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Abstract

The property of "monotonicity" is necessary, and in many contexts, sufficient, for a solution to be Nash implementable (Maskin, 1977). In this paper, we follow Sen (1995) and evaluate the extent to which a solution may fail monotonicity by identifying the minimal way in which it has to be enlarged so as to satisfy the property. We establish a general result relating the "minimal monotonic extensions" of the intersection and the union of a family of solutions to the minimal monotonic extensions of the members of the family. We then calculate the minimal monotonic extensions of several solutions in a variety of contexts, such as classical exchange economies, with either individual endowments or a social endowment, economies with public goods, and one-commodity economies in which preferences are single-peaked. For some of the examples, very little is needed to recover monotonicity, but for others, the required enlargment is quite considerable, to the point that the distributional objective embodied in the solution has to be given up altogether.

Key-words: Maskin-monotonicity. Nash-implementation. Minimal monotonic extension. Fair allocation.

Contents

1 Introduction

A "solution" is a mapping that associates with each economy in some admissible domain a non-empty subset of its feasible set. Solutions are mathematical representations of the objectives of a social planner. They are typically specified so as to select efficient allocations, and also to satisfy some distributional requirement or participation constraint. In order to attain the allocations selected by a solution, information about the agents' preferences is necessary in almost all interesting cases. The theory of implementation (see Moore, 1992, and Corchón, 1994, for surveys) was developed to deal with the strategic problems due to the fact that preferences are private information and that an agent often gains by unilateral misrepresentation. It exploits the fact that for some solutions this problem can be circumvented by confronting agents with well-chosen "game forms": a game form consists of a list of "strategy spaces", one for each agent, and an "outcome function"; each agent is supposed to choose and announce an element of his strategy space, and the outcome function associates with every one of the possible resulting profiles of strategies a feasible allocation. A solution is "(Nash)-implementable" if there exists a game form such that for each admissible economy the set of Nash equilibrium outcomes of the game form played in that economy coincides with the set of outcomes that the solution would select for it.

A property was shown to be necessary, and in many contexts, sufficient, for a solution to be implementable (Maskin, 1977); it is "monotonicity": if an allocation is selected by the solution for some profile of preferences, and preferences change in such a way that the allocation does not fall in anybody's estimation relative to any other feasible allocation, then it is still selected for the new profile. Unfortunately, many interesting solutions are not monotonic and therefore not implementable. Here, we follow Sen (1995) who proposed a method of evaluating the extent to which a solution may fail to be monotonic. Specifically, we look for the minimal way in which the solution has to be enlarged so as to satisfy the property. There is indeed a "minimal monotonic" solution that contains any solution. In his work on the subject, Sen limited his attention to the Arrovian model of abstract social choice. Our main objective here is to study the concept of minimal monotonic extension in economic models instead.

First however, we establish a general result relating the minimal monotonic extensions of the intersection and the union of a family of solutions to the minimal monotonic extensions of the members of the family. Then we turn to examples and calculate the minimal monotonic extensions of several solutions in a variety of economic contexts. We consider classical exchange economies, with either individual endowments or a social endowment, economies with public goods, and one-commodity economies in which preferences are single-peaked. We chose the examples so as to illustrate the wide range of possibilities. For some of them, very little is needed to recover monotonicity, but for others, the required enlargment is quite considerable, to the point that the distributional objective embodied in the solution has to be given up altogether. However, efficiency is generally preserved, since under minor assumptions on preferences, the solution that associates with each economy its set of efficient allocations is monotonic.

In each of the examples we specified domains so as to obtain as simple a statement as possible. This typically required assumptions eliminating boundary situations. In the case of the Walrasian solution, violations of monotonicity occur only on the boundary (under convexity of preferences) and of course we did not make any such assumptions.

2 Minimal monotonic extensions

We start with a general statement of the problem. Let A be a set of feasible alternatives and $\mathcal{R}$ a class of preference relations defined on A. Let $N = \{1, \ldots, n\}$ be a set of agents whose preferences belong to $\mathcal{R}$. Given $i \in N$, $R_i \in \mathcal{R}$, and $a \in A$, let $L(R_i, a) = \{b \in A: aR_ib\}$ be the ***lower contour set of R_i at a***. A ***solution*** is a correspondence $\varphi: \mathcal{R}^n \to A$, which associates with each profile $R = (R_i)_{i \in N} \in \mathcal{R}^n$ a non-empty subset of A. Given $i \in N$, $R_i \in \mathcal{R}$, and $a \in A$, we say that ***$R'_i \in \mathcal{R}$ is obtained from R_i by a Monotonic Transformation at a*** if $L(R_i, a) \subseteq L(R'_i, a)$. Let $\boldsymbol{MT(R_i, a)} \subseteq \mathcal{R}$ denote the class of all such R'_i. Given $R, R' \in \mathcal{R}^n$ with $R'_i \in MT(R_i, a)$ for all $i \in N$, we write $R' \in MT(R, a)$.

The following property of solutions is fundamental: it says that if an alternative is selected for some profile of preferences, and preferences change in such a way that the alternative does not fall in anybody's estimation relative to any other feasible alternative, then it is still selected for the new profile.

(Maskin)-Monotonicity: (Maskin, 1977) For all R, $R' \in \mathcal{R}^n$, and all $a \in \varphi(R)$, if $R' \in MT(R, a)$, then $a \in \varphi(R')$.

It is easy to check that an arbitrary intersection of *monotonic* solutions, if ***well-defined***, that is, if non-empty for each economy in its domain, is *monotonic*. Moreover, the solution that associates with each economy its feasible set is *monotonic*. Therefore, the intersection of all the *monotonic* solutions that contain a given solution φ,[1] is a well-defined *monotonic* solution that contains φ, and it is the smallest solution with these properties. These observations suggest the following definition:

Minimal monotonic extension: (Sen, 1995) Given $\varphi: \mathcal{R}^n \to A$, the ***minimal monotonic extension of φ, $mme(\varphi)$,*** is defined by

$$mme(\varphi) = \cap\{\psi: \psi \supseteq \varphi, \psi \text{ is monotonic}\}$$

Our first lemma, although straigthforward, will be of great help in our calculations:

Lemma 1 *Let $\varphi: \mathcal{R}^n \to A$. For all $R \in \mathcal{R}^n$, $mme(\varphi)(R) = \{a \in A$: there exists $R' \in \mathcal{R}^n$ such that $a \in \varphi(R')$ and $R \in MT(R', a)\}$.*

Proof: Let φ^* be the solution defined in the statement of the Lemma. It obviously contains φ. Also, any *monotonic* solution containing φ contains it. Finally, it is *monotonic*. Indeed, let $R \in \mathcal{R}^n$, $a \in \varphi^*(R)$, and $R'' \in MT(R, a)$. To show that $a \in \varphi^*(R'')$, observe that by definition of φ^*, there exists $R' \in \mathcal{R}^n$ such that $a \in \varphi(R')$ and $R \in MT(R', a)$. But since $R'' \in MT(R, a)$ and $R \in MT(R', a)$, we have $R'' \in MT(R', a)$, so that indeed $a \in \varphi^*(R'')$. □

Sen studies the concept of *minimal monotonic extension* in the context of abstract social choice when the number of alternatives is finite and under the "unrestricted domain" assumption. However, the case of concretely specified economic models seems to have been left open. The purpose of Section 3 is to examine such models. We consider several examples of commonly discussed solutions and calculate their *minimal monotonic extensions.*

The general economic model and notation are as follows. Let $\ell \in \mathbb{N}$ be the number of goods; preferences are defined over $\mathbb{R}^\ell_+$. As before, for each $i \in N$, R_i denotes agent i's preference relation. Let P_i be the strict preference relation associated with R_i and I_i the corresponding indifference

[1] This is the solution that associates with each economy the set of allocations that are chosen by all the *monotonic* solutions that contain φ.

relation. Let $\mathcal{R}_{cl}$ be the domain of "classical", *i.e.*, continuous, convex, and monotone (this means that $z_i > z_i'$ implies $z_i P_i z_i'$)[2] preferences. Let $\mathcal{R}_{cl,B}$ be the subdomain of $\mathcal{R}_{cl}$ of preferences such that no indifference surface containing a positive point meets any of the coordinates subspaces (Cobb-Douglas preferences satisfy this property and so do Leontieff preferences. Linear preferences do not). We refer to this boundary condition as "condition B":

Condition B: A preference relation R_i defined on $\mathbb{R}^\ell_+$ satisfies ***condition B*** if for all $z_i \in \mathbb{R}^\ell_{++}$ and all $z_i' \in \mathbb{R}^\ell_+$, if $z_i I_i z_i'$, then $z_i' \in \mathbb{R}^\ell_{++}$.

Each agent $i \in N$ is endowed with a vector $\omega_i \in \mathbb{R}^\ell_+$ of goods. We refer to the list $\omega = (\omega_i)_{i \in N}$ as the ***initial allocation***. Endowments are given once and for all, and therefore we simply denote an ***economy*** by a list $R = (R_i)_{i \in N}$ of preference relations. The ***feasible set*** is $Z = \{z \in \mathbb{R}^{\ell n}: \sum_N z_i = \sum_N \omega_i\}$. From each individual i's preferences over his consumption space $\mathbb{R}^\ell_+$ we can derive a preference relation $\tilde{R}_i$ defined over the feasible set in the usual way by comparing his components of allocations: given $z = (z_i)_{i \in N}$ and $z' = (z_i')_{i \in N} \in Z$, we write $z \tilde{R}_i z'$ if and only if $z_i R_i z_i'$. If $z, z' \in Z$ are such that $z_i I_i z_i'$ for all $i \in N$, we write $z I z'$. When we discuss the problem of fair division, we specify a social endowment $\Omega \in \mathbb{R}^\ell_{++}$ instead of the individual endowments.

Before we turn to the examples, we present a lemma which describes a simple relationship between the *minimal monotonic extensions* of two solutions and those of their intersection and union. The lemma is an exact counterpart of a result concerning a certain property of "consistency" of solutions (Thomson, 1994). To prove the strict inclusion that it states, we find it convenient to consider the following domain of quasi-linear economies: preferences are defined over $\mathbb{R} \times \mathbb{R}^{\ell-1}_+$, and they admit representations that are separable additive in the first good, and linear in that good. This means that there is a function $v_i: \mathbb{R}^{\ell-1}_+ \to \mathbb{R}$ such that, denoting by $x_i \in \mathbb{R}$ agent i's consumption of the first good and $y_i \in \mathbb{R}^{\ell-1}_+$ his consumption of the remaining goods, his preference relation can be represented by the function assigning value $u_i(x_i, y_i) = x_i + v_i(y_i)$ to the bundle (x_i, y_i). In a quasi-linear economy, if an allocation is efficient, then so is any other allocation obtained

[2] Vector inequalities: given $a, b \in \mathbb{R}^\ell$, $a \geqq b$ means $a_k \geq b_k$ for all k ; $a \geq b$ means $a \geqq b$ and $a \neq b$; $a > b$ means $a_k > b_k$ for all k.

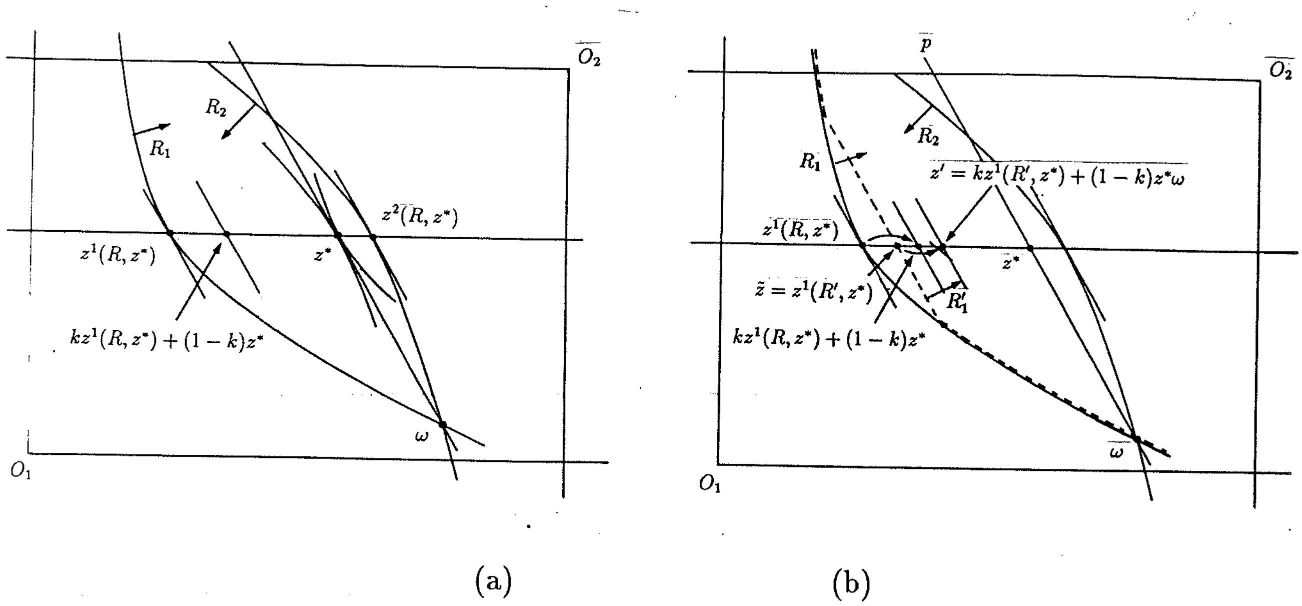


Figure 1: Example showing that the *minimal monotonic extension* of the intersection of a family of solutions may be strictly contained in the intersection of the *minimal monotonic extensions* of the elements of the family (Proof of Lemma 2). (a) Definition of the solution. (b) By "flattening" his indifference curve through his endowment, agent 1 can push the chosen allocation in a direction that is favorable to him.

by arbitrary redistributions of the first good. Let $\mathcal{R}_{ql}$ be the class of all such preference relations. We are now ready for the statement of the lemma.

Lemma 2 *Given two solutions φ and φ',*

$$mme(\varphi \cup \varphi') = mme(\varphi) \cup mme(\varphi')$$

Also, if $\varphi \cap \varphi'$ is a well-defined solution,

$$mme(\varphi \cap \varphi') \subseteq mme(\varphi) \cap mme(\varphi')$$

The inclusion may be strict.[3]

Proof: The proofs of the equality and of the inclusion are identical to their counterparts pertaining to consistency, and we refer the reader to Thomson (1994) for details. Indeed, the only property of consistency used there is that it is preserved under arbitrary intersections and unions.

To show that the inclusion may be strict, we construct an example. Let $\ell = 2$ and $\mathcal{R}^t_{ql} \subseteq \mathcal{R}_{ql}$ be the class of quasi-linear preferences defined on

[3]The result holds for any domain. The same statements also apply to arbitrary, finite or not, unions and intersections.

$\mathbb{R} \times \mathbb{R}_+$ such that at every point z_i, there is a unique line of support to the upper-contour set at z_i (this implies that at every point z_i on the x-axis, the only line of support to the indifference curve passing through z_i is horizontal, so that indifference curves reach the x-axis tangentially.) Let W denote the Walrasian solution. Given $R \in (\mathcal{R}^t_{ql})^2$, and $z^* \in W(R)$, let $z^1(R, z^*)$ be the individually rational[4] and efficient allocation that differs from z^* only in the distribution of the x-good and that is worst for agent 1 (Figure 1a). Now, given $k \in [0,1]$, let $\varphi^k(R) = \{z^*, kz^1(R, z^*) + (1-k)z^*: z^* \in W(R)\}$.[5] If there is a unique Walrasian allocation, the generic case, $\varphi^k(R)$ contains at most two points, one of which is that allocation.

Under our domain assumptions, the Walrasian solution W is *monotonic* (more on this issue in Section 3.2). However, φ^k is not *monotonic* except for $k = 0$, in which case it coincides with W. Finally, the *minimal monotonic extension* of φ^k is the solution ψ^k defined by $\psi^k(R) = \{[z^*, kz^1(R, z^*) + (1-k)z^*]: z^* \in W(R)\}$. This says that in order to obtain *monotonicity*, and in the case when there is a unique Walrasian allocation, say, we need to add to the two points that the solution would have selected, the whole interval joining them. To prove this, it suffices to observe that (i) ψ^k is *monotonic* and that (ii) by Lemma 1, given any $R \in (\mathcal{R}^t_{ql})^2$, and any $z' \in \psi^k(R)$, there are $z^* \in W(R)$ such that $z' \in [kz^1(R, z^*) + (1-k)z^*]$ and $R' \in (\mathcal{R}^t_{ql})^2$ such that $z' \in \varphi^k(R')$ and $R \in MT(R', z')$. Indeed, let $\tilde{z} \in [z^1(R, z^*), z^*]$ be such that $z' = k\tilde{z} + (1-k)z^*$. The preference relation $R'_1 \in \mathcal{R}^*_{ql}$ can be defined by specifying one upper-contour set. For upper-contour set at $\tilde{z}$, choose the intersection of the upper-contour set at $z^1(R, z^*)$ for the preference relation R_1 with the line normal to p passing through that point, where p are prices supporting z^*. This construction is illustrated in Figure 1b (the dashed line). Now, given k, $k' \in]0,1]$, with $k \neq k'$, we have $\varphi^k \cap \varphi^{k'} = W$, and by *monotonicity* of the Walrasian solution on the domain under consideration, we obtain $mme(\varphi^k \cap \varphi^{k'}) = W$. However, $mme(\varphi^k) \cap mme(\varphi^{k'}) = \psi^k \cap \psi^{k'} = \psi^{\min\{k,k'\}} \neq W$. □

[4]An allocation is individual rational if it Pareto-dominates the initial allocation.

[5]The set $\varphi^k(R)$ consists of the Walrasian allocations and the allocations that are obtained as linear combinations, with weights $k-1$ and k of a Walrasian allocation and the worst individually rational allocation for agent 1 at which the distribution of the second good is the same.

3 Applications

In order to illustrate the notion of *minimal monotonic extension*, we start with a private good model, in economies with individual endowments first, then in economies with a social endowment. We discuss two fundamental solutions, the Pareto solution and the Walrasian solution. Then we turn to various solutions to the problem of fair division. Our final example pertains to one-commodity economies with single-peaked preferences. Our motivation in this selection of examples was to illustrate the wide range of possibilities: for some of the solutions, a very minor enlargment is needed to recover *monotonicity*; for others, the required enlargment is considerable, to the point that the distributional objective embodied in the solution is essentially lost; the remaining cases fall somewhere in between. For applications of the notion of *minimal monotonic extensions* to classes of matching problems, see Kara and Sönmez (1994, 1996).

3.1 The Pareto solution

The Pareto solution is the solution that associates with each economy its set of feasible allocations such that there is no other feasible allocation that all agents prefer and at least one agent strictly prefers.

Pareto solution, $\boldsymbol{P}$: Given $R \in \mathcal{R}^n$, $\boldsymbol{P(R)} = \{z \in Z\colon$ there is no $z' \in Z$ such that $z'_i R_i z_i$ for all $i \in N$ and $z'_i P_i z_i$ for some $i \in N\}$.

The fact that this solution is not *monotonic* on the classical domain is illustrated in Figure 2.[6] In the economy $R \in \mathcal{R}^2_{cl}$ it depicts, agent 1's preferences can be represented by the function u_1 defined by $u_1(z_1) = \sum_k z_{1k}$ for all $z_1 \in \mathbb{R}^2_+$, and agent 2 has the same preferences. Let $z = ((\Omega_1, \Omega_2/2), (0, \Omega_2/2))$, and note that $z \in P(R)$. Now, let $R'_1 \in \mathcal{R}_{cl}$ be the preference relation that can be represented by the function u'_1 defined by $u'_1(z_1) = z_{11}$ for all $z_1 \in \mathbb{R}^2_+$, (now, agent 1 cares only about the first good), and let $R'_2 = R_2$. Let $R' = (R'_1, R'_2)$. Note that $R' \in MT(R, z)$ and that z is Pareto-dominated in R' by $z' = ((\Omega_1, 0)(0, \Omega_2))$, proving the claim.

This lack of *monotonicity* of the Pareto solution is due to the fact that preferences are not strictly monotone (this term means that $z_i \geq z'_i$ implies

[6] This is a point worth clarifying since it is a common misconception that the Pareto solution is *monotonic* on this domain.

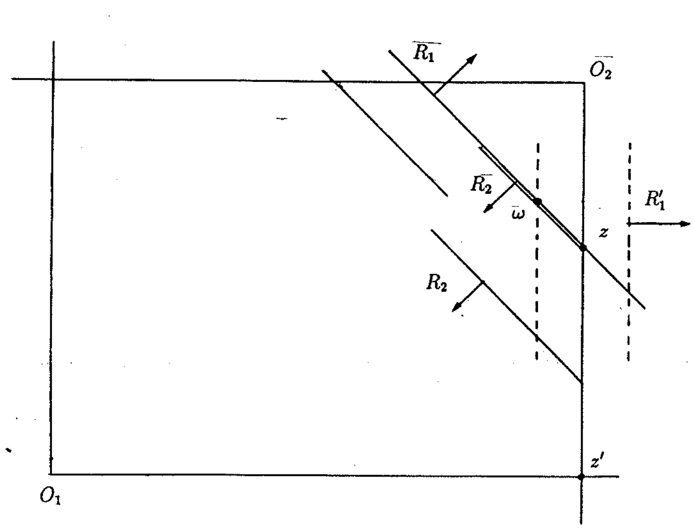


Figure 2: The Pareto solution is not *monotonic.* On the classical domain violations of *monotonicity* occur on the boundary of the feasible set.

$z_iP_iz_i'$). When the domain is required to contain only strictly monotone preferences, the Pareto solution coincides with the solution that picks all the feasible allocations to which there is no other feasible allocation that all agents strictly prefer, the "weak Pareto solution":

Weak Pareto solution, $\boldsymbol{WP}$: Given $R \in \mathcal{R}^n$, $\boldsymbol{WP(R)} = \{z \in Z$: there is no $z' \in Z$ such that $z_i'P_iz_i$ for all $i \in N\}$.

This solution is *monotonic* on the classical domain. We state without proof the following result, which relates the Pareto and weak Pareto solutions.

Theorem 1 *On $\mathcal{R}^n_{cl}$, the* minimal monotonic extension *of the Pareto solution is the weak Pareto solution.*

A consequence of this result and of Lemma 2 together is that on the classical domain, the *minimal monotonic extension* of a subsolution of the Pareto solution is a subsolution of the weak Pareto solution.

3.2 The Walrasian solution

Our next example is the Walrasian solution. Hurwicz, Maskin, and Postlewaite (1985) noted, and this fact is now well-known, that the Walrasian solution is not *monotonic* on the classical domain. A violation of *monotonicity* is illustrated in Figure 3a for an economy $R \in \mathcal{R}^2_{cl}$. We have $z \in W(R)$ with supporting prices $p \in \Delta^{\ell-1}$, $R_1' \in MT(R_1, z)$ and $R_2' = R_2$, but $z \notin W(R')$. Indeed, z_1 fails to maximize R_1' in agent 1's budget set at prices p, these

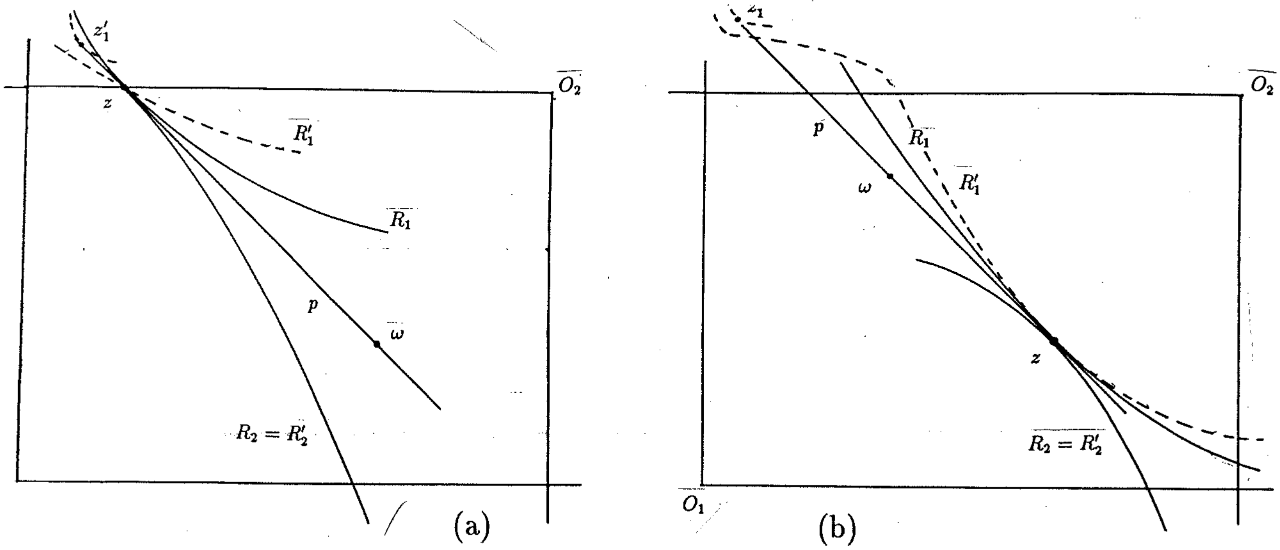


Figure 3: (a) The Walrasian solution is not *monotonic*. Violations of *monotonicity* occur on the boundary of the feasible set. Its *minimal monotonic extension* is the constrained Walrasian solution (Theorem 3.2). (b) If preferences are not convex, violations of *monotonicity* may occur in the interior of the Edgeworth box.

prices being the only candidate equilibrium prices. Under convexity of preferences, violations of *monotonicity* can only occur at boundary Walrasian allocations. Figure 3b shows that if preferences are not convex, violations of *monotonicity* can also occur in the interior of the feasible set. There, $z \in W(R)$, $R_1' \in MT(R_1, z)$, and $R_2' = R_2$, but $z \notin W(R')$, since at the only prices at which z_1 is a local maximizer of R_1' on the resulting budget set, the point z_1' is affordable by agent 1 and $z_1' P_1' z_1$.

Hurwicz, Maskin, and Postlewaite defined the concept of the "constrained Walrasian" solution by having each agent maximize his preferences in the intersection of his Walrasian budget set with the projection of the feasible set onto his consumption set, and showed that it is *monotonic* and that it contains the Walrasian solution. The result following the definition provides a formal justification for the introduction of this solution in terms of the concept of *minimal monotonic extension.*

Constrained Walrasian solution, CW: Given $R \in \mathcal{R}^n$, $CW(R) = \{z \in Z$: there exists $p \in \Delta^{\ell-1}$ such that for all $i \in N$, and all $z_i' \in \mathbb{R}^\ell$ such that $[pz_i' \leq p\omega_i$, and for some $z_{-i}' \in \mathbb{R}_+^{\ell(n-1)}$, $(z_i', z_{-i}') \in Z]$, we have $z_i R_i z_i'\}$.

Theorem 2 *On $\mathcal{R}_{cl}^n$, the* minimal monotonic extension *of the Walrasian solution is the constrained Walrasian solution.*

Proof: Given $R \in \mathcal{R}_{cl}^n$ and $z \in CW(R)$ with associated supporting price $p \in \Delta^1$, it is trivial to construct $R' \in \mathcal{R}_{cl}^n$ such that $z \in W(R')$ and $R \in$

$MT(R', z)$. For instance, for all $i \in N$, let R_i' be a linear preference relation whose indifference surfaces are hyperplanes normal to p. Then, appeal to Lemma 1. □

Another proof of this result, noted by Hurwicz, Maskin, and Postlewaite (1995), can be obtained along the lines of Hurwicz (1979).[7] Incidentally, on the classical domain, the constrained Walrasian solution is not a subsolution of the Pareto solution (on this domain, the Walrasian solution is). Indeed, in the example depicted in Figure 2, $z \in CW(R_1', R_2)$ (the supporting prices are normal to the indifference curves through ω) but z is not efficient. Any point of the segment $[z, z']$ is a constrained Walrasian allocation for R, but only z', which is Walrasian, is efficient.[8]

Note that on the domain $\mathcal{R}^n_{cl,B}$, the Walrasian solution is *monotonic* since the corner problems illustrated in Figure 3 do not occur.

At this point, it may also be useful to observe that on the classical domain, the *minimal monotonic extension* of any solution satisfying the following minor condition (*Property P* in Thomson, 1983; *non-discrimination between Pareto-indifferent allocations* in Gevers, 1986), and selects allocations that Pareto dominate the initial allocation, contains the constrained Walrasian solution. The condition says that if an allocation is chosen, then so is any allocation that is Pareto-indifferent to it:

Pareto-indifference: For all $R \in \mathcal{R}^n$ and all $z, z' \in Z$, if $z \in \varphi(R)$ and $z'Iz$, then $z' \in \varphi(R)$.

The same statement can be made about solutions satisfying the next condition (Thomson, 1987), which says that if the initial allocation is efficient, then the solution selects all the allocations that are Pareto-indifferent to it (perhaps others):[9]

[7]Hurwicz shows that if an upper semi-continuous subsolution of the individual rationality and Pareto solution is implementable over a sufficiently rich domain (the domain has to contain linear preferences as well as preferences that are arbitrarily close to being linear), then it contains all Walrasian allocations. By the same argument, one shows that in fact it has to contain all constrained Walrasian allocations. The result follows then from the fact that the constrained Walrasian solution is implementable.

[8]Of course, a constrained Walrasian allocation that is not Walrasian may be efficient. In Figure 2, if R_1 is replaced by a linear preference map with steeper indifference curves, z remains constrained Walrasian without being Walrasian, but now it is efficient.

[9]In the first case, the conclusion follows from the fact that on the classical domain

Condition α: For all $R \in \mathcal{R}^n$, if $\omega \in P(R)$, then $\varphi(R) \supseteq \{z \in Z: zI\omega\}$.

For public good economies, a similarly defined extension of the Lindahl solution, known as the constrained Lindahl solution, can be given a similar justification. Also, the ratio equilibrium solution (Kaneko, 1977), and recently introduced solutions such as the balanced linear cost share equilibrium solution (Mas-Colell and Silvestre, 1989), are not *monotonic*, but their *minimal monotonic extensions* can be easily obtained by requiring maximization of each agent's preferences over the intersection of his budget set as specified in the original definitions of these solutions, with the projection of the feasible set onto his consumption space (Diamantaras, 1993).

3.3 The essential no-envy solution

The next example shows that the extension required to obtain *monotonicity* can bring about a considerable change in a solution. It pertains to an extension of the no-envy solution, the solution that selects for each economy its set of allocations such that no agent would rather receive someone else's bundle to his own. In private good economies with non-convex preferences, and even if all other standard properties of preferences are maintained, there may be no envy-free and efficient allocations (Varian, 1974). To remedy this difficulty, Vohra (1991) suggested to select the allocations such that for each agent, there is a Pareto-indifferent allocation at which he envies no-one. He proved the non-emptiness of this solution under very general conditions. Let $\tilde{\mathcal{R}}$ be the class of preferences defined on $\mathbb{R}^\ell_+$ that are continuous, strictly monotone on $\mathbb{R}^\ell_{++}$ and satisfy condition B (convexity is not assumed any longer). On this domain the Pareto solution is *monotonic*.[10]

Essential no-envy solution, V: (Vohra, 1991) Given $R \in \tilde{\mathcal{R}}^n$, $\boldsymbol{V(R)} = \{z \in Z$: for all $i \in N$, there is $z^i \in Z$ such that $z^i I z$ and for all $j \in N$, $z^i_i R_i z^i_j.\}$

A disadvantage of the essential no-envy solution as compared to the no-envy solution is that it is not *monotonic*. We show next that the *minimal*

any *monotonic* subsolution of the solution that selects allocations that Pareto-dominate the initial allocation and satisfies *Pareto-indifference* contains the constrained Walrasian solution. In the second case, it follows from the fact that any *monotonic* solution satisfying satisfying *Condition α* contains the constrained Walrasian solution.

[10] So is the Walrasian solution on the subdomain on which it is well-defined.

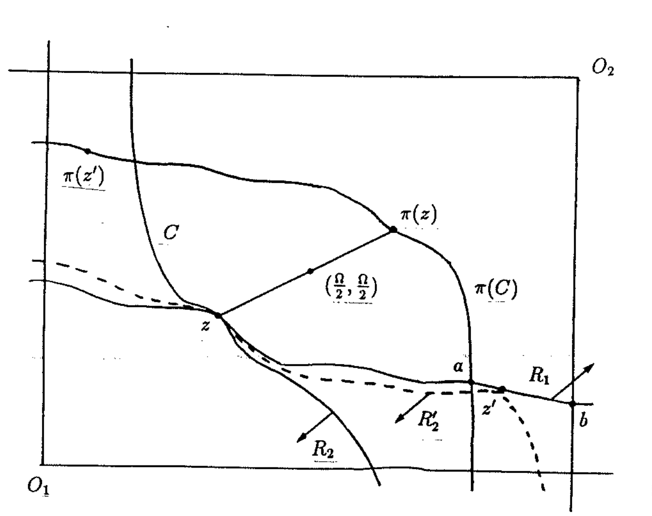


Figure 4: *Minimal monotonic extension* of the essential no-envy solution (Proof of Theorem 3).

monotonic extension of its intersection with the Pareto solution can be easily calculated, at least in the two-agent case. The result is that this extension almost coincides with the Pareto solution! On the domain considered in Theorem 3, the only Pareto-optimal allocations that are excluded are the origins. Let P^0 denote this "interior" Pareto solution.

Theorem 3 *On $\tilde{\mathcal{R}}^2$, the* minimal monotonic extension *of the essential no-envy and Pareto solution is the interior Pareto solution.*

Proof: (Figure 4) Let $\varphi = mme(V \cap P)$. Let $R = (R_1, R_2) \in \tilde{\mathcal{R}}^2$, and $z \in P^0(R)$ be given. If $z \in F(R)$, and since $\varphi \supseteq V \cap P \supseteq F \cap P$, then $z \in \varphi(R)$. Suppose now that $z \notin F(R)$ and to fix the ideas, that agent 1 envies agent 2 at z: $z_2 P_1 z_1$. Since $z \in P(R)$, then agent 2 does not envy agent 1 at z (Varian, 1974).[11] Let $\pi: \mathbb{R}^2 \to \mathbb{R}^2$ denote the symmetry operator with respect to the middle of the Edgeworth box. Let C be agent 1's indifference curve passing through z, and $C' = \pi(C)$. Now, select $z' \in C$ such that $\pi(z')$ be below C. By condition B, such a z' exists (any allocation on the curvi-linear segment connecting a to b would do, where a is the point of intersection of C with $\pi(C)$ to the South-West of z and b is the point of intersection of C with the right side of the Edgeworth box). Let $R_1' = R_1$ and $R_2' \in \tilde{\mathcal{R}}$ be such that $z, z' \in P(R')$ and $R \in MT(R', z)$. Since at z', which is Pareto-indifferent to z in R', agent 1 does not envy agent 2, and at

[11]Varian shows that at an efficient allocation, there is at least one agent that envies no-one. In our two-person economy, if agent 2 envied agent 1, we could exchange their bundles and make them both better off.

z, agent 2 does not envy agent 1, we have $z \in (V \cap P)(R')$. Since $\varphi \supseteq V \cap P$, $z \in \varphi(R')$. The proof concludes by Lemma 1. $\square$

Note that by contrast with the no-envy solution, Vohra's solution satisfies *Pareto-indifference.* One could ask how much the no-envy solution, or its intersection with the Pareto solution, differ from what could be called their *minimal Pareto-indifferent* extensions: *Pareto-indifference* being closed under arbitrary intersections, this concept can be defined just as we defined the notion of *minimal monotonic extension.* If preferences are strictly convex, then of course, the no-envy and Pareto solution does satisfy *Pareto-indifference.* It turns out that otherwise, we may have to add points at which non only envy is violated, but even the weaker condition of no-domination, which says that no agent should consume more of every good than any other agent (Thomson, 1983; Moulin and Thomson, 1988). In fact, given any $\epsilon > 0$, an example can be constructed so that one may have to add points at which some agent's bundle multiplied by the factor ϵ may still dominate the bundle of some other agent (this is a violation of the condition of ϵ-no-domination of Moulin and Thomson, 1988). Since the essential no-envy solution satisfies *Pareto-indifference*, one can deduce from these observations that the considerable enlargment of the essential no-envy solution that is required to obtain *monotonicity* is at least in part due to the enlargment of no-envy needed to obtain *Pareto-indifference.* The *minimal monotonic extension* of the *Pareto-indifferent* extension of the no-envy and Pareto solution can be easily determined, at least for $n = 2$, by adapting the argument of Theorem 3.

3.4 Egalitarian-equivalence and Ω-egalitarianism

Next we consider another solution that has played an important role in the theory of fair allocation. Say that an allocation is "egalitarian-equivalent" if there exists a "reference bundle" such that every agent is indifferent between his assigned consumption bundle and the reference bundle.

Egalitarian-equivalence solution, $\boldsymbol{E}$: (Pazner and Schmeidler, 1978) Given $R \in \mathcal{R}^n$, $\boldsymbol{E(R)} = \{z \in Z$: there exists $z_0 \in \mathbb{R}^\ell_+$ such that for all $i \in N$, $z_i I_i z_0\}$.

It is easy to see that the egalitarian-equivalence solution is not *monotonic* and that neither is its intersection with the Pareto solution, with which Theorem 4 is concerned. For that result, we will find it convenient to consider

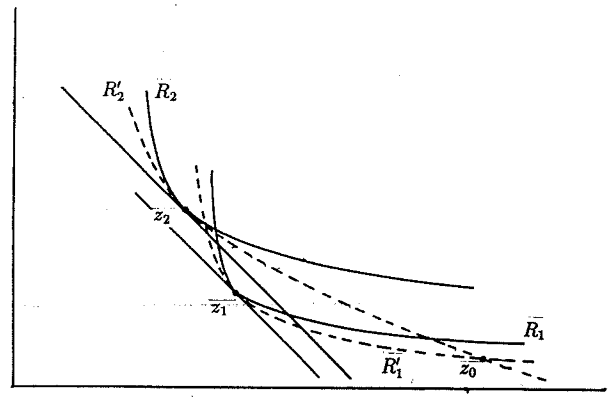


Figure 5: *Minimal monotonic extension* of the egalitarian-equivalence solution (Proof of Theorem 4).

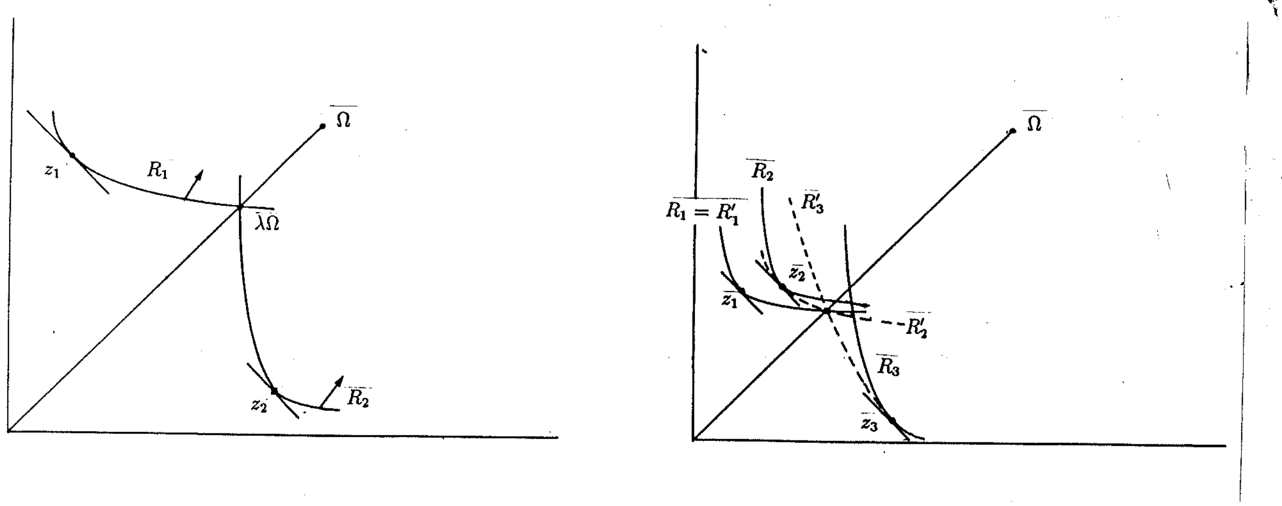
the class $\mathcal{R}^*$ of preferences that are continuous, convex, strictly monotonic in $\mathbb{R}^\ell_{++}$, and satisfy condition B:

Theorem 4 *On $\mathcal{R}^{*n}$, the* minimal monotonic extension *of the egalitarian-equivalence and Pareto solution is the interior Pareto solution.*

Proof: (Figure 5) First, note that if $R \in \mathcal{R}^{*n}$ and $z \in (E \cap P)(R)$, then $z \in P^0(R)$. Let $\varphi = mme(E \cap P)$. Now, let $R \in \mathcal{R}^{*n}$ and $z \in P^0(R)$. Also, let $p \in \Delta^{\ell-1}$ be prices supporting z (they exist by convexity of preferences). Since $z \in P^0(R)$, then $p > 0$. Let $z_0 \in \cap_N L(R_i, z_i)$ be such that $pz_0 \geqq \max_{i \in N} pz_i$. For each $i \in N$, let $R_i' \in \mathcal{R}^*$ be such that (i) $R_i \in MT(R_i', z_i)$, (ii) $z_i' I_i z_0$, and (iii) p supports the upper-contour set at z_i, $\{z_i' \in \mathbb{R}^\ell_+ : z_i' R_i' z_i\}$. Note that $z \in (E \cap P)(R')$, with reference bundle z_0. The proof concludes by appealing to Lemma 1. □

If condition B is not imposed, the result is a little more difficult to state but the proof is essentially the same. Then, $mme(E \cap P)(R)$ consists of all the allocations z at which the intersection of all the lower contour sets contains at least one point whose value at some prices supporting z is greater than the maximal value of any of the components of z. Then a profile $R' \in \mathcal{R}^{*n}$ satisfying (i), (ii), and (iii) as in the proof can easily be found.

A useful selection from the egalitarian-equivalence solution is obtained by requiring that the reference bundle be proportional to the social endowment. Again, we will also consider its intersection with the Pareto solution (Figure 6a). It is suggested by Pazner and Schmeidler and it has played an important role in a number of recent studies.



(a) (b)

Figure 6: (a) Definition of the Ω-egalitarian solution. (b) Its *minimal monotonic extension* (Proof of Theorem 5).

The Ω-egalitarian solution, E_Ω: Given $R \in \mathcal{R}^n$, $E_\Omega(R) = \{z \in Z\colon$ there exists $\lambda \in \mathbb{R}_+$ such that for all $i \in N$, $z_i I_i(\lambda\Omega)\}$.

We refer to this solution as "egalitarian" since it involves equating numerical representations of preferences but note that this differs from standard usage of this term where the utility functions capture notions of intensity of satisfaction. By contrast, our definition is ordinal.

We will need the following piece of notation: given $R \in \mathcal{R}^{*n}$, $z \in P(R)$ and $i \in N$, let $\lambda(R_i, z_i) \in \mathbb{R}_+$ be such that $z_i I_i \lambda(R_i, z_i)\Omega$.

Theorem 5 *On $\mathcal{R}^{*n}$, the* minimal monotonic extension *of the Ω-egalitarian and Pareto solution is the solution φ defined by: $\varphi(R) = \{z \in P(R)$: there exist supporting prices $p \in \Delta^{\ell-1}$ such that* $\max_{j\in N} pz_j \leq \min_{i\in N} p\lambda(R_i, z_i)\Omega\}$.

Proof: First, we show that φ is *monotonic.* Let $R \in \mathcal{R}^{*n}$ and $z \in \varphi(R)$, with $p \in \Delta^{\ell-1}$ as supporting prices. Also, let $R' \in \mathcal{R}^{*n} \in MT(R, z)$. First, we note that $z \in P(R')$ with supporting prices p. Also, for each $i \in N$, $\lambda(R_i', z_i) \geq \lambda(R_i, z_i)$. Therefore, $\max_{j\in N} pz_j \leq \min_{i\in N} p\lambda(R_i', z_i)z_i$, so that $z \in mme(E_\Omega \cap P)(R')$.

Next, given $R \in \mathcal{R}^{*n}$ and $z \in \varphi(R)$, with $p \in \Delta^{\ell-1}$ as supporting prices, we show that there is $R' \in \mathcal{R}^{*n}$ such that $z \in (E_\Omega \cap P)(R')$ and $R \in MT(R', z)$. Indeed the only additional requirement is that that agent i's indifference curve through z_i passes through the point $[\min_{j\in N} \lambda(R_i, z_i)]\Omega$ (Figure 6b). The proof concludes by appealing to Lemma 1. $\square$

3.5 An equal-gains solution for economies with quasi-linear preferences

We now turn to an examination of the domain of quasi-linear economies and for this domain, we focus on the solution that divides the gains from trade equally among all agents. These gains can be unambiguously measured in terms of the good with respect to which the representations of preferences are linear. Let $s(R) \in \mathbb{R}_+$ be the gains from trade achievable in $R \in \mathcal{R}^n_{ql}$. Using the notation introduced when we first defined quasi-linearity of preferences, $s(R) = \max\{\sum u_i(x_i): \sum z_i = \sum \omega_i\} - \sum u_i(\omega_i)$. The definition is illustrated in the two-person case in Figure 7a. Note that the solution is a subsolution of the Pareto solution.

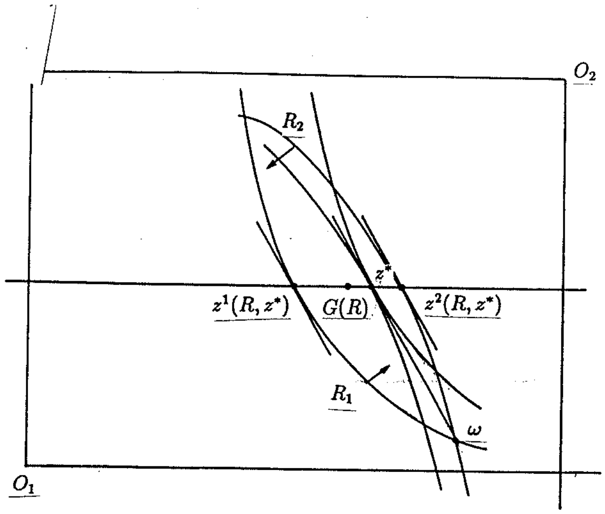
Equal-gains solution, G: For all $R \in \mathcal{R}^n_{ql}$, $\boldsymbol{G(R)} = \{z \in Z: u_i(z_i) = u_i(\omega_i) + s(R)/n\}$.

For $n = 2$, the *minimal monotonic extension* of the equal-gains solution can be informally described as the solution that selects for each economy the set of allocations at which each agent's gain from his endowment is equal to 1/2 times his gain at a constrained Walrasian allocation. This result is illustrated in Figure 7b by means of an example $R \in \mathcal{R}^2_{ql}$ for which the Walrasian allocation is unique. The worst allocation for agent 1 in $mme(G)(R)$ is half-way between the Walrasian allocation and the efficient allocation that is indifferent for him to his endowment.

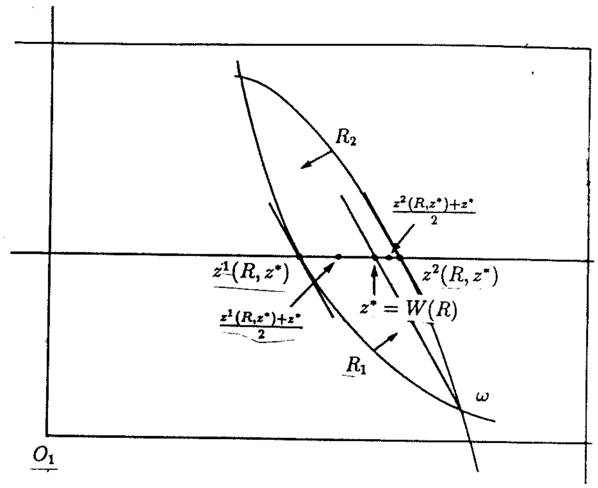
For $n \geq 2$, the construction is as follows (I owe the complete argument for that case to Eiichi Miyagawa). Let $R \in \mathcal{R}^n_{ql}$, and z^* be a constrained Walrasian allocation for R, with supporting prices $p \in \Delta^{\ell-1}$. For all $i \in N$, let $a_i(R, z^*) \in \mathbb{R}$ be such that $(a_i(R, z^*), y_i^*) I_i \omega_i$, and $B_i(R, z^*) = [a_i(R, z^*), x_i^*]$. Now, let $\varphi(R) = \{z \in Z$: there exist $z^* \in CW(R)$ and $b \in \Pi B_i(R, z^*)$ such that $x_i = b_i + \sum(x_i^* - b_i)/n$ and $y = y^*\}$. Note that φ contains CW (choose $b_i = x_i^*$ for all $i \in N$) and φ contains G (choose $b_i = a_i(R, z^*)$ for all $i \in N$).

Theorem 6 *On $\mathcal{R}^n_{ql}$, the* minimal monotonic extension *of the equal-gains solution is the solution φ^*.*

Proof: First, we show that φ is *monotonic*. Let $R \in \mathcal{R}^n$ and $z \in \varphi(R)$, let $R' \in MT(R, z)$. By definition of φ, there exists $z^* \in CW(R)$ and $b \in$



(a)



(b)

Figure 7: (a) The definition of the equal-gains solution. (b) In the two-person case, its *minimal monotonic extension* consists of half of the set of efficient allocations that Pareto-dominate equal division.

$\Pi B_i(R, z^*)$ such that $x_i = b_i + \sum(x_i^* - b_i)/n$ and $y = y^*$. Since CW is *monotonic*, $z^* \in CW(R')$. Also, note that for all $i \in N$, $B_i(R', z^*) \supseteq B_i(R, z^*)$. Therefore, $b \in \Pi B_i(R', z^*)$ and we are done.

Finally, we show that for all $R \in \mathcal{R}^n$ and all $z \in \varphi(R)$, there exists $R' \in \mathcal{R}^n$ such that $z \in G(R')$ and $R \in MT(R', z)$. Let p be a price of support to z. For all $i \in N$, let $R_i' \in \mathcal{R}_{ql}$ be the preference relation whose upper contour set at z_i is the intersection of the upper contour set of R_i at ω_i and the upper contour set at z_i of the linear preference relation whose indifference curves are all normal to p. The proof concludes by Lemma 1. □

Note that in the two-person case, the equal-gains solution coincides with the Shapley value. Therefore, Theorem 6 tells us how much the Shapley value has to be modified in that case so as to recover *monotonicity*. It is an open question whether a simple formula exists for the Shapley value when there are more than two agents.

3.6 Two solutions to the problem of fair division in economies with single-peaked preferences

Finally, we consider the problem of fair division in the one-commodity case ($\ell = 1$) when preferences are single-peaked (Sprumont, 1991). Let $\mathcal{R}_{sp}$ be the class of continuous preference relations R_i satisfying the following property:

there is a number $p(R_i) \in \mathbb{R}_+$ such that for all $z_i, z'_i \in \mathbb{R}_+$, if $z'_i < z_i \leq p(R_i)$, or $p(R_i) \leq z_i < z'_i$, then $z_i P_i z'_i$. This number is agent i's ***peak***. Let $\Omega \in \mathbb{R}_+$ denote the total amount to be allocated. Here too, the social endowment is kept fixed and ignored in the notation, so that an ***economy*** is a list $R = (R_i)_{i \in N} \in \mathcal{R}^n_{sp}$. The feasible set is defined as $Z = \{z \in \mathbb{R}^n_+ \colon \sum z_i = \Omega\}$.

Our first example of a solution is the "proportional solution", which allocates the commodity proportionally to the peaks.

Proportional solution, *Pro*: Given $R \in \mathcal{R}^n_{sp}$, $z =$ ***Pro(R)*** if $z \in Z$ and there exists $\lambda \in \mathbb{R}_+$ such that $z_i = \lambda p(R_i)$ for all $i \in N$; if no such λ exists, $z = (\Omega/n, \ldots, \Omega/n)$.

The solution defined below is based on comparing distances from peaks ***unit for unit*** as opposed to proportionally. It selects the allocation at which all agents' consumptions are equally far from their peaks except when boundary problems occur, in which case those agents whose consumptions would be negative are given zero instead (Thomson, 1994):

Equal-distance solution, *D*: Given $R \in \mathcal{R}^n_{sp}$, $z =$ ***D(R)*** if $z \in Z$ and (i) when $\Omega \leq \sum p(R_i)$, there exists $d \geq 0$ such that $z_i = \max\{0, p(R_i) - d\}$ for all $i \in N$, and (ii) when $\sum p(R_i) \leq \Omega$, there exists $d \geq 0$ such that $z_i = p(R_i) + d$ for all $i \in N$.

Note that both solutions are selections from the Pareto solution. Neither is *monotonic*. To describe their *minimal monotonic extensions*, we need to introduce the solution that selects all the Pareto-optimal allocations except for those allocations at which one agent receives the whole endowment (if it is efficient). Let us refer to this solution as the strong Pareto solution and denote it by P^*. For convenience we will consider the class $\mathcal{R}_{sp,++}$ of preferences having a positive peak.

Theorem 7 *On* $\mathcal{R}^n_{sp,++}$, *the* minimal monotonic extension *of the proportional solution contains the interior Pareto solution. On* $\mathcal{R}^n_{sp}$, *the* minimal monotonic extension *of the equal-distance solution contains the strong Pareto solution.*

Proof: Let $R \in \mathcal{R}^n_{sp,++}$ and $x \in P^*(R)$. First, we assume that $\Omega \leq \sum p(R_i)$. Let $\lambda > 1$ be such that $\lambda \max x_i \leq \Omega$. For each $i \in N$, let $R'_i \in \mathcal{R}_{sp,++}$ be such that $p(R'_i) = \lambda x_i$ and $\Omega P_i x_i$. Note that $x = Pro(R')$ and that $R \in MT(R', z)$. The argument for the case $\Omega \geq \sum p(R_i)$ is similar and we omit it. The proof for the equal-distance solution is similar. □

4 Concluding comments

We applied the notion of *minimal monotonic extension* to a variety of economic models and showed that the *minimal monotonic extensions* of a number of solutions can be easily calculated. We chose the examples so as to illustrate the wide range of possibilities: in some cases the enlargment needed to obtain *monotonicity* is quite small (the Walrasian solution, the Pareto solution). In some other cases, it is considerable (the essential no-envy solution, the egalitarian-equivalence solution). The extensions needed to recover *monotonicity* of other examples fall somewhere in between (the Ω-egalitarian solution). Therefore, the cost of implementability can be very low or very high depending on the particular solution that is being considered. It would be interesting to identify general properties of solutions under which each of these cases occurs.

Gevers (1986) defined a property related to *Maskin-monotonicity*: it says that if z is chosen by a solution for some profile of preferences and preferences change in such a way that for every agent, z does not fall with respect to any other allocation in the space over which his preferences are defined, then z is still chosen for the new profile. Any *Maskin-monotonic* solution is *Gevers-monotonic.* The difference with *Maskin-monotonicity* lies in the fact that in economic models, preferences are defined over a set that is not the feasible set, and that *Maskin-monotonicity* only pays attention to the way preferences change over the feasible set. This property was analyzed by Nagahisa (1994) and Maniquet (1994). *Gevers-monotonicity* is also closed under arbitrary intersections so that we could define the concept of the *minimal Gevers-monotonic extension* of a solution in a similar way to the way we defined its *minimal Maskin-monotonic extension.* On the classical domain, the Pareto solution and the Walrasian solution are *Gevers-monotonic.*

Finally, we note that instead of enlarging a non-monotonic solution in order to obtain the property, we may restrict it. Here, we would of course like to restrict it in a minimal way. That this can be done is a consequence of the fact that *monotonicity* is preserved under arbitrary unions. Therefore, if a solution contains at least one *monotonic* solution, it has a maximal *monotonic* subsolution, which is simply the union of all of its *monotonic* subsolutions.[12] As examples of application, we can show that in the two-person

[12]Thomson (1994) considers and studies the similarly defined notion of the "maximal consistent subsolution" of a given solution.

case, the *maximal monotonic subsolution* of the egalitarian-equivalence and Pareto solution is the no-envy solution. In the case of three or more agents, the egalitarian-equivalence and Pareto solution contains no *monotonic* subsolution, and therefore it has no *maximal monotonic subsolution.*

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