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ON THE OPTIMALITY OF INTEREST RATE SMOOTHING*

Sergio Rebelo† and Danyang Xie‡

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Abstract

In this paper, we study some continuous-time cash-in-advance models in which interest rate smoothing is optimal. We consider both deterministic and stochastic models. In the stochastic case we prove two results which are of independent interest: (i) we study what is, to our knowledge, the only version of the neoclassical model under uncertainty that can be solved in closed form in continuous time; and (ii) we discuss how to characterize the competitive equilibrium of a stochastic continuous time model that cannot be computed by solving a planning problem. We also discuss the scope for monetary policy to improve welfare in an economy in which the real competitive equilibrium is suboptimal, focusing on the particular example of an economy with externalities.

Key words: inflation, growth, interest rate smoothing monetary policy.

JEL Classification: E31, E43, E52, O42.

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1 Introduction

The US Federal Reserve System is often described as following a policy of interest rate smoothing (Goodfriend, 1987). In this paper, we discuss some continuous-time cash-in-advance models in which interest rate smoothing is optimal: the optimal monetary policy requires a constant nominal interest rate. We first prove this result in a deterministic model. In this setting we also show that monetary policy is generally non-neutral along the transition to the steady state, even though the cash-in-advance constraint applies only to consumption. This non-neutrality is preserved even when the rate of monetary expansion is constant. Monetary policy can influence significantly the behavior of real variables: we show that monetary policy can support *any* path for the capital stock that is feasible and that converges to a level no greater than the steady state of the analogous real economy (where transactions can be carried out without the use of money).

We then extend the model to incorporate stochastic shocks and show that the optimal monetary policy continues to involve a constant nominal interest rate, despite the fact that the real interest rate is stochastic. To prove this we develop two sets of results that are of independent interest. First, we describe what is, to our knowledge, the only version of the neoclassical model under uncertainty that can be solved in closed form in continuous time. Second, we show how to characterize a competitive equilibrium that cannot be computed as a solution to a planning problem. While the computation of this type of equilibrium in deterministic continuous time models has become familiar since the work of Brock (1975), extending this characterization to environments with uncertainty is non-trivial.

Finally, we discuss the scope for monetary policy to improve welfare in an economy in which the real competitive equilibrium is suboptimal, focusing on the particular example of an economy with externalities. We fully characterize the optimal monetary policy and show that interest rate smoothing is no longer optimal in this setting.

The paper is organized as follows. Section 2 contains our results for the deterministic model. Section 3 discusses optimal monetary policy in an economy with externalities. The version of our model that incorporates uncertainty is studied in Section 4.

2 Money and Growth in the Neoclassical Model

2.1 The Real Competitive Equilibrium

Our analysis will proceed within the confines of the Cass-Koopmans growth model. It is well-known that the competitive equilibrium for this economy can be obtained by solving the following optimization problem, which we formulate using standard notation:

$$\max \int_0^\infty \frac{c^{1-\sigma}-1}{1-\sigma} e^{-\rho t}\, dt, \quad \sigma > 0, \ \rho > 0 \tag{1}$$

$$\text{subject to: } \dot{k} = f(k) - c, \ \text{ with } k_0 > 0 \text{ given.} \tag{2}$$

The production function $f(k)$ is assumed to be strictly increasing, strictly concave, continuously differentiable, and to satisfy the conditions: $\lim_{k \to 0} f'(k) = \infty$, and $\lim_{k \to \infty} f'(k) = 0$. We refer to this as the Real Competitive Equilibrium (RCE), because it pertains to an economy in which transactions can be made in frictionless markets without the use of money.

The Hamiltonian for this problem is given by,

$$H_{RCE} = \left[\frac{c^{1-\sigma}-1}{1-\sigma}\right] + \theta\left[f(k) - c\right] \tag{3}$$

The first order conditions are:

$$c^{-\sigma} = \theta \tag{4}$$

$$\dot{\theta} = \rho\theta - \theta f'(k) \tag{5}$$

The transversality condition is $\lim_{t\to\infty} \theta k e^{-\rho t} = 0$.

The optimal path characterized by these conditions has the property that k increases monotonically from k_0 to the steady state capital stock, k^*_{RCE}, defined by the condition: $f'(k^*_{RCE}) = \rho$. To simplify the exposition we assume throughout the paper that $k_0 < k^*_{RCE}$.

2.2 The Monetary Competitive Equilibrium

We now consider an economy in which money is valued because consumption has to be purchased using money.[1] The resulting Monetary Competitive Equilibrium (MCE) can be obtained by maximizing the objective function (1) subject to the following constraints:

$$\dot{k} = f(k) - c - \pi m^d - z + v \tag{6}$$

$$c \leq m^d \tag{7}$$

$$\dot{m}^d = z \tag{8}$$

Equation (7) is the continuous time analogue of a cash-in-advance constraint. It specifies that consumption in any period t can be no greater than the stock of money held by the agent in that period. In the resource constraint m^d denotes the demand for real money balances and π is the rate of inflation. The government rebates seignorage through lump sum transfers whose real value we denote by v.[2] These transfers can be negative, whenever seignorage revenue is also negative.

[1] Feenstra (1986) and Wang and Yip (1992) compare this cash-in-advance approach with other alternatives for introducing money in this class of models.

[2] Equation (7) implicitly assumes that government transfers take place after consumption purchases are carried out. This assumption simplifies our analysis. Changing the cash-in-advance constraint to $c \leq m^d + v$ does not affect any of our results, or yield additional insights, but makes the proofs more cumbersome.

The Hamiltonian for our maximization problem is:

$$H_{MCE} = \left[\frac{c^{1-\sigma}-1}{1-\sigma}\right] + \lambda_1\left[f(k) - c - \pi m^d - z + v\right] + \lambda_2\left[m^d - c\right] + \lambda_3 z \quad (9)$$

The first order conditions are:

$$c^{-\sigma} = \lambda_1 + \lambda_2 \quad (10)$$

$$\lambda_1 = \lambda_3 \quad (11)$$

$$\dot{\lambda}_1 = \rho\lambda_1 - \lambda_1 f'(k) \quad (12)$$

$$\dot{\lambda}_3 = \rho\lambda_3 + \pi\lambda_1 - \lambda_2 \quad (13)$$

$$\lambda_2 \geq 0,\ \lambda_2\left[m^d - c\right] = 0 \quad (14)$$

with transversality conditions: $\lim_{t\to\infty} \lambda_1 k e^{-\rho t} = 0$ and $\lim_{t\to\infty} \lambda_3 m^d e^{-\rho t} = 0$.

Let μ denote the growth rate of the money supply, $\mu \equiv \dot{M}^s/M^s$. In equilibrium, the nominal price level (P) has to be such that real money demand equals real money supply, $m^d = M^s/P \equiv m$. This implies:

$$\dot{m}/m = \mu - \pi \quad (15)$$

and

$$v = \dot{M}^s/P = \mu m \quad (16)$$

Thus we can substitute $\pi = \mu - \dot{m}/m$ and $v = \mu m$ into equation (6) to arrive at,

$$\dot{k} = f(k) - c \quad (17)$$

Equations (11) to (13) imply that

$$\lambda_2 = \lambda_1 R \quad (18)$$

where $R = \pi + f'(k)$ is the nominal interest rate.

We will limit our attention to the cases where the CIA constraint is binding (including just-binding). Using the fact that $m = c$, together with equations (10), (15) and (18) we arrive at:

$$\dot{c} = [1 + \mu + f'(k)]\, c - c^{1-\sigma}/\lambda_1 \tag{19}$$

The MCE is the solution to the system of differential equations composed of equations (12), (17) and (19) with three boundary conditions $k(0) = k_0$, $\lim_{t\to\infty} \lambda_1 k e^{-\rho t} = 0$ and $\lim_{t\to\infty} \lambda_1 c e^{-\rho t} = 0$. Once we obtain the solution we need to verify that $\lambda_2 \geq 0$ for all t.

Proposition 1 *In order for the MCE to reproduce the RCE, it is necessary and sufficient that the nominal interest rate be constant.*

Proof. It is well-known that in the RCE, the growth rate of consumption is given by

$$\frac{\dot{c}}{c} = \frac{f'(k) - \rho}{\sigma}. \tag{20}$$

This equation, in combination with the resource constraint (2) and the boundary conditions $k(0) = k_0$ and $\lim_{t\to\infty} c^{-\sigma} k e^{-\rho t} = 0$, determines the equilibrium allocation.

To establish sufficiency, suppose that in the MCE, the nominal interest rate R is constant. From equations (10) and (18), we have

$$c^{-\sigma} = \lambda_1 (1 + R) \tag{21}$$

With R constant, equations (21) and (12) imply that the growth rate of consumption is also governed by equation (20). Since equation (17) is the same as the resource constraint (2), and since $\lim_{t\to\infty} \lambda_1 k e^{-\rho t} = 0$ is identical to $\lim_{t\to\infty} c^{-\sigma} k e^{-\rho t} = 0$ for constant R, the MCE reproduces the RCE.

To establish necessity, note that in order for the MCE to reproduce the RCE, the marginal utility of consumption must be equal in the two equilibria.

That is $\lambda_1(1+R) = \theta$. An inspection of equations (5) and (12) reveals that θ and λ_1 grow at the same rate and thus R has to be constant for $\lambda_1(1+R)$ to equal θ. ■

REMARK: In the economy described here, the RCE is Pareto Optimal. Proposition 1 shows that the Friedman rule $R = 0$ is just one of the many rules that can lead to Pareto optimality in a MCE. The key feature of optimal monetary policy that we find is that it makes the effective price of consumption in the MCE ($\lambda_1(1+R)$) identical to the price of consumption (θ) in the RCE. It is important to note that this result is not an artifact of our continuous time setting. Carlstrom and Fuerst (1995) show that a similar result holds in discrete time.

Proposition 2 *The money growth path that reproduces the RCE in an MCE must be time varying unless the economy is already at the steady state or $\sigma = 1$. During the transition to the steady state it is optimal to increase the growth rate of money when $\sigma < 1$ and to decrease it when $\sigma > 1$. Regardless of the value of σ, optimal monetary policy implies that inflation increases during the transition to the steady state.*

Proof. We have already shown that R has to be constant. When R is positive, we have $\lambda_2 > 0$, so the CIA constraint is strictly binding. When $R \equiv 0$, the CIA constraint is just binding. In other words, we always have $c = m$. Using the fact that c and m grow at the same rate we can write the nominal interest rate as $R = \mu + (1-\sigma)f'(k)/\sigma + \rho/\sigma$. Since k rises during the transition this expression implies that to keep R constant μ must rise when $\sigma < 1$ and fall when $\sigma > 1$. The real interest rate ($f'(k)$) always falls as the economy moves toward the steady state, thus inflation must rise to keep R constant. ■

Note that when the economy is at the steady state, $f'(k)$ is constant. Hence a constant μ leads to a time invariant nominal interest rate. When $\sigma = 1$, a constant rate of money growth generates a constant R even during the transition. What is special about $\sigma = 1$? During the transition to the steady state, the real interest rate declines as the capital stock expands. When the growth rate of money is constant, the rate of inflation increases during the transition. This is due to the fact that the growth rate of consumption is falling in response to declining real interest rate and that $\pi = \mu - \dot{c}/c$. In the case of $\sigma = 1$, the fall in the real interest rate is exactly offset by the increase in the rate of inflation, leaving the nominal interest rate time invariant.

Proposition 3 *Any monotonic capital path $\{k(t)\}_{t=0}^{\infty}$ that satisfies, (1) $c(t) = f(k) - \dot{k} > 0$ and (2) $\lim_{t\to\infty} k(t) = k^* \leq k^*_{RCE}$, can be supported in a MCE by choosing an appropriate path for the money supply. The paths for the money supply that support capital stock paths that converge to $k^* < k^*_{RCE}$ involve hyperinflation.*

Proof. Since $\dot{k} > 0$ and $k(t)$ converges to k^*, we have $c(t) \leq c^* \equiv f(k^*)$ for all t. Equation (12) implies that the values of $\lambda_1(t)$ associated with such a capital path, are monotonically decreasing over time. Thus by solving equation (12) for $\lambda_1(t)$ with $\lambda_1(0) = (c^*)^{-\sigma}$, we can make sure that $\lambda_1(t) \leq (c^*)^{-\sigma} \leq c(t)^{-\sigma}$. Hence, $\lambda_2(t) \geq 0$ is guaranteed for all t. From equation (19), we can back out the money growth rate $\{\mu(t)\}_{t=0}^{\infty}$ that supports this path. It is easy to verify that all the transversality conditions hold. Closer inspection of the equilibrium equations show that in the case where $k^* < k^*_{RCE}$, it must be true that $\lim_{t\to\infty} \lambda_1(t) = 0$, $\lim_{t\to\infty} \lambda_2(t) = (c^*)^{-\sigma}$. Thus, equation (18) implies that $R \to \infty$ and hyperinflation is always involved in this circumstance.

REMARK: No money growth path can induce a capital stock path that converges to $k^* > k^*_{RCE}$. Equation (12) implies that the λ_1 trajectory associated with such a capital path converges to ∞. Thus $\lambda_2(t) = c(t)^{-\sigma} - \lambda_1(t) < 0$ when t is large enough because $c(t) \to f(k^*)$. As a result, $R(t) = \lambda_2(t)/\lambda_1(t)$ will eventually be negative. Hence, $\{k_t\}_{t=0}^{\infty}$ cannot be supported as a competitive equilibrium with money.

Proposition 3 implies that monetary policy can be quite powerful in improving welfare in economies in which the competitive equilibrium is suboptimal. We will examine this issue in detail in Section 3.

Our results so far involve situations in which the growth rate of the money supply varies over time. We now focus on the case of constant rates of monetary expansion and examine the standard question of whether money is neutral. The following result is similar to that obtained by Stockman (1981) in a discrete time version of the model we are considering.

Proposition 4 *(Steady state neutrality) When money expands at a constant rate μ and the economy is at the steady state, money is neutral, i.e. the values taken by the different real variables are independent of μ.*

Proof. Equations (12) and (17) imply the usual steady state conditions $f'(k) = \rho$ and $c = f(k)$. Hence the steady state values of k and c are independent of μ. The growth rate of money affects only the steady state value of λ_1 (see (19)) and the nominal interest rate.

Proposition 5 *(Non-neutrality during the transition) When money expands at a constant rate μ and $\sigma \neq 1$, money is not neutral in the transition toward the steady state.*

Proof. We show by contradiction. Suppose the real allocations are independent of the money growth rate. Equation (19) implies that $\partial\lambda_1(t,\mu)/\partial\mu = -c^\sigma\lambda_1^2$. Equation (12) implies

$$\begin{aligned} \frac{\partial^2\lambda_1}{\partial\mu\partial t} &= [\rho - f'(k)]\,\partial\lambda_1(t,\mu)/\partial\mu \\ &= -\,[\rho - f'(k)]\,c^\sigma\lambda_1^2 \end{aligned} \tag{22}$$

But we also have

$$\begin{aligned} \frac{\partial^2\lambda_1}{\partial\mu\partial t} &= \frac{\partial}{\partial t}\left(\frac{\partial\lambda_1(t,\mu)}{\partial\mu}\right) \\ &= -\sigma c^{\sigma-1}\dot{c}\lambda_1^2 - 2c^\sigma\lambda_1\dot{\lambda}_1 \\ &= -\sigma c^{\sigma-1}\lambda_1^2\dot{c} - 2c^\sigma\lambda_1^2\,[\rho - f'(k)] \end{aligned} \tag{23}$$

Comparing (22) with (23), we find that

$$\frac{\dot{c}}{c} = \frac{f'(k) - \rho}{\sigma}. \tag{24}$$

Hence this MCE reproduces the RCE. When $\sigma \neq 1$ and under the assumption that the economy is not yet at the steady state, Proposition 2 implies that any money growth path that reproduces the RCE must be time varying. A contradiction. ■

REMARK: When $\sigma = 1$, the proof of Proposition 2 shows that any constant μ will ensure a constant nominal interest rate. Provided that the nominal interest rate is non-negative, this μ allows the MCE to reproduce the RCE. Hence money is neutral on the transition when $\sigma = 1$. It is interesting to note that our results on neutrality are identical to those originally stressed by Cohen (1985) in a continuous time model in which money enters the utility function.[3] A different conclusion is reached in Abel (1985) in a discrete time

[3]Cohen's (1985) utility function is $u = \left[\left(c^\alpha m^{1-\alpha}\right)^{1-\sigma} - 1\right]/(1-\sigma)$.

CIA model. There, he finds that when the CIA constraint applies only to consumption, money is always neutral along the transition path independently of the form taken by momentary utility. Our study reveals that the different conclusions between Cohen (1985) and Abel (1985) are not the outcome of difference in MIUF and CIA, instead they are the outcome of difference in time frame. Clearly, if the length of time required for monetary settlement is infinity, the CIA constraint is ineffective and any money rule can reproduce the RCE and thus money is neutral. As the length of time becomes finite and remains positive, Abel showed that constant money growth rules are still neutral but fail to reproduce the RCE. When the length of time shrinks to zero such as in our model where instant monetary settlement is required, money stops being neutral unless $\sigma = 1$ or the economy is already at steady state.

REMARK: To study the effects of monetary policy on long run growth we adopted the standard assumption that labor supply is exogenous. When labor supply is endogenous, or money is introduced through a transactions technology in which real balances allow agents to economize on "shopping" effort (as in Kimbrough (1986)), interest rate smoothing continues to be optimal. However, in order to maximize welfare, the value of the nominal interest rate must be zero. Only the Friedman rule guarantees that monetary policy does not distort the labor-leisure choice and minimizes "shopping" time. The results in Correia and Teles (1996) suggest that, surprisingly, the optimality of the Friedman rule survives even when the revenue necessary to finance the negative inflation tax is obtained through distorting taxes on capital and labor. Unfortunately, currently available estimates of the elasticity of labor supply are very imprecise, which prevents us from gauging the welfare loss that results from maintaining a constant positive value of the interest rate, instead of setting $R = 0$.

3 Money and Growth in an Economy with Externalities

In an economy with no market imperfection or tax distortions, the RCE is Pareto optimal. We have seen that for any policy that maintains a constant nominal interest rate, the MCE reproduces the RCE, and is thus optimal. However, when the RCE is suboptimal and monetary policy is the sole policy instrument, it is no longer desirable to engage in interest rate smoothing. In this section we show how the principles in Proposition 3 can be used to construct a money growth rule that dominates interest rate smoothing. For a concrete example, we study an economy with a production externality.

Suppose the production function exhibits a positive externality,

$$y = f(k, k_a)$$

where k_a is the per capita capital stock in the whole economy. The production function $f(k, k_a)$ is strictly increasing in both k, and k_a, and f is concave in k and continuously differentiable. We assume that $\lim\limits_{k\to 0} f_i(k,k) = \infty$, and $\lim\limits_{k\to\infty} f_i(k,k) = 0$, $i = 1, 2$. We also assume that $g(k) = f(k,k)$ is concave in k so that the social planner's problem is well-defined.

3.1 The RCE

Following Kehoe, Levine, and Romer (1992), we obtain the RCE by solving the following Pseudo-Planner's problem: maximize (1) subject to the constraint,

$$\dot{k} = f(k, k_a) - c, \text{ with } k_0 \text{ given.} \tag{25}$$

The Hamiltonian for this problem is given by,

$$H_{RCE} = \left[\frac{c^{1-\sigma} - 1}{1-\sigma}\right] + \theta\left[f(k, k_a) - c\right] \tag{26}$$

The first order conditions are:

$$c^{-\sigma} = \theta \tag{27}$$

$$\dot{\theta} = \rho\theta - \theta f_1(k,k) \tag{28}$$

where the equilibrium condition $k_a = k$ is substituted in equation (28) only after the derivative of H_{RCE} with respect to k is taken. Let the steady state be denoted by k^{**}_{RCE}: $f_1(k^{**}_{RCE}, k^{**}_{RCE}) = \rho$. The transversality condition is $\lim\limits_{t\to\infty} \theta k e^{-\rho t} = 0$.

3.2 The Pareto Optimum

The Pareto Optimum is the solution to the following central planner's problem: maximize (1) subject to the constraint,

$$\dot{k} = f(k,k) - c, \text{ with } k_0 \text{ given.} \tag{29}$$

The Hamiltonian for this problem is given by,

$$H_{PO} = \left[\frac{c^{1-\sigma} - 1}{1-\sigma}\right] + \eta\,[f(k,k) - c] \tag{30}$$

The first order conditions are:

$$c^{-\sigma} = \eta \tag{31}$$

$$\dot{\eta} = \rho\eta - \eta\,[f_1(k,k) + f_2(k,k)] \tag{32}$$

The transversality condition is $\eta k e^{-\rho t} \to 0$ as t approaches infinity. Let the steady state be denoted by k^{**}_{PO}: $f_1(k^{**}_{PO}, k^{**}_{PO}) + f_2(k^{**}_{PO}, k^{**}_{PO}) = \rho$.

Inspection of the equations characterizing the RCE and the Pareto Optimum shows that there is only one difference between these: in equation (32) the marginal product of capital is $f_1(k,k)+f_2(k,k)$, whereas in equation (28) it is $f_1(k,k)$. The difference arises because the central planner internalizes the externality.

3.3 The MCE

The MCE can be obtained by maximizing (1) subject to the following constraints, expressed using the same notation defined in Section 2.2:

$$\dot{k} = f(k, k_a) - c - \pi m^d - z + v \tag{33}$$

$$c \leq m^d \tag{34}$$

$$\dot{m}^d = z \tag{35}$$

All the Propositions in the previous section hold for this economy.

Even though monetary policy is not the most obvious policy instrument for correcting the effects of the externality, it is interesting to determine how far it can lead us, since the answer is likely to apply to many other sources of suboptimality. Can monetary policy improve welfare? Can it achieve Pareto optimality? We will see that the answer to the first question is yes, while the answer to the second is no.

Our Proposition 3 provides the key to characterizing the best outcome that monetary policy can achieve in this economy. We have shown that it is possible to use monetary policy to support any path that satisfies the economy wide resource constraint and for which $\underset{t\to\infty}{lim}\ k(t) = k^* \leq k^{**}_{RCE}$. The question is then, which of these supportable paths is the best. This amounts to solve the following problem: maximize (1), subject to the constraints,

$$\begin{aligned} \dot{k} &= f(k,k) - c \text{ with } k_0 \text{ given.} \\ \text{and} \qquad k &\leq k^{**}_{RCE} \end{aligned} \tag{36}$$

The Hamiltonian for this problem is given by,

$$H = \left[\frac{c^{1-\sigma} - 1}{1-\sigma}\right] + \xi\left[f(k,k) - c\right] + \varphi(k^{**}_{RCE} - k) \tag{37}$$

The first order conditions are:

$$c^{-\sigma} = \xi \tag{38}$$

$$\dot{\xi} = \rho\xi - \xi\left[f_1(k,k) + f_2(k,k)\right] + \varphi \tag{39}$$

$$\varphi \geq 0,\ k^{**}_{RCE} - k \geq 0,\ \varphi(k^{**}_{RCE} - k) = 0 \tag{40}$$

The transversality condition is $\lim_{t\to\infty} \xi k e^{-\rho t} = 0$. Note that it is impossible for φ to be zero all the time because in this case $k \to k^{**}_{PO} > k^{**}_{RCE}$. The possible solutions must take the form:

$$\varphi(t) = \begin{cases} 0 \text{ when } t \leq T \\ \varphi^{*} \text{ when } t > T \end{cases} \tag{41}$$

for some T. For each T, let us call the allocation that satisfies equations (38)–(41), $\{k^T(t),\ c^T(t)\}_0^\infty$. Let $\bar{T}$ be the maximum of such Ts. The trajectory corresponding to $\bar{T}$ is marked BMCE in Figure 1.

In Figure 1, the vertical line on the left ($k = k^{**}_{RCE}$) represents the RCE condition $\dot{\theta} = 0$. The vertical line on the right ($k = k^{**}_{PO}$) represents the analogous condition ($\dot{\eta} = 0$) that corresponds to the planning problem. Since the central planner internalizes the production externality, $k^{**}_{PO} > k^{**}_{RCE}$. The path PO illustrates the Pareto optimal path of capital, which we have shown to be unreachable by an MCE. The Friedman rule allows us to reproduce the RCE but it is not the best monetary rule. We show in the next proposition that the best achievable path in a monetary economy is the BMCE $\{k^{\bar{T}}(t)\}_0^\infty$. By construction, the BMCE satisfies the first order conditions in the central planner's problem up to time $\bar{T}$. At time $\bar{T}$ the capital stock becomes equal to k^{**}_{RCE} and remains at that point thereafter.

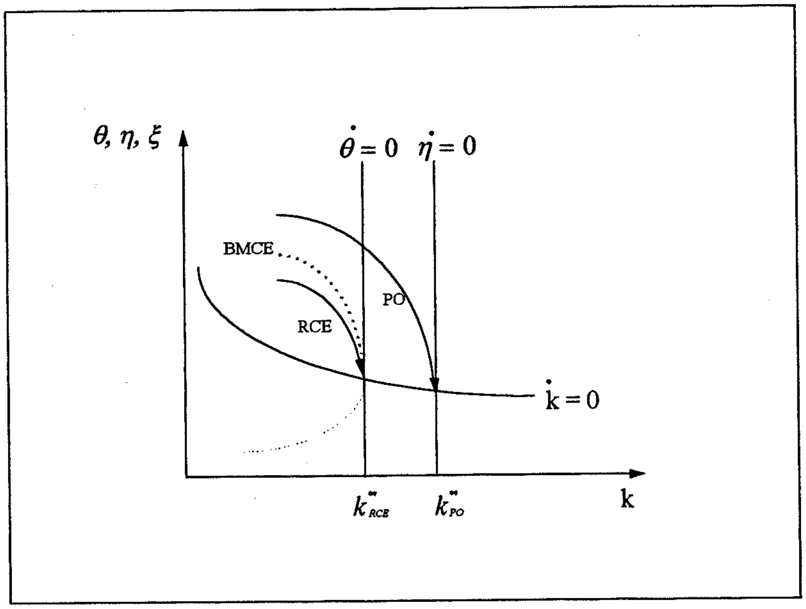

Figure 1

Proposition 6 *$\{k^{\bar{T}}(t),\ c^{\bar{T}}(t)\}_0^\infty$ is superior to $\{k^T(t),\ c^T(t)\}_{t=0}^\infty$ for $T < \bar{T}$.*

Proof. To begin with, for any $t > \bar{T}$, $c^T(t) = c^{**}_{RCE}$. Thus, we only need to compare the period $[0, \bar{T}]$. However, by construction, it is obvious that $\{k^{\bar{T}}(t),\ c^{\bar{T}}(t)\}_0^{\bar{T}}$ solves

$$\max \int_0^{\bar{T}} \frac{c^{1-\sigma} - 1}{1 - \sigma} e^{-\rho t}\, dt \tag{42}$$

$$\begin{aligned} \dot{k} &= f(k, k) - c \text{ with } k_0 \text{ given.} && (43) \\ \text{and} \qquad k(\bar{T}) &= k^{**}_{RCE} \end{aligned}$$

and $\{k^T(t),\ c^T(t)\}_0^{\bar{T}}$ satisfies the two constraints above. Thus, $\{k^{\bar{T}}(t),\ c^{\bar{T}}(t)\}_0^\infty$ is superior to $\{k^T(t),\ c^T(t)\}_0^\infty$. ■

These results show that monetary policy can improve on the RCE of this economy, but falls short of achieving Pareto Optimality.

4 Money and Growth in a Stochastic Economy

We will start by describing two closed form solutions to the real competitive equilibrium of a stochastic version of the neoclassical growth model. Even though to make progress we will have to make specific assumptions about functional forms, these restrictions pay off, in the sense that the model displays an elegant, insightful closed form solution. We then use one of these closed form solutions to show that optimal monetary policy entails maintaining a constant nominal interest rate.

4.1 The Real Competitive Equilibrium

The RCE can be computed as a solution to a problem where a representative agent maximizes expected lifetime utility subject to the economy's resource constraint:

$$\max E_0 \int_0^\infty e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} dt, \quad \rho > 0, \sigma > 0 \tag{[P1]}$$

$$dk = (Ak^\alpha - c)dt + k\varepsilon dz + kdq$$

The capital stock follows a generalized Ito process. This process comprises two forms of uncertainty: dz is the increment of a Wiener process, while dq is the increment of a Poisson process with arrival rate λ: $dq = 0$ with probability $1 - \lambda dt$ and $dq = -u$ with probability λdt, where u is a random variable. The two increments, dz and dq are assumed to be independent. There are two cases in which we can solve this model in closed form. The first, which is well-known, corresponds to the case of $\alpha = 1$. Strictly speaking this is not a version of the neoclassical model. It is the ubiquitous "AK" model, popular in the recent growth literature.

Proposition 7 *When $\alpha = 1$ the solution to [P1] is given by the policy function* $c = \left[\rho - A(1-\sigma) + \frac{1}{2}\sigma(1-\sigma)\varepsilon^2 + \lambda - \lambda E_u(1-u)^{1-\sigma}\right] k/\sigma$

Proof. The Hamilton-Jacobi-Bellman equation for this problem is:

$$\begin{aligned} 0 &= \max_c \frac{c^{1-\sigma}}{1-\sigma} - \rho J(k) + J'(k)(Ak - c) + \\ &\quad \lambda E_u\left[J(k(1-u)) - J(k)\right] + \frac{1}{2}J''(k)\varepsilon^2 k^2 \end{aligned} \tag{44}$$

Consider the following guess for the value function $J(k)$, which we will later verify to be correct: $J(k) = b\frac{k^{1-\sigma}}{1-\sigma}$. The optimum condition for c implies

$$c^{-\sigma} = J'(k) = bk^{-\sigma}. \tag{45}$$

Combining (45), (44) and our guess for the value function we can solve for b, verifying at the same time that the value function has the form that we postulated:

$$b = \left[\frac{\rho - A(1-\sigma) + \frac{1}{2}\sigma(1-\sigma)\varepsilon^2 + \lambda - \lambda E_u(1-u)^{1-\sigma}}{\sigma}\right]^{-\sigma}$$

Using this solution for b in equation (45) we obtain the policy function for c. ■

The second solution requires the same restriction used in Xie (1991) in a deterministic context: $\alpha = \sigma$.

Proposition 8 *When $\alpha = \sigma$ the solution to [P1] is given by the policy function* $c = \left[\rho + \frac{1}{2}\sigma(1-\sigma)\varepsilon^2 + \lambda - \lambda E_u(1-u)^{1-\sigma}\right] k/\sigma.$

Proof. The Hamilton-Jacobi-Bellman equation for this problem is:

$$\begin{aligned} 0 &= \max_c \frac{c^{1-\sigma}}{1-\sigma} - \rho J(k) + J'(k)(Ak^\sigma - c) + \\ &\quad \lambda E_u\left[J(k(1-u)) - J(k)\right] + \frac{1}{2}J''(k)\varepsilon^2 k^2 \end{aligned} \tag{46}$$

Consider the following guess for the value function $J(k)$, which we will later verify to be correct: $J(k) = a + b\frac{k^{1-\sigma}}{1-\sigma}$. The optimum condition for c implies

$$c^{-\sigma} = J'(k) = bk^{-\sigma}. \tag{47}$$

Combining (47), (46) and our guess for the value function we can solve for a and b, verifying at the same time that the value function has the form that we postulated:

$$b = \left[\frac{\rho + \frac{1}{2}\sigma(1-\sigma)\varepsilon^2 + \lambda - \lambda E_u(1-u)^{1-\sigma}}{\sigma}\right]^{-\sigma}$$

$$a = bA/\rho$$

Using this solution for b in equation (47) we obtain the policy function for c. ■

Notice that the only difference between the consumption decision rules when $\alpha = \sigma$ and $\alpha = 1$ is that b is independent of A in the former case.

It is possible to generate other closed form solutions for the stochastic neoclassical model by postulating a decision rule for consumption and deriving the utility function that makes this rule optimal, as in Chang (1988). However, these closed form solutions often imply stringent restrictions tying, not only the parameters of preferences and technology, but also the parameters of the shock process. One example of a solution that requires these restrictions is a version of the closed form discussed in Barro and Sala-i-Martin (1995, page 78) which incorporates uncertainty. Barro and Sala-i-Martin show that if $\sigma = \frac{\rho+\delta}{\alpha\delta} > 1$, where δ is the rate of depreciation, the optimal policy function for consumption is: $C = \frac{\sigma-1}{\sigma}Ak^{\alpha}$. When we incorporate uncertainty we need to require the following awkward condition to obtain a closed form:

$\rho + \delta(1 - \alpha\sigma) = -\frac{1}{2}\alpha\sigma(1 - \alpha\sigma)\varepsilon^2$. When this condition holds the decision rule retains the same form as in the deterministic case.

We will now study a monetary version of this model. Since introducing money will complicate the problem considerably we will restrict ourselves to the case in which $\alpha = 1$ and ignore the jump process component of the stochastic shock ($\lambda = 0$).

4.2 A Monetary Economy

As in previous sections money will be introduced in the model through a cash-in-advance constraint.[4] Our first task is to derive the representative agent's budget constraint. We formulate the agent's problem as having to choose, at every point in time, how much of his nominal wealth (W) to invest in productive capital, nominal bonds (B), which yield a nominal interest rate of R, and money holdings (M). This formulation is exactly equivalent to the one adopted in Sections 2 and 3 but it is more convenient, from a notation standpoint, to handle the presence of uncertainty. Nominal bonds will be in zero net supply in equilibrium, but including them at this stage of the analysis will allow us, later on, to derive the expression for the nominal interest rate. The budget constraint can be written as:

$$Pdk + dM + dB = P(Ak - c)dt + Pk\varepsilon dz + Vdt + RBdt$$

where P represents the price level. The nominal lump sum transfers which rebate the proceeds of the inflation tax are denoted by V.

Written in real terms the budget constraint is:

$$dk + \frac{dM}{P} + \frac{dB}{P} = (Ak - c)dt + k\varepsilon dz + vdt + \frac{RB}{P}dt$$

[4]Eaton (1981, section 6) studies a similar model in which money is held despite the absence of a cash-in-advance constraint. In Eaton's monetary economy there are no bonds, only money and capital. Money is held because it has a lower risk than capital investments.

We conjecture that the equilibrium law of motion for P is the following geometric brownian motion process:

$$dP = P\pi dt - P\varepsilon dz$$

We will verify that this law of motion holds in equilibrium and that the average rate of inflation, π, is constant whenever the money supply expands at a constant rate μ. The change in the agent's real wealth (denoted by w) is, by definition:

$$dw = dk + d\left(\frac{M}{P}\right) + d\left(\frac{B}{P}\right)$$

Notice that $d\left(\frac{M}{P}\right)$ and $d\left(\frac{B}{P}\right)$can be written as:

$$\begin{aligned} d\left(\frac{M}{P}\right) &= \frac{dM}{P} + Md\left(\frac{1}{P}\right) \\ d\left(\frac{B}{P}\right) &= \frac{dB}{P} + Bd\left(\frac{1}{P}\right) \end{aligned}$$

Since P follows a geometric Brownian motion, $d\left(\frac{1}{P}\right)$ has to be computed using Ito's lemma:

$$d\left(\frac{1}{P}\right) = -\left(\frac{\pi}{P} - \frac{\varepsilon^2}{P}\right)dt + \frac{\varepsilon}{P}dz$$

This allows us to write the budget constraint in real terms as:

$$dw = (A\phi_1 w - c)dt - (1-\phi_1)\, w\pi dt + (1-\phi_1)\, w\varepsilon^2 dt + vdt + R\phi_2 wdt + w\varepsilon dz$$

where ϕ_1 and ϕ_2 denote, respectively, the fractions of real wealth devoted to physical capital and to nominal bonds, and $v = V/P$.

The cash-in-advance can be written as:

$$c = (1 - \phi_1 - \phi_2)w$$

The rate of inflation, dP/P and the lump sum transfer that rebates the proceeds of the inflation tax (v) both depend on the evolution of per capita wealth in the economy, which we denote by $\bar{w}$. This variable is outside the control of an individual agent but, in equilibrium, since all agents are identical, $w = \bar{w}$. In a deterministic environment it is easy to take the path of $\bar{w}$ as exogenous when we compute the conditions that characterize the individual optimum. In a stochastic environment computing such an equilibrium is more complex. The optimization conditions do not involve computing derivatives with respect to w in a way that would make it easy to treat the process for $\bar{w}$ as exogenous. To make progress we first have to write the value function J as depending on both w and $\bar{w}$. Second, we have to conjecture a law of motion for $\bar{w}$ and guarantee that this law of motion will coincide in equilibrium with that of w.

Our conjecture for the law of motion of $\bar{w}$ is:

$$d\,\bar{w} = g(\bar{w})dt + h(\bar{w})dz$$

The functions $g(.)$ and $h(.)$ will later be selected so that in equilibrium, $\bar{w} \equiv w$.

The Hamilton-Jacobi-Bellman equation for this problem is:

$$\begin{aligned} 0 &= \max_{\phi_1, \phi_2} \frac{[(1-\phi_1-\phi_2)w]^{1-\sigma}}{1-\sigma} - \rho J(w,\bar{w}) \\ &\quad + J_1\left[A\phi_1 w - (1-\phi_1-\phi_2)w - (1-\phi_1)\,w\left(\pi-\varepsilon^2\right) + v + R\phi_2 w\right] \\ &\quad + J_2 g(\bar{w}) + \frac{1}{2}J_{11}w^2\varepsilon^2 + J_{12}w\varepsilon h(\bar{w}) + \frac{1}{2}J_{22}[h(\bar{w})]^2 \end{aligned}$$

We will use the following guess for the form of the value function, which we will later verify to be correct:

$$J(w,\bar{w}) = b\frac{(w+\beta\,\bar{w})^{1-\sigma}}{1-\sigma}$$

The optimality condition for ϕ_1 is:

$$(1-\phi_1-\phi_2)w = (w+\beta\,\bar{w})\left[b(A+1+\pi-\varepsilon^2)\right]^{-1/\sigma} \tag{48}$$

The optimality condition for ϕ_2 is:

$$(1-\phi_1-\phi_2)w = (w+\beta\,\bar{w})\left[b\,(1+R)\right]^{-1/\sigma} \tag{49}$$

These two equations imply that the equilibrium nominal rate is:

$$R = A+\pi-\varepsilon^2 \tag{50}$$

Values of the nominal interest rate greater than this would imply an infinite demand for nominal bonds (which in equilibrium are in zero net supply). Lower values of the nominal interest rate would drive the demand for nominal bonds to $-\infty$.

Equation (50) is interesting because it is a modification of the standard Fisherian equation for nominal interest rate determination that takes uncertainty explicitly into account. Now that we determined the expression for R, we can economize on notation by setting ϕ_2 equal to its equilibrium value of zero and redefining ϕ_1 as ϕ:

$$\begin{aligned}\phi_2 &= 0\\ \phi &= \phi_1\end{aligned}$$

Equation (48) defines a function $\phi(w,\bar{w})$. In order to have $w=\bar{w}$ in equilibrium, we must select $g(.)$ and $h(.)$ as follows:

$$g(\bar{w}) = A\phi(\bar{w},\bar{w})\,\bar{w} - (1-\phi(\bar{w},\bar{w}))\,\bar{w} - (1-\phi(\bar{w},\bar{w}))\,\bar{w}\,(\pi-\varepsilon^2)+v$$

and

$$h(\bar{w}) = \bar{w}\,\varepsilon$$

Using equation (48) and the fact that in equilibrium the lump sum transfers v are given by:

$$v = \mu(1 - \phi(\bar{w}, \bar{w}))\, \bar{w}$$

we can write:

$$g(\bar{w}) = \Big[A - (A+1+\pi-\varepsilon^2-\mu)\left[b(A+1+\pi-\varepsilon^2)\right]^{-1/\sigma}(1+\beta)\Big]\,\bar{w}$$

Using these equations for $h(\bar{w})$ and $g(\bar{w})$ and re-arranging terms, we can re-write the Hamilton-Jacobi-Bellman equation as:

$$\begin{aligned}
0 \;=\; & \left[b(A+1+\pi-\varepsilon^2)\right]^{-(1-\sigma)/\sigma}(w+\beta\,\bar{w}) - \rho b(w+\beta\,\bar{w}) \\
& + b(1-\sigma)\left[\begin{array}{c} Aw - [b(A+1+\pi-\varepsilon^2)]^{-1/\sigma}(w+\beta\,\bar{w})(A+1+\pi-\varepsilon^2) \\ +\mu\,[b(A+1+\pi-\varepsilon^2)]^{-1/\sigma}(1+\beta)\,\bar{w} \end{array}\right] \\
& + b\beta(1-\sigma)\Big[A - (A+1+\pi-\varepsilon^2-\mu)\left[b(A+1+\pi-\varepsilon^2)\right]^{-1/\sigma}(1+\beta)\Big]\,\bar{w} \\
& - \frac{1}{2}(1-\sigma)\sigma\varepsilon^2 b(w+\beta\,\bar{w})
\end{aligned}$$

For the above equation to hold for any w and $\bar{w}$, it is necessary and sufficient that:

$$\begin{aligned}
0 \;=\; & \left[b(A+1+\pi-\varepsilon^2)\right]^{-(1-\sigma)/\sigma} - \rho b \\
& + b(1-\sigma)\Big[A - \left[b(A+1+\pi-\varepsilon^2)\right]^{-1/\sigma}(A+1+\pi-\varepsilon^2)\Big] \\
& - \frac{1}{2}(1-\sigma)\sigma\varepsilon^2 b
\end{aligned} \tag{51}$$

and

$$\begin{aligned}
0 \;=\; & \left[b(A+1+\pi-\varepsilon^2)\right]^{-(1-\sigma)/\sigma}\beta - \rho b\beta \\
& + b(1-\sigma)\left[b(A+1+\pi-\varepsilon^2)\right]^{-1/\sigma}\left[\mu(1+\beta) - \beta(A+1+\pi-\varepsilon^2)\right] \\
& + b\beta(1-\sigma)\Big[A - (A+1+\pi-\varepsilon^2-\mu)\left[b(A+1+\pi-\varepsilon^2)\right]^{-1/\sigma}(1+\beta)\Big] \\
& - \frac{1}{2}(1-\sigma)\sigma\varepsilon^2 b\beta
\end{aligned} \tag{52}$$

It is important to note that when we compare coefficients to guarantee that the laws of motion for w and $\bar{w}$ coincide we need to replace ϕ, making use of equation (48). Otherwise we would be treating ϕ as a constant, which would yield incorrect results.

Equation (51) implies that:

$$\left[b(A+1+\pi-\varepsilon^2)\right]^{-1/\sigma}(A+1+\pi-\varepsilon^2)=\frac{\rho-(1-\sigma)A}{\sigma}+\frac{1}{2}(1-\sigma)\varepsilon^2 \quad (53)$$

Also, (51) and (52) imply that:

$$\mu=\beta\left(A+1+\pi-\varepsilon^2-\mu\right) \quad (54)$$

The cash-in-advance constraint implies that, in equilibrium:

$$M/P=(1-\phi)w=\left[b(A+1+\pi-\varepsilon^2)\right]^{-1/\sigma}(1+\beta)\,\bar{w}$$

Differentiating this expression using Ito's lemma yields,

$$\begin{aligned}
\mu-\pi+\varepsilon^2 &= g(\bar{w})/\,\bar{w} \\
&= A-(A+1+\pi-\varepsilon^2-\mu)\left[b(A+1+\pi-\varepsilon^2)\right]^{-1/\sigma}(1+\beta) \\
&= A-(A+1+\pi-\varepsilon^2)\left[b(A+1+\pi-\varepsilon^2)\right]^{-1/\sigma}, \text{ using (54)} \\
&= A-\frac{\rho-(1-\sigma)A}{\sigma}-\frac{1}{2}(1-\sigma)\varepsilon^2
\end{aligned}$$

So

$$A+\pi-\varepsilon^2-\mu=\frac{\rho-(1-\sigma)A}{\sigma}+\frac{1}{2}(1-\sigma)\varepsilon^2 \quad (55)$$

and thus,

$$\begin{aligned}
\frac{c}{w} &= \frac{(1-\phi)w}{w} \\
&= \left[b(A+1+\pi-\varepsilon^2)\right]^{-1/\sigma}(1+\beta) \\
&= \left[b(A+1+\pi-\varepsilon^2)\right]^{-1/\sigma}\left[\frac{A+1+\pi-\varepsilon^2}{A+1+\pi-\varepsilon^2-\mu}\right]
\end{aligned}$$

$$= \left[\frac{\rho-(1-\sigma)A}{\sigma}+\frac{1}{2}(1-\sigma)\varepsilon^2\right]\frac{1}{A+\pi-\varepsilon^2-\mu+1}$$
$$= \left[\frac{\rho-(1-\sigma)A}{\sigma}+\frac{1}{2}(1-\sigma)\varepsilon^2\right]\frac{1}{\left[\frac{\rho-(1-\sigma)A}{\sigma}+\frac{1}{2}(1-\sigma)\varepsilon^2\right]+1} \quad (56)$$

It is straightforward to verify, at this point, that whenever μ is constant, the average inflation π is also constant and that the form of the value function and the laws of motion of P and of $\bar{w}$ all conform with our conjectures.

We are now ready to state our main result in this section:

Proposition 9 *Whenever the rate of money growth is constant, the nominal interest rate is also constant and the MCE coincides with the RCE.*

Proof. In the real economy the optimal consumption function is:

$$c = \left[\frac{\rho-(1-\sigma)A}{\sigma}+\frac{1}{2}(1-\sigma)\varepsilon^2\right]k$$

From (56), we see that

$$\phi = \frac{1}{\left[\frac{\rho-(1-\sigma)A}{\sigma}+\frac{1}{2}(1-\sigma)\varepsilon^2\right]+1}$$

Hence, (56) and the fact that $k=\phi w$ imply that, regardless of the value of μ, the decision rule for consumption coincides with that of the real economy. The constancy of the nominal interest rate can be verified by using equations (50) and (55):

$$R = \mu + \frac{\rho-(1-\sigma)A}{\sigma}+\frac{1}{2}(1-\sigma)\varepsilon^2$$

■

This result shows that a monetary rule that keeps the nominal interest rate constant allows the MCE to display the same relative prices of consumption as the RCE, both over time and across different states of nature.

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