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Abstract

for

Smooth Valuation Functions and Determinancy with Infinitely Lived Consumers

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In this paper, we consider a production economy with a finite number of heterogeneous, infinitely lived consumers. We show that for almost all endowments, equilibria that converge to a nondegenerate stationary state or cycle are locally unique. We do so by converting the infinite dimensional fixed point problem stated in terms of prices and commodities into a Negishi problem involving individual weights in a social value function. The key step in the analysis is to add a set of artificial fixed factors to utility and production functions. Then the equilibrium conditions equating spending and income for each consumer can be stated entirely in terms of time 0 factor endowments and derivatives of the social value function.

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1. INTRODUCTION

In this paper, we consider a production economy with a finite number of heterogeneous, infinitely lived consumers. We show that for almost all endowments, equilibria that converge to a nondegenerate stationary state or cycle are determinate, that is, locally unique. This result stands in stark contrast to results established for overlapping generations models: Kehoe and Levine [1985c] give a simple example of an overlapping generations model that has no cycles, is not chaotic, and has a robust continuum of Pareto efficient equilibria that converge to the same Pareto efficient stationary state. This example also shows that indeterminacy has nothing to do with whether or not equilibrium prices lie in the dual of the commodity space. More strongly, Kehoe, Levine, Mas-Colell and Zame [1986] show that robust indeterminacy can arise when both the commodity and price spaces are the same Hilbert space, provided there are infinitely many consumers. Consequently, it is the assumption of finitely many consumers that drives our results in this paper.

Our results extend those of Muller and Woodford [1983], who consider production economies with both finitely and infinitely lived agents. They show that there can be no indeterminacy if the infinitely lived agents are sufficiently large. Their results are local however, and concern only equilibria that converge to a particular stationary state. We prove a global theorem: for a given starting capital stock, there are only finitely many equilibria that converge to any nondegenerate stationary state. One particular implication is that when the discount factor is sufficiently close to one, which implies that there is a global turnpike, then equilibria are determinate.

The case of chaotic equilibrium paths remains largely unexplored: as far as we know, there are no examples of robust indeterminacy in this case. Some of our results suggest that if such examples exist, they must be qualitatively different from the robust indeterminacy that occurs in overlapping generations models.

We assume that markets are complete and that the technology and preferences are convex. Consequently, the behavior of equilibria in our model can be characterized by the properties of a value function. This is because the second theorem of welfare economics holds; that is, any Pareto efficient allocation can be decentralized as a competitive equilibrium with transfer payments. If the preferences of consumers can be represented by concave utility functions, then an equilibrium with transfers can be calculated by maximizing a weighted sum of the individual utility functions subject to the feasibility constraints implied by the aggregate technology and the initial endowments. Showing that an equilibrium exists is equivalent to showing that there exists a vector of welfare weights such that the transfer payments needed to decentralize the resulting Pareto optimal allocation are zero. This approach has been pioneered by Negishi [1960] and applied to dynamic models in a series of papers by Bewley [1980, 1982]. Using this approach, Kehoe and Levine [1985a] have considered the regularity properties of an infinite horizon economy without production.

In general, calculating the transfers associated with a given set of weights requires the complete calculation of equilibrium quantities and prices. In a dynamic model with an infinite number of commodities, this can be awkward. To simplify the calculation, we adopt an alternative strategy based on the simple geometrical observation that any convex set in R^n can be interpreted as the cross-section of a cone in R^{n+1}. To exploit this fact, we add a set of artificial fixed factors to the economy and include them as arguments of the weighted social value function. These factors are chosen so that the augmented utility and production functions are homogeneous of degree one. Thus, the usual problem of choosing a point on the frontier of a convex utility possibility set is converted into a problem of choosing a point from a cone of feasible values for utility. This extension has theoretical advantages analogous to those that arise when a strictly concave production function is converted into a homogeneous of degree one function by the addition of a fixed factor. When the technology for the firm is a cone, profits and revenues are completely accounted for by factor payments. Analogously, making the social value function homogeneous of degree one simplifies the accounting necessary to keep track of the transfers associated with any given Pareto efficient allocation. The present value of income and expenditure for each individual can be calculated directly from an augmented list of endowments and from the derivatives of the augmented social value function without explicitly calculating the dynamic paths for prices or quantities. This is the framework for studying multi-agent intertemporal equilibria developed by Kehoe and Levine [1985b].

In such a setting, equilibria are equivalent to zeros of a simple finite dimensional system of equations involving the derivatives of the social value function and the endowments. Intuition says that since the number of equations and the number of unknowns in this system are both equal to the number of agents, equilibria ought to be determinate. To do the usual kind of regularity analysis, however, we require that the system of equations that determines the equilibria be continuously differentiable. Because these equations involve derivatives of the social value function, they are C^1 if the value function is C^2.

There are four relevant cases. If for each set of welfare weights there is a unique globally stable turnpike, as will be true for social discount factors near one, then results of Araujo and Scheinkman [1977] show that the value function is C^2, and determinacy follows. The extent to which the value function is C^2 outside the basin of stable turnpikes is not known. However, if for each set of welfare weights every optimal path converges to a turnpike (or cycle) with no roots on the unit circle, we should expect that the boundary between the basins of sinks is of low dimension, and consequently, that for most initial capital stocks, the value function is C^2. In this case we can indeed show that equilibria are generically determinate, and that the value function is C^2 near every equilibrium.

If the dimension of the stable or unstable manifold of a stationary state changes as the welfare weights change, or if the total number of stationary states changes, then the system bifurcates. In this case stationary states must have unit roots, and our theorem does not apply. Moreover, little is known about whether value functions are C^2 in systems which are chaotic and have paths which do not converge to cycles. Consequently our results may be summarized by saying that in the class of economies which has no bifurcations and no chaos, determinacy is generic. In the case of bifurcating or chaotic systems we do not know if determinacy is generic. We should remark, however, that our general methods apply to stochastic as well as deterministic complete contingent claims economies (see Kehoe and Levine [1985b]), and as Blume, Easley and O'Hara [1982] have shown, a small amount of the right kind of uncertainty leads to smooth value functions. This provides an alternative direction for proving determinacy results, although we do not pursue this line here.

In the next section, we provide an overview of the method used to study equilibria in the context of a simple two person dynamic growth model. Section 3 establishes preliminary mathematical details about concave functions. Section 4 describes the economy. Section 5 defines and analyzes the savings function of the economy. Section 6 studies competitive equilibrium and regularity, taking as given differentiability of the savings function. In

Section 7 we prove our major results on the genericity of regularity. Section 8 concludes with an overview of our results.

2. OVERVIEW AND AN EXAMPLE

Consider a simple two person growth model. Assume that preferences take the usual additively separable form, discounted at the common rate β. For each of the individuals $i = 1,2$, let u_i denote the strictly concave momentary utility function. Let $k_o \in R$ be the time 0 endowment of the single productive factor; let $\theta_i \in R$ be the share belonging to individual i. Obviously, $\theta_1 + \theta_2 = 1$, and i's total endowment is $\theta_i k_o$. Assume that the technology is described by a concave aggregate production function $f\colon R_+ \to R$. Let $\alpha_i \in R$ denote the individual weights in a social optimization problem. Given an aggregate endowment of capital k_o and a vector of nonnegative welfare weights (α_1, α_2), define a value function $V(k_o, \alpha_1, \alpha_2)$ as the maximum of

$$\sum_{t=0}^{\infty} \beta^t \sum_{i=1}^{2} \alpha_i u_i(c_{it})$$

subject to the constraint

$$\sum_{i=1}^{2} c_{it} + k_{t+1} \leq f(k_t).$$

Using standard results from capital theory, we can treat the derivative $D_1V(k_o, \alpha_1, \alpha_2)$ as a time zero price for capital and use it to calculate the value of the endowment $\theta_i k_o$ for each individual. To calculate the transfers associated with these weights, we must also calculate the profits of the firm (if any) and the expenditure of each individual. For profits this is straightforward: If f is not homogeneous of degree one, introduce a fixed factor $x \in R$ and define $F(k,x) = xf(k/x)$. Specify endowments ϕ_i of this fixed factor equal to the ownership shares of the individuals in the aggregate firm. (Of course $\phi_1 + \phi_2 = 1$.) Then define $V(k_o,x,\alpha_1,\alpha_2)$ as the maximum of the weighted objective function subject to the constraint $\sum_{i=1}^{2} c_{it} + k_{t+1} \leq F(k_t,x)$. In equilibrium, the aggregate endowment of the factor x is be equal to 1, but it is useful to allow for the hypothetical possibility that it takes on other values so that we can calculate derivatives. Formally, we can treat x as a factor of production analogous to k and conclude that the share of the profits for agent i is ϕ_i multiplied by the price $D_2V(k_o,x,\alpha_1,\alpha_2)$. Since $F(k,x)$ is homogeneous of degree one, profits net of the new factor payments are zero. McKenzie [1959] has observed that any convex production possibility set could be interpreted as a cross-section of a cone in precisely this fashion and suggested that x be interpreted as an "entrepreneurial" factor.

The next step is to show that strictly concave utility functions can also be made homogeneous of degree one. If we interpret the fixed factor x as an accounting device used to keep track of producer surplus -- the difference between revenue and expenditure — it is clear that a similar factor can be used to account for consumer surplus -- the difference between utility and expenditure. Introduce an additional, person specific fixed utility factor w_i for each agent, and endow agent i with the entire aggregate supply of one unit of factor i. (For simplicity, we make no distinction in the notation between i's holdings of factor w_i and the aggregate endowment.) Just as we do for production, define an augmented utility function $U_i(c,w_i) = w_i u_i(c/w_i)$. In the next section, we show that this augmented utility function can always be defined and is well behaved even when u_i is unbounded from below. Now define a value function $V(k_o,x,w_1,w_2,\alpha_1,\alpha_2)$ as the maximum of the weighted sum of the augmented utility functions subject to the augmented technology.

If we let c_{it} denote the optimal consumption of agent i at time t, the first order conditions from the maximization imply the equality

$$\beta^t \alpha_i D_1 U_i(c_{it}, w_i) = \beta^t \alpha_j D_1 U_j(c_{jt}, w_j).$$

As a result, discounted marginal utility for either consumer can be used as a time zero price for consumption at time t. The only difference from the usual representative consumer framework is that the weights α convert the individual marginal utility prices at time 0 into a social marginal value price at time 0. We can then evaluate the expenditure of consumer i in period t as c_{it} multiplied by this price. Using the properties of homogeneous functions, we can decompose period t utility for consumer i into the sum of a term of this form and an analogous term involving the added utility factor:

$$U_i(c_{it}, w_i) = c_{it} D_1 U_i(c_{it}, w_i) + w_i D_2 U_i(c_{it}, w_i).$$

If the term involving the utility factor is interpreted as a measure of consumer surplus, expenditure on goods in period t is simply utility minus consumer surplus. By using a version of the envelope theorem, we can calculate the present value of consumer surplus for agent 1 as the derivative of the social value function $V(k_o, x, w_1, w_2, \alpha)$ with respect to w_1 multiplied by the endowment w_1:

$$w_1 D_3 V(k_o, x, w_1, w_2, \alpha_1, \alpha_2) = \sum_{t=0}^{\infty} \beta^t \alpha_1 w_1 D_2 U_1(c_{1t}, w_1).$$

Similarly, we can calculate the discounted sum of utility for consumer 1, measured in social value units, as the derivative of the social value function with respect to α_1 multiplied by α_1:

$$\alpha_1 D_5 V(k_o, x, w_1, w_2, \alpha_1, \alpha_2) = \sum_{t=0}^{\infty} \beta^t \alpha_1 U_1(c_{1t}, w_1).$$

Then the present value of expenditure by agent 1 is simply the difference

$$\alpha_1 D_5 V(k_o,x,w_1,w_2,\alpha_1,\alpha_2) - w_1 D_3 V(k_o,x,w_1,w_2,\alpha_1,\alpha_2).$$

The transfer to agent 1 necessary to support this equilibrium is be zero if and only if this expenditure is equal to the time zero value of the agent's endowment

$$\theta_1 k_o D_1 V(k_o,x,w_1,w_2,\alpha_1,\alpha_2) + \phi_1 D_2 V(k_o,x,w_1,w_2,\alpha_1,\alpha_2).$$

Formally, this equality can be interpreted in terms of an augmented economy where trade in the production factor x and the utility factors w_i actually takes place. In this case, this equality can be interpreted as a requirement that the value of the augmented endowment for consumer 1, $\theta_1 k_o D_1 V + \phi_1 D_2 V + w_1 D_3 V$, equals the amount of social utility purchased, $\alpha_1 D_5 V = \alpha_1 \sum_{t=0}^{\infty} \beta^t U_1$.

It is useful to define a net savings function s_1 for agent 1, as follows:

$$s_1(k_o,\theta,\phi,\alpha) = \theta_1 k_o D_1 V(k_o,1,1,1,\alpha_1,\alpha_2) + \phi_1 D_2 V(k_o,1,1,1,\alpha_1,\alpha_2)$$
$$+ D_3 V(k_o,1,1,1,\alpha_1,\alpha_2) - \alpha_1 D_5 V(k_o,1,1,1,\alpha_1,\alpha_2).$$

The savings function for consumer 2 is defined symmetrically. For a given set of welfare weights α, the transfer for each individual needed to support the social optimum as a competitive equilibrium is the negative of the net savings for that individual. A competitive equilibrium is therefore a vector of weights α such that $s(k_o,\theta,\phi,\alpha) = 0$. In general, if m is the number of individuals, $s(k_o,\theta,\phi,.)$ is a map from R^m into R^m, and the existence of an equilibrium can be established using a standard fixed point argument in a finite dimensional space.

This characterization of equilibria as zeros of an equation involving endowments and the derivatives of an augmented value function is quite general. All that is required is that the second welfare theorem hold and that the preferences of the consumers can be represented using concave utility functions. To do the regularity analysis, s must be C^1. If it is, we can develop conditions under which the inverse function theorem and the implicit function theorem show respectively that equilibria are locally unique and that they vary continuously with parameters of the economy. For s to be C^1, V must be C^2 in the weights α and in the state variables k_o. To show this, we rely heavily on the additional restrictions, like additively separable utility, that we impose on the class of economies that we study. This could, however, be weakened to allow for the kind of intertemporal dependence suggested by Kydland and Prescott [1982], which can be described in terms of the evolution of some finite dimensional state variable.

3. HOMOGENEOUS EXTENSION OF CONCAVE FUNCTIONS

To support the claim that the approach outlined above is broadly applicable, we must first establish that it is possible to convert any concave utility function into a homogeneous function by adding a fixed factor. This does not follow immediately from results for production functions because utility functions need not be bounded from below. The analysis that follows would be considerably simpler if we restricted attention to utility functions that are bounded, but functions like logarithmic utility and isoelastic utility $u(c) = -c^{-\sigma}$ where $\sigma > 0$ are widely used in applications of this kind of model. In the formal analysis, we accommodate these functions using the concepts and terminology from convex analysis for dealing with extended real valued functions; see Rockafellar [1970] for a complete treatment.

If n_c denotes the number of consumption goods in this economy, a utility function u is a function that is defined on the nonnegative orthant $R_+^{n_c}$ in commodity space, and that takes on values in $R \bigcup \{-\infty\}$. On the strictly positive orthant $R_{++}^{n_c}$, u is finite, but to accommodate functions like logarithmic utility, we want to allow for the possibility that u(c) is equal to $-\infty$ if one of the components of c is equal to zero. We can define a topology on $R \cup \{-\infty\}$ by adding the open intervals $[-\infty, a)$ to the base for the usual topology of R. Note that $(-\infty, \infty)$ is not a closed set in this topology and that convergence to $-\infty$ has the usual interpretation: a sequence $\{b^n\}$ in R converges to $-\infty$ if, for all $M \in R$, there exists an N such that $n \geq N$ implies $b^n \in [-\infty, M)$. With this topology, the natural assumption on preferences is that $u\colon R_+^{n_c} \to R \cup \{-\infty\}$ be continuous. For example, the utility functions $u(c) = c^{1/2}$ and $u(c) = \ln(c)$ can both be represented as continuous functions from R_+ to $R \cup \{-\infty\}$.

The extension of u(c) to the homogeneous function $U(c,w) = wu(c/w)$ does not preserve continuity on the nonnegative orthant in R^{n_c+1}. A discontinuity can arise at the point $(c,w) = (0,0)$. This extension does, however, preserve a weaker notion of continuity. Recall that for a function $g\colon R^\ell \to R$, g is <u>upper-semi-continuous</u> (u.s.c) if the inverse image $g^{-1}([a,\infty))$ is always a closed set. If we allow the function g to take values in $R \cup \{-\infty\}$ instead of R, we can make an identical definition. Equivalently, g is u.s.c. if, for any sequence $\{y^n\}$ in R^ℓ converging to y, $\lim_{n\to\infty} \sup g(y^n) \leq g(y)$. Since an u.s.c. function has a maximum over a compact set, upper-semi-continuity is strong enough for our purposes. If the function g is concave, define the <u>recession function</u> of g, $r_g\colon R_+^\ell \to R \cup \{-\infty\}$, as

$$r_g(y) = \lim_{t\to\infty} \frac{g(z+ty) - g(z)}{t}$$

where z is any point such that g(z) is finite. Since g is concave, it can be shown that r_g is homogeneous of degree one and does not depend on the choice of z in the definition. Roughly speaking, $r_g(y)$ describes the asymptotic average slope of g along a ray from the origin passing through the point y.

Given these definitions, we can now state the key lemma for our construction. See Rockafellar [1970, p. 67] for a proof.

LEMMA 1: Let $g\colon R_+^{\ell} \to R \cup \{-\infty\}$ be concave and continuous. Let $G\colon R^{\ell} \times R \to R \cup \{-\infty\}$ be defined by

$$G(y,\rho) = \begin{cases} \rho g(y/\rho) & \text{if } \rho > 0 \text{ and } y \in R_+^{\ell}, \\ r_g(y) & \text{if } \rho = 0 \text{ and } y \in R_+^{\ell}, \\ -\infty & \text{otherwise.} \end{cases}$$

Then G is concave, u.s.c., and homogeneous of degree one.

If g is a production function, hence nonnegative, $G(y,\rho)$ is increasing in ρ. If g represents a utility function that takes on negative values, $G(y,\rho)$ is decreasing in ρ for some values of y. In the artificial equilibrium where we allow for trade in the utility factors, this may imply that the price associated with the utility factor is negative. Implicitly, the strategy here is to consider first an equilibrium with explicit markets in all goods, including the fixed factors in the utility functions. Prices are such that each individual consumes his endowment (equal to 1) of the utility factor. Then prices and quantities for all other goods do not depend on whether or not trade in the utility factors is possible. The possibility of negative prices for utility factors in the complete markets equilibrium poses no problem for proving existence because it is not necessary to assume free disposal of the utility factors. So long as each individual is endowed at time zero with a positive amount of capital or some other factor with positive value, strictly positive consumption of all true consumption goods is feasible. The total value of any individual's endowment may be negative, but it is always possible to use up the utility factor (that is, consume it), leaving strictly positive income to be spent on the true consumption goods.

4. FORMAL EQUILIBRIUM MODEL

Assume that there are m consumers in the economy. Let n_k denote the number of reproducible capital stocks, n_c the number of consumption goods. Let k_0 denote the n_k vector of initial aggregate capital stocks. Let ϕ denote the m vector of ownership shares for the single aggregate firm. Each agent has a utility function $u_i: R^{n_c} \to R \cup \{-\infty\}$; let $U_i: R^{n_c} \times R \to R \cup \{-\infty\}$ denote the homogeneous extension of u_i as defined in Lemma 1. Naturally $U_i(c_{it},1) = u_i(c_{it})$.

Again, we assume that all consumers have the same discount factor $\beta < 1$. Conceptually, there is no difficulty with different consumers having different discount factors. Kehoe and Levine [1986] show how to integrate this into the formal model. Moreover, the proof of the existence of an equilibrium remains straightforward. We should note, however, that insofar as some individuals with low discount factors asymptotically consume zero, we do not know whether or not the value function is C^2, and are unsure of whether our determinacy results extend to this case.

For simplicity, assume that the technology that relates period t to period t+1 can be described in terms of an aggregate production function. Let c_t, k_t, and k_{t+1} denote aggregate consumption at time t, and capital at time t and t+1. Then the technology is described by the constraint $f(k_t,k_{t+1},c_t) \geq 0$, where $f: R_+^{n_k} \times R^{n_k} \times R_+^{n_c} \to R$. Formally, it is convenient to allow f to be defined when the terminal stock k_{t+1} lies outside the nonnegative orthant. Hypothetically, if it were possible to leave negative capital for next period, f describes the additional current consumption that would be possible. In the intertemporal optimization problems we explicitly impose the constraint that k_{t+1} be nonnegative.

In this specification of the aggregate technology, we have not made explicit the dependence of output on factors of production that are in fixed supply. Formally it is as if we have given ownership of all such factors to the aggregate firm. Individuals sell any endowments of land and labor for an increased ownership share in the firm. To consume a specified amount of leisure or of consumption services from land, an individual must purchase these like any other consumption good. This is merely a notational convenience. To make these factors explicit, we would simply need to augment the argument list for the production function and specify individual endowments in these additional factors.

By Lemma 1, there exists a homogeneous function $F(k_t, k_{t+1}, c_t, x)$ such that $F(k_t, k_{t+1}, c_t, 1) = f(k_t, k_{t+1}, c_t)$. Given the additional fixed factor x, the aggregate technology set is a cone. Its representation in terms of an aggregate production function is convenient because it allows a simple specification of the smoothness properties of the technology. If F is smooth, the boundary of the cone is smooth. A more general treatment along the lines of Bewley [1982] would start from assumptions about the separate technologies available to individual firms, but our interest here lies not so much with the specification of the technology, but rather with the specification of preferences and endowments.

We can now specify the properties assumed for the preferences and technology. The assumptions concerning continuity and smoothness are standard. For convenience, the usual monotonicity assumptions are strengthened, but this is not essential. A more important restriction is that the utility function be strictly concave and that the production possibility set for output capital stocks be strictly convex when the input capital stocks and aggregate consumption are held constant.

ASSUMPTION 1: For all i, the utility function u_i: $R_+^{n_c} \to R \cup \{-\infty\}$ is concave, strictly increasing, and continuous. On the strictly positive orthant $R_{++}^{n_c}$, u is C^2 and has a negative definite Hessian.

ASSUMPTION 2: The production function f: $R_+^{n_k} \times R^{n_k} \times R_+^{n_c} \to R$ is concave and continuous, with $f(0,0,0) = 0$. On the interior of its domain, f is C^2, strictly increasing in its first argument, and strictly decreasing in the second and third arguments. Also the matrix of second derivatives with respect to the vector of terminal stocks, $D_{22}f(k_t,k_{t+1},c_t)$, is negative definite.

In his discussion of the von Neumann facet, McKenzie [1983] has emphasized that it is restrictive to assume that f is strictly concave. If fixed factors in production can be allocated between different constant returns to scale industries (for example labor in the multi-sector neoclassical growth model), there can exist an affine set of initial and terminal capital stocks that produce the same consumption goods vector. In the usual case where consumption and next period capital can be exchanged one for one, the weaker assumption that $D_{22}f(k_t,k_{t+1},c_t)$ is negative definite requires that, given k_t, the set of possible output combinations have a production possibility frontier with positive curvature.

Because some of the factors of production are in fixed supply, output exhibits diminishing returns as a function of the initial capital stock. The next assumption strengthens the diminishing returns so that feasible output stocks are bounded.

ASSUMPTION 3: (Boundedness) There exists $k_{max} \varepsilon R_{+}^{n_k}$ and a bound $b < 1$ such that $k_t \geq k_{max}$ and $k_{t+1} \geq bk_t$ implies that k_{t+1} is not feasible.

This assumption states that capital stocks larger than k_{max} cannot be sustained. By the definition of F, this bound also holds when $f(k_t, k_{t+1}, c)$ is replaced by its homogeneous extension $F(k_t, k_{t+1}, c, x)$ for any value of x less than or equal to 1. This boundedness assumption is stronger than is needed for existence of a social optimum or an equilibrium, but it is required to rule out unbounded growth paths in the proof of the existence of an optimal stationary value for the capital stock.

Proving the existence of an optimal stationary state also requires the other half of a set of Inada-type conditions on production. Assumption 4 ensures that there exist strictly positive feasible paths for capital and consumption and that at least one such path does not converge asymptotically to zero consumption and capital. Recall that $R_{++}^{n_k}$ denotes the strictly positive orthant in R^{n_k} and that $\beta < 1$ is the discount rate.

ASSUMPTION 4: (Feasibility) For all $k_t \varepsilon R_{++}^{n_k}$, there exists $k_{t+1} \varepsilon R_{++}^{n_k}$ and $c \varepsilon R_{++}^{n_c}$ such that (k_t, k_{t+1}, c) is feasible, that is, $f(k_t, k_{t+1}, c) \geq 0$. Furthermore, for some point $k_t \varepsilon R_{++}^{n_k}$, k_{t+1} and c can be chosen so that $c \varepsilon R_{++}^{n_c}$ and $\beta k_{t+1} > k_t$.

The smoothness arguments that follow require that the optimal values of the capital stock and consumption lie in the interior of the domain of the production function and the consumption function respectively. This is guaranteed here by infinite steepness conditions on the boundary of the domains. For a concave function $g: R^{\ell} \to R \cup \{-\infty\}$, the generalization of a derivative is a subgradient. The set of <u>subgradients</u> of g at y, denoted $\partial g(y)$, is defined by

$$\partial g(y) = \{p \in R^{\ell}: g(z) - g(y) \leq p(z-y) \text{ for all } z \in R^{\ell}\}.$$

Note that we follow the unfortunate, but well established, convention of letting a term like subgradient have a different meaning for concave and convex functions. For a convex function $h(z)$, the definition of $\partial h(z)$ is given by reversing the direction in the inequality in the definition given here. Let $\{y^n\}$ be a sequence in R^{ℓ}_{++}. Suppose g is a differentiable function with the property that one of the components of the gradient $\partial g(y^n)$ has a limit equal to ∞ as y^n approaches a point y. Then $\partial g(y)$ is empty. By the assumption of concavity, a point like y can arise only on the boundary of the domain of g. For a function like $g(y_1,y_2) = y_1^{0.3}y_2^{0.3}$, the limit of the gradient as (y_1^n,y_2^n) goes to $(0,0)$ can not be defined, but it is still the case that $\partial g(0,0)$ is empty.

ASSUMPTION 5: (Infinite steepness on the boundary)

(a) If c is an element of the boundary of the domain of u_i, the set of subgradients $\partial u_i(c)$ is empty.

(b) If any component of k_t is 0, the set of subgradients of f with respect to its first argument, $\partial_1 f(k_t,k_{t+1},c)$, is empty.

Part (a) implies that the marginal utility of any good is infinite starting from zero consumption of that good. As stated, it allows u to be finite or to equal $-\infty$ on the boundary. It implies that every individual consumes some amount of every good in equilibrium, but weaker conditions could be used. All that is necessary is that a strictly positive amount of each good be produced in equilibrium. Part (b) is the usual assumption of infinite marginal productivity of each capital good starting from zero usage.

5. SOCIAL RETURN AND SAVINGS FUNCTIONS

We now define the return and savings functions derived from social optimization. These are then used to define an equilibrium. For expositional convenience, all proofs in this section may be found in the Appendix. Given the underlying preferences and technology, we define a weighted momentary social return function $v\colon R_+^{n_k} \times R^{n_k} \times R_+ \times R_+^m \times R_+^m \to R \cup \{-\infty\}$ as follows: If $F(k_t, k_{t+1}, 0, x) \geq 0$, that is, if nonnegative aggregate consumption is feasible,

$$v(k_t, k_{t+1}, x, w, \alpha) = \max \sum_{i=1}^{m} \alpha_i U_i(c_i, w_i)$$

$$\text{s.t. } F(k_t, k_{t+1}, \sum_{i=1}^{m} c_i, x) \geq 0,$$

$$c_i \geq 0.$$

If $F(k_t, k_{t+1}, 0, x) < 0$,

$$v(k_t, k_{t+1}, x, w, \alpha) = -\infty.$$

If we were to work only with utility functions that are bounded from below on a suitably chosen domain, v would be a familiar, real valued saddle function. It is concave and homogeneous of degree one in (k_t, k_{t+1}, x, w), convex and homogeneous of degree one in α. It would also be continuous in the usual sense, instead of u.s.c. as established below.

<u>PROPOSITION 1</u>: Under Assumptions 1-5, we have the following results:

(a) v is well defined.

(b) For $\alpha \in R_+^m$, $v(\cdot,\cdot,\cdot,\cdot,\alpha)$ is concave, u.s.c. and homogeneous of degree one, with the same monotonicity properties as F.

(c) For any $(k_t,k_{t+1},x,w) \in R_+^{n_k} \times R^{n_k} \times R_+ \times R_+^m$, the function $v(k_t,k_{t+1},x,w,\cdot)\colon R_{++}^m \to R \cup \{-\infty\}$ is convex and homogeneous of degree one.

(d) For any $(x,w,\alpha) \in R_+ \times R_+^m \times R_+^m$, the set of subgradients of the concave function $v(\cdot,\cdot,x,w,\alpha)$ is empty at every point on the boundary of its domain.

(e) On the interior of its domain, v is C^2.

(f) Evaluated at any point in the interior of the domain of v, $D_{22}v(k_t,k_{t+1},x,\alpha)$ is negative definite.

Let $V\colon R_+^{n_k} \times R_+ \times R_+^m \times R_+^m \to R \cup \{-\infty\}$ denote the social present value function derived from the return function v:

$$V(k_o,x,w,\alpha) = \max \sum_{t=0}^{\infty} \beta^t v(k_t,k_{t+1},x,w,\alpha).$$

Here, the maximization is over all nonnegative sequences $\{k_t\}$ having an initial value equal to k_o. The constraint that the sequence be feasible is implicit in the maximization problem since v must take on the value $-\infty$ at some point along an infeasible sequence. Let $P(k_o,x,w,\alpha)$ denote the constrained maximization problem that defines V. The next proposition establishes the basic properties of the problem $P(k_o,x,w,\alpha)$ and its value function $V(k_o,x,w,\alpha)$. The proof that $P(k_o,x,w,\alpha)$ has a solution is complicated by the possibility that the utility functions u_i can take on the value $-\infty$. If all the functions u_i were assumed to be bounded, this follows from the fact that the objective function is continuous in the product topology. Here we use Fatou's lemma (as stated for example in Rudin [1966]) to show that the extended real valued objective functional is upper-semi-continuous.

PROPOSITION 2: Let $k_o \varepsilon R_+^{n_k}$, let $x \varepsilon R_+$, let $w \varepsilon R_+^m$, and let $\alpha \varepsilon R_+^m$. Under Assumptions 1-5 the following results hold:

(a) The problem $P(k_o,x,w,\alpha)$ has a unique solution; $V(k_o,x,w,\alpha)$ is well defined and finite.

(b) The function $V(\cdot,\cdot,\cdot,\alpha)$ is concave, homogeneous of degree one with the same monotonicity properties as F.

(c) The function $V(k_o,x,w,\cdot)$ is convex and homogeneous of degree one.

(d) There exists an optimal stationary value $k^{ss} = k^{ss}(x,w,\alpha)$; that is, the sequence defined by $k_t = k^{ss}$ solves the problem $P(k^{ss},x,w,\alpha)$; every optimal stationary value lies strictly in the interior.

(e) On the interior of its domain, V is C^1; moveover, if $\{k_t\}_{t=o}^{\infty}$ is the solution to $P(k_o,x,w,\alpha)$, then

$$D_1 V(k_o,x,w,\alpha) = D_1 v(k_o,k_1,x,w,\alpha), \text{ and}$$

$$D_j V(k_o,x,w,\alpha) = \sum_{t=0}^{\infty} \beta^t D_{j+1} v(k_t,k_{t+1},x,w,\alpha) \quad \text{for } j = 2,3,4$$

The value function for the social optimization problem depends only on aggregate quantities, but to define the savings functions we need to specify the matrices of individual endowments. Let θ denote the $n_k \times m$ matrix of nonnegative capital shares. Naturally $\sum_{j=1}^{m} \theta_{ij} = 1$. Let K_o denote the $n_k \times n_k$ diagonal matrix of capital stock corresponding to an n_k vector k_o, and let A denote the $m \times m$ diagonal matrix of welfare weights corresponding to an m vector α. As before, ϕ denotes the m vector of endowment shares of the fixed factor for production. We say that the endowment shares θ and ϕ, and initial stock k_o are <u>admissible</u> if all of the components are nonnegative, if the aggregate supplies k_o, are strictly positive, if every individual is endowed with a positive amount of some capital good, and if the shares sum to one. If we let θ^T denote the transpose of θ and interpret all the following products as matrix products, we can define the savings function for any admissible k_o,θ,ϕ and $\alpha \in R^m_+$ as follows:

$$s(k_o,\theta,\phi,\alpha) = \theta^T K_o D_1 V(k_o,1,1,\alpha) + \phi D_2 V(k_o,1,1,\alpha) + D_3 V(k_o,1,1,\alpha)$$
$$- AD_4 V(k_o,1,1,\alpha).$$

Note that in defining the savings function, we have set $x = 1$ and $w_i = 1$ for $i = 1,\ldots,m$. At these values the augmented functions $U_i(c_{it},w_i)$ and $F(k_t,k_{t+1},c_t,x)$ reduce to the original specifications u_i and f.

To prove the existence of an equilibrium and demonstrate its properties, the next proposition establishes that the savings function is formally similar to the standard pointwise product of excess demand with prices in an economy with m goods.

PROPOSITION 3: Let k_o,θ,ϕ w be admissible:

(a) $s(k_o,\theta,\phi,\cdot)$ is continuous at all points α in R^m_+ different from 0.

(b) $s(k_o,\theta,\phi,\cdot)$ is homogeneous of degree 1 in α.

(c) $\sum_{i=1}^{m} s_i(k_o,\theta,\phi,\alpha) = 0$ for all $\alpha \in R^m_+$.

(d) If $\alpha_i = 0$, then $s_i(k_o,\theta,\phi,\alpha) > 0$.

6. COMPETITIVE EQUILIBRIUM

In this framework, an economy $E = \{k_o,\theta,\phi,f,u_1,\ldots,u_m\}$ consists of a specification of the production function f, the utility functions $u_1,\ldots,u_m$, an admissible capital shares matrix θ, and an admissible vector of ownership shares ϕ. Following the remarks in Section 2, we define a competitive equilibrium for E as a vector of weights $\alpha \in R^m$ such that the transfers associated with the solution to the problem $P(k_o,1,1,\alpha)$ are zero, that is, such that $s(k_o,\theta,\phi,\alpha) = 0$.

To see that this definition is equivalent to the usual definition of an equilibrium, one need only verify that this economy satisfies conditions like those given in Debreu [1959] for the second welfare theorem to hold in a general linear space. (The existence of an interior point follows immediately from the use of the sup norm in the usual Banach space of bounded sequences.) Then using the necessary conditions from the concave maximization problem $P(k_o,x,w,\alpha)$, it is easy to verify that discounted marginal utilities are indeed time zero prices for consumption goods and that a zero of our savings function does correspond to an equilibrium with no transfers.

Given the properties of the savings function described in Proposition 3, the proof of the existence of an equilibrium follows by repeating the standard textbook proof of the existence of a competitive equilibrium with m goods. (See for example, Varian [1984].) Consequently, we merely state our existence result.

PROPOSITION 4: For any economy there exists an equilibrium.

We can now describe the conditions under which equilibria are locally unique and vary continuously with the parameters of the economy. Exactly as in the theory of regular economies as developed by Debreu [1970], these results follow if we can apply the inverse function theorem and the implicit function theorem to an equation like

$$s(k_o,\theta,\phi,\alpha) = 0.$$

We cannot apply these theorems directly to this equation for the $m \times m$ matrix of derivatives of s with respect to α is singular. The difficulty arises because we have one too many variables (because of the homogeneity of degree one of s with respect to α) and one too many equations (because the sum of the components of s is identically equal to zero.) Because the savings function is strictly positive on the boundary of R^m_+, we can choose a normalization of α — analogous to he usual normalization for prices — such that $\alpha_m = 1$. Just as one typically uses Walras's law to discard one of the equations, we can use the adding up constraint on s to discard the equation $s_m = 0$. Thus, it is sufficient to consider the resulting $m-1$ equations in the $m-1$ unknowns $\alpha_1,\ldots,\alpha_{m-1}$. Then we can apply the inverse function theorem and the implicit function theorem if the function s is continuously differentiable and if the following matrix J is nonsingular:

$$J = \begin{bmatrix} \frac{\partial s_1}{\partial \alpha_1}, & \cdots & \frac{\partial s_{m-1}}{\partial \alpha_1} \\ \frac{\partial s_1}{\partial \alpha_{m-1}} & \cdots & \frac{\partial s_{m-1}}{\partial \alpha_{m-1}} \end{bmatrix}$$

Following Debreu, we say that an economy E is <u>regular</u> if these two conditions hold. In this case the index theorem developed by Dierker [1972] allows us to use the sign of the determinant of this matrix to count the number of equilibria and establish conditions ensuring uniqueness.

Under the assumption of C^1 differentiability, the analogue of J in finite economies is almost always nonsingular; that is, regular economies are generic. As we show below, this is true as well for J, provided that s is C^1. The difficulty, as we shall see later, lies in verifying that s is C^1, or equivalently, that V is C^2.

<u>PROPOSITION 5</u>: Suppose that, for given k_o, there exists admissable θ and ϕ so that $s(k_o,\theta,\phi,.)$ is C^1. Then, for the given values of k_o and ϕ and almost all θ, the economy is regular.

Proof: Since s is linear in θ it is jointly C^1 in (θ,α). We must show that, for almost all θ and given k_o and ϕ, the matrix J is nonsingular when the derivatives of s are evaluated at a point (k_o,θ,ϕ,α) such that $s_1(k_o,\theta,\phi,\alpha) = 0, \ldots, s_{m-1}(k_o,\theta,\phi,\alpha) = 0$. As noted prior to the definition of J, the last equation in $s = 0$ can be discarded and α can be chosen so that $\alpha = (\alpha_{-m},1) \in R_+^{m-1} \times R$. The transversality theorem of differential topology [Guillemin and Pollack 1974, p. 67] states that a sufficient condition for J to be of full rank $m-1$ at such a point for almost all θ is that changes in θ can cause changes in s in $m-1$ dimensions. More precisely, the matrix of partial derivatives of the first $m-1$ components of s with respect to θ should itself be of rank $m-1$. But, from the definition of s, we can calculate (the transpose of) this matrix to be a $(m-1)n_k \times (m-1)$ matrix of the form,

$$\begin{bmatrix} K_oD_1V & 0 & \cdots & 0 \\ 0 & K_oD_1V & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & K_oD_1V \end{bmatrix}$$

where D_1V stands for the derivatives $D_1V(k_o,1,1,\alpha)$. Since k_o and the derivative D_1V must be strictly positive, this matrix has $m-1$ linearly independent rows and the result follows. Q.E.D.

The simplicity of this theorem comes from the fact that we consider changes in the specification of the economy that do not affect the aggregate variables that enter as arguments in the social value function.

One important implication of Proposition 5 follows from standard turnpike theorems, of the type proven by McKenzie [1986]. These theorems assert that if β is sufficiently close to one, relative to the curvature of v, then there is a unique stationary capital stock k^{ss} to which all optimal paths converge. Araujo and Scheinkman [1977] show that in this case the value function V is C^2 with respect to k_o and β. With minor modifications, indicated below in Section 8, their method of proof shows that V is also C^2 with respect to other parameters, such as α. In our context these two results, together with the fact that the curvature of v can be bounded uniformly in α, imply that for β sufficiently close to one V is C^2, and the conclusion of Proposition 5 is immediate.

While Proposition 5 assumes that s is C^1 globally in α, in fact s need only be C^1 in the neighborhood of equilibrium values of α. Moreover, in this case we can use the derivatives of V and the implicit function theorem to do local comparative statics with respect to θ, ϕ, or other parameters.

7. REGULARITY

We do not know, either by theorem or counterexample, about the differentiability of s outside the basin of a stationary state. In a non-chaotic system the area between the basins of stationary states is a low dimensional set, consisting of the stable manifolds of unstable stationary states or cycles. If differentiability fails only on a low dimensional subset, however, then it ought to be merest coincidence that it fails in equilibrium.

In what follows, it is often convenient to let x and w_1 be equal to 1, and write $v(k_t,k_{t+1},\alpha)$, $V(k_o,\alpha)$, $P(k_o,\alpha)$, and so forth. From the Assumption 5 of infinite steepness on the boundary, we may assume that (k_t,k_{t+1},α) lies in the interior. Let $\ell_\infty^{n_k}$ denote the Banach space of bounded sequences in R^{n_k} under the sup norm, $|k| = \sup_t |k_t|$, where $|k_t|$ denotes any norm equivalent to the usual norm on R^{n_k}; let $(\ell_\infty^{n_k})_+$ denote the positive orthant in $\ell_\infty^{n_k}$. For convenience, assume that the first component of any sequence in $\ell_\infty^{n_k}$ has an index $t=1$. Define the mapping associated with the Euler equation, $\xi\colon (\ell_\infty^{n_k})_{++} \times R_{++}^{n_k} \times R_{++}^{m} \to \ell_\infty^{n_k}$, by the rule

$$\xi(k,k_o,\alpha)_t = D_2v(k_{t-1},k_t,\alpha) + \beta D_1v(k_t,k_{t+1},\alpha) \quad t \geq 1.$$

By the usual sufficient conditions for concave maximization problems, any path k_t that remains bounded and satisfies the Euler equation $\xi(k,k_o,\alpha) = 0$ is an optimal path for the problem $P(k_o,\alpha)$. Conversely, any optimal path k starting at an interior $k_o \in R_{++}^{n_k}$ satisfies this equation and remains bounded. Consequently, $k \in (\ell_\infty^{n_k})_+$ is optimal if and only if $\xi(k,k_o,\alpha) = 0$.

Lemma 2 below shows that ξ is C^1. Define $D_h\xi = [D_1\xi \; D_2\xi]$. Let $D_{ij}v_t$ denote $D_{ij}v(k_t,k_{t+1},\alpha)$. Then we can write the component t of $D_h\xi(k,k_o,\alpha)h$ as

$$\beta D_{12}v_t h_{t+1} + (\beta D_{11}v_t + D_{22}v_{t-1})h_t + D_{21}v_{t-1}h_{t-1}.$$

In other words, $D_h\xi h = 0$ gives rise to a linear dynamical system. At a stationary state, the coefficients are time independent, so we omit the time subscript. For emphasis we write $D_h\xi^{ss}$ to emphasize we are considering a stationary state. By the roots of $D_h\xi^{ss}$ we mean the eigenvalues of the associated linear dynamical system. We call a stationary state <u>nondegenerate</u> if $D_h\xi^{ss}$ has no roots on the unit circle and if $D_{12}v$ is nonsingular.

The fact that $D_{12}v$ is nonsingular ensures that the dynamical system can be solved both forwards and backwards. In economic terms, this means that the model must be stated using a minimal set of capital goods. To see what this rules out, consider a Cobb-Douglas neoclassical growth model stated in terms of two capital goods, two consumption goods and a fixed endowment of labor that must be allocated between identical production functions for the two consumption goods. Let two individuals have identical preferences $\ln(c_1) + \ln(c_2)$. This economy satisfies Assumptions 1 through 5. It is straightforward to show by direct algebraic manipulation that independent of the initial stocks of capital, the subsequent aggregate stocks of the capital goods in the social maximization problem for any set of weights α are always chosen in fixed proportions. (The consumption goods are also consumed in fixed proportions.) The model is not in any relevant sense two dimensional; the two capital goods and the two consumption goods can be combined into single composite capital and consumption goods. The dynamical system associated with the social maximization problem for this economy always maps k_{t+1} onto a line in R^2. One can also show directly that $D_{12}v(k_t,k_{t+1},\alpha)$ is everywhere singular in this case. This kind of collapse in the dimensionality of the model is prevented, even locally, by assuming that $D_{12}v$ is nonsingular.

At a nondegenerate steady state, it is well known that R^{2n_k} can be written as the direct sum of a stable and unstable manifold. We refer to n_k minus the dimension of the stable manifold as the <u>index</u> of $D_h\xi^{ss}$. In Lemma 3 below we show that $D_1\xi^{ss}$ is one-to-one. It follows directly that the index is nonnegative. We call a path k <u>nondegenerate</u> for α and k_o if k_t converges to a nondegenerate stationary state $k^{ss}(\alpha)$ and if, whenever index $k^{ss}(\alpha) \geq 1$, $D_{12}v(k_t,k_{t+1},\alpha)$ is nonsingular for $t = 0,1,\ldots.$

Note that these definitions may easily be extended to allow cycles, in place of stationary states. Consider a cycle with period p. We simply redefine periods with $n_c p$ commodities and $n_k p$ types of capital per period so that all cycles appear as stationary states. This kind of trick is frequently used with overlapping generations models. Consequently the subsequent propositions apply equally to paths conversity to cycles.

Let $\bar{E}(k_o)$ denote the set of pairs $(k,\alpha) \in (\ell_\infty^{n_k})_+ \times R^{m-1}$ such that k is nondegenerate for α and k_o. In other words, we restrict attention to paths that converge to a non-degenerate stationary state. Our goal in this section is to prove:

PROPOSITION 6: For fixed ϕ and a full measure subset of k_o and θ there are finitely many equilibria in $\bar{E}(k_o)$ and if $(k,\alpha) \in \bar{E}(k_o)$ is an equilibrium for ϕ and θ, then $V(k_o,x,w,\alpha)$ is C^2 in a neighborhood of $(k_o,1,1,\alpha)$.

In particular, in this full measure set, $s(k_o,\theta,\phi,\alpha)$ is C^1 and Proposition 5 implies a further full measure subset in which the economy is regular. Notice incidentally, that by Fubini's theorem, the fact that Proposition 6 holds for fixed ϕ and a full measure subset of k_o and θ implies that it holds for a full measure subset of k_o, θ and ϕ.

Our strategy of proving Proposition 6 is to expand the dimensionality of the system of equations to include the dynamic path of capital. This means we are dealing with an infinite dimensional equation system, but one that is globally C^1. We first show that generically equilibria in $\bar{E}(k_o)$ are regular in an infinite dimensional sense. Proposition 6 is then an immediate corollary.

We first examine the savings functions. By Part (e) of Proposition 2 each of the derivatives in the definition of s may be written as a sum of derivatives along the optimal path. Define

$$\sigma(k,k_o,\alpha,\theta,\phi) = \theta^T K_o D_1 v(k_o,k_1,1,1,\alpha) + \phi \sum_{t=0}^{\infty} \beta^t D_3 v(k_t,k_{t+1},1,1,\alpha)$$

$$+ \sum_{t=0}^{\infty} \beta^t D_4 v(k_t, k_{t+1}, 1, 1, \alpha)$$

$$- A \sum_{t=0}^{\infty} \beta^t D_5 v(k_t, k_{t+1}, 1, 1, \alpha).$$

It follows that for given k_o and θ, k and α are an equilibrium if and only if

$$\xi(k, k_o, \alpha) = 0$$

$$\sigma(k, k_o, \alpha, \theta, \phi) = 0.$$

As in Proposition 5, we fix $\alpha_m = 1$ and delete one redundant equation from σ. For notational simplicity, we assume throughout that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$ and that σ consists of σ_1 to σ_{m-1} only; in other words, we take σ to be the reduced system. A proof of the following result can be found in the Appendix.

LEMMA 2: The system of equations ξ, σ is C^1.

We can now define an equilibrium to be regular if the operator

$$\Sigma = \begin{bmatrix} D_1 \xi & D_3 \xi \\ D_1 \sigma & D_3 \sigma \end{bmatrix}$$

is nonsingular. This definition, although it differs from the earlier one, has precisely the same consequences since the inverse function theorem and implicit function theorem work as well in infinite dimensions. In particular, since for each α, the optimal k is unique, it follows that a regular economy has only finitely many equilibria.

Our preliminary goal is to study the circumstances under which Σ is nonsingular.

LEMMA 3: $D_1\xi$ is one-to-one.

Proof: Araujo and Scheinkman [1977] provide a proof under the assumption that $v(.,.,\alpha)$ is strictly concave but this is stronger than is necessary. Proposition 2 demonstrates the uniqueness of solutions for this model and this is all that is required for their argument. Q.E.D.

With this preliminary, we can now give a sufficient condition for Σ to be nonsingular.

LEMMA 4: If Σ is onto, then it is non singular.

Proof: We must show Σ is one to one. Let

$$\kappa = \{h_\alpha \in R^{m-1} \mid \text{ there exists } h_\kappa \in \ell_\infty^{n_k} \text{ such that } D_1\xi h_k + D_3\xi h_\alpha = 0\}.$$

Since $D_1\xi$ is one-to-one by Lemma 3, there is a unique linear operator $B: \kappa \to \ell_\infty^{n_k}$ such that $D_1\xi Bh_\alpha + D_3\xi h_\alpha = 0$. Notice that, since B has finite dimensional domain, it is a continuous operator.

Suppose $h \in \ker \Sigma$. Then $h_\alpha \in \kappa$ and $h_k = Bh_\alpha$, which implies that $(D_1\sigma B + D_3\sigma)h_\alpha = 0$. On the other hand, $D_1\sigma B + D_3\sigma$ is onto. Let $y_\alpha \in R^{m-1}$ and let $0 \in \ell_\infty^{n_k}$ with $y = (0,y_\alpha)$. Since Σ is onto, let $\bar{h}$ be a solution of $\Sigma\bar{h} = y$. Then $\bar{h}_\alpha \in \kappa$ and $\bar{h}_k = B\bar{h}_\alpha$. This implies that $(D_1\sigma B + D_3\sigma)\bar{h}_\alpha = y_\alpha$, which implies that $D_1\sigma B + D_3\sigma$ is onto. Finally, since $D_1\sigma B + D_3\sigma$ is a finite dimensional square matrix, it is also one-to-one. We conclude that if $h \in \ker \Sigma$, since $(D_1\sigma B + D_3\sigma)h_\alpha = 0$, then $h_\alpha = 0$. Since $h_k = Bh_\alpha$, we find that $h = 0$. Q.E.D.

We should emphasize that the picture is already very different than that with infinitely many agents. If we follow standard practice in infinite dimensional transversality theory, we would call an equilibrium regular if Σ

is onto and its kernal has closed complement. We just showed that this definition of regularity implies that Σ is nonsingular. This should be contrasted with the robust indeterminacy that occurs with infinitely many agents. In this case the fact that Σ is regular, that is, onto, does not imply that it is one-to-one. The kernal of Σ is simply the tangent space to the manifold of equilibria. Since the manifold deforms smoothly with respect to small perturbations, they change neither the fact that Σ is regular, nor the dimension of the kernal. The indeterminacy is robust. For a more detailed discussion of this point, the reader is referred to Kehoe, Levine, Mas-Colell and Zame [1986].

LEMMA 5: At a nondegenerate steady state $D_h\xi^{ss}$ is onto and $\dim \ker D_h\xi^{ss} = n_k - \text{index } D_h\xi^{ss}$.

<u>Proof</u>: That $\dim \ker D_h\xi^{ss} = n_k - \text{index } D_h\xi^{ss}$ means that $\dim \ker D_h\xi^{ss}$ has the same dimension as the stable manifold; since multiple solutions to $D_h\xi^{ss}h = 0$ are indexed by pairs (h_o,h_1) on the stable manifold, this follows. That $D_h\xi^{ss}$ is onto follows from the fact that the stable manifold is robust at a nondegenerate stationary state with respect to small non-stationary perturbations; see the proof of the local stable manifold theorem in Irwin [1980]. Consequently, $D_h\xi^{ss}h = b$ has nonempty stable manifold for small enough b, and since it is linear, for all b. In particular $D_h\xi^{ss}h = b$ has at least one solution. Q.E.D.

The next task is to show that, if k converges to a nondegenerate stationary state, then $D_h\xi(k,k_o,\alpha)$ is onto.

<u>PROPOSITION 7</u>: If k is a nondegenerate path for α and k_o, then $D_h\xi(k,k_o,\alpha)$ is onto, and has index equal to that at $k^{ss}(\alpha)$.

Proof: First we show that $D_h\xi$ is onto. Araujo and Scheinkman [1977] give a proof for the case where index $D\xi^{ss} = 0$. We examine only the case where index $D\xi^{ss} \geq 1$. Let $F: R^{n_k} \times \ell_\infty^{n_k} \to R^{n_k} \times \ell_\infty^{n_k}$ be defined by the rule $Fk = (k_1, (k_2, k_3, \ldots))$. Since $k_t \to k^{ss}(\alpha)$ and small perturbations of $D_h\xi^{ss}$ are also onto, for some finite T, $D_h\xi(F^T k, k_T, \alpha)$ is onto. Given b, find $h \in R^{n_k} \times \ell_\infty^{n_k}$ such that $D_h\xi(F^T k, k_T, \alpha)F^T h = F^T b$. Then since $D_{12}v_t$ is by assumption nonsingular, we simply solve recursively backwards to find

$$h_{t-1} = -D_{21}^{-1}v_{t-1}\,[(\beta D_{11}v_t + D_{22}v_{t-1})h_t + \beta D_{12}v_t h_{t+1} - b_t].$$

Since only a finite number of steps are involved, $h \in \ell_\infty^{n_k}$.

The fact that $D_h\xi$ and $D_h\xi^{ss}$ have the same index follows from the fact (shown, for example, in Araujo and Scheinkman) that they differ by a compact operator, and the fact that the index of a Fredholm operation is invariant under the addition of a compact operator. Q.E.D.

Let $\bar{E}_i(k_o)$ denote the set of pairs $(k,\alpha) \in (\ell_\infty^{n_k})_+ \times R^{m-1}$ such that k is nondegenerate for α and k_o and is of index i. Recall now that $0 \leq i \leq n_k$. We are interested in $\bar{E}(k_o) \equiv \bigcup_{i=0}^{n_k} \bar{E}_i(k_o)$.

PROPOSITION 8: For any fixed ϕ and a full measure subset of k_o and θ every equilibrium in $\bar{E}(k_o)$ is regular.

Proof: In steps 1-4 we consider a fixed index i, and $\bar{E}_i(k_o)$.

Step 1: We must find an open domain for ξ in order to do calculus. If $(k,\alpha) \in \bar{E}_i(k_o)$, then there is an open neighborhood $E_i(k_o)$ of (k,k_o,α) such that if $(k',k_o',\alpha') \in E_i(k_o)$ and $\xi(k',k_o',\alpha') = 0$, then $(k',\alpha') \in \bar{E}_i(k_o')$; in other words, locally paths either converge to a nondegenerate stationary state, or leave; they do not remain bounded nearby without converging. This is shown in the proof of the robustness of the stable manifold; see for example Irwin [1980]. We may also assume that, in $E_i(k_o)$, $D_h\xi(k',k_o',\alpha')$ is onto and has kernal of dimension n_k-i. This follows from the facts that the set of operators of this type is an open set (see Abraham and Robbin [1967]), and that $D_h\xi$ is a continuous function of its arguments by Lemma 2. Finally let $E_i = \bigcup_{k_o} E_i(k_o)$. This open set we take to be the domain of ξ.

Step 2: Consider the matrix function on $E_\ell \times R^{n_k(m-1)}$

$$\mu = \begin{bmatrix} \underline{k} & \underline{k_o} & \underline{\alpha} & \underline{\theta} \\ D_1\xi & D_2\xi & D_3\xi & 0 \\ D_1\sigma & D_2\sigma & D_3\sigma & D_4\sigma \end{bmatrix}$$

Proposition 5 says that $D_4\sigma$ is onto; and by construction $[D_1\xi\ D_2\xi] = D_h\xi$ is onto. It follows that μ is onto. Moreover, since $D_4\sigma$ is nonsingular, and $[D_1\xi\ D_2\xi]$ is onto with a n_k-i dimensional kernal, it is clear that $\dim\ker\mu = n_k - i + m - 1$. The implicit function theorem then implies that the set of (k,k_o,α,θ) such that (α,k) is an equilibrium is an $n_k - \ell + m - 1$ dimensional C^1 manifold.

Step 3: To apply the parameteric transversality theorem in step 4 below, we must show that the equilibrium manifold is second countable; that is, that every open covering has a countable subcovering. Since $\ell_\infty^{n_k}$ is not separable, it is not itself second countable. It is clearly sufficient, however, that the set of (k,k_o,α) in E_i with $\xi(k,k_o,\alpha) = 0$ is second countable. By construction such k converge to a nondegenerate stationary state, and such convergence must be exponential, so it suffices to show that the space of convergent sequences converging at the rate $1/t$ is second countable. This is the product of the second countable space R^{n_k}, containing the limits, and the space of sequences converging to zero at the rate $1/t$. The latter space of sequences is second countable because it is the union of sequences dominated by N/t as $N \to \infty$, and each of these spaces is compact. Finally, we observe that the product of second countable spaces is second countable.

Step 4: This step is identical to the finite dimensional proof of the parameteric transversality theorem. Consider the projection $\Pi(k,k_o,\alpha,\theta) = (k_o,\theta)$ restricted to the equilibrium manifold. This is a C^1 map between second countable $n_k - i + (m - 1)$ and $n_k + (m - 1)$ dimensional C^1 manifolds; moreover, the point (k,k_o,α,θ) is a regular equilibrium if and only if it is a regular value of Π. By Sard's theorem, however, the set of regular values (k_o,θ) are of full measure. This shows regular equilibria are full measure for each i.

Step 5: Since the countable union of measure zero sets has measure zero, the intersection of the full measure sets for each i has full measure.

Q.E.D.

Observe that in step 4 the map to which Sard's theorem applies is from an $n_k - i + m - 1$ dimensional manifold to an $n_k + m - 1$ dimensional one. It follows directly that if $i \geq 1$ then the equilibria in $E_i(k_o)$ are regular by virtue of not existing at all. Moreover, as we remarked above, Araujo and Scheinkman show that $D_1\xi$ is onto at a steady state with index 0. Consequently, under the hypothesis of Proposition 6, we may assume that there are finitely many equilibria, and every equilibrium has $D_1\xi$ nonsingular. By the implicit function theorem it follows that locally near equilibria we may solve to find $k(k_o,\alpha)$ a C^1 function. We already know that V is C^1. Moreover, as noted in the proof of Lemma 2 in the Appendix,

$D_i V(k_o,x,w,\alpha) = \psi_i(k_o,x,w,\alpha,k)$ evaluated at $k = k(k_o,\alpha)$, where ψ is C^1. Consequently, since $k(k_o,\alpha)$ is C^1, $D_i V$ is C^1, and V is C^2. This yields the final conclusion of Proposition 6.

8. CONCLUSION

We can sum up this way: For almost all k_o and θ there are finitely many equilibria with nondegenerate capital paths. In particular, if the dynamics of the optimization problem are such that for each α the only solutions to the problem $P(k_o,\alpha)$, are paths converging to a nondegenerate stationary state, then there are finitely many equilibria.

If the discount factor β is close to one, there is for each α a global non-degenerate turnpike. It follows that there are finitely many equilibria. For β far from one, there may be multiple stationary states or cycles. However, there are still finitely many equilibria converging to non-degenerate cycles. It is also possible that there are a continuum of equilibria at values of α for which the cycles bifurcate, and we cannot say whether or not these are robust. Moreover, Boldrin and Montrucchio [1985] and Deneckere and Pelikan [1985] have demonstrated that when the discount factor β is far from one, chaotic behavior can arise. More strongly, Boldrin and Montrucchio [1986] has shown that any dynamical system, including any chaotic system, can be the solution to the type of dynamic optimization problem we consider. Because of the methods used to construct these examples, the value function in each case is at least C^2, so chaos does not necessarily imply that regularity analysis fails. It remains an open question whether these examples are exception, or whether robust indeterminacy is possible.

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APPENDIX

Proof of Proposition 1: Since F is concave and strictly decreasing in c, the set of feasible, nonnegative aggregate consumption vectors is compact for given (k_t, k_{t+1}, x, w), so v is well defined in either case. Concavity and homogeneity of degree one for $v(k_t, k_{t+1}, x, w, .)$ follows immediately from these properties for U_i and F.

To show that v is u.s.c., let $z^j = (k_t^j, k_{t+1}^j, x^j, w^j, \alpha^j)$ be a sequence in the domain of v converging to a point z. Corresponding to each z^j, either $v = -\infty$ or there exists an optimal nonnegative consumption vector for the maximization problem in the definition of v. Suppose first that the sequence $\{z^j\}$ has an infinite subsequence with the property that nonnegative consumption is feasible. Since $\{z^j\}$ is bounded in R^ℓ, any infinite subsequence of optimal consumption vectors is contained in a compact set in $R_+^{n_c}$ and hence has a convergent subsequence. Then using the upper-semi-continuity of U_i and F, it follows that $v(z) \geq \lim\sup v(z^j)$. If only a finite number of the elements in $\{z^j\}$ allow nonnegative consumption, this inequality is trivial.

If (k_t, k_{t+1}, x) allows finite utility for each agent, $v(k_t, k_{t+1}, x, w, .)$ is the support function for the convex set of feasible utilities, hence is convex and homogeneous of degree one (Rockafellar [1970]). If it does not allow positive consumption then $(\lambda k_t, \lambda k_{t+1}, \lambda x)$ does not either and convexity and homogeneity follow once again. By the assumption that f is strictly increasing in c, the boundary of the set of points where $v(., ., x, w, \alpha)$ is finite consists of those pairs (k_t, k_{t+1}) such that k_t has a component equal to 0 or such that the implied optimal value for consumption is 0 because k_{t+1} is as large as possible or k_t is as small as possible. In any of these cases, Assumption 5 guarantees that the set of subgradients is empty.

Since v is concave-convex, it is continuous on the interior of its domain. To see that it is C^2 in this region, define a Lagrangian

$$\sum_{i=1}^{m} \alpha_i U_i(c_i, w_i) + \lambda F(k_t, k_{t+1}, \sum_{i=1}^{m} c_i, x).$$

Using the interiority assumption and the fact that $D^2 u_i$ is invertible for each individual, we can apply the implicit function theorem to express the optimal values for c_i and λ as C^1 functions of k_t, k_{t+1}, x, w, and α. By an application of the envelope theorem, all of the first partial derivatives of v can be written as the composition of the C^1 functions $U_i(c, w_i)$, $F(k_t, k_{t+1}, c, x)$, $c_i(k_t, k_{t+1}, x, w, \alpha)$, and $\lambda(k_t, k_{t+1}, x, \alpha)$, so v is C^2.

In particular,

$$D_2 v(k_t, k_{t+1}, x, w, \alpha) = \lambda D_2 F(k_t, k_{t+1}, \sum_{i=1}^{m} c_i, x),$$

where we have suppressed the arguments of the functions c_i and λ. Then $D_{22}v$ includes a term of the form, $\lambda D_{22}F$. Because v is concave, $D_{22}v$ is negative semidefinite. Since $D_{22}F$ is negative definite, a simple argument by contradiction shows that $D_{22}v$ must also be negative definite.

Q.E.D.

Proof of Proposition 2: Parts (b) and (c) of the proposition follows immediately from the properties of v from above. A proof of part (d) is given in McKenzie [1983]; the interiority follows from part (c) of Proposition 1. Part (e) is shown by Benveniste and Scheinkman [1975].

We turn to part (a). Define Ω to be the objective functional as defined on the sequence space for x, w, and α held constant:

$$\Omega(k) = \sum_{t=0}^{\infty} \beta^t v(k_t, k_{t+1}, x, w, \alpha).$$

To prove that there exists a solution to the maximization problem $P(k_o,x,w,\alpha)$ it suffices to show that in some topology the problem is to maximize a u.s.c. function over a compact set. The finiteness of V follows from Assumption 4, guaranteeing a feasible solution. Let M denote the set of nonnegative sequences in R^∞ that are bounded componentwise by $\bar{m} = \max \{|k_o|, |k_{max}|\}$, where k_o is the initial value for the problem and k_{max} is the upper bound specified in Assumption 3. By Assumptions 2 and 3 and the definition of v, any sequence yielding a finite value for the objective function is contained in this set. Moreover, M is compact in the product topology. It remains to show that the objective function Ω exhibits sequential upper semi-continuity. Suppose $k^n \in M$ and $k^n \to k$ in the product topology. Consequently $k_t^n \to k_t$. Let v_t^n denote $v(k_t^n, k_{t+1}^n, x, w, \alpha)$ and let v_t denote $v(k_t, k_{t+1}, x, w, \alpha)$. Since v_α is u.s.c., $\lim\limits_{n\to\infty} \sup v_t^n \leq v_t$ for all t. Since $v_t^n \leq \max \{v(\tilde{k}, 0, x, w, \alpha): |\tilde{k}| \leq \bar{m}\}$, the sequence v^n is uniformly bounded from above by a (constant) sequence that is summable with respect to β^t. Interpreting summations as integrals with respect to a measure concentrated on the integers, an application of Fatou's lemma implies that

$$\limsup_{n\to\infty} \sum_{t=0}^{\infty} \beta^t v(k_t^n, k_{t+1}^n, x, w, \alpha) \leq \sum_{t=0}^{\infty} \beta^t v(k_t, k_{t+1}, x, w, \alpha).$$

or $\limsup\limits_{n\to\infty} \Omega(k^n) \leqslant \Omega(k)$. Since the sequence $\{k^n\}$ can be chosen so that $\limsup\limits_{n\to\infty} \Omega(k^n)$ equals the supremum of the problem $P(k_o,x,w,\alpha)$ it must have a solution.

The uniqueness of the solution follows from the assumption that $v(k_t, \cdot, x, \alpha)$ is strictly concave. This implies a kind of conditional strict concavity; given the fixed initial value, the objective functional defines a strictly concave function over the sequence space. To see this, assume that $k^1, k^2 \in M$, and let k^λ denote the convex combination $\lambda k^1 + (1-\lambda)k^2$. Let

F denote the forward operator on the sequence space: for any sequence $k = \{k_0, k_1, k_2, \ldots\}$, $F^t k$ is the sequence $\{k_t, k_{t+1}, \ldots\}$. Let τ be the largest component at which k^1 and k^2 agree. Since they must have the same initial value, $\tau \geq 0$. Then we can express the difference

$$\lambda\Omega(k^1) + (1-\lambda)\ \Omega(k^2) - \Omega(k^\lambda) = \beta^\tau[\lambda v(k^1_\tau, k^1_{\tau+1}, x, w, \alpha) + (1-\lambda)\ v(k^2_\tau, k^2_{\tau+1}, x, w, \alpha)$$

$$- v(k^\lambda_\tau, k^\lambda_{\tau+1}, x, w, \alpha)] + \lambda\beta^{\tau+1}\ [\Omega(F^\tau k^1) + (1-\lambda)\ \Omega(F^\tau(k^2) - \Omega(F^\tau k^\lambda)].$$

Since v is (weakly) concave in its first two arguments, Ω is (weakly) concave. Consequently the second term in this expression is nonpositive. Given that $k^1_\tau = k^2_\tau$, that $k^1_{\tau+1} \neq k^2_{\tau+1}$, and that v is strictly concave in its second component, the first term is strictly negative. This shows that k^λ dominates k^1 and k^2. In other words, $\Omega(k)$ is strictly concave for fixed k_0. Q.E.D.

<u>Proof of Proposition 3</u>: From Part (e) of Proposition 2 it follows that the derivatives of V exist and are continuous and, consequently, that s is well defined and continuous everywhere on the interior of R^m_+. The definition on the boundary is discussed below.

By an application of the envelope theorem in finite dimensional space to the definition of v, $D_j v(k_t, k_{t+1}, x, w, \alpha)$ is homogeneous of degree 1 for $j = 1, \ldots, 4$. Also, $D_4 V(k, x, w, \alpha)$ is homogeneous of degree 0 since V is homogeneous of degree 1 in α. Thus, s is homogeneous of degree 1. That the sum of the components of s is equal to zero follows by applying twice the observation that a function that is homogeneous of degree 1 is equal to the sum of its arguments times its derivatives.

The only remaining issue is the continuity of s on the boundary. The difficulty arises when one of the components of α goes to 0. Let α^ℓ be a sequence in the interior converging to a point α such that $\alpha_i = 0$ and

$\alpha_j \neq 0$. If c_i^ℓ and c_j^ℓ denote the corresponding consumption choices in the definition of v, infinite steepness on the boundary of the utility functions implies c_i^ℓ, c_j^ℓ, and c_j are strictly positive and that the equality

$$\alpha_i^\ell Du(c_i^\ell) = \alpha_j^\ell Du_j(c_j^\ell)$$

hold for all ℓ. By an application of the maximum theorem (see, for example, Hildenbrand [1974]), since the mapping that sends α to the vector of optimal consumptions is single valued, it is a continuous function. From the definition of v, it is clear that the optimal value for c_i is 0 since $\alpha_i = 0$. By continuity, c_i^ℓ converges to zero. Using the equality noted above, this implies that

$$\lim_{\ell \to \infty} \alpha_i^\ell Du_i(c_i^\ell) c_i^\ell = 0.$$

That is, the expenditure on the usual consumption goods allowed consumer i goes to 0 as his weight in social utility goes to 0.

By the envelope theorem, we know that the last term in the definition of s is simply the product of the utility weights times the present discounted utility for each consumer. Using the homogeneity of the augmented utility function $U_i(c, w_i)$, we can combine the last two terms in s and express s_i as

$$s_i(k_o, \theta, \phi, \alpha) = \theta_i K_o D_1 V(k_o, 1, 1, \alpha) + \phi_i D_2 V(k_o, 1, 1, \alpha)$$
$$- \alpha_i \sum_{t=0}^{\infty} \beta^t Du_i(c_{it}) c_{it}$$

By the argument above, the last term in this expression goes to 0 as α_i^ℓ goes to 0; from the definition of v and the envelope theorem, the first and second derivatives of V can be expressed in terms of the marginal utility of agent j and hence is continuous as $\ell \to \infty$. Therefore, s is continuous on all of $R_+^m \setminus \{0\}$ if we define it so that at a boundary point like α,

$$s_i(k_o,\theta,\phi,\alpha) = \theta_i K_o D_1 V(k_o,1,1,\alpha) + \phi_i D_2 V(k_o,1,1,\alpha).$$

Note that this last term is simply the value of consumer i's endowment capital and share of profits. By the definition of admissible endowments, $\theta_i \neq 0$. Since f and u_j were assumed to be strictly increasing, every component of D_1V is strictly positive and s_i is greater than 0. Q.E.D.

Proof of Lemma 2: Consider any function $\psi_t = w(k_t,k_{t+1},\alpha)$ where w is C^1. Since (k_t,k_{t+1},α) may be restricted to a compact domain, the operator defined by

$$(D\psi(k,\alpha)h)_t = D_1 w(k_t,k_{t+1},\alpha)h_t + D_2 w(k_t,k_{t+1},\alpha)h_{t+1} + D_3 w(k_t,k_{t+1},\alpha)h_\alpha$$

is bounded and therefore continuous. Moreover,

$$\sup_{\substack{|k-k'|,|\alpha-\alpha'|\leq\varepsilon \\ |h_t|\leq 1}} \left|[D\psi(k,\alpha) - D\psi(k',\alpha')]h\right|$$

$$\leq 3\sup \left|D_j w(k_t,k_{t+1},\alpha) - D_j w(k'_t,k'_{t+1},\alpha')\right|$$

The compactness of the domain implies that $D_j w$ is uniformly continuous; so, as $\varepsilon \to 0$, $|D\psi(k,\alpha) - D\psi(k',\alpha')| \to 0$, in other words, $D\psi$ varies continuously.

Finally, we can show that $D\psi$ is actually the derivative of ψ by again using the uniform continuity of $D_j w$ to show that the integral form of the remainder in period t, which is made of terms of the form

$$\int_0^1 (1-s)[D_j w(k_{t-1}+sh_{t-1},k_t+sh_t,\alpha+sh_\alpha) - D_j w(k_{t-1},k_t,\alpha)]ds$$

vanishes uniformly across periods as $h \to 0$.

This shows that ξ is C^1. Moreover, the mapping $B: \ell_\infty^{n_k} \to R$ by $B(k) = \sum_{t=0}^{\infty} \beta^t k_t$ is continuous linear, and thus C^∞. Since σ is then a composition of the form $B(\psi)$, it too is C^1. Q.E.D.

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