken from some infinite set $\mathcal{A}$. We require $|N| = |A|$. An amount $\Omega \in \mathbb{R}_+$ of an infinitely divisible good, called ***money***, is also available for distribution. Each agent $i \in N$ is equipped with a ***preference relation*** R_i defined over the space $A \times \mathbb{R}$. This relation is continuous and strictly monotonic in its second argument, and such that for all $(\alpha, m_i) \in A \times \mathbb{R}$, and all $\alpha' \in A$, there is $m' \in \mathbb{R}$ such that $(\alpha, m_i) I_i (\alpha', m')$. Let $\mathcal{R}_{ind}$ be the class of these preference relations. Each agent should consume one and only one object. A ***fair allocation problem with indivisible goods*** is a triple $(R, A, \Omega) \in \mathcal{R}_{ind}^N \times \mathcal{A} \times \mathbb{R}$. A ***feasible allocation*** for (R, A, Ω) is a pair $z = (\sigma, m)$ of a bijection $\sigma \colon N \to A$ specifying which object each agent receives, and a vector $m \in \mathbb{R}^N$ such that $\sum_N m_i = \Omega$ specifying how much money each agent receives. Let $\mathcal{O}^N$ be the class of these problems, $\mathcal{O} = \bigcup_{N \in \mathcal{N}} \mathcal{O}^N$, and $X_{\mathcal{O}} = \bigcup_{N \in \mathcal{N}} (A \times \mathbb{R})^N$. A ***solution*** is a correspondence that associates with every $N \in \mathcal{N}$ and every $(R, A, \Omega) \in \mathcal{O}^N$ a non-empty subset of its feasible set.[76]

The primary solution here is the ***no-envy solution***, already introduced. Under the assumptions made above, the no-envy solution is non-empty (see Maskin, 1987, Alkan, Demange, and Gale, 1991, Aragones, 1995, for existence results). It also turns out to be a subsolution of the ***Pareto solution*** (Svensson, 1983). Moreover, the ***group no-envy solution*** and the ***equal-income Walrasian solution***, first encountered in the context of classical economies, and which are easily adapted to the current model, coincide with the no-envy solution (Svensson, 1983). Another interesting solution is the solution that associates with each economy the set of allocations that each agent prefers to the unique (up to Pareto-indifference) envy-free allocation that would exist if everyone had the same preferences as his (Moulin, 1990). We will refer to it as the ***identical-preferences lower bound solution***. We also have the ***egalitarian-equivalence and Pareto solution*** and the selections from it obtained by requiring the reference bundle to contain a specific object. Finally, we will consider the intersections of these last two solutions with the Pareto solution.

[76]This model has been considered by Svensson (1983), Maskin (1987), Alkan, Demange, and

Gale (1991), Tadenuma and Thomson (1991,1993,1995), Moulin (1991), Bevia (1993, 1996), Alkan (1994), and Aragones (1995).

As discussed earlier, we only allow solutions to depend on agents' preferences over the bundles containing the objects that are actually present: when some agents leave with their assigned objects, the preferences of the remaining agents over consumptions containing the vanished objects are ignored in deciding whether the resources they have collectively received should or should not be reallocated among them. It is for that reason that the egalitarian-equivalence solution and its selections discussed above are not *consistent*. In the statement of *consistency*, we use the preference relations so restricted, the "reduced" preference relations. If $N' \subset N$, let $R_{i_{|\sigma(N') \times \mathbb{R}}}$ denote the restriction of agent i's preferences from $A \times \mathbb{R}$ to $\sigma(N') \times \mathbb{R}$.

Consistency for allocation problems with indivisible goods: The solution $\varphi: \mathcal{O} \rightarrow X_{\mathcal{O}}$ is ***consistent*** if for all $N, N' \in \mathcal{N}$ with $N' \subset N$ and all $(R, A, \Omega) \in \mathcal{O}^N$, if $z \in \varphi(R, A, \Omega)$, then $z_{N'} \in \varphi((R_{i|\sigma(N') \times \mathbb{R}})_{i \in N'}, \sigma(N'), \sum_{N'} m_i)$.

We omit the formal statement of *converse consistency*, which is straightforward. Just as on the classical domain, the Pareto solution and the no-envy solution are *consistent*. The former is not *conversely consistent* but the latter still is.

To state the results, we need the following very mild condition: if an allocation obtained by exchanges of bundles from one that is chosen by the solution leaves unaffected the welfares of all agents, then it should also be chosen:

Neutrality: If $z \in \varphi(R, A, \Omega)$, z' is obtained from z by switching around its components, and $z_i I_i z'_i$ for all $i \in N$, then $z' \in \varphi(R, A, \Omega)$.

Theorem 41 (Tadenuma and Thomson, 1991) If a subsolution of the no-envy solution satisfies *neutrality* and *consistency*, then it coincides with the no-envy solution.

There is an infinity of subsolutions of the no-envy solution satisfying *neutrality* and *bilateral consistency*, or *neutrality* and *converse consistency*, but they can be completely described. When all three conditions are used together, we obtain the following additional characterization of the no-envy solution:

Theorem 42 (Tadenuma and Thomson, 1991) If a subsolution of the no-envy solution satisfies *neutrality*, *bilateral consistency*, and *converse consistency*, then it coincides with the no-envy solution.

Bevia (1996) investigates the existence of *consistent* selections from the identical-preferences lower bound solution. This solution contains the no-envy solution and it coincides with it in the two-person case. It is not *consistent* but it is *conversely consistent.* Given that the no-envy solution is *conversely consistent*, a consequence of these observations and Theorem 41 is that the no-envy solution is its only *neutral* and *consistent* subsolution.

A special case of the model just described is when the objects are all identical. For instance, imagine the allocation of jobs on an assembly line when there are more workers than jobs, all extra workers remaining unemployed. Of course, on this subdomain, the Pareto solution and the no-envy solution remain *consistent* but now the Pareto solution is *conversely consistent.* Moreover, the domain is still sufficiently rich to allow characterizations of the no-envy solution along the lines of Theorems 41 and 42 (Tadenuma and Thomson, 1991). Bevia (1996) shows that her results also extend to that case.

(ii) One-object case. If we consider the even more specialized situation in which there is a single object to allocate, the results change in a significant way.

To give a geometrical representation of this case, we need only two axes, one indexed by the object, which is received by exactly one person, the "winner" and the other indexed by what we will call the "null object", which is received by all other agents, the "losers". The null object is denoted by ν on the figure. For the losers not to envy each other, they should receive the same amount of money.

This respecification of the domain has important consequences. Indeed, *consistency* is now satisfied by many proper subsolutions of the no-envy solution. An example is the solution ***F^* that selects the envy-free allocation(s) such that the winner is indifferent between what he receives and what the losers receive***, as represented in Figure 2.13a: the winner's indifference curve through his bundle passes through the losers' common bundle. We also need to consider the case when there is only money to allocate, or when there is only one agent. In any one of these cases, there

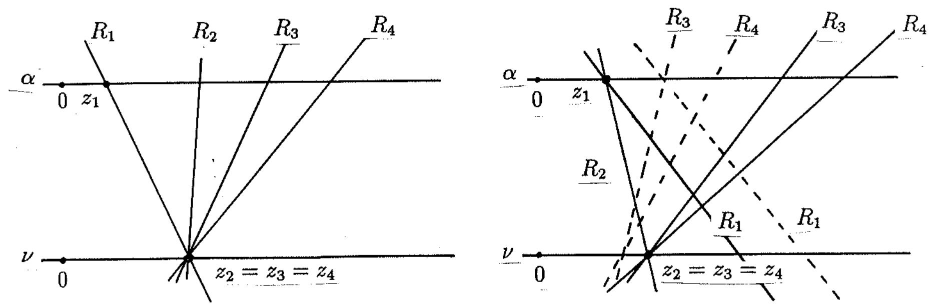


Figure 2.13: ***Consistency*** **in economies with a single indivisible good.** (a) The solution F^*, which selects the envy-free allocation such that the winner is indifferent between his bundle and the losers' common bundle, is *consistent.* (b) The "dual" solution, which selects the envy-free allocation at which one loser is indifferent between his bundle and the winner's bundle, is not *consistent.* In this example, if agent 2 leaves, the winner could be moved further to the right (consider the dashed indifference curves).

is a unique envy-free allocation, which of course the solution should pick. Now, for the solution so defined, if some but not all of the losers leave the scene with their assignments of money, the required indifferences remain in the resulting reduced economy. The degenerate cases, when all the losers leave or when the winner leaves, are equally easy to check. Therefore, the solution F^* is *consistent.* It is also *conversely consistent.* On this restricted domain, both the Pareto solution and the egalitarian-equivalence and Pareto solution are *conversely consistent* too.

The following theorem is the central result for this model:

Theorem 43 (Tadenuma and Thomson, 1993) One-object case. If a subsolution of the no-envy solution satisfies *neutrality* and *bilateral consistency,* then it contains the solution F^*.

As corollaries of Theorem 43, we obtain characterizations of the classes of subsolutions of the no-envy solution satisfying *neutrality* and *consistency,* or *neutrality* and *bilateral consistency.* A straighforward consequence is that

F^* is the only *neutral* and *consistent* subsolution of the no-envy solution satisfying the following requirement:

Single-valuedness up to indifferent exchanges: If z, $z' \in \varphi(R, A, \Omega)$, then z' is obtained from z by switching around its components and $z_i I_i z'_i$ for all $i \in N$.

(iii) Related models. Bevia (1993) studies a more general model in which each agent may receive more than one object. The assumption brings about a considerable enlargment of the space of admissible preferences, and the results presented above do not extend: it is not true any more that the no-envy solution is a subsolution of the Pareto solution; more significant is that the no-envy and Pareto solution is not *conversely consistent*; finally there are no counterparts of Theorems 41 and 42.[77]

Fleurbaey (1995a) considers a model in which agents differ with respect to a variable that can be interpreted as talent or handicap. Then, preferences are defined over the cross-product with $\mathbb{R}$ of the set of possible levels of this variable. The level of the variable is fixed for each agent. Therefore, his model can be interpreted as a special case of the model examined above in which the indivisible goods cannot be reassigned. He focuses on solutions satisfying *consistency* and *converse consistency*, and studies the compatibility of these properties with other normatively appealing desiderata.

Of particular interest are certain logical relations that he notes between *consistency* and various monotonicity properties. These relations, also pointed out by Chun (1985) in the context of bankruptcy problems, extend beyond the framework of the particular model that they study. For instance, any *consistent* solution that is *resource-monotonic* (all agents benefit from an increase in the amount of resources to divide) is *population-monotonic* (all agents initially present lose if the number of agents increases but resources stay fixed; see Section 2.2.1 for an application of the idea).

2.5 OTHER MODELS

Here, we examine two additional models. The first one, apportionment, is familiar to political scientists. The second one is a model of matching,

[77]The same conclusions hold on the subdomain of quasi-linear economies.

which has many applications. Both models are characterized by their discrete mathematical structure.

2.5.1 Apportionment problems

One of the oldest problems in political science is that of attributing seats to states in order to get as close to proportional representation as possible. The problem arises because rounding is necessary, and it is important because which rounding method is used may dramatically affect the representation of small states. Consider the three-state apportionment problem described in Table 3, and let us solve it according to two well-known methods: for ***Jefferson's method***, choose a divisor of the populations of the states so that the whole numbers contained in the quotients sum to the total number of seats. Then, give to each state its whole number. For ***Hamilton's method***, define the "quota" of each state to be the ratio of its population to the aggregate population times the total number of seats. Give to each state the whole number contained in its quota. Assign the remaining seats to those states having the largest fractions.

Note that in the example, under Jefferson's method states 1 and 3 have been allocated 1 and 5 seats respectively, for a total of 6. Applying this method to the problem of allocating 6 seats to these states produces exactly the same apportionment (1,5). This is because Jefferson's method is *consistent.* However, Hamilton's method is not since initially, it would assign 1 seat and 5 seats to states 1 and 3 respectively, for the same total of 6 seats, but applying the method to the allocation of 6 seats among them produces the apportionment (2,4).

We now turn to the general definitions. An ***apportionment problem*** is a pair $(s, H) \in \mathbb{N}^N \times \mathbb{N}$: the members of $N \in \mathcal{N}$ are states with ***populations*** given by the coordinates of s; H is the number of seats in the ***house***. The objective is to allocate seats to states as close to proportionally to their populations as possible. Let $\mathcal{A}^N$ be the class of these problems, $\mathcal{A} = \bigcup_{N \in \mathcal{N}} \mathcal{A}^N$, and $X_{\mathcal{A}} = \bigcup_{N \in \mathcal{N}} (\mathbb{N} \cup \{0\})^N$. A ***solution*** is a correspondence that associates with every $N \in \mathcal{N}$ and every $(s, H) \in \mathcal{A}^N$ a vector in $(\mathbb{N} \cup \{0\})^N$ whose coordinates add up to H.

Balinski and Young (1982) carry out an extensive analysis of apportionment. One of their main axioms is *consistency.* It corresponds directly to

States' Population	Jefferson's method applied to		Hamilton's method applied to	
s	$(s, 10)$	$(s_1, s_3, 6)$	$(s, 10)$	$(s_1, s_3, 6)$
$s_1 = 200$	1	1	1	2
$s_2 = 500$	4		4	
$s_3 = 590$	5	5	5	4

Table 2.3: *Consistency* of apportionment methods. Jefferson's method is *consistent* but Hamilton's method is not. In the example, $N = \{1, 2, 3\}$, $s = (200, 500, 590)$, and $H = 10$.

what we have already seen a number of times. The second condition they use was introduced in Subsection 1.7.1. under the name of *flexibility*. It says if $x \in \varphi(s, H)$ and $y \in \varphi(r^x_{N'}(s, H))$, then (s, H) admits as solution outcome the juxtaposition of y with $x_{N \setminus N'}$, namely $(y, x_{N \setminus N'})$.[78]

Consistency for apportionment problems: The solution $\varphi\colon \mathcal{A} \to X_{\mathcal{A}}$ is ***consistent*** if for all N, $N' \in \mathcal{N}$ with $N' \subset N$, all $(s, H) \in \mathcal{A}^N$, and all $x \in \varphi(s, H)$, we have $x_{N'} \in \varphi(r^x_{N'}(s, H))$, where $r^x_{N'}(s, H) = (s_{N'}, \sum_{N'} x_i)$.

Flexibility for apportionment problems: The solution $\varphi\colon A \to X_{\mathcal{A}}$ is ***flexible*** if for all N, $N' \in \mathcal{N}$ with $N' \subset N$, all $(s, H) \in \mathcal{A}^N$, all $x \in \varphi(s, H)$, and all $y \in \varphi(r^x_{N'}(s, H))$, we have $(y, x_{N \setminus N'}) \in \varphi(s, H)$.

In addition to *consistency*, we will require of solutions that they satisfy two properties. For this model, as in most models with a discrete structure, there is sometimes no way of treating identical agents identically. Instead, we will first demand that if two states have equal populations, their apportionments should not differ by more than one seat:

Balancedness: $s_i = s_j$ implies $|\varphi_i(s, H) - \varphi_j(s, H)| \leq 1$.

Next, we will ask that the the solution should be invariant under exchanges of the names of the states:

[78]What they call "uniformity" is the conjunction of *consistency* and *flexibility*.

Anonymity: If $|N| = |\tilde{N}|$, and $\pi: N \rightarrow \tilde{N}$ is a permutation, then $\varphi(\pi(s), H) = \pi(\varphi(s, H))$.

Let $r: \mathbb{C} \times \mathbb{N} \cup \{0\}) \rightarrow \mathbb{R}$ be a monotone decreasing function of its second argument, and let $\mathcal{F}$ be the class of all functions $f: \mathbb{C}^N \times (\mathbb{N} \cup \{0\}) \rightarrow (\mathbb{N} \cup \{0\})^N$ defined recursively as follows:

(i) for $H = 0$, $f(s, H) = (0, \ldots, 0)$

(ii) if $f(s, H) = x$, then $f(s, H+1)$ is obtained by giving $x_i + 1$ seats to some state $i \in N$ such that $r(s_i, x_i) \geq r(s_j, x_j)$ for all $j \in N \backslash \{i\}$, and x_j seats to all $j \in N \backslash \{i\}$.

Then, the ***rank index solution based on r*** is defined by $\varphi(s, H) = \{x \in (\mathbb{N} \cup \{0\})^N : x = f(s, H) \text{ for some } f \in \mathcal{F}\}$.[79]

Theorem 44 (Balinski and Young, 1982) The rank index solutions are the only solutions satisfying *balancedness, anonymity, consistency*, and *flexibility.*

By imposing two further conditions, an interesting refinement of Theorem 44 can be obtained. It identifies the family defined as follows: given a monotone increasing function $d = \mathbb{N} \cup \{0\} \rightarrow \mathbb{N} \cup \{0\}$ such that $a \leq d(a) \leq a+1$, the ***divisor solution based on d*** is defined by $\varphi(s, H) = \{x \in (\mathbb{N} \cup \{0\})^N : d(x_i - 1) \leq s_i/\lambda \leq d(x_i) \text{ and } \sum_N x_i = H$ for some $\lambda \in \mathbb{R}_+\}$.

The first additional condition says that if the populations of the states change in the same proportions, the apportionment should be unchanged. Let $\mathbb{C}$ denote the set of rational numbers:

Homogeneity: $\varphi(\lambda s, H) = \varphi(s, H)$ for all $\lambda \in \mathbb{C}_{++}$, the set of positive rational numbers.

If there is a feasible apportionment proportional to the populations, it should be the only one recommended by the solution:

Proportionality when possible: If there exists $x \in (\mathbb{N} \cup \{0\})^N$ such that $\sum_N x_i = H$ and $x = \lambda s$ for some $\lambda \in \mathbb{C}_{++}$, then $\varphi(s, H) = \{x\}$.

[79] Jefferson's method is the member of the family obtained for r defined by $r(s_i, x_i) = s_i/(x_i + 1)$.

The second requirement is that if a state has a greater population than another one, it should receive at least as many seats at any apportionment recommended by the solution:

Order preservation: If $s_i > s_j$, and $x \in \varphi(s, H)$, then $x_i \geq x_j$.

Theorem 45 (Balinski and Young, 1982) The divisor solutions are the only ones to satisfy *homogeneity, anonymity, proportionality when possible, order preservation*, *consistency*, and *flexibility.*

> **Bibliographic note.** The more general problem of allocating seats "proportionally to both populations and parties" has been examined by Balinski and Demange (1989a,b). *Consistency* is one of the axioms they use. Balinski (1995) also invokes *consistency* in his discussion of the problem of rounding.

2.5.2 Matching and assignment problems

There are three men and three women. Each of the men has preferences over the women and each of the women has preferences over the men. These preferences are represented by the functions graphed in Figure 2.14: a score of 0 is assigned to the least preferred mate, a score of 2 to the most preferred, and a score of 1 to the intermediate one. In the example, man 1 prefers woman 2 to woman 3, and woman 3 to woman 1 (scores are indicated horizontally).

The objective is to find a way of matching men and women. One method consists in selecting matches such that no pair of a man and a woman prefer each other to their assigned mates. Say that such a match is ***divorce-proof***. There always is at least one such match (Gale and Shapley, 1962). In the example of Figure 2.14, the match indicated by the arrows passes the test. Now, note that m_3 and w_3 have been paired. Let us imagine them to leave the scene. Would the restriction of the match to the four remaining agents be divorce-proof? The answer is of course yes. This is because this solution is *consistent.*

On the other hand, the solution that selects the match that minimizes the sum of rankings of all agents within the core (McVitie and Wilson, 1971) is not *consistent.*

Another application of the above model is to the assignment of medical residents to positions in hospitals. A generalization is when each agent on

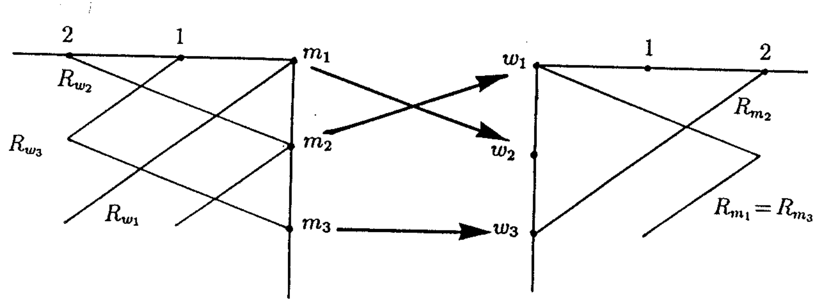


Figure 2.14: *Consistency* for matching problems. A six-person matching problem. If the match is divorce-proof, then its restriction to any subgroup obtained by deleting married pairs, is also divorce-proof.

one side of the market can be matched to several agents on the other side, as would be the case for the matching of workers to firms. An application of this more general model that has been the object of much literature is to the so-called "college admission" problem. Here, we limit ourselves to one-to-one matches.

Here are the formal definitions. We consider groups $N \in \mathcal{N}$ that are the union of two subgroups N_M and N_W of ***men*** and ***women*** respectively. A ***matching problem*** is a list $R = (R_i)_{i \in N}$, where for each $i \in N$, R_i denotes agent i's ***strict preference relation*** over the members of the opposite sex: if $i \in N_M$, R_i is defined over N_W; if $i \in N_W$, R_i is defined over N_M. We assume equality of the numbers of men and women: $|N_M| = |N_W|$.[80] The objective is to pair men and women, *i.e.* to find a ***match***, simply a bijection $b: N_M \to N_W$. Let $\mathcal{B}^N$ be the set of all such bijections. Let $\mathcal{M}^N$ be the class of these problems, $\mathcal{M} = \bigcup_{N \in \mathcal{N}} \mathcal{M}^N$, and $X_{\mathcal{M}} = \bigcup_{N \in \mathcal{N}} \mathcal{B}^N$. A ***solution*** is a correspondence assigning with every $N \in \mathcal{N}$ and every $R \in \mathcal{M}^N$, a non-empty subset of $\mathcal{B}^N$. This model, first formulated by Gale and Shapley (1962), has been the object of a now considerable literature, reviewed by

[80] See below for a discussion of the case when the two groups may have unequal cardinalities (and situations where not all agents have to be matched even when equality of cardinalities holds). Formally, this situation is accommodated by the introduction of a "single" status. In the application to assignments of workers to firms, this corresponds to unemployment for a worker and closing down for a firm.

Roth and Sotomayor (1990).

The ***stable solution*** is the solution that chooses for all $N \in \mathcal{N}$, and all $R \in \mathcal{M}^N$, the set of matches $b \in \mathcal{B}^N$ such that for no $(m, w) \in N_M \times N_W$, $wR_m b(m)$ and $mR_w b^{-1}(w)$, with at least one strict preference. This solution actually coincides with ***the core***, the solution that chooses the set of matches that cannot be improved upon by any subgroup, and from here on, we will refer to it by that name.

Just as in the case of the assignment of indivisible goods, it is appealing in specifying a reduced problem to redefine the preferences of the remaining agents, here by simply restricting them to the set of possible remaining partners. Given $N \in \mathcal{N}$, $R \in \mathcal{M}^N$, $b \in \mathcal{B}^N$, and $N' \subset N_M$, let $b(N') = \{w \in N_W : w = b(m) \text{ for some } m \in N'\}$. Given $m \in N'$, let $R_{m|b(N')}$ be the restriction of man m's preferences to the women in $b(N')$ and let $R_{w|N'}$ be symmetrically defined. In the reduced problem the preference profile is $(R_{m|b(N')})_{m \in N'} \cup (R_{w|N'})_{w \in b(N')}$.

Consistency for matching problems: The solution $\varphi : \mathcal{M} \to X_{\mathcal{M}}$ is ***consistent*** if for all N, $N' \in \mathcal{N}_M$ with $N' \subset N$ and all $R \in \mathcal{M}^N$, if $b \in \varphi(R)$, then $b_{|N'} \in \varphi((R_{m|b(N')})_{m \in N'} \cup (R_{w|N'})_{w \in b(N')})$.

It is worth noting here that a reduced problem is formally identical to a subproblem, as it is simply obtained by restricting the original data to a subset of the agents. However, the conclusion of *consistency* is not written for all subsets of the original set of players, but only for subsets that are defined by applying the solution to the original problem.

Here is the natural form taken by *converse consistency* for this model: consider some matching problem and a match for that problem. Check whether the restriction of the match to each subgroup of two matched pairs is among the recommendations made by the solution for this four-person reduced problem. If the answer is yes for all such subgroups, the match should be one of the recommendations for the original problem.

Apart from the standard requirement of *Pareto-optimality*, which we will not state explicity, we will use the following condition, which says that the names of the participants should not matter. Let $N, \tilde{N} \in \mathcal{N}$ be such that $|N| = |\tilde{N}|$:

Anonymity: For all $R \in \mathcal{M}^N$ and $(R_i)_{i \in \tilde{N}} \in \mathcal{M}^{\tilde{N}}$ with $|N| = |\tilde{N}|$, let $\pi : N \to \tilde{N}$ be a bijection such that

(a) $\pi(N_M) = \tilde{N}_M$ and $\pi(N_W) = \tilde{N}_W$

(b) For all $m \in N_M$ and all w, $w' \in N_W$, wP_mw' if and only if $\pi(w)\tilde{P}_{\pi(m)}\pi(w')$

(c) A similar statement holds when the roles of men and women are exchanged

Furthermore, for each $b \in \mathcal{B}^N$, let π_b be defined by
$\pi_b(m') = \pi(b(\pi^{-1}(m'))$ for all $m' \in \tilde{N}_M$.

Then, if $b \in \varphi(R)$, $\pi_b \in \varphi((R_i)_{i\in\tilde{N}})$.

We are now ready for the main result of this section, which is a characterization of the core:

Theorem 46 (Sasaki and Toda, 1992) The core is the only solution satisfying *Pareto-optimality*, *anonymity*, *consistency*, and *converse consistency.*

In the model as specified above, no agent ever remains single. Let us now introduce a "single" status. The preferences of each agent are defined over the members of the opposite sex and remaining single. For this model, examined by Toda (1993a), the axioms of Theorem 46 do not identify a unique solution any more.[81] However, by adding two additional but mild conditions, the core emerges once again. The first condition, ***individual rationality***, simply says that no agent is assigned a mate to whom he prefers remaining single. The second condition, ***weak strategy-proofness***, says that in economies in which there is only one man or only one woman, no one gains by misrepresenting his or her preferences.

Theorem 47 (Toda, 1993a) On the domain of matching problems in which remaining single is a possibility, the core is the only solution satisfying *Pareto-optimality*, *individual rationality*, *anonymity*, *weak strategy-proofness*, *consistency*, and *converse consistency.*

One may wonder about the implications of a *consistency* notion in the spirit of the one used by Davis and Maschler (see Subsection 2.2.2.(i)). Recall that this notion is based on a certain maximization exercise carried out by coalitions: in the present model, the options open to a man, say, in the

[81]Not surprisingly however, Theorem 46 applies if preferences are restricted by the condition that for each agent the single status is necessarily the least preferred alternative.

reduced problem associated with a certain match and a particular subgroup that contains him, would be defined by letting him search for the best wife among those women in the complementary subgroup that would want to elope with him; a woman would want to do that if she prefers him to the husband assigned to her by the match under consideration. We will refer to this notion as ***max consistency***. Note that the same woman may be the solution to the maximization problems of several men, whereas the definition we considered earlier raises no such feasibility issue. Toda (1993b) looks for solutions satisfying in addition the requirement of ***anti-monotonicity***: if a match is chosen for some profile, then it is still chosen for the profile obtained by possibly lowering the single status of anyone who is matched. As compared to Theorem 46, it is of interest that the characterization of the core stated next relies on neither *anonymity* nor any notion of *converse consistency*. The possibility of moving the single status allows a considerably simpler proof.[82]

Theorem 48 (Toda, 1993b) On the domain of matching problems in which remaining single is a possibility, the core is the only solution satisfying *Pareto-optimality*, *individual rationality*, *max consistency*, and *anti-monotonicity*.

Next, we consider matching problems with money, commonly called "assignment" problems (Shapley and Shubik, 1972). In this model, studied by Sasaki (1995), and to which the next theorems pertain, the formation of each couple produces a certain worth. Each agent only cares about how much money he or she receives, not to whom he or she is matched. The issue is to determine which couples to form and how to distribute the worths their formations produce. Sasaki considers the weakening of *consistency*, which we will call ***separation independence***, obtained by limiting its applications to situations where the restriction to a subgroup of the payoff vector that is being evaluated is feasible for that subgroup: this is when the sum of the payoffs received by the members of the subgroup is equal to the sum of the worths that can be generated by forming couples in the subgroup.[83] He also uses a condition of ***worth monotonicity*** which says that given any outcome chosen by the solution, if the worth of a pair of a man and a woman increases, then there exists an outcome chosen by the solution for the new

[82]It is based on the well-known "decomposition lemma".

[83]It is the counterpart for this model of the property of the same name discussed in connection with the problem of fair division (Section 2.4.1.)

problem at which the sum of their payoffs is at least as large.[84] Other requirements are *individual rationality*, which here says that each agent should get a payoff greater than 0, the common value of their reservation payoffs, ***couple rationality***, which says that each matched pair should receive in total at least as much money as its formation produces, and a standard ***continuity*** condition with respect to the worths of the pairs.

Theorem 49 (Sasaki, 1995) On the domain of matching problems with money, the core is the only solution satisfying *Pareto-optimality*, *continuity*, *individual rationality*, *couple rationality*, *separation independence*, and *worth monotonicity*. A similar statement holds with *continuity* replaced by *Pareto-indifference*.[85]

For this model, Toda (1993c) explored the implications of a *consistency* property that here too is the counterpart of the Davis and Maschler condition. However, in defining the reduced game relative to a subgroup of agents to which a given agent, say a man, belongs and a payoff vector x, only his reservation payoff is adjusted. This adjustment is effected by means of a maximization exercise with respect to his potential partners in the complementary group: his revised reservation payoff is defined to be the greater of the following two numbers: first, his original reservation payoff and second, the surplus that would remain after paying his partner her payoff at x, after the partner for whom the surplus is the greatest is identified. Toda characterizes the core by means of the corresponding notion of *consistency*, to which we again refer as *max consistency*. This result involves as an auxiliary condition a counterpart of the ***super-additivity*** condition used by Peleg (1986; Subsection 2.2.2): consider a problem that is obtained from two component problems by adding their worth vectors on the one hand and their reservation functions on the other. Then, if a payoff vector of the sum game is obtained by adding payoff vectors chosen by the solution for the component games, then it should be chosen for the sum game.

[84]This is a weakening of a condition used by Zhou (1991) in the context of coalitional form games.

[85]Recall that this is the condition that if an allocation is chosen, any allocation that is Pareto-indifferent to it is also chosen.

Theorem 50 (Toda, 1993c) On the domain of matching problems with money, the core is the only solution satisfying *Pareto-optimality*, *individual rationality*, *couple rationality*,[86] *max consistency*, and *super-additivity*.

In a previous study of the problem, Rochford (1984) had proposed the following solution: consider a payoff vector and a match that makes it feasible. For each man, identify the woman for which the difference between the worth of the pair and her payoff is maximal. For each woman, perform a similar calculation. Then, say that a payoff vector is ***symmetrically bargained*** if there exists a match that makes it feasible, and such that in each matched pair, the payoff of each partner is equal to his or her maximal surplus at the match plus one half of the difference between the worth of the pair and the sum of their maximal surpluses at the match. We name the solution that selects those payoff vectors for each problem the ***symmetrically bargained solution***.[87] In the characterization of this solution presented next, the axiom of ***homogeneity*** is used. It says that the addition of constants to the reservation payoffs and the multiplication of the worths of all pairs by the same positive number should affect the payoffs in exactly the same way (see the condition of the same name in Section 2.2.2 on coalition form games). Finally, we will use ***weak symmetry***, which says that for a problem consisting of only one man and one woman with equal reservation payoffs, their payoffs should be equal (note that this implies *single-valuedness* for that case).

Theorem 51 (Toda, 1993c) On the domain of matching problems with money, the symmetrically bargained solution is the only solution satisfying *Pareto-optimality*, *homogeneity*, *symmetry*, *couple rationality*, *max-consistency*, and *max converse consistency*.

Toda also considers an extension of the model to the case when preferences are defined over pairs consisting of a mate and some amount of money, and

[86] This condition does not appear explictly in Toda's theorem. Instead it is a consequence of *Pareto-optimality* together with the stronger feasibility condition that he uses. The formulation we have chosen should facilitate the comparison of Theorems 50 and 49.

[87] Rochford (1984) shows the solution is non-empty and that it coincides with the intersection of the core and the kernel of the associated coalitional form game. She also constructs an algorithm converging to it. In fact, for this model, the kernel is a subset of the core, as shown by Granot (1995) and Driessen (1995). Driessen (1995) provides useful geometric illustrations of the core of the assignment game and of the symmetrically bargained solution.

provides a characterization of the core along the lines of Theorem 50: it is the only solution satisfying *Pareto-optimality*, *individual rationality*, *couple rationality*, *max consistency*, and ***weak anti-monotonicity*** (if a payoff vector is chosen for some problem, it should still be chosen after a possible decrease of the reservation payoffs of all matched agents).

Moldovanu also (1990a) extends the model to the non-transferable utility case, when the set of options available to each pair is defined as a subset of a two-dimensional utility space satisfying some regularity assumptions. Without going into the details, let us refer to the domain as "regular". He defines an NTU version of the symmetrically bargained solution and proves its non-emptiness on the regular domain. He also establishes the following characterization:

Theorem 52 (Moldovanu, 1990a) On the regular domain of NTU matching problems, the NTU extension of the symmetrically bargained solution is the only solution satisfying *bilateral max consistency*, *max converse consistency*, and coinciding with the Nash solution for the two-person case.

Part 3

OTHER ISSUES AND DIRECTIONS FOR FURTHER RESEARCH

In Part 3, we discuss a number of directions for further research. In each case, some results are already available, but much remains to be done. First, we discuss *consistency* in a model with a large number of agents modelled as a continuum. We then propose and formalize two ways of evaluating how far a non-*consistent* solution is from satisfying the property. We point out the difficulties in formulating notions of *consistency* in private good economies in which agents are differentially endowed, and suggest a resolution of these difficulties. We present the few results available for economies with public goods. We show how the *converse consistency* of a solution can be used in the calculation of the outcomes the solution recommends. Finally, we consider the issue of *consistency* in the context of intertemporal allocation.

3.1 MODELS WITH A CONTINUUM OF AGENTS

First, we consider economies with a large number of infinitesimal agents. Following Aumann (1964), it is now standard to model such economies by specifying the set of agents as a measure space. An analysis of *consistency* in such a setting is undertaken by Thomson and Zhou (1993) who work on a domain of exchange economies with possibly satiated preferences, and characterize the extension to such a domain of the solution proposed by Mas-Colell (1992) for finite economies under the name of ***Walrasian solution with slack***. An equal-income Walrasian allocation with slack is supported by equilibrium prices, that is, prices at which demand equals supply, but some prices may be zero or negative, and agents maximize their preferences in a modified budget set — it is the set of consumptions whose value at the equilibrium prices does not exceed the value of the mean endowment by more than some number whose specification is also part of the definition of the equilibrium. At equilibrium, the slack generated by interior maximization of some agents is thereby redistributed to the others. When specialized to economies with monotonic preferences, the equal-income Walrasian solution with slack coincides with the equal-income Walrasian solution (Subsection 2.4.1). When specialized to the one-commodity case and preferences are strictly convex, it coincides with the uniform rule (Subsection 2.4.2).

Our next result is a characterization of the Walrasian solution with slack. A feature of this theorem worth noting is that it applies to a solution defined on any domain consisting of a given economy e and all of the economies in which the set of agents is an arbitrary measurable subset of the set of agents of e, and the social endowment is an arbitrary share of the social endowment of e. It is proved under standard continuity and measurability assumptions, and the requirements on preferences that they be locally non-satiated at non-satiated points, and smooth at non-satiated points. Let $\mathcal{E}_*$ denote any such domain.

Theorem 53 (Thomson and Zhou, 1993) On a domain $\mathcal{E}_*$, if a subsolution of the equal division lower bound and Pareto solution is *consistent*, then it is a subsolution of the equal income Walrasian solution with slack.

As corollaries of Theorem 53 we derive characterizations of the equal-income Walrasian solution and of the uniform rule. They are obtained by imposing the additional restrictions on the domain that preferences be monotonic on the one hand, or that there be only one commodity on the other hand.

Most of the domains of problems reviewed in part II can be reformulated by modelling the set of potential agents as a continuum but much work remains to be done to find out whether and how the results presented earlier can be rewritten for such formulations, in particular how the auxiliary conditions can be weakened.

The analysis of *converse consistency* in such models also remains to be carried out. A possible formulation for fair division problems is as follows. Consider a solution and let x be an efficient allocation for some economy in its domain of definition. Suppose that for all $\epsilon > 0$, the restriction of x to all measurable subsets of the set of agents of measure at most ϵ is one of the recommendations the solution would make for the associated reduced economy. Then, x should be recommended by the solution for the original economy.

Bibliographic note. Diamantaras (1991) applies to public good economies techniques similar to those developed in Theorem 53 and provides a characterization of Foley's (1967) public competitive equilibrium. Winter and Wooders (1994) offer a characterization of the cores of games of TU or NTU games in which the set of players may be a continuum but only finite coalitions are allowable. Their axioms are *max consistency* and its *converse*, and the requirement that the solution coincides with the core for two-player games. This result should be compared with Peleg (1985)'s own characterization of the core for finite games (Theorem 6).

3.2 MINIMAL CONSISTENT EXTENSION. MAXIMAL CONSISTENT SUBSOLUTION

When a solution is not *consistent*, we would like to know how serious the violations of *consistency* are. One way to evaluate them is to find out the extent to which the solution would have to be modified in order to satisfy the property. Two procedures for doing this are developed in Thomson (1994b), on which this section is based.

3.2.1 Minimal consistent extension.

We first propose to minimally enlarge the solution so as to recover *consistency.* That this can be done follows from the following observations: it is a direct consequence of the definition of *consistency* that if all the members of a family Ψ of solutions with common domain and range are *consistent,* and the intersection $\bar{\varphi}(e) = \bigcap_{\psi\in\Psi}\psi(e)$ is non-empty for each economy e in the domain, then the well-defined solution $\bar{\varphi}$ also is *consistent.* Now, given a solution φ, let Ψ be the family of *consistent* solutions containing φ; that is, $\Psi = \{\psi : \psi \supseteq \varphi,\ \psi$ is *consistent*$\}$. The solution that associates with each economy its whole feasible set is of course *consistent.* Therefore $\Psi \neq \emptyset$. Let $\bar{\varphi} = \bigcap_{\psi\in\Psi}\psi$. Since $\bar{\varphi} \supseteq \varphi$, $\bar{\varphi}$ is a well-defined solution. Therefore $\bar{\varphi}$ can be described as the *minimal consistent extension of* φ. The "size" of the difference $\bar{\varphi}\backslash\varphi$ is the price we have to pay to recover *consistency* if we insist that all the allocations picked by φ be included.

Minimal consistent extension: Given a solution φ, its ***minimal consistent extension,*** $\boldsymbol{mce(\varphi)}$, is defined by

$$mce(\varphi) = \bigcap_{\psi\in\Psi}\psi, \text{ where } \Psi = \{\psi : \psi \supseteq \varphi, \psi \text{ is } \textit{consistent}\}$$

Given two solutions φ and φ', we have $mce(\varphi \cup \varphi') = mce(\varphi) \cup mce(\varphi')$. Also, if φ and φ' are such that $\varphi \cap \varphi'$ is a well-defined solution, we have $mce(\varphi \cap \varphi') \subseteq mce(\varphi) \cap mce(\varphi')$. The inclusion may be strict.

We illustrate these notions by first examining the problem of fair allocation in classical economies (See Subsection 2.4.1 for the definitions of the solutions used as examples below). To simplify the statement of the first result, consider the solution that selects for each economy the subset of its efficient allocations admitting supporting prices such that the value of all consumptions be positive. We refer to it as the "strong" Pareto solution. Note that the Pareto and the strong Pareto solutions are "very close". Clearly, the strong Pareto solution is *consistent.* The following theorem reveals that if the point of departure is the equal division lower bound and strong Pareto solution, which as we saw is not *consistent,* a considerable enlargment of it is necessary in order to obtain the property:

Theorem 54 (Thomson, 1992c) On the domain of economies with classical preferences, the *minimal consistent extension* of the equal division lower bound and strong Pareto solution is the strong Pareto solution itself.

The *minimal consistent extension* of the often used selection from the egalitarian-equivalence solution that is obtained by requiring the reference bundle to be proportional to the social endowment (the Ω-egalitarian-equivalence solution of Subsection 2.4.1) can also be calculated in a simple way: it essentially coincides with the egalitarian-equivalence solution.

In the context of fair allocation in the presence of indivisible goods, recall the solution that selects the efficient allocations that each agent prefers to what he would receive under *equal treatment of equals* if everyone had preferences identical to his — this is the identical-preferences lower bound solution (see Subsection 2.4.3 for the definition). This solution is not *consistent* but here too, the extent of the violations can be calculated. Once again, a considerable enlargment is necessary to recover *consistency*:

Theorem 55 (Bevia, 1996) On the domain of fair allocation problems with indivisible goods, the *minimal consistent extension* of the identical-preferences lower bound and Pareto solution is the Pareto solution. The same result holds on the subdomain of economies in which the indivisible goods are identical, and in the one-object case.[1]

Similar calculations can be carried out in the context of the abstract bargaining model when solutions are allowed to be *multi-valued.* For instance, on the domain of strictly comprehensive problems, the *minimal consistent extension* of the Kalai-Smorodinsky solution can be shown to bear a close relation to the solution that selects for each problem the relative interior of the Pareto set (Thomson, 1995a).

[1] In that case however, if preferences are such that receiving the object is always preferable to not receiving it, then the *minimal consistent extension* of the identical-preferences lower bound and Pareto solution is the solution that selects the set of efficient allocations such that each agent prefers his assigned bundle to the bundle consisting of the null object and an equal share of the social endowment of money.

3.2.2 Maximal consistent subsolution

The procedure discussed in the preceding paragraphs is certainly not the only way of evaluating the extent to which a solution φ may fail to be *consistent*. Alternatively we could delete from, instead of adding to, the φ-optimal set, and ask how much should be deleted in order to recover the property. Here, of course, we would like to delete as little as possible. This operation will be well-defined only if φ does contain a *consistent* subsolution, but there is no other precondition. Indeed, if all the members of a non-empty family Ψ of solutions are *consistent*, then so is the union $\underline{\varphi} = \bigcup_{\psi\in\Psi} \psi$. If $\psi \subseteq \varphi$ for all $\psi \in \Psi$, then of course $\underline{\varphi} \subseteq \varphi$, so that $\underline{\varphi}$ can be described as the *maximal consistent subsolution of* φ.

Maximal consistent subsolution: Given a solution φ containing a *consistent* subsolution, its ***maximal consistent subsolution,*** $\boldsymbol{Mcs(\varphi)}$, is defined by

$$Mcs(\varphi) = \bigcup_{\psi\in\Psi} \varphi, \text{ where } \Psi = \{\psi: \psi \subseteq \varphi, \psi \text{ is } \textit{consistent.}\}$$

Given two solutions φ and φ', each of which contains a *consistent* solution, we have $Mcs(\varphi \cup \varphi') \supseteq Mcs(\varphi) \cup Mcs(\varphi')$. The inclusion may be strict. Also, if $\varphi \cap \varphi'$ contains a *consistent* solution, we have $Mcs(\varphi \cap \varphi') = Mcs(\varphi) \cap Mcs(\varphi')$.

Note that $Mcs(\varphi)$ is equal to the solution $\underline{\varphi}$ defined, for all $N \in \mathcal{N}$ and all $e \in \mathcal{E}^N$, by

$$(*) \quad \underline{\varphi}(e) = \{z \in Z(e): z_{N'} \in \varphi(r^z_{N'}(e)) \text{ for all } N' \subseteq N\}.$$

Indeed, $\underline{\varphi}$ is *consistent*, and of course $\underline{\varphi} \subseteq \varphi$ (set $N' = N$ in the definition). Maximality follows from the fact that these conditions are necessary.

The concept of a *maximal consistent subsolution* can be used to relate various solutions that have been discussed separately in the literature. For instance, for fair division problems in classical economies, the solution that selects the allocations such that every agent prefers his consumption to any point in the convex hull of the various consumptions assigned to all the agents (Kolm, 1973), can be understood as the *maximal consistent* and *replication invariant*[2] subsolution of the equal division lower bound. Similarly,

[2] *Replication invariance* is stable under union too, so the concept is well-defined.

the strict no-envy solution (Zhou, 1992) can be described as the *maximal consistent subsolution* of the solution that selects the set of allocations at which each agent prefers what he receives to the average of what the others receive (Thomson, 1982; Baumol, 1986). On the domain of fair allocation problems with indivisible goods, Bevia (1996) shows that the *maximal consistent subsolution* of the identical-preferences lower bound solution is the no-envy solution. Other results presented earlier in this survey can be seen as describing the *maximal consistent* subsolutions of solutions of interest.

Similar operations to those described in the previous paragraphs can be defined starting from *bilateral consistency.*

It remains to be determined whether the notions of *minimal consistent extensions* and *maximal consistent subsolutions* can help in the understanding of other domains.

3.2.3 Minimal conversely consistent extension. Minimal flexible extension

It is also possible to define the notion of the *minimal conversely consistent extension* of a solution, or that of its *minimal flexible extension.* Indeed, the intersection of an arbitrary family of *conversely consistent* solutions is *conversely consistent*, and the feasibility solution has that property. Similarly, the intersection of an arbitrary family of *flexible* solutions, if well-defined, is *flexible*, and so is the feasibility solution.[3]

3.3 ECONOMIES WITH INDIVIDUAL ENDOWMENTS

In the standard formulation of the problem of fair division, there is a social endowment on which agents have equal rights. In richer specifications, each agent starts out with his own endowment, that is, the social endowment is initially distributed among the agents, not necessarily evenly. A constraint

[3]However, the union of two *conversely consistent* solutions may not have the property and the union of two *flexible* solutions may not have the property either. Therefore, there does not seem to be a natural counterpart of the concept of *maximal consistent subsolution* when the point of departure is either *converse consistency* or *flexibility.*

in the design of allocation rules is that they recognize these possibly different rights. This is the usual model of the theory of general equilibrium.

Most of the central concepts that have been formulated to solve the problem of distributing a social endowment can be adapted to the problem of ***redistributing*** individual endowments. The question addressed here is whether, and how, *consistency* and its *converse* can be adapted to such situations. The central issue is defining the concept of a "reduced economy". We will discuss the difficulties that come up in formulating a definition and propose a resolution. This resolution involves enlarging the class of problems under consideration by specifying, in addition to the agents' preferences and endowments, a net trade vector in commodity space; a positive coordinate is interpreted as a surplus of the corresponding good that has to be distributed among the agents, whereas a negative coordinate is to be interpreted as a shortfall that has to be absorbed. In this context, a solution provides a way of redistributing endowments and accommodating a net trade vector among agents, taking into proper account their preferences and endowments. Equipped with these definitions, *consistency* is then straightforward. We close by providing simple examples of *consistent* solutions. This section is based on Thomson (1992b). Dagan (1992) has also addressed the same issue. Additional papers are by van den Nouweland, Peleg, and Tijs (1994), and Serrano and Volij (1995).

First, we describe the model. There are $\ell \in \mathbb{N}$ goods. Each agent $i \in \mathbb{N}$ is characterized by a ***preference relation*** defined over $\mathbb{R}^\ell_+$, denoted by R_i, and an ***individual endowment***, a point in $\mathbb{R}^\ell_+$, denoted by ω_i. Given a class $\mathcal{R}$ of admissible preferences, an ***economy*** is a pair $(R,\omega) \in \mathcal{R}^N \times \mathbb{R}^{\ell N}$, where $N \in \mathcal{N}$ is a group of agents, $R = (R_i)_{i \in N}$ is the profile of their preferences and $\omega = (\omega_i)_{i\in N}$ the profile of their endowments. For each $N \in \mathcal{N}$, let $\mathcal{E}^N$ be a class of admissible economies, that is, economies in which preferences satisfy some regularity conditions, and let $\mathcal{E} = \bigcup_{N \in \mathcal{N}} \mathcal{E}^N$ and $X = \bigcup_{N\in\mathcal{N}} \mathbb{R}^{\ell N}_+$. A ***solution*** is a mapping that associates with every $N \in \mathcal{N}$ and every economy $e = (R,\omega) \in \mathcal{E}^N$ a non-empty subset of its feasible set $Z(e) = \{z \in \mathbb{R}^{\ell N}_+ : \sum_N z_i = \sum_N \omega_i\}$.

Let φ be a solution. Given $e = (R,\omega) \in \mathcal{E}^N$, $N' \subset N$, and $z \in \varphi(e)$, a first choice for the ***reduced economy of e relative to N′ and z*** is $e' = (R_{N'}, z_{N'})$; that is, the endowments of the members of N' are simply ***replaced*** by the consumptions assigned to them by φ. Once the reduced economy is defined, *consistency* would simply say that $z_{N'} \in \varphi(R_{N'}, z_{N'})$. We feel that this definition limits too much the options open to the remaining agents. Indeed, if a subsolution of the Pareto solution chooses the profile of endowments whenever that profile is an efficient allocation, and most solutions of interest do, it automatically satisfies *consistency* as so formulated. Consequently, the property is too weak to be of much help in distinguishing among solutions. An alternative formulation would be to keep the profile of endowments unchanged. But then, the requirement $z_{N'} \in \varphi(R_{N'}, \omega_{N'})$ would be unreasonably strong since the allocation $z_{N'}$ would only exceptionally be feasible for the economy $(R_{N'}, \omega_{N'})$. Writing that $z_{N'} \in \varphi(R_{N'}, \omega_{N'})$ ***if*** $z_{N'}$ is feasible would give us the counterpart of the condition of *separation independence* discussed in Subsection 2.4.1, a condition that is in general rather weak.[4]

More interesting is a formulation in which the endowment profile is ***adjusted***, instead of replaced, so as to reflect the constraints placed on the remaining agents by the departure of some members of the initial group. This respecification of the endowment profile should of course be carried out in such a way that $z_{N'}$ be feasible in the resulting reduced economy. To achieve this, it is necessary to distribute the net trade vector $\sum_{N \setminus N'}(\omega_i - z_i)$ among the remaining agents, the members of N'. Equal distribution comes to mind, but this might produce revised endowments with negative coordinates. A distribution as close to equal division as possible taking these non-negativity constraints into account is a natural alternative, although we do not feel that there is a strong justification for it. For the problem of reallocating a private good among agents with single-peaked preferences (Section 2.4.2), this formulation was recently investigated by Klaus, Peters, and Storcken (1996b), who based on it several characterizations of an extension of the uniform rule, characterizations patterned after results obtained for the fair division version of the model.

Other procedures of redistributing net trades could be devised but whatever method is adopted, it seems appropriate to require that it be itself *consistent*. A formulation of the problem along these lines would then lead us to searching for pairs (ψ, φ) where ψ is a method of redefining profiles of endowments so as to absorb a given trade vector, and φ is an allocation

[4]Yet, we saw that in some models, such as matching (Subsection 2.5.2), this condition is actually quite useful.

method. *Consistency* would be required of both ψ and φ.

Instead of this two-part process, we propose a unified formulation based on the generalization of the concept of an economy. The generalization is obtained by simply adding a net trade vector to the original data. This generalization was independently proposed by Dagan (1992), and his notion of *consistency* is the same as the one to be stated shortly:

A ***generalized economy*** is a list $e = (R, \omega, T)$ where $N \in \mathcal{N}$, $R = (R_i)_{i \in N} \in \mathcal{R}^N$, and $\omega = (\omega_i)_{i \in N} \in \mathbb{R}^{\ell N}$ are profiles of preferences and endowments respectively, and $T \in \mathbb{R}^\ell$ is a net trade vector satisfying $\sum_N \omega_i \geqq T$. Let $\mathcal{G}^N$ be a class of such problems, $\mathcal{G} = \bigcup_{N \in \mathcal{N}} \mathcal{G}^N$ and $X_{\mathcal{G}} = \bigcup_{N \in \mathcal{N}} \mathbb{R}^{\ell N}$. A ***solution*** on $\mathcal{G}$ associates, with each $N \in \mathcal{N}$ and each $(R, \omega, T) \in \mathcal{G}^N$, a non-empty subset of $Z(e) = \{z \in \mathbb{R}_+^{\ell N} : \sum_N z_i = \sum_N \omega_i + T\}$.

A positive coordinate of T corresponds to a ***surplus*** of the good, and a negative coordinate to a ***deficit***. Solutions are designed to allocate these combinations of surpluses and deficits in a desirable way, taking preferences and endowments into proper consideration.

With the concept of a generalized economy, *consistency* can now be given the following very natural formulation:

Consistency for generalized economies: The solution $\varphi : \mathcal{G} \to X_{\mathcal{G}}$ is ***consistent*** if for all N, $N' \in \mathcal{N}$ with $N' \subseteq N$, all $e = (R, \omega, T) \in \mathcal{G}^N$, and all $z \in \varphi(e)$, we have $z_{N'} \in \varphi(r^z_{N'}(e))$, where $r^z_{N'}(e) = (R_{N'}, \omega_{N'}, T + \sum_{N \setminus N'} (\omega_i - z_i))$.

Two special subclasses of generalized economies that are of interest are the class of ***surplus-sharing*** problems, obtained when $T \geq 0$, and the class of ***deficit-sharing*** problems, obtained when $T \leq 0$. We discussed these classes in Subsection 2.3.1 in the one-good case. Note that monotonic increasing preferences over a one-dimensional commodity space are identical, so that in that special case, $|N| + 1$ numbers are all that is needed to specify a problem, the $|N|$ agents' endowments (or claims, or investments, depending upon the interpretation given to the model), and one additional number, representing a surplus if positive, and a deficit if negative.

Now, returning to the ℓ-dimensional case, and given a net trade vector $T \in \mathbb{R}^\ell$ that only has non-negative coordinates, the vector $T + \sum_{N \setminus N'} (\omega_i - z_i)$ may nevertheless have negative coordinates. Similarly, if T happens to only have negative coordinates, the vector $T + \sum_{N \setminus N'} (\omega_i - z_i)$ may very well have some positive coordinates. Therefore, if our point of departure had been the class of surplus-sharing problems, the "reduction" would sometimes take us outside of the class. And if we wanted to limit ourselves to the class of deficit-sharing problems, once again, the reduction would sometimes take us outside of the class.

We could of course require of the solution that it produce recommendations with respect to which the reduction operation is closed. Alternatively, we could restrict the application of *consistency* to situations where the reduced game remains admissible. However, we find it more natural to allow net trade vectors with coordinates that are unrestricted in sign.

Two *consistent* solutions are defined next. Both can be described as "partially" Walrasian since they are based on concepts of prices. Since there is no *a priori* reason why such concepts should play a role, we do not have a strong justification, except that they seem to be natural extensions of solutions that have played an important role in related models. For the first one, prices are chosen so that after adjusting agents' incomes by a fraction of the value of the net trade proportional to the values of the endowments, maximization of preferences yields an equilibrium.

Definition. Given $N \in \mathcal{N}$ and $e = (R, \omega, T) \in \mathcal{R}^N \times \mathbb{R}_+^{\ell N} \times \mathbb{R}^\ell \in \mathcal{G}^N$, let $\boldsymbol{M(e)} = \{z \in Z(e)$: there exists $p \in \Delta^{\ell-1}$ s.t. for all $i \in N$ and all $z'_i \in \mathbb{R}_+^\ell$ with $pz'_i \leq p\omega_i + [(p\omega_i)/\sum_N p\omega_j]pT$, $z_i R_i z'_i\}$.

For the second example, the value of the net trade is distributed evenly, but an adjustment is made to prevent bankruptcy.

Definition. Given $N \in \mathcal{N}$ and $e = (R, \omega, T) \in \mathcal{R}^N \times \mathbb{R}_+^{\ell N} \times \mathbb{R}^\ell \in \mathcal{G}^N$, let $\boldsymbol{N(e)} = \{z \in Z(e)$: there exists $p \in \Delta^{\ell-1}$ s.t. agents can be divided into two groups N_1 and N_2; for all $i \in N_1$, $p\omega_i + pT/|N_2 + 1| < 0$ and $z_i = 0$; for all $i \in N_2$ and all $z'_i \in \mathbb{R}_+^\ell$ with $pz'_i \leqq p\omega_i + pT/|N_2|$, $z_i R_i z'_i\}$.

Note that if $T \geq 0$, then for all $p \in \Delta^{\ell-1}$ we have $p\omega_i + pT/|N| \geq 0$ so that $N_1 = \emptyset$ and $N_2 = N$.

At this point, Dagan (1992)'s formulation departs from ours as he does not insist on non-emptiness of solutions. He defines a Walrasian allocation as one at which there exist prices such that each agent maximizes his preferences over his standard budget set at his component of the allocation. Together with the distributional requirements that he formulates, he shows that *consistency* leads to the equal-income Walrasian solution on the subdomain of economies for which the value of the net trade vector at the equilibrium prices is zero. Non-emptiness is not required by van den Nouweland, Peleg, and Tijs (1994) either, and they too obtain a characterization of the Walrasian solution on the same subdomain. The axioms they use are *consistency* and its *converse*, together with the condition that the rule coincides with the Walrasian solution for one-person economies.

Another contribution on the subject is Serrano and Volij (1995), who consider several reduction operations. For the first one, and given a recommendation z made for some economy with agent set N, the production set of the grand coalition is set equal to all the vectors that can be written as the difference between a production plan that was initially available to the grand coalition of the original game minus any sum $\sum_{N\setminus N'} z'_i$ of consumptions with the property that each agent in $N\setminus N'$ prefers z'_i to z_i. For the proper subsets of N', the reduced production set is defined to be the union of all the sets obtained in this way when cooperation with any subset of $N\setminus N'$ is allowed. This is as in the Davis-Maschler max reduced game of the theory of coalition form without transferable utility (Section 2.2.3). The second definition differs from the first only in that the grand coalition of the reduced game can also search for the best subset of $N\setminus N'$ with which to get together. The last notion is defined for economies in which only the one-person subgroups have production opportunities, and it involves infinite iterations of the previous construction. Using these conditions Serrano and Volij successively characterize the Pareto solution, the core and the Walrasian solution.

Understanding the implications of the condition of *consistency* formulated above for the model considered here, and further extending its definition and applications to other models, seem to be interesting open questions. We also add that in economies with production, the definition that we proposed above can be easily adapted by similarly enlarging the class of economies under consideration, and working with ***generalized production economies with individual endowments***, that is, lists of the form (R, ω, Y, T), where a production set Y has been added to the data defining what we called a generalized economy. Then, the reduced economy of (R, ω, Y, T) with agent set N, with respect to some subset $N' \subset N$ and some recommended allocation z would be $(R_{N'}, \omega_{N'}, Y, \sum_{N\setminus N'}(\omega_i - z_i))$.

In the context of a one-dimensional model with single-peaked preferences (Section 2.4.2), the following results concerning the implications of *consistency* are available. First, we respecify economies by adding to the list of preference relations a profile of endowments $\omega \in \mathbb{R}_+^N$ and a "net trade" $T \in \mathbb{R}$ satisfying the inequality $\sum \omega_i + T \geq 0$ (once again, this is to guarantee the existence of feasible allocations). For an economy $(R, \omega, T) \in \mathcal{R}_{sp}^N \times \mathbb{R}_+^N \times \mathbb{R}_+$ so defined, the no-envy idea is applied to ***reallocations*** of endowments in the following staighforward way: for no pair $\{i, j\} \in N$, agent i should prefer the difference between agent j's final consumption and endowment to the corresponding difference for himself (this is as in Schmeidler and Vind (1972)). Since envy-free reallocations may not exist, we use the slightly less demanding notion obtained by adding the proviso ***"if agent j 's consumption is positive"***. We also generalize our earlier definition of the uniform rule, under the name of "generalized uniform rule", by dropping the sign restriction on the parameter λ, an adjustment being made to ensure that all agents receive non-negative consumptions. The main result here is that if a subsolution of the weak no-envy and Pareto solution satisfies *consistency* and *continuity*, then it contains the generalized uniform rule (Thomson, 1995b). The corollaries of this result are parallel to the corollaries of the main theorem for the fair division version of the model.

3.4 ECONOMIES WITH PUBLIC GOODS

We now turn to the public good economies, which have not been the object of much work either. Here too, it is not immediately obvious how *consistency* should be defined.

One result is available that pertains to the very simple problem of choosing the level of a unique public good in some interval, each agent being equipped with a single-peaked preference relation defined over the interval. Once the level of the public good is chosen, let us imagine some agents to leave. If we think of a commitment to a "physical decision" having been made, then of course, the only way to guarantee that the departing agents receive what they were promised is to maintain the initial decision, and *consistency* is vacuously satisfied: in the reduced economy, there is only one permissible decision. However, an interesting alternative formulation is possible. Moulin (1984) enriches the model by allowing the interval of permissible decisions itself to vary, solutions being required to provide a recommendation not only when population changes but also when the interval changes. Now, imagine the departure of some agents, and in specifying the options open to the remaining agents, impose the requirement that the agents that departed be made ***at least as well off*** as they were initially. It is this constraint that defines the reduced problem. When agents have single-peaked preferences over the initial interval and when the preferred levels of all the departing agents are on the same side of the level initially chosen, there is in general a non-degenerate interval of levels that all of them would find at least as good as the initial choice.

We will search for solutions that are ***anonymous***, that is, invariant under permutations of agents, satisfy ***contraction independence***, which says that for each pair of intervals related by inclusion, if the choice made for some profile of preferences defined over the larger interval belongs to the smaller interval, then the choice made for the restriction of that profile to the smaller interval should be the same; and finally ***independence with respect to preferences over unfeasible alternatives***: the decision from some interval should depend only on the restriction of preferences over the interval.

These conditions lead to a characterization of the following family of ***alternating generalized Condorcet-winner*** solutions. For each population size, select a number of parameters in the interval of possible public good levels equal to the number of agents minus one. Between any two parameters pertaining to the list relative to populations of size $n+1$ is located one and only one parameter pertaining to the list relative to populations of size n. Then, given a preference profile for some group of agents, the chosen level is the median of the agents' preferred levels and the list of parameters chosen relative to that size.

Theorem 56 (Moulin, 1984) The alternating generalized Condorcet-winner solutions are the only subsolutions of the Pareto solution satisfying *anonymity*, *contraction independence*, *independence with respect to preferences over unfeasible alternatives*, and *consistency.*

This proof of this result is a simple corollary of the characterization of the class of solutions satisfying all of the listed properties except for *consistency.* An open question here concerns the extent to which these properties could be dispensed with, without the class of admissible solutions becoming unmanageably large.

A formulation of *consistency* is explored by Tadenuma and Thomson (1989) for a two-dimensional model in which there are crowding effects. The technology is linear and crowding effects are modelled by making the technology depend on the number of agents; specifically, if k is the number of agents, 1 unit of the input yields $1/k$ units of the public good. In such an economy, when some agents leave, the options available to the remaining agents change. Another contribution is Diamantaras (1991) who considers a model with a continuum of agents (also, see Section 3.2).

Van den Nouweland, Tijs, and Wooders (1995) formulate other notions of *consistency* and characterize two important solutions, the Lindahl and ratio solutions. Their reduced games involve notions of prices and ratios respectively, and an interesting open question is whether these solutions would emerge from considerations of consistency free of such notions.

3.5 COMPUTATIONAL IMPLICATIONS OF CONVERSE CONSISTENCY

Here, we apply the computational algorithm based on the notion of *converse consistency with respect to a graph* that we developed in Subsection 1.6.4.

We start with bargaining problems. Recall (Subsection 2.2.1) that the Nash solution is *conversely consistent* on the class of smooth problems. So is the egalitarian solution even if the smoothness restriction is not imposed. In fact these properties hold for any connected graph. Does this help in the calculation of the desired outcomes. The next proposition gives a positive answer.[5]

[5]The convergence of dynamic processes leading to solution outcomes of coalitional form games has been studied by several authors. Stearns (1968) defines a process that leads to points in the kernel, and one that converges to the bargaining set. Justman (1977) extends his results. The most general theorems are due to Maschler and Peleg (1976). Butnariu and Censor (undated) study convergence to the core. None of these works is explicitly based on *converse consistency* notions however, and it would be interesting to see whether a connection exists.

Theorem 57 (Thomson, 1992a) On the class of smooth bargaining problems, the Nash solution and the egalitarian solution are *decentralizable* with respect to any connected graph.[6]

The second example pertains to fair allocation in classical economies, and to the main solution for this domain, namely the Walrasian solution operated from equal division. Here, we find that *decentralizability* does not hold for two natural examples of graphs (hub-and-spoke, and circular) with respect to which, under smoothness of preferences, the solution is *conversely consistent* (Thomson, 1992a). Similarly, the no-envy solution is not *decentralizable* for the problem of allocating indivisible goods and money.

The examples above illustrate the possible usefulness of the notion of *decentralizability* but also its limitations. In contexts where the algorithm does converge, important further questions concern its speed of convergence. Can it be estimated? Can variants of the algorithm be identified that speed convergence? In contexts where the algorithm does not converge, can it be modified so as to generate sequences that do converge? Can interesting domain restrictions be imposed that permit convergence? In most interesting applications, we should use a connected graph. Complete graphs or graphs in which links are visited randomly, with each link being visited with positive probability, might be sufficient for probabilistic convergence when minimal connected graphs are not.

Goldman (1984) and Ostroy and Starr (1974) ask whether step-by-step Walrasian exchange processes involving only two agents at a time lead to Pareto-optimal allocations. Goldman (1984) considers connected graphs. Their results pertain to the *decentralizability* of the Pareto solution as we defined this term. Lainé (1986,1987a,b) derives additional results on this issue, and in particular studies the range of allocations that can be reached at the limit. Bell (1996) studies sequences of allocations obtained by bilateral trading when the set of agents is endowed with a graph structure, and establishes sufficient conditions for convergence to a Pareto-optimal allocation.

Finally, we recall the convergence results due to Maschler and Owen (1989) and Orshan (1992). These results pertain to the version of *aver-*

[6]Even though smoothness is not needed to guarantee the *converse consistency* of the egalitarian solution, cycles may occur in the adjustment process associated with a circular graph if the restriction is not imposed.

age consistency on which Maschler and Owen based their extension of the Shapley value from the class of TU games to the class of (NTU) hyperplane games (Section 2.2.2). Maschler and Owen show that their value can be obtained by a dynamic adjustment process involving only 2-person coalitions. Orsham shows that the Maschler-Owen value can be reached by a dynamic adjustment process involving all coalitions. Neither convergence result is based on *converse consistency* however.

3.6 INTERTEMPORAL ALLOCATION

Consider an intertemporal allocation problem in which agents are interpreted as generations. In such a situation, it is natural to formulate *consistency* by only allowing for reductions obtained by setting aside the first t generations, for t arbitrary. Would a decision made at time 0 remain the right one after having been implemented for t periods?

A special feature of the situation is that reduced problems belong to the same class as original problems, so that a decision rule has only one component.

There are few contributions on this subject. Epstein (1986a) looks for orderings on a space of infinite consumption plans required to satisfy a certain feasibility condition, and he establishes the non-existence of orderings satisfying a number of basic requirements, including *consistency*: in this context, this is the requirement that if a consumption plan maximizes the ordering from time 0, then its restriction maximizes the ordering over the space of consumption plans starting at time t obtained by taking as given the consumptions up to that time. Shinotsuka (1994a) provides additional information on the independence of the axioms in Epstein's theorem.

The model considered by Blackorby, Bossert, and Donaldson (1994) is richer in that it pertains to the ranking of infinite utility matrices. However, they impose no feasibility conditions.

When the objective is changed from defining an ordering on the set of consumption plans to obtaining a decision rule, positive results become available: indeed, Epstein (1986b) develops a joint characterization of egalitarian and utilitarian solutions. Again, the main axiom is *consistency*.

The analysis of a model formulated in commodity space and in which each generation may comprise several agents as in Blackorby, Bossert, and

Donaldson, and feasibility constraints are made explicit, remains to be carried out.[7] For such a model, consistency conditions obtained by imagining the departure of all but a connected (finite or infinite) set of generations would be particularly meaningful. Consider for instance the group N' of agents whose lifespans overlap with the interval between t and t', where $t' > t$; given some solution and a recommendation made by this solution, calculate the resources made available to N' at the initial date t under the requirement that the commitments to the agents alive until t be honored; similarly, calculate the obligations incurred by N' so as to make it possible to honor the commitments to the agents alive after t'. Now, identify the alternatives available to N' subject to these initial and final constraints. This is the reduced problem they face. Solve the problem by applying to it the relevant component of the solution. The solution is *consistent* if the recommendation it makes for the reduced problem produces the same consumption plans over the interval $[t, t']$ for the members of N' as the ones they were initially assigned.

Note that for this new test to be meaningful, it is necessary that the concept of a solution be redefined: the domain should include the set of all infinite and ***finite*** allocation problems defined by, (i) a pair of dates t and t', where $t' > t$, t' being possibly infinite, (ii) a pair of a vector of resources made available to a group at time t — this is the group's endowment — and a vector of resources to be bequeathed by the group at time t' — the group's obligations to the future generations, (iii) the preferences of each of the members of the group over the subinterval of $[t, t']$ during which he is alive, and finally, (iv) the production possibilities of the group. For this exercise to make sense, preferences also should be redefined so as to take into account any consumption that may have taken place, or will take place, outside of the interval. Indeed, an agent alive in the interval (i) may also have been alive before t and according to the initial plan, he may already have consumed, or (ii) he may still be alive after t' and he may have to consume after that date, or (iii) both.

3.7 CONCLUDING COMMENT

We hope to have convinced the reader that the *consistency* principle is interesting, powerful, and versatile, and that so is its *converse.*

[7]The following discussion is based on Thomson (1995b).

Much remains to be investigated, however, and we also hope that the present review will help motivate further explorations and applications of these ideas.

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