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BEYOND BALANCED GROWTH*

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Abstract

One of the most striking regularities of the growth process is the massive reallocation of labor from agriculture into industry and services. Balanced growth models are commonly used in macroeconomics because they are consistent with the well-known Kaldor facts about economic growth. These models are, however, inconsistent with the structural change dynamics that are a central feature of economic development. This paper discusses models with generalized balanced growth paths. These paths retain some of the key features of balanced growth but are consistent with the observed labor reallocations dynamics.

Key words: Growth, Structural Change.

JEL Classification: O14, O41.

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1 Introduction

The U.S. economy is often described as following a balanced growth path, that is, a trajectory along which all the relevant variables grow at a constant rate. Balanced growth models have the advantage of being generally consistent with the well-known Kaldor facts about economic growth. They display a constant growth rate of output, capital-output ratio, real interest rate, and labor income share.

Just as important as these regularities stressed by Kaldor are the facts about structural change uncovered by authors such as Clark (1940), Kuznets (1957), and Chenery (1960). These researchers documented the massive reallocation of labor from agriculture into manufacturing and services. A few numbers help to put the importance of these reallocations in perspective. In 1870 the U.S. share of employment in agriculture was 40%. One hundred years later, agriculture accounted for only 4% of employment. Services, which accounted for 20% of employment in 1870, absorbed 40% of the labor force by 1970.

Is there a growth model that is consistent with the Kaldor facts and accounts for this massive sectoral reallocation? At first sight the answer to this question is no. After all, one property of balanced growth models is that the shares of capital and labor allocated to different industries remain constant over time. We show that there is a class of models that retain some of the key features of balanced growth and are consistent with the dynamics of structural change. In this paper we characterize these *generalized balanced growth* (GBG) paths, which feature a constant real interest rate but time-varying allocations of inputs across sectors.

Balanced growth paths are easy to study. The fact that all variables grow at a constant rate transforms the system of difference equations that describes the economy's competitive equilibrium into a system of static equations that can be easily solved. This system also delivers the vector of initial values for the capital stocks that is consistent with balanced growth. Our generalized balanced growth paths share this attractive feature. These paths and the corresponding vector of initial conditions can be characterized analytically, even though some variables have time-varying growth rates.

Our paper is organized as follows. Section 2 presents the main empirical facts about structural change. Section 3 discusses a simple example that introduces the concept of generalized balanced growth. Section 4 is devoted to our benchmark model. Section 5 studies an extension of this model which allows for time-varying relative prices. A final section provides some concluding thoughts.

2 The Empirical Facts

Until the 18th century agriculture was the most important sector of the economy, employing most of the labor force and accounting for most of the production. Its importance was such that the influential French economist Francois Quesnai theorized that land was the only source of wealth and that agriculture was the only sector that generated profits. In Quesnai's eyes manufacturing, trade, and commerce were all sterile activities.

During the last century the world as a whole experienced rapid rates of economic expansion. But economic development did not reinforce the pivotal role of agriculture. Instead it led to a gradual erosion of its importance. At the same time we witnessed a relentless rise in the importance of the service sector. We will, in a moment, describe the key elements of this process of structural change. But let us start by reviewing the Kaldor facts about economic growth:

The Kaldor Facts

Besides observing that the rate of economic expansion differs widely across countries, Kaldor proposed the following well-known set of empirical regularities:

1 - Per capita output grows at a rate that is roughly constant;

2 - The capital-output ratio is roughly constant;

3 - The real rate of return to capital is roughly constant;

4 - The shares of labor and capital in national income are roughly constant.

These stylized facts, which suggest that several aggregate "great ratios" evolve smoothly over time, have had an enormous impact on the construction of growth models. Are they too stylized to be facts? Figure 1, which depicts the logarithm U.S. per capital real GNP from 1889 to 1989, shows that the growth rate of output is indeed remarkably stable. This visual impression is confirmed by formal statistical tests which do not reject the hypothesis that the mean rate of growth has been the same in the first and in the second parts of the sample (see Stokey and Rebelo (1995)). Figure 2 shows the U.S. capital-output ratio. While this ratio rose during the Great Depression, it has been remarkably stable during the post-war period. Table 1 reports data compiled by Siegel (1995) for the real rate of return on the U.S. stock market. There is remarkably little variation in this real rate of return over different historical periods.[1]

[1] Siegel (1995) also reports the real return on bonds, which was higher in the beginning of the sample than in the end. This time variation may, however, reflect the fact that the bond portfolio used in the first part of the sample comprised riskier securities (municipal and utility bonds), than the Treasury bills featured in the second part of the sample.

Table 1

Evolution of U.S. Real Interest Rate

Period	Real, Geometrically Compounded Return to U.S. Stock
1802-1992	6.7
1871-1992	6.6
1802-1870	7.0
1871-1925	6.6
1926-1992	6.6
1946-1992	6.6

Finally, Figure 3 shows that the variation in the share of labor income in the period 1959-1994 is relatively small. In summary, there is good reason to hold on to the Kaldor facts as a short-hand description of the U.S. growth process. But while these facts suggest that the growth process is smooth, this impression is quickly shattered once we move beyond these aggregate statistics to the simplest level of industry disaggregation.

Structural Change

Table 2 summarizes the main elements of the process of structural change that has taken place in the U.S. and in other growing economies during the last 100 years.

Table 2

The Stylized Facts of Structural Change

	Share of Total Employment	Share of GDP	Share of Total Consumption Expenditures
Agriculture	declines	declines	declines
Manufacturing	stable[2]	stable	stable
Services	increases	increases	increases

We will now illustrate the facts described in this Table through a sequence of Figures. Figure 4 depicts the evolution of U.S. employment shares in Agriculture, Manufacturing, Services, Transportation and Communications and Construction from 1869 to 1970. The decline in agricultural workforce and the rise of the service sector are salient in this figure. Figure 5 shows the evolution of the shares of these different sectors in GDP during the same time period.[3] Once again we see the impressive fall in agriculture and the rise of the service sector. Figure 6 shows the evolution of aggregate consumption expenditure shares. Over time the average U.S. consumer has increased the share of expenditure devoted to services and reduced the share devoted to agricultural products. These dynamics are clearly associated with the rise in per capita income observed over this period. The fact that as income rises there is an increase in the share of expenditures devoted to the consumption of services and a decline in the share devoted to agricultural goods has been well documented in panel data studies of consumption patterns (Houthakker

[2] This description reflects our focus on the last one hundred years. Prior to this period there has been a rise in the importance of manufacturing, see Laitner (1994).

[3] This information in this figure has to be interpreted with some caution. Since there are no consistent deflators for the entire sample, the shares reported are nominal shares.

and Taylor (1970), Bils and Kleenow (1994)). The low income elasticity of the demand for farm products was clearly anticipated by Adam Smith in *The Wealth of Nations*, where he stressed that "The desire of food is limited in every man by the narrow capacity of the human stomach".[4]

Are the sectoral movements documented in Figures 4 and 5 peculiar to the U.S. economy, or are they a general feature of economic development? We use both an historical data set with data for 22 countries and a cross-section of 123 non-socialist countries with data for the period 1970-1989 to answer this question.

The historical data panel comprises all countries for which we were able to obtain sectoral data with a minimum of 50 yearly observations.[5] We pooled the data for the different countries and regressed the shares of employment in agriculture, manufacturing and services against the level of income and the square of income. Since there are several outlier observations in the data set we ran least absolute deviation regressions (LAD) as well as OLS regressions. LAD regressions are robust to the presence of outliers since they estimate conditional medians instead of conditional means. Our estimates are reported in Table 3.

[4] Cited in Fuchs (1968).

[5] The data was obtained from the following sources: Mitchell (1975, 1982, 1983), Maddison (1982) and Liesner (1989). We used the same long term series for real per capita GDP as Easterly and Rebelo (1993). These series were we constructed by using the Summers and Heston (1991) data for the period 1950-1970 and extending it backwards in time using the growth rate of real per capita GDP implied by the historical sources.

Table 3

Estimation Results, Long Run Data

Dependent Variable: Fraction of Labor Allocated to Each Sector

Independent Variables: Per capita Real GDP (Y) and

Per capita Real GDP Squared (Y^2)

Sector	Estimation Technique	Y (t stat)	Y^2 (t stat)	Const. (t stat)	# obs.	Adjusted/ Pseudo R^2
Agric.	OLS	−0.682 (−6.380)	0.027 (4.185)	4.089 (9.244)	444	0.850
Agric.	LAD	−1.067 (−14.850)	0.050 (11.528)	5.694 (19.143)	444	0.667
Manuf.	OLS	1.443 (12.604)	−0.083 (−12.125)	−5.914 (−12.461)	430	0.386
Manuf.	LAD	1.541 (13.722)	−0.090 (−13.350)	−6.250 (−13.437)	430	0.235
Services	OLS	−0.255 (−2.125)	0.028 (3.846)	0.593 (1.192)	437	0.777
Services	LAD	−0.742 (−7.741)	0.057 (9.883)	2.607 (6.569)	437	0.567

Figure 7 shows the resulting regression line. This figure displays the same structural change dynamics that are evident in U.S. data. It is clear that as income rises we observe a decline in the share of agriculture and a rise of the service sector.

Table 4

Estimation Results, Cross Section of 123 Countries, 1970-89

Dependent Variable: Real Output Shares

Independent Variables: Per capita Real GDP (Y)

and Per capita Real GDP Squared (Y^2)

Sector	Estimation Technique	Y (t stat)	Y^2 (t stat)	Const. (t stat)	# of Countries	Adjusted/ Pseudo R^2
Agric.	OLS	−0.342 (−2.630)	0.014 (1.730)	1.994 (3.923)	79	0.755
Agric.	LAD	−0.379 (−3.370)	0.016 (2.300)	2.154 (4.896)	79	0.590
Manuf.	OLS	0.255 (1.650)	−0.014 (−1.447)	−0.963 (−1.570)	63	0.182
Manuf.	LAD	0.211 (1.015)	−0.011 (−0.884)	−0.781 (−0.947)	63	0.152
Services	OLS	−0.003 (−0.018)	−0.004 (0.396)	0.229 (0.357)	77	0.344
Services	LAD	−0.023 (−0.103)	0.005 (0.347)	0.347 (0.407)	77	0.215

To study our 1970-89 data set we averaged the level of income and the shares of the different sectors in real GDP to obtain a single observation per country for each of these variables. Figure 8 displays LAD regression lines for our cross-section of countries, while Table 4 shows the regression coefficients associated with both LAD and OLS estimates. Once again we observe the same sectoral dynamics, with this data suggesting that the importance of manufacturing initially rises with income. Figure 8 provides analogous information for the evolution of the real output shares in the three sectors.

Taken together, both the historical and the cross-section data set confirm the development regularities that we documented for the U.S. Growth in per capita income tends to be accompanied by a rise of services and by a decline in the agricultural sector, both in terms of labor employment and of weight in GDP.

In a well-known book Fuchs (1968) proposed three explanations for the expansion of the service sector: (i) the income elasticity of demand for services is greater than 1; (ii) as income rises it becomes more efficient to contract out services that were once produced in the household or firm; and (iii) productivity growth is slower in the service sector. We have seen evidence of factor (i) in Figure 6. It is difficult to obtain direct evidence on factor (ii). Table 5 (extracted from Jorgenson and Gollop (1992)'s Table 1), reports the average growth rate of total factor productivity for the period 1947-1985, and sheds light on the third factor proposed by Fuchs.

Table 5 is consistent with the view that productivity growth has been high in agriculture and low in the service sector. However, productivity estimates are very sensitive to measurement error in the deflators used to compute real output. These deflators often fail to take into account the changes in the quality of the different goods that occur over time. The quality of farm produce has, arguably, declined. Organic methods have been largely replaced by chemically-based agriculture. Also, the ability to withstand long-haul transportation has replaced taste and aroma as the primary criterion in choosing which varieties to cultivate. In contrast, there has been a substantial rise in the quality of services which is not taken into account in the construction of price deflators.[6]

[6] See Shapiro and Wilcox (1996) for a discussion of the biases introduced by the increases in the quality of medical services. Hornstein and Krusell (1996) review the problems associated with productivity measurement in the service sector.

Table 5

Average Annual Total Factor Productivity Growth, 1947-1985

Agriculture	1.58
Manufacturing	0.72
Transportation	0.96
Communications	2.04
Utilities	0.87
Trade	0.90
Fire	0.24
Other Services	-0.13

An issue closely related to the evolution of sectoral productivity is whether there are secular movements in the relative prices of agriculture and services vis-a-vis manufacturing. It is often asserted that there is a secular decline in the relative price of agricultural goods in terms of manufacturing goods. However, this trend is far from evident in the data for the two countries for which we have long term series for agricultural prices, the U.S. and Canada. This data is depicted in Figures 9 and 10. U.S. data for the relative price of services in terms of both durable and non-durable manufacturing goods is plotted in Figure 11. This Figure suggests that the price of services has risen over time, but problems with quality adjustments mar our confidence in this trend.

3 A Simple Example

There is a theoretical literature that deals with some of the facts about structural adjustment that we just reviewed but ignores the Kaldor facts, in part because it often focuses on a longer time period for which these facts may not apply (e.g. Baumol (1967), Laitner (1994), Echevarria (1995), and Park (1995)). In contrast, most of the growth literature revolves around the Kaldor facts but closes its eyes to the massive structural change that virtually all expanding economies have experienced.

In this paper we will focus on the extent to which the Kaldor facts can be compatible with the presence of structural change. This analysis is relevant to interpret the last one hundred years of U.S. history. As we have seen, there was significant structural change in this period but the Kaldor facts are a good description of the growth process at an aggregate level.

In order to build our intuition about generalized balanced growth, we start with a simple example in which the three sectors of activity–agriculture, industry and services–share production functions that are identical up to a constant of proportionality. In all the models that we consider we abstract from the presence of land and of international trade, to maintain a structure that is as close as possible to that of standard growth models.[7] We cast our models in continuous time but analogous results can be obtained in discrete time.

Production and Accumulation Technology

There are only two factors of production, capital (K_t) and labor. We normalized to one the total amount of labor available in the economy at every point in time. The production structure is as follows:

$$A_t = B_A F(\phi_t^A K_t, N_t^A X_t), \tag{1}$$

[7] For an analysis of structural change in which land plays a key role in the analysis see Goodfriend and McDermott (1995). Kongsamut (1995) finds little evidence that openness to trade influences the behavior of sectoral shares.

$$M_t + \dot{K}_t + \delta K_t = B_M F(\phi_t^M K_t, N_t^M X_t), \tag{2}$$

$$S_t = B_S F(\phi_t^S K_t, N_t^S X_t), \tag{3}$$

$$\phi_t^A + \phi_t^M + \phi_t^S = 1, \tag{4}$$

$$N_t^A + N_t^M + N_t^S = 1, \tag{5}$$

$$\dot{X}_t = X_t g, \tag{6}$$

$$K_0, X_0 > 0, \text{ given.} \tag{7}$$

The function $F(.,.)$ is assumed to be of class C^2, homogenous of degree one, concave, and increasing in both arguments. The variables ϕ^i and N^i denote, respectively, the fraction of capital and labor devoted to sector i. Capital and labor are freely mobile across sectors. The variable X_t denotes the level of technological progress which we assumed to be labor augmenting. Balanced growth paths require technical progress to take this form. Since we seek to generalize the concept of balanced growth, we retained the labor augmenting character of technical progress.

Output of agriculture (A_t) and services (S_t) can be used for consumption, while the output of the manufacturing sector can be consumed (M_t) or invested. This assumption is consistent with the evidence provided by U.S. input-output matrices summarized in Table 6, which suggests that most investment goods are supplied by manufacturing and construction.

Table 6

Fraction of Investment Inputs Accounted for by Different Sectors

	1958	1967	1977	1982	1987
Agriculture	0	0	0	0	0
Mining	0	0	0	0	0
Construction	59	49	48	49	51
Manufacturing	33	43	42	44	40
Services	8	8	10	7	9
Government	0	0	0	0	0

Efficiency in Production

An efficient allocation of capital and labor requires that the marginal rate of transformation be equated across these three sectors, which implies:

$$\frac{\phi_t^A}{N_t^A} = \frac{\phi_t^M}{N_t^M} = \frac{\phi_t^S}{N_t^S} = 1. \tag{8}$$

Since the production functions of the different sectors are proportional, the relative prices of the agriculture and services in terms of manufacturing goods are given by:

$$\begin{aligned} P_A &= B_M/B_A, \\ P_S &= B_M/B_S. \end{aligned}$$

Using these relative prices and the efficiency condition (8) we can rewrite the economy's resource constraint as:

$$M_t + \dot{K}_t + \delta K_t + P_A A_t + P_S S_t = B_M F(K_t, X_t). \tag{9}$$

Preferences

Given our assumptions about the economy's production structure, sectoral movements have to originate in differences in the income elasticity of demand for the different goods. We assume that preferences are time-separable. Our momentary utility specification embeds different income elasticities in a parsimonious way that is familiar from work on the linear expenditure system:

$$U = \int_0^\infty e^{-\rho t} \frac{[(A_t - \bar{A})^\beta M^\gamma (S_t + \bar{S})^\theta]^{1-\sigma} - 1}{1-\sigma} dt. \tag{10}$$

We assume that σ, β, γ, ρ, $\bar{A}$, $\bar{S}$, are all strictly positive and that $\beta + \gamma + \theta = 1$. These preferences imply that the income elasticity of the demand for agricultural goods is less than one, that for manufacturing goods is equal to one, and that for services is greater than one. The variable $\bar{A}$ can be interpreted as the level of subsistence consumption, while one possible interpretation for $\bar{S}$ is that it reflects home production of services.

The Competitive Equilibrium

The competitive equilibrium for this economy coincides with the Pareto Optimum solution. This solution can be characterized by maximizing (10) subject to (9). The economy's real interest rate is given by:

$$r = B_M F_1(k, 1) - \delta, \tag{11}$$

where

$$k = K/X.$$

The optimal allocation of consumption across sectors requires:

$$\frac{P_A(A_t - \bar{A})}{\beta} = \frac{M_t}{\gamma}, \tag{12}$$

$$\frac{P_S(S_t+\bar{S})}{\theta}=\frac{M_t}{\gamma}. \tag{13}$$

The optimal allocation of consumption of manufacturing goods over time is given by:

$$\frac{\dot{M}}{M}=\frac{r-\rho}{\sigma}. \tag{14}$$

A Balanced Growth Path

Suppose, for the moment that $\bar{A}=\bar{S}=0$. It is clear from (9) that the only path along which all variables expand at a constant rate requires that A_t, M_t, S_t and K_t grow at rate g. Equations (11) and (14) determine the steady state value of k:

$$B_M F_1(k,1)=\sigma g+\delta+\rho. \tag{15}$$

This equation has a simple interpretation. The only constant rate of growth that is feasible for this economy to adopt is g. The economy will follow a balanced growth path whenever the value of k is consistent with a real interest rate that leads households to choose to expand their consumption of the three goods at rate g.

A Generalized Balanced Growth Path

Let us now return to the case in which $\bar{A}$ and $\bar{S}$ are strictly positive. In this case a balanced growth path does not exist. Equations (12), (13) and (14) imply that even when the real interest rate is constant households do not choose to expand A and S at constant rates.

We will search for a path along which the real interest rate is constant. We choose this feature of the balanced growth path as our starting point because, unlike some of its other properties (e.g. the constancy of the growth rate of output), it has clear, tractable implications in multi-sector models.

Definition 1 *A Generalized Balanced Growth Path is a trajectory along which the real interest rate is constant.*

Equation (11) implies that k has to be constant in order for the real interest rate to be constant. The economy's resource constraint can be written as:

$$M_t + \dot{K}_t + \delta K_t + P_A A_t + P_S S_t = B_M F(k,1) X_t. \tag{16}$$

The right hand side of this equation expands at rate g. On the left hand side M_t, $\dot{K}_t$, and δK_t grow at rate g, but A_t and S_t do not. It looks as if the requirement that the real interest rate be constant is simply incompatible with the system of differential equations that describes the competitive equilibrium. Suppose, however, that the following restriction holds:

$$\bar{A} B_S = \bar{S} B_A. \tag{17}$$

This implies that $P_S \bar{S} - P_A \bar{A} = 0$, which allows us to re-write the resource constraint as:

$$M_t + \dot{K}_t + \delta K_t + P_A(A_t - \bar{A}) + P_S(S_t + \bar{S}) = B_M F(k,1) X_t. \tag{18}$$

Since both $A_t - \bar{A}$ and $S_t + \bar{S}$ grow at rate g, all the terms in this expression grow at a constant rate. We just proved the following proposition:

Proposition 1 *A Generalized Balanced Growth path exists whenever $\bar{A}B_S = \bar{S}B_A$. Besides having a constant real interest rate, the GBG path for this model features constant relative prices, a constant aggregate labor income* *share, a constant growth rate for capital and aggregate output, and time-varying sectoral growth rates and employment shares in the three sectors. The employment share declines in agriculture, rises in services, and is stable in manufacturing. The initial value of k consistent with the GBG path is given by equation (15).*

The growth rates of output in agriculture and services are given by:

$$\begin{aligned}\frac{\dot{A}_t}{A_t} &= g\frac{A_t - \bar{A}}{A_t},\\ \frac{\dot{S}_t}{S_t} &= g\frac{S_t + \bar{S}}{S_t}.\end{aligned}$$

Using efficiency condition (8) we find that:

$$\begin{aligned}\dot{N}_t^A &= -g\frac{\bar{A}}{B_A X_t F(k,1)},\\ \dot{N}_t^M &= 0,\\ \dot{N}_t^S &= g\frac{\bar{S}}{B_S X_t F(k,1)}.\end{aligned}$$

The share of labor in agriculture declines, while the one in services expands. The rates of change of both factor shares converge to zero in the long run. The two panels of Figure 12, show the evolution of the employment shares and the output share of the three sectors for a numerical example.[8] The fact that the employment shares coincide with the output shares is an implication of the production functions being identical up to a constant of proportionality.

[8] The parameter values used in this example were: $\bar{A} = 400$, $\bar{S} = 250$, $\beta = 0.1$, $\gamma = 0.15$, $\theta = 0.75$, $\sigma = 3$, $\rho = 0.01$, $\delta = 0.05$, $g = 0.018$, $X_0 = 100$, $B_A = 4$, $B_M = 1$, $B_S = 2.5$. The function $F(.)$ was assumed to be Cobb Douglas with a capital share of 0.4.

Transition Dynamics

By analogy to a balanced growth model, this economy will feature some transitional dynamics whenever the initial value of k is incompatible with the GBG path. For this simple economy these transition dynamics are easy to characterize, because the model has the same stability properties as the one sector neoclassical growth model. To see this use equations (12) and (13) to substitute $A_t - \bar{A}$ and $S_t + \bar{S}$ into the utility function and in the resource constraint (9) and re-write the planning problem for this economy as:

$$\max U = \int_0^\infty e^{-\rho t} \frac{M_t^{1-\sigma} - 1}{1-\sigma} dt.,$$

subject to the constraint:

$$M_t + \dot{K}_t + \delta K_t = B_M F(K_t, X_t),$$

and the initial conditions K_0 and X_0.

The GBG restriction

The GBG path exists only when the restriction (17) holds. We are used to the many restrictions that are necessary for models to display balanced growth: labor-augmenting technical progress, isoelastic momentary utility and, in multisector models, cross-parameter restrictions similar to (17), tying together parameters of preferences and technology (see e.g. Evans, Honkapohja and Romer (1996)). The need to assume that (17) holds is a cost that, in our view, is compensated by the fact that the dynamics of structural change can be characterized analytically in a clear and sharp manner. However, it is natural to ask what are the model's implications when (17) does not hold. In the Appendix we discuss the dynamics in the real interest rate that occur when the GBG constraint does not hold and show that the model's trajectory converges to the GBG path.

4 The Basic Model

The model that we just studied is an attractive extension of the one sector neoclassical growth model that displays structural change while preserving all the properties of balanced growth. But is it empirically reasonable to assume that these different sectors share the same production functions? One simple way to answer this question is to compute the labor share of income in different sectors of activity. Figure 13 displays this information for the period 1959-1994. It is clear that the structure of production differs across sectors. For this reason we now generalize our simple model to the case where the three sectors have different production functions.

Production and Accumulation Technology

The production structure is as follows:

$$A = B_A Q(\phi^A K, N^A X), \tag{19}$$

$$\dot{K} + M + \delta K = B_M F(\phi^M K, N^M X), \tag{20}$$

$$S = B_S G(\phi^S K, N^S X). \tag{21}$$

Equations (4)-(7), which include the adding up constraints for labor and capital, the law of motion for X and the initial conditions for K_0 and X_0, complete the specification of the technology.

The functions $Q(.), F(.)$,and $G(.)$ are assumed to be of class C^2, homogenous of degree one, concave, and increasing in both arguments. They are also assumed to satisfy the Inada conditions.

Preferences

The preferences of the representative agent are, as before, given by (10).

The Competitive Equilibrium

The conditions that characterize the competitive equilibrium can be easily derived using familiar optimal control techniques. To economize on space we will omit this derivation and write the resulting system of differential equations in a form that can be easily interpreted.

Efficient consumption decisions continue to be characterized by (12), (13) and (14). Efficiency in the allocation of factors across sectors implies that the marginal unit of capital and labor must be equally productive in all three sectors:

$$\begin{aligned} B_M F_1 &= P_A B_A Q_1, \\ B_M F_1 &= P_S B_S G_1, \\ B_M F_2 &= P_A B_A Q_2, \\ B_M F_2 &= P_S B_S G_2. \end{aligned}$$

Computing ratios of these equations we obtain the familiar requirement that the marginal rate of transformation must be the same in all three sectors:

$$\frac{F_1}{F_2} = \frac{Q_1}{Q_2} = \frac{G_1}{G_2}. \tag{22}$$

Since all production functions are constant returns to scale each of the ratios in the expression above is solely a function of the factor intensity in that industry. For example, F_1/F_2 is solely a function of $\frac{\phi^M K}{N^M X}$. Thus, we can define the functions Λ_A, Λ_M, and Λ_S as:

$$\begin{aligned} \frac{Q_1 K}{Q_2 X} &= \frac{N^A}{\phi^A} \Lambda_A \left(\frac{\phi^A K}{N^A X} \right), \\ \frac{F_1 K}{F_2 X} &= \frac{N^M}{\phi^M} \Lambda_M \left(\frac{\phi^M K}{N^M X} \right), \end{aligned} \tag{23}$$

$$\frac{G_1 K}{G_2 X} = \frac{N^S}{\phi^S}\Lambda_S\left(\frac{\phi^S K}{N^S X}\right).$$

These will be useful in a moment when we characterize the GBG path.

The Generalized Balanced Growth Path

We now follow the same steps taken in the simple model to search for a GBG path. We start by exploring the requirement that the real interest rate must be constant. Just like in the simple model, we will find that a GBG path exists provided that a restriction across parameters of preferences and technology holds.

The real interest rate for this economy is given by:

$$r = B^M F_1\left(\frac{\phi^M K}{N^M X}, 1\right) - \delta.$$

In order for the real interest rate to be constant we need $\frac{\phi^M K}{N^M X}$ to be constant. Equation (22) implies that $\frac{\phi^A K}{N^A X}$ and $\frac{\phi^S K}{N^S X}$ must be constant as well.

It is useful to define the variable q, as follows:

$$q = \frac{F_2 X}{F_1 K}.$$

Using equations (23) we can write q in a form that will be useful later on:

$$q = \frac{1}{\Lambda_M - (\Lambda_M - \Lambda_A)N^A - (\Lambda_M - \Lambda_S)N^S}. \tag{24}$$

We omitted the argument in the functions Λ_A, Λ_M and Λ_S because factor intensities are constant along the GBG path.

We can express F_1 as a function of q:

$$F_1\left(\frac{\phi^M K}{N^M X}, 1\right) = F_1\left(\Lambda_M q\frac{K}{X}, 1\right) \equiv f(\Lambda_M q\frac{K}{X}).$$

The constancy of the real interest rate requires that qK grow at the same rate as X:

$$\frac{\dot{q}}{q} + \frac{\dot{K}}{K} = \frac{\dot{X}}{X} = g. \tag{25}$$

The growth rate of q can be computed using equation (24). The only time-varying variables in the resulting expression for $\dot{q}/q$ are N^A and N^S. We will now work out the law of motion for these two variables. Equation (12) implies that:

$$\frac{\dot{A}}{A - \bar{A}} = g.$$

At the same time, equation (19) implies that along a GBG the growth rate of agricultural production is:

$$\frac{\dot{A}}{A} = \frac{\dot{N}^A}{N^A} + g.$$

These two differential equations can be integrated to obtain the law of motion for N^A:

$$N_t^A = N_0^A + \Gamma_A - \Gamma_A e^{-gt},$$

where:

$$\Gamma_A = -\frac{\bar{A}}{B_A Q\left(\Lambda_A \frac{q_0 K_0}{X_0}, 1\right) X_0} = -\frac{\bar{A}}{B_A Q\left(\Lambda_A f^{-1}(\rho + \delta + \sigma g)/\Lambda_M, 1\right) X_0}.$$

The analogous equations for the service sector are:

$$\frac{\dot{S}}{S - \bar{S}} = g,$$

and

$$\frac{\dot{S}}{S} = \frac{\dot{N}^S}{N^S} + g.$$

Integrating these two equations we obtain the law of motion for N_t^S:

$$N_t^S = N_0^S + \Gamma_S - \Gamma_S e^{-gt},$$

$$\Gamma_S = \frac{\bar{S}}{B_S G\left(\Lambda_S \frac{q_0 K_0}{X_0}, 1\right) X_0} = \frac{\bar{S}}{B_S G\left(\Lambda_S f^{-1}(\rho + \delta + \sigma g)/\Lambda_M, 1\right) X_0}.$$

The path of N_t^M can be computed as:

$$N_t^M = 1 - N_t^A - N_t^S.$$

Now that we characterized the allocation of labor across sectors we turn our attention to the law of motion for the stock of capital. Equation (24) can be used to compute the growth rate of q, while equation (25) can be integrated to yield the following equation for K:

$$\begin{aligned} K &= K_0 e^{gt} \left\{1 - q_0 \left[\Lambda_M - \Lambda_A\right] \Gamma_A - q_0 \left[\Lambda_M - \Lambda_S\right] \Gamma_S\right\} \\ &\quad + q_0 K_0 \left\{\left[\Lambda_M - \Lambda_A\right] \Gamma_A + \left[\Lambda_M - \Lambda_S\right] \Gamma_S\right\}. \end{aligned} \tag{26}$$

The capital accumulation equation (20) can, making use of (23), be rewritten as:

$$\dot{K} = \left[1 - N^A - N^S\right] X \cdot B_M F \left[\Lambda_M \frac{q_0 K_0}{X_0}, 1\right] - M - \delta K. \tag{27}$$

The GBG Restriction

A GBG path exists whenever (26), which embodies the law of motion for N^A and N^S, is compatible with (27). Since in equation (27) $\dot{K}$, X and M grow at the rate g this requires that:[9]

[9] Note that when the production functions are identical ($Q = F = G$) $\Lambda_A = \Lambda_M = \Lambda_S$ and (28) reduces to $\Gamma_A + \Gamma_S = 0$. This in turn implies that $\bar{A} B_S = \bar{S} B_A$, which is the GBG path restriction for the simple model.

$$(\Gamma_A + \Gamma_S) \cdot B_M F\left[\Lambda_M \frac{q_0 K_0}{X_0}, 1\right] = \delta \frac{q_0 K_0}{X_0}\left\{[\Lambda_M - \Lambda_A]\Gamma_A + [\Lambda_M - \Lambda_S]\Gamma_S\right\}.$$

Or, in terms of deep parameters:

$$\begin{aligned} & (\Gamma_A + \Gamma_S) \cdot B_M F\left(\Lambda_M f^{-1}(\rho + \delta + \sigma g), 1\right) \\ = \; & \delta f^{-1}(\rho + \delta + \sigma g)\left\{[\Lambda_M - \Lambda_A]\Gamma_A + [\Lambda_M - \Lambda_S]\Gamma_S\right\}. \end{aligned} \tag{28}$$

Our final task is to find the initial condition for k, that is, the value of K/X compatible with GBG path for a given X_0. The following system of equations allows us to jointly determine K_0, M_0, q_0, N_0^A, N_0^S :[10]

$$\frac{1}{q_0} = \Lambda_M - [\Lambda_M - \Lambda_A]\, N_0^A - [\Lambda_M - \Lambda_S]\, N_0^S, \tag{29}$$

$$q_0 K_0 = X_0 \frac{f^{-1}(\rho + \delta + \sigma g)}{\Lambda_M}, \tag{30}$$

$$\begin{aligned} M_0 \;=\; & (1 - N_0^A - N_0^S) X_0 B_M F\left(f^{-1}(\rho + \delta + \sigma g),\, 1\right) - \delta K_0 \\ & - g K_0 \left\{1 - q_0 [\Lambda_M - \Lambda_A]\Gamma_A - q_0 [\Lambda_M - \Lambda_S]\Gamma_S\right\}, \end{aligned} \tag{31}$$

$$N_0^A X_0 B_A Q\left(\frac{\Lambda_A}{\Lambda_M} f^{-1}(\rho + \delta + \sigma g), 1\right) = \left[\frac{B_A Q_1\left(\frac{\Lambda_A}{\Lambda_M} f^{-1}(\rho + \delta + \sigma g), 1\right)}{\rho + \delta + \sigma g}\right] \frac{\beta M_0}{\gamma} + \bar{A}, \tag{32}$$

$$N_0^S X_0 B_S G\left(\frac{\Lambda_S}{\Lambda_M} f^{-1}(\rho + \delta + \sigma g), 1\right) = \left[\frac{B_S G_1\left(\frac{\Lambda_S}{\Lambda_M} f^{-1}(\rho + \delta + \sigma g), 1\right)}{\rho + \delta + \sigma g}\right] \frac{\theta M_0}{\gamma} - \bar{S}. \tag{33}$$

This system of equations can be easily solved. First of all, equation (32) and (33) can be used to write N_0^A and N_0^S as functions of M_0 alone. Then

[10] Equation (32) results from the fact that: $A_0 - \bar{A} = \frac{1}{P_A}\frac{\beta M_0}{\gamma} = \frac{B_A}{B_M}\frac{Q_1}{F_1}\frac{\beta M_0}{\gamma} = \frac{B_A Q_1}{\rho + \delta + \sigma g}\frac{\beta M_0}{\gamma}$.

equation (29) can be used to write q_0 as a function of M_0. Therefore equation (30) says K_0 can be rewritten as a function of M_0. Finally, using equation (31) we can solve first for M_0 and then for all the remaining variables.

All these results can be summarized in the following proposition.

Proposition 2 *This economy follows a Generalized Balanced Growth Path whenever the cross-parameter restriction (28) holds and the initial value of the capital stock conforms with (29)-(32). Besides displaying a constant real interest rate this path features constant relative prices. The shares of labor allocated to the difference sectors vary over time. Employment in agriculture contracts while the share of labor in services expands.*

Notice that the growth rate of output and the capital labor ratio, which were constant in our simple model, now vary along the GBG path. However, for plausible parameterizations both of these variables are approximately constant. Figure 14 depicts the evolution of the rate of growth, the capital-output ratio, the consumption-output ratio and the shares of employment in the three sectors for a simple numerical example.[11]

Unfortunately, the transition dynamics of this model cannot be easily characterized analytically. The main difficulty is in the fact that whenever the cross parameter restriction (28) does not hold, the system of differential equations becomes non-autonomous.

[11]The parameter values that underlie this example are: $\bar{A} = 400$, $\bar{S} = 200$, $\beta = 0.1$, $\gamma = 0.15$, $\theta = 0.75$, $\sigma = 3$, $\rho = 0.01$, $\delta = 0.05$, $g = 0.018$, $X_0 = 100$, $B_M = 1$, $B_S = 2.5$. The value of B_A was set equal to 2.73 so that equation (28) holds. The three production functions were assumed to be Cobb-Douglas with capital shares: $\alpha_A = 0.5$, $\alpha_M = 0.4$, and $\alpha_S = 0.3$.

5 Introducing Relative Price Movements

The model that we just studied features constant relative prices. Even though there is uncertainty about whether there has been substantial movements in relative prices during the past century, it is worthwhile to study whether GBG paths can encompass changing relative prices. To explore this we amend our benchmark model to include sector-specific rates of technical progress. In the basic model that we just reviewed all the sectors grow at a constant rate in the long run, once all the sectoral dynamics have been played out. To preserve this property in the presence of sector-specific rates of technical progress we need to assume that the production functions are Cobb-Douglas. Our production structure is thus:

$$A = B_A(\phi^A K)^{\alpha_A}(N^A X_A)^{1-\alpha_A}, \tag{34}$$

$$\dot{K} + M + \delta K = B_M(\phi^M K)^{\alpha_M}(N^M X_M)^{1-\alpha_M}, \tag{35}$$

$$S = B_S\left[\phi_S K\right]^{\alpha_S}\left[N^S X_S\right]^{1-\alpha_S}, \tag{36}$$

$$\begin{aligned}\frac{\dot{X}^i}{X^i} &= g_i, \quad i = A, M, S,\\ X_0^i &> 0, \quad i = A, M, S.\end{aligned}$$

Efficiency in the allocation of capital and labor across the three sectors requires:

$$\alpha_M B_M\left(\frac{\phi^M K}{N^M X_M}\right)^{\alpha_M-1} = P_i\alpha_i B_i\left(\frac{\phi^i K}{N^i X_i}\right)^{\alpha_i-1} \qquad i = A, S, \tag{37}$$

$$(1-\alpha_M)B_M\left(\frac{\phi^M K}{N^M X_M}\right)^{\alpha_M} X_M = P_i(1-\alpha_i)B_i\left(\frac{\phi^i K}{N^i X_i}\right)^{\alpha_i} X_i \qquad i = A, S. \tag{38}$$

These conditions readily imply that the marginal rate of transformation must be identical across sectors. This requirement is expressed in the following equation, which also defines q:

$$\frac{\alpha_M}{(1-\alpha_M)}\frac{N^M}{\phi^M} = \frac{\alpha_A}{(1-\alpha_A)}\frac{N^A}{\phi_A} = \frac{\alpha_S}{(1-\alpha_S)}\frac{N^S}{\phi_S} \equiv \frac{1}{q}. \tag{39}$$

We can solve ϕ_A and ϕ_S in the equation above as a function of q to obtain:

$$\phi_A = \frac{\alpha_A}{(1-\alpha_A)}N^A q, \tag{40}$$

$$\phi_S = \frac{\alpha_S}{(1-\alpha_S)}N^S q. \tag{41}$$

This, in turn, implies:

$$\frac{\alpha_M}{(1-\alpha_M)} - \left[\frac{\alpha_M}{(1-\alpha_M)} - \frac{\alpha_A}{(1-\alpha_A)}\right]N^A - \left[\frac{\alpha_M}{(1-\alpha_M)} - \frac{\alpha_S}{(1-\alpha_S)}\right]N^S = \frac{1}{q}. \tag{42}$$

The real interest rate associated with lending one unit of the manufacturing good and being repaid in manufacturing goods one period later is given by:[12]

$$r = \alpha_M B_M\left(\frac{\phi^M K}{N^M X_M}\right)^{\alpha_M - 1} - \delta.$$

Just as before, the requirement that the real interest rate must be constant implies the constancy of $(\phi^M K)/(N^M X^M)$. Expressed in terms of q the real interest rate is given by:

[12]Since relative prices vary over time this real interest rate is different from one that applies to, e.g. borrowing one unit of the agricultural good and repaying the loan one period later also in agricultural goods.

$$r = \alpha_M B_M \left(\frac{\alpha_M q}{(1-\alpha_M)} \frac{K}{X_M} \right)^{\alpha_M - 1} - \delta.$$

Thus, a constant r implies that:

$$\frac{\dot{q}}{q} + \frac{\dot{K}}{K} = g_M. \tag{43}$$

Given that r is constant, equations (37) and (39) imply that the rate of change of the price of agricultural goods is given by:

$$\frac{\dot{P}_A}{P_A} = (1-\alpha_A)(g_M - g_A).$$

The growth rate of the price of services can also be computed using (37) and (39):

$$\frac{\dot{P}_S}{P_S} = (1-\alpha_S)(g_M - g_S).$$

In order for P_S to rise and P_A to fall, which is the pattern suggested by the data, we need to assume that:

$$g_A > g_M > g_S.$$

This ranking of rates of technical progress is in accordance with the evidence described in Table 5.

The efficient allocation of consumption across goods is characterized by (12), (13) and (14) just as in our previous models. Equation (12) implies that

$$\frac{\dot{A}}{A - \bar{A}} = \alpha_A g_M + (1-\alpha_A) g_A.$$

At the same time two equations from the production side of the model, (34) and (39) allow us to compute the growth rate of A:

$$\frac{\dot{A}}{A} = \frac{\dot{N}^A}{N^A} + \alpha_A g_M + (1-\alpha_A) g_A.$$

Thus, we have

$$\begin{aligned}\frac{\dot{N}^A}{N^A} &= -\frac{\bar{A}}{A}\left[\alpha_A g_M + (1-\alpha_A)g_A\right] \\ &= -\frac{\bar{A}}{B_A\left[\frac{\alpha_A}{(1-\alpha_A)}\frac{qK}{X_A}\right]^{\alpha_A} N^A X_A}\left[\alpha_A g_M + (1-\alpha_A)g_A\right].\end{aligned}$$

Let us write

$$\dot{N}^A = \Gamma_A\left[\alpha_A g_M + (1-\alpha_A)g_A\right]e^{-[\alpha_A g_M+(1-\alpha_A)g_A]t},$$

with

$$\Gamma_A = -\frac{\bar{A}}{B_A\left[\frac{\alpha_A}{(1-\alpha_A)}\frac{q_0 K_0}{X_{A0}}\right]^{\alpha_A} X_{A0}}.$$

The solution to this equation is:

$$N^A = N_0^A - \Gamma_A(e^{-[\alpha_A g_M+(1-\alpha_A)g_A]t} - 1). \tag{44}$$

Similarly, the law of motion for N_S is:

$$N^S = N_0^S - \Gamma_S(e^{-[\alpha_S g_M+(1-\alpha_S)g_S]t} - 1). \tag{45}$$

Equation (43) can be solved to obtain K:

$$K_t = K_0 q_0 e^{g_M t}/q_t.$$

Using (42), (44) and (45) we obtain:

$$\begin{aligned}K_t &= K_0 e^{g_M t}(1 - q_0\mu_A\Gamma_A - q_0\mu_S\Gamma_S) + \\ &\quad q_0 K_0\left\{\mu_A\Gamma_A e^{-[(1-\alpha_A)(g_A-g_M)]t} + \mu_S\Gamma_S e^{-[(1-\alpha_S)(g_S-g_M)]t}\right\}, \end{aligned}\tag{46}$$

where:

$$\mu_i = \frac{\alpha_M}{(1-\alpha_M)} - \frac{\alpha_i}{(1-\alpha_i)}, \qquad i = A, S.$$

Equation (46) characterizes the optimal path for the stock of capital whenever the real interest rate (denominated in terms of manufacturing goods) is constant. The two issues that remain to be settled are: (i) what is the analogue of the cross parameter restriction (28); and (ii) what is the value of K_0 that is consistent with the GBG path. Both of these problems are solved in the Appendix. The properties of this economy can be summarized as follows:

Proposition 3 *This economy follows a Generalized Balanced Growth path whenever the restrictions across parameters (48)-(49) and initial condition for the capital stock defined by (50)-(54) in the Appendix hold. Along the GBG path the share of employment in agriculture declines and the share of employment in services expands. At the same time, the relative price in terms of manufacturing goods falls for agriculture and rises for services.*

Note that different rates of technical progress across sectors are necessary to produce movements in the relative prices of the three goods. But by themselves they cannot generate structural change: when $\bar{A}$, $\bar{S} = 0$ the economy has a standard balanced growth path which features no sectoral reallocations. Hence, it is essential to have different elasticities of demand ($\bar{A}$, $\bar{S} > 0$) in order to generate movements in the shares of labor employed by the different sectors.

6 Conclusions

This paper described several models that display generalized balanced growth paths, which are trajectories along which the real interest rate remains constant. In the examples that we discussed these paths can be characterized analytically and preserve many of the features of balanced growth. At the same time they are consistent with the dynamics of structural change that are intimately linked to the development process. We focused on the dramatic sectoral dynamics that have taken place in agriculture, manufacturing and services. But these dynamics are also present within the manufacturing and service sector. As macroeconomists venture into more disaggregated models of the economy these dynamics will be difficult to ignore. The concept of generalized balanced growth may prove to be a very useful tool in this context.

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7 Appendix

7.1 Model of Section 3: What happens if the GBG constraint does not hold?

In this appendix, we use a phase diagram with moving isoclines to illustrate the dynamics in the simple model when $\varepsilon = B_M(\bar{S}/B_S - \bar{A}/B_A) \neq 0$. Without loss of generality, let us assume that the constant ε is less than zero.

Because the three sectors have similar production functions, we can aggregate these sectors so that the competitive equilibrium is the outcome of the following optimization problem:

$$\max \int_0^\infty \frac{M^{1-\sigma}}{1-\sigma} e^{-\rho t} dt,$$

$$\text{subject to: } \dot{K} = B_M K^\alpha X^{1-\alpha} - \frac{1}{\gamma} M - \delta K + \varepsilon. \tag{47}$$

where X grows at exogenous rate of g.

Let λ be the co-state variable associated with K. The first order conditions are:

$$M^{-\sigma} = \frac{\lambda}{\gamma},$$

$$\dot{\lambda} = \rho\lambda - \lambda\left(\alpha B_M K^{\alpha-1} X^{1-\alpha} - \delta\right).$$

The transversality condition is $\lim_{t\to\infty} K\lambda e^{-\rho t} = 0$. Define the following transformed variables:

$$\tilde{K} = K/X,$$

$$\tilde{\lambda} = \lambda X^\sigma.$$

We can rewrite the first order conditions in terms of $\tilde{K}$ and $\tilde{\lambda}$ as follows:

$$\frac{d\tilde{K}}{dt} = B_M \tilde{K}^\alpha - \frac{1}{\gamma}\left[\frac{\tilde{\lambda}}{\gamma}\right]^{-1/\sigma} - \delta\tilde{K} + \frac{\varepsilon}{X},$$

$$\frac{d\tilde{\lambda}}{dt} = \tilde{\lambda}\left[\rho + \delta + \sigma g - \alpha B_M \tilde{K}^{\alpha-1}\right].$$

We cannot employ a standard phase diagram because X is time dependent and hence the system of differential equations is not autonomous. We can, however, use a phase diagram with moving isoclines to describe the dynamics because X depends on time in an explicit fashion, namely $X = X_0 e^{gt}$.

As an example, we use a phase diagram with moving isoclines to depict the equilibrium trajectory when $K_0/X_0 = \left[(\rho + \delta + \sigma g) / (\alpha B_M)\right]^{1/(\alpha-1)}$.

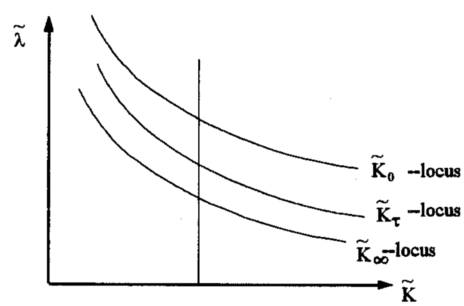


Figure 15: Time varying loci

In Figure 15, the vertical line represents the equation $d\tilde{\lambda}/dt = 0$. The downward sloping curve on the top is the locus along which $d\tilde{K}/dt = 0$ when $t = 0$. The curve in the middle represents the same locus when $t = \tau$ (we will abuse language and call it the $\tilde{K}_\tau$—locus). The curve at the bottom represents the points along which $\tilde{K}$ is constant when $t = \infty$.

At any time τ the following phase diagram indicates the directions of movement for $\tilde{K}$ and $\tilde{\lambda}$.

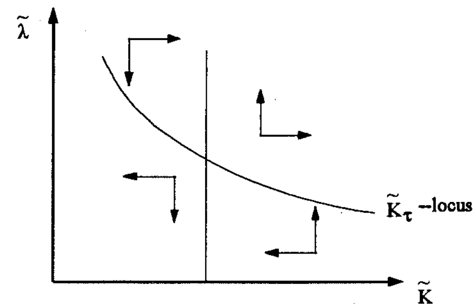


Figure 16: Directions for Immediate Movements at time τ

With the two Figures above understood, we can now draw three representative paths. The trajectory in the middle is the equilibrium trajectory.

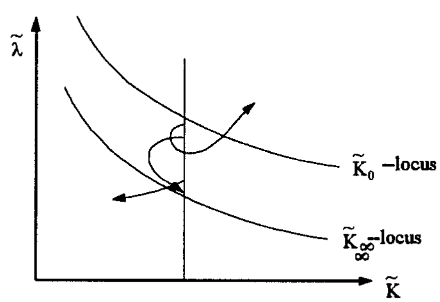


Figure 17: Representative Trajectories

To understand the top trajectory, note that the initial $(\tilde{K},\ \tilde{\lambda})$ is below the $\tilde{K}_0$—locus and therefore the immediate direction of movement is toward the left. But since $(\tilde{K},\ \tilde{\lambda})_0$ is close to both the $\tilde{K}_0$—locus and the vertical line, the magnitude of the movement is small. After a short while, say at $t = \tau$, $(\tilde{K},\ \tilde{\lambda})_\tau$ will be above $\tilde{K}_\tau$—locus and therefore the trajectory moves down to the right. This movement continues until the path hits the vertical line. After this point the trajectory moves up to the right.

To understand the bottom trajectory, note that the initial $(\tilde{K}, \tilde{\lambda})$ is far below the $\tilde{K}_0$—locus and therefore the leftward movement is strong at the beginning. In fact, the movement is so strong that $(\tilde{K}, \tilde{\lambda})_t$ lies under $\tilde{K}_t$—locus for all t. Once the trajectory passes $\tilde{K}_\infty$—locus, there is no turning back.

The equilibrium locus is the unique path which starts at an appropriate $(\tilde{K}, \tilde{\lambda})$ so that as time elapses, $(\tilde{K}, \tilde{\lambda})_t$ comes to be above and stays above the $\tilde{K}_t$—locus and in the meantime $(\tilde{K}, \tilde{\lambda})_t$ stays to the left of the vertical line.

To see the property of the equilibrium path with $\varepsilon < 0$, note that K_t/X_t first declines and then increases and approaches the steady state level. Therefore, the real interest rate increases first and then comes back to the original level. The opposite is true for $\varepsilon > 0$.

7.2 Model of Section 5: the GBG constraint and the initial condition for K

The GBG restriction is the cross-parameter relation that makes (46) compatible with the capital accumulation equation:

$$\dot{K} + \delta K = (1 - N^A - N^S) X_M B_M F\left(\frac{(1-\phi_A-\phi_S)K}{(1-N^A-N^S)X_M}, 1\right) - M.$$

Note that X_M and M grow at the rate g_M. The term $\dot{K} + \delta K$ has one component which grows at g_M, and two other components, one which grows at rate $-(1-\alpha_A)(g_A - g_M)$, and other which grows at rate $-(1-\alpha_S)(g_S - g_M)$. Also, note from the solutions to N^A and N^S, that $(N^A + N^S)X_M B_M F\left(\frac{(1-\phi_A-\phi_S)K}{(1-N^A-N^S)X_M}, 1\right)$ has one component that grows at g_M, and two other components one of which grows at rate $-(1-\alpha_A)(g_A - g_M)$, and the other at rate $-(1-\alpha_S)(g_S-g_M)$. Therefore, in order for the capital accumulation equation to hold, we need

$$\begin{aligned} x &= -\left[\Gamma_A e^{-[\alpha_A g_M+(1-\alpha_A)g_A]t}+\Gamma_S e^{-[\alpha_S\, g_M+(1-\alpha_S)g_S]t}\right] X_M B_M F\left(\frac{(1-\phi_A-\phi_S)K}{(1-N^A-N^S)X_M},1\right) \\ &\quad +q_0K_0\left[\delta-(1-\alpha_A)(g_A-g_M)\right]\mu_A\Gamma_A e^{-(1-\alpha_A)(g_A-g_M)t} \\ &\quad +q_0K_0\left[\delta-(1-\alpha_S)(g_S-g_M)\right]\mu_S\Gamma_S e^{-(1-\alpha_S)(g_S-g_M)t}. \end{aligned}$$

to grow at rate g_M. This requires,

$$X_0^M B_M F\left(\frac{\alpha_M}{(1-\alpha_M)}\frac{q_0K_0}{X_0^M},1\right)=\mu_A q_0K_0\left[\delta-(1-\alpha_A)(g_A-g_M)\right],$$

and

$$X_0^M B_M F\left(\frac{\alpha_M}{(1-\alpha_M)}\frac{q_0K_0}{X_0^M},1\right)=\mu_S q_0K_0\left[\delta-(1-\alpha_S)(g_S-g_M)\right].$$

Note that,

$$\begin{aligned} r &= B_M F_1\left(\frac{\alpha_M}{(1-\alpha_M)}\frac{q_0K_0}{X_0^M},1\right) \\ &= \rho+\delta+\sigma g_M+\beta(1-\sigma)(1-\alpha_A)(g_M-g_A)-\theta(1-\sigma)(1-\alpha_S)(g_M-g_S). \end{aligned}$$

We know that

$$B_M F_1\left(\frac{\alpha_M}{(1-\alpha_M)}\frac{q_0K_0}{X_0^M},1\right)=\alpha_M\frac{B_M F\left(\frac{\alpha_M}{(1-\alpha_M)}\frac{q_0K_0}{X_0^M},1\right)}{\frac{\alpha_M}{(1-\alpha_M)}\frac{q_0K_0}{X_0^M}}.$$

Thus,

$$\begin{aligned} \frac{B_M F\left(\frac{\alpha_M}{(1-\alpha_M)}\frac{q_0K_0}{X_0^M},1\right)}{\frac{q_0K_0}{X_0^M}} &= \frac{1}{(1-\alpha_M)}r \\ &= \frac{1}{(1-\alpha_M)}\left[\begin{array}{l} \rho+\delta+\sigma g_M \\ +\beta(1-\sigma)(1-\alpha_A)(g_M-g_A) \\ -\theta(1-\sigma)(1-\alpha_S)(g_M-g_S) \end{array}\right]. \end{aligned}$$

Therefore, we need the following restrictions on parameters to hold;

$$\frac{1}{(1-\alpha_M)}\begin{bmatrix} \rho+\delta+\sigma g_M \\ +\beta(1-\sigma)(1-\alpha_A)(g_M-g_A) \\ -\theta(1-\sigma)(1-\alpha_S)(g_M-g_S) \end{bmatrix} = [\delta-(1-\alpha_A)(g_A-g_M)]\left[\frac{\alpha_M}{(1-\alpha_M)}-\frac{\alpha_A}{(1-\alpha_A)}\right. \tag{48}$$

$$\frac{1}{(1-\alpha_M)}\begin{bmatrix} \rho+\delta+\sigma g_M \\ +\beta(1-\sigma)(1-\alpha_A)(g_M-g_A) \\ -\theta(1-\sigma)(1-\alpha_S)(g_M-g_S) \end{bmatrix} = [\delta-(1-\alpha_S)(g_S-g_M)]\left[\frac{\alpha_M}{(1-\alpha_M)}-\frac{\alpha_S}{(1-\alpha_S)}\right] \tag{49}$$

When these restrictions hold we can determine the initial value of K that will put the economy on a GBG path given the values of X_0^M, X_0^A and X_0^M. The following are the equations that will jointly determine K_0, M_0, q_0, N_0^A and N_0^S:

$$\frac{\alpha_M}{(1-\alpha_M)}-\left[\frac{\alpha_M}{(1-\alpha_M)}-\frac{\alpha_A}{(1-\alpha_A)}\right]N_0^A-\left[\frac{\alpha_M}{(1-\alpha_M)}-\frac{\alpha_S}{(1-\alpha_S)}\right]N_0^S=\frac{1}{q_0}, \tag{50}$$

$$\begin{aligned} B_M F_1\left(\frac{\alpha_M}{(1-\alpha_M)}\frac{q_0K_0}{X_0^M},1\right) &= \rho+\delta+\sigma g_M \\ &\quad +\beta(1-\sigma)(1-\alpha_A)(g_M-g_A) \\ &\quad -\theta(1-\sigma)(1-\alpha_S)(g_M-g_S). \end{aligned} \tag{51}$$

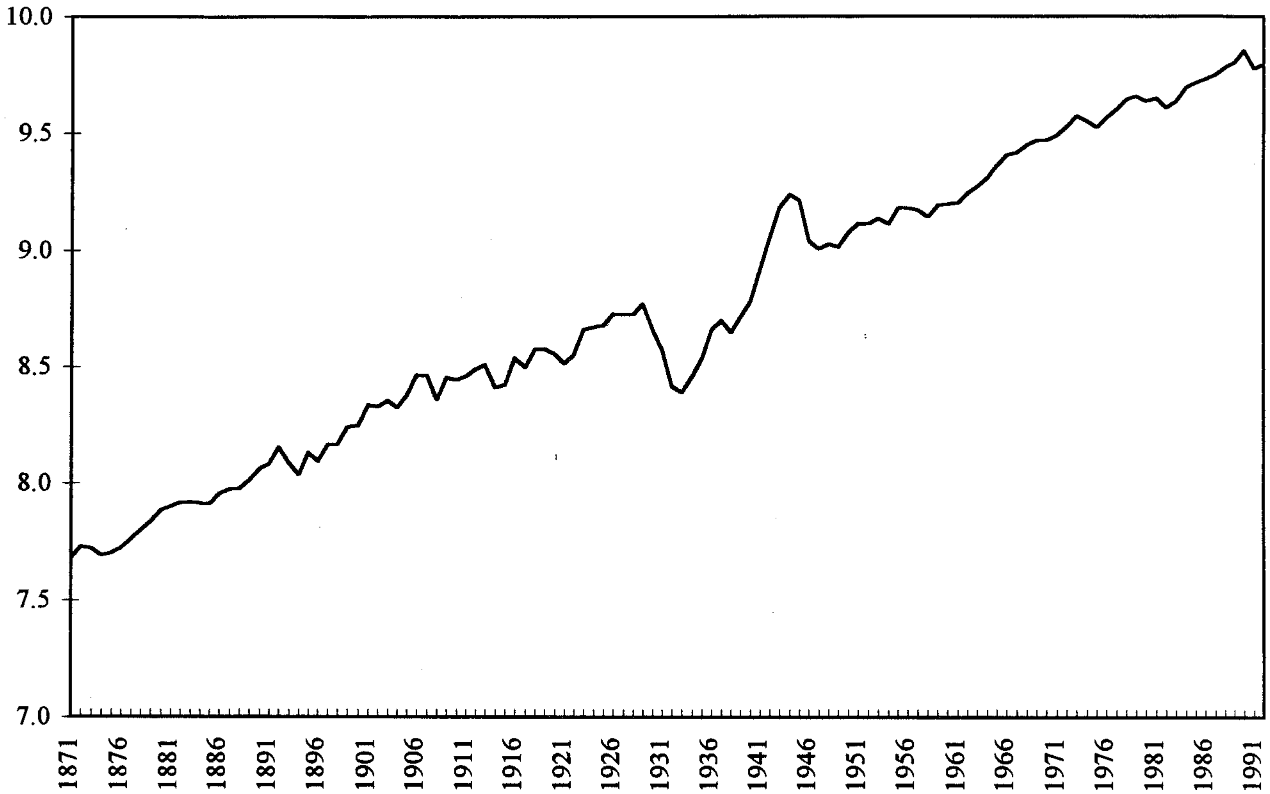
$$\begin{aligned} M_0 &= (1-N_0^A-N_0^S)X_0^M B_M F\left(\frac{\alpha_M}{(1-\alpha_M)}\frac{q_0K_0}{X_0^M},1\right)-\delta K_0 \\ &\quad -K_0\left\{\begin{array}{l} g_M-q_0\left[\frac{\alpha_M}{(1-\alpha_M)}-\frac{\alpha_A}{(1-\alpha_A)}\right]\Gamma_A\left[\alpha_A g_M+(1-\alpha_A)g_A\right] \\ -q_0\left[\frac{\alpha_M}{(1-\alpha_M)}-\frac{\alpha_S}{(1-\alpha_S)}\right]\Gamma_S\left[\alpha_S g_M+(1-\alpha_S)g_S\right] \end{array}\right\} \end{aligned} \tag{52}$$

$$N_0^A X_0^A B_A Q\left(\frac{\alpha_A}{(1-\alpha_A)}\frac{q_0K_0}{X_0^A},1\right)=\left[\frac{B_A Q_1\left(\frac{\alpha_A}{(1-\alpha_A)}\frac{1-\alpha_M}{\alpha_M}f^{-1}(\rho+\delta+\sigma g_M),1\right)}{\rho+\delta+\sigma g_M}\right]\frac{\beta M_0}{\gamma}+\bar{A}, \tag{53}$$

$$N_0^S X_0^S B_S G\left(\frac{\alpha_S}{(1-\alpha_S)}\frac{q_0 K_0}{X_0^S}, 1\right) = \left[\frac{B_S G_1\left(\frac{\alpha_S}{(1-\alpha_S)}\frac{1-\alpha_M}{\alpha_M} f^{-1}(\rho+\delta+\sigma g_M), 1\right)}{\rho+\delta+\sigma g_M}\right]\frac{\theta M_0}{\gamma} - \bar{S}. \tag{54}$$

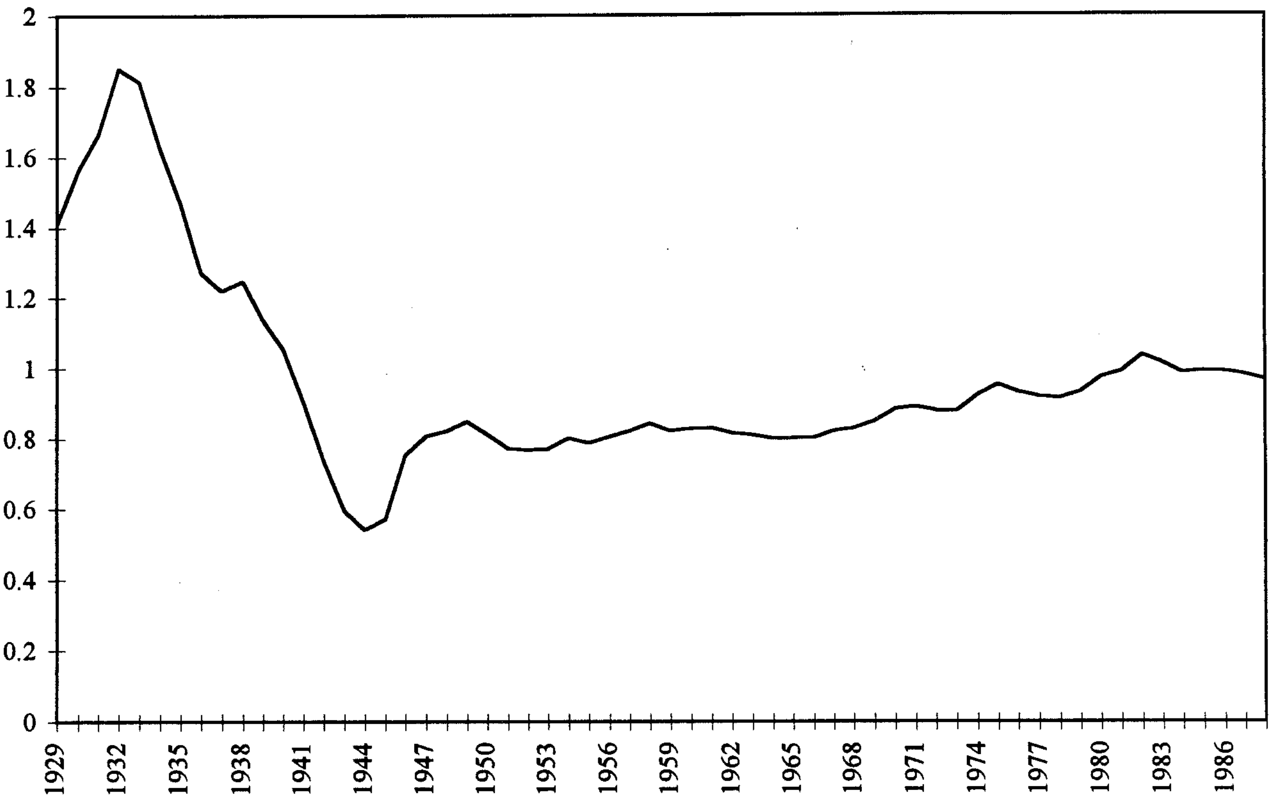
These equations can be solved easily. First of all, equation (53) and (54) can be used to write N_0^A and N_0^S as functions of M_0 alone. Then equation (50) can be used to write q_0 as a function of M_0. Therefore equation (51) says K_0 can be rewritten as a function of M_0. Finally, substituting all these functions into equation (52) we can solve for M_0 and for all the remaining variables.

Figure 1
Logarithm of U.S. per capita real GDP



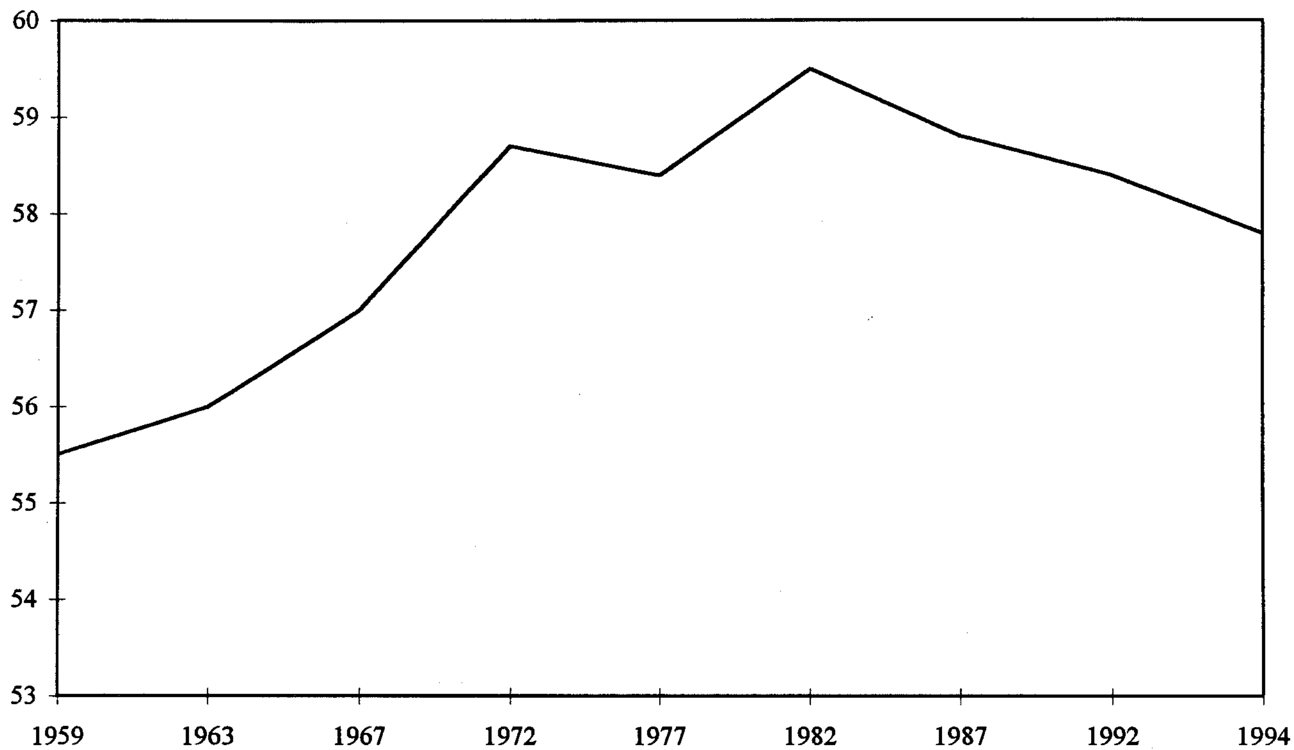
Sources: Penn World Tables, Easterly and Rebelo (1993).

Figure 2
U.S. Capital-Output Ratio



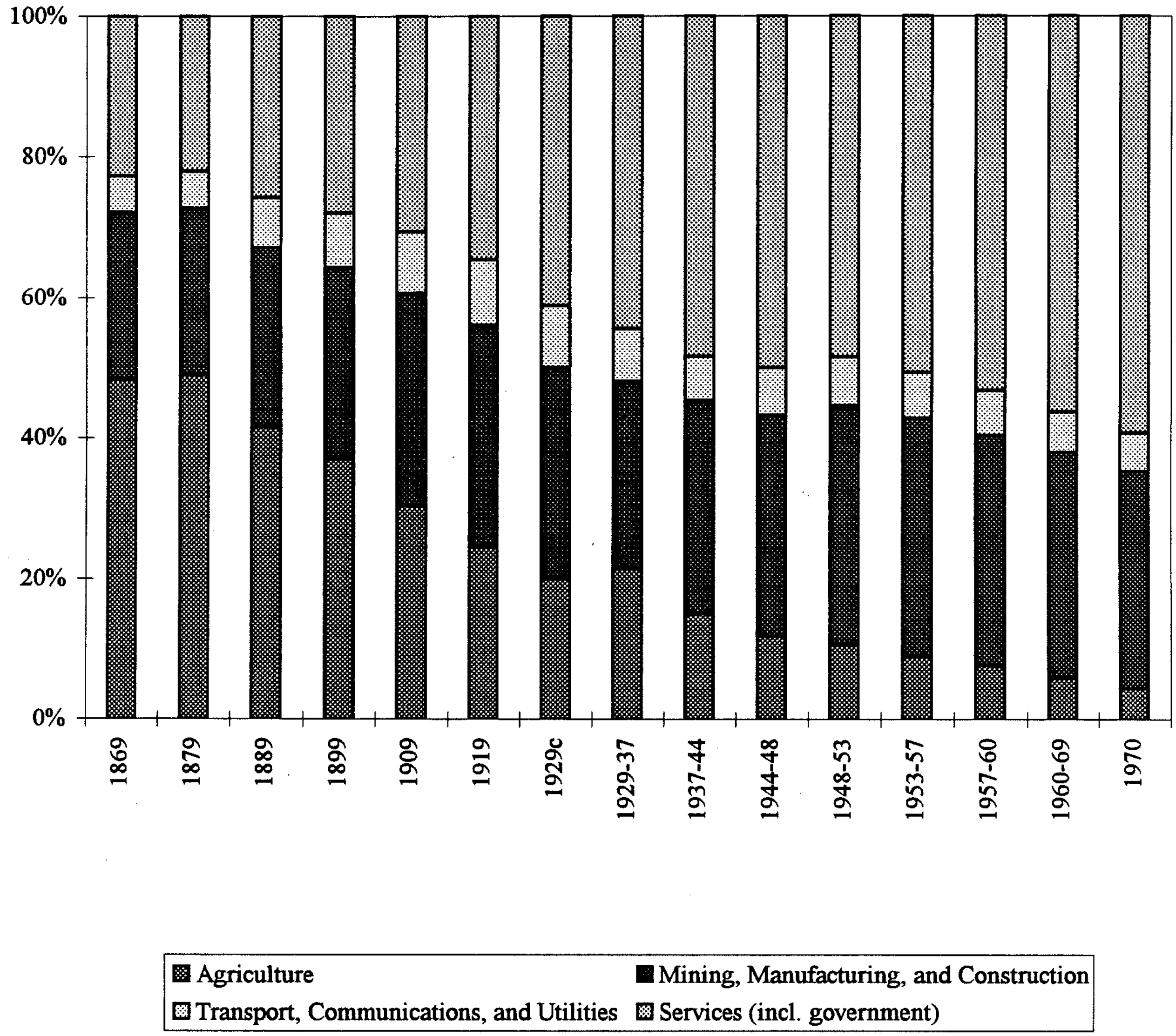
Sources: Fixed Reproducible Tangible Wealth in the U.S., 1925-1989; NIPA,1929-1988.

Figure 3
U.S. Labor Income Share



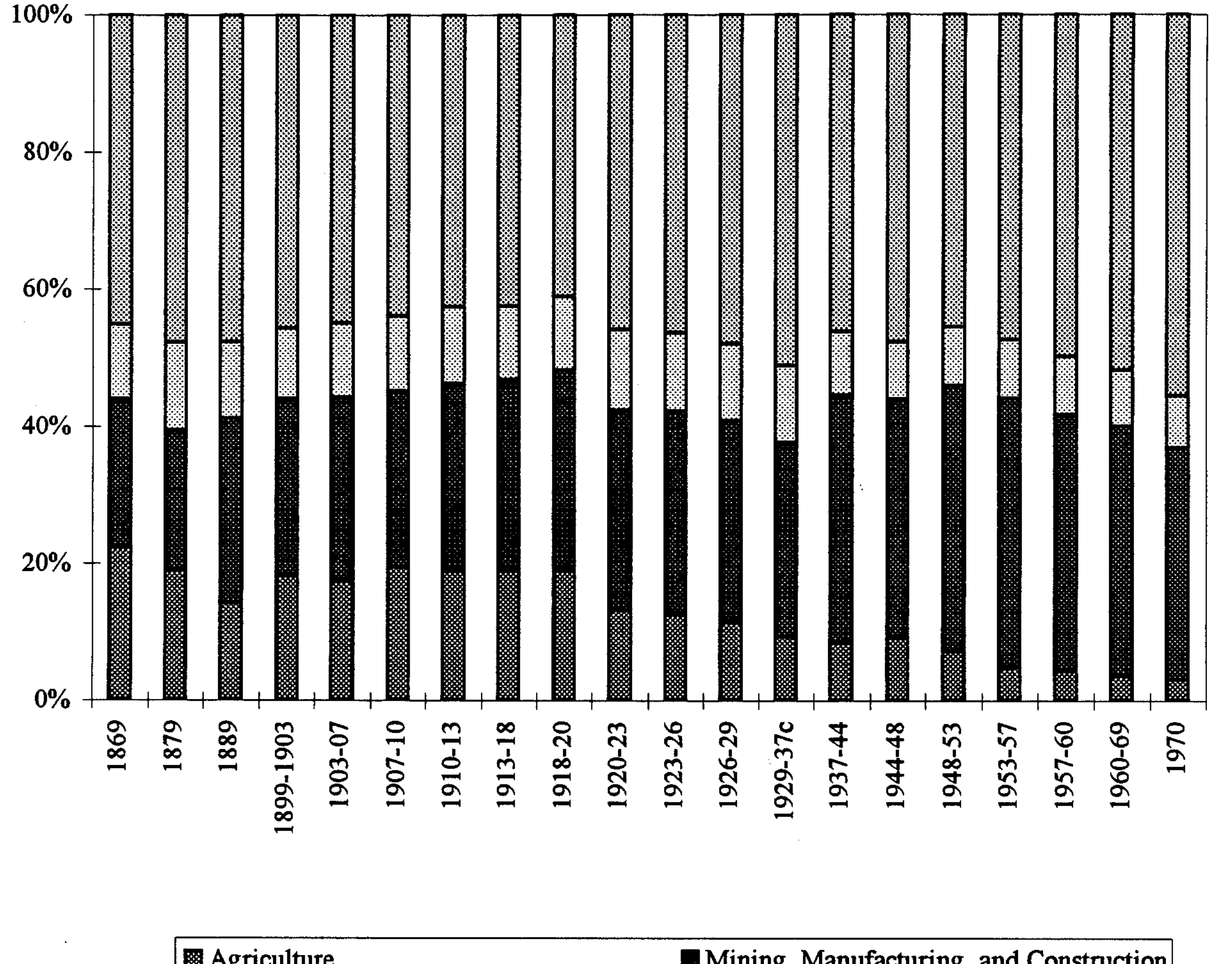
Source: Survey of Current Business, August 1996.

Figure 4
U.S. Employment share by sector



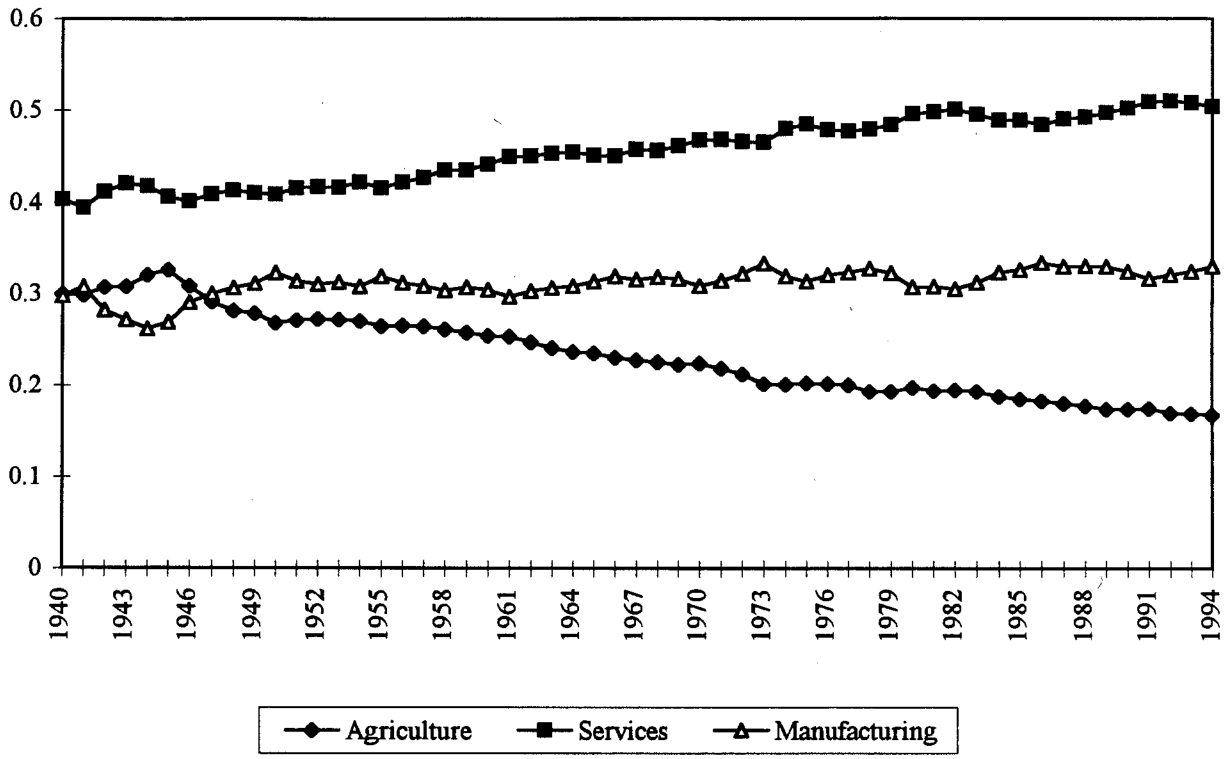
Source: Historical Statistics of the United States from Colonial Times.

Figure 5
U.S. Sectoral National Income Shares



Source: Historical Statistics of the United States from Colonial Times.

Figure 6
U.S. Consumption Share by Sector



Source: Economic Report of the President, various years.

Figure 7
LAD Regression Line, Employment Share by Sector, Long Run Data

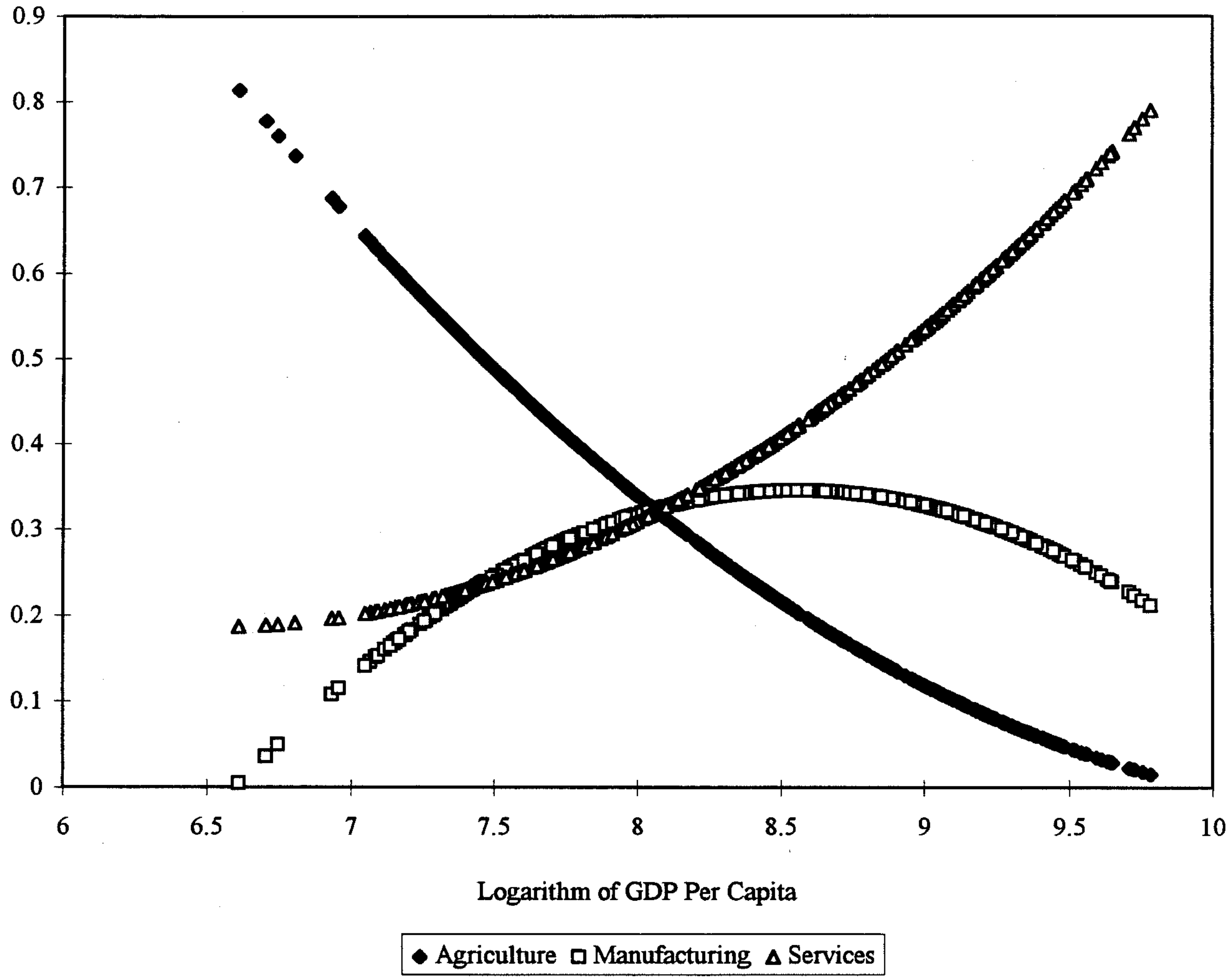


Figure 8
LAD Regression Line, Real Sectoral GDP Shares, Cross-Section of Countries, 1970-1989

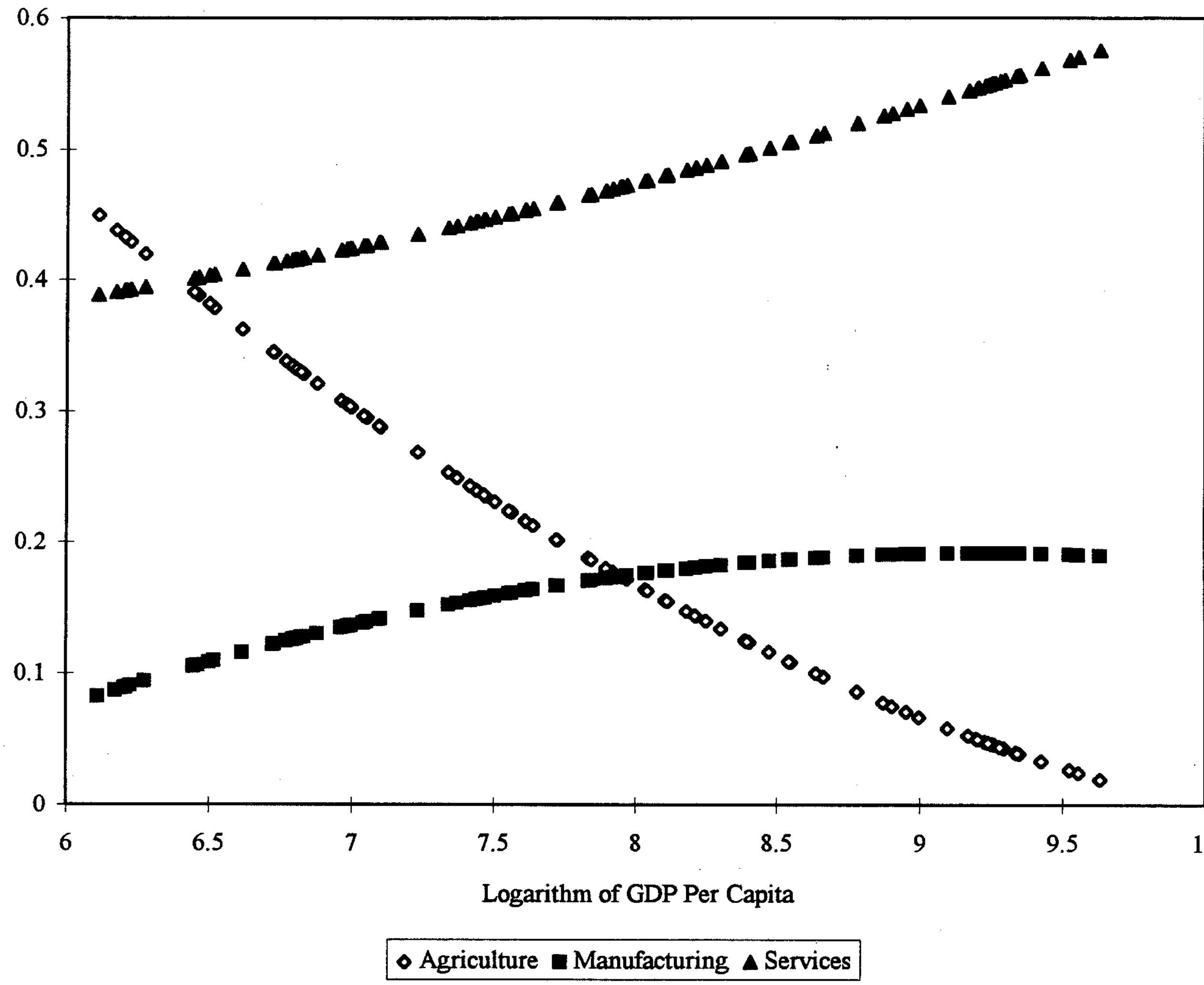
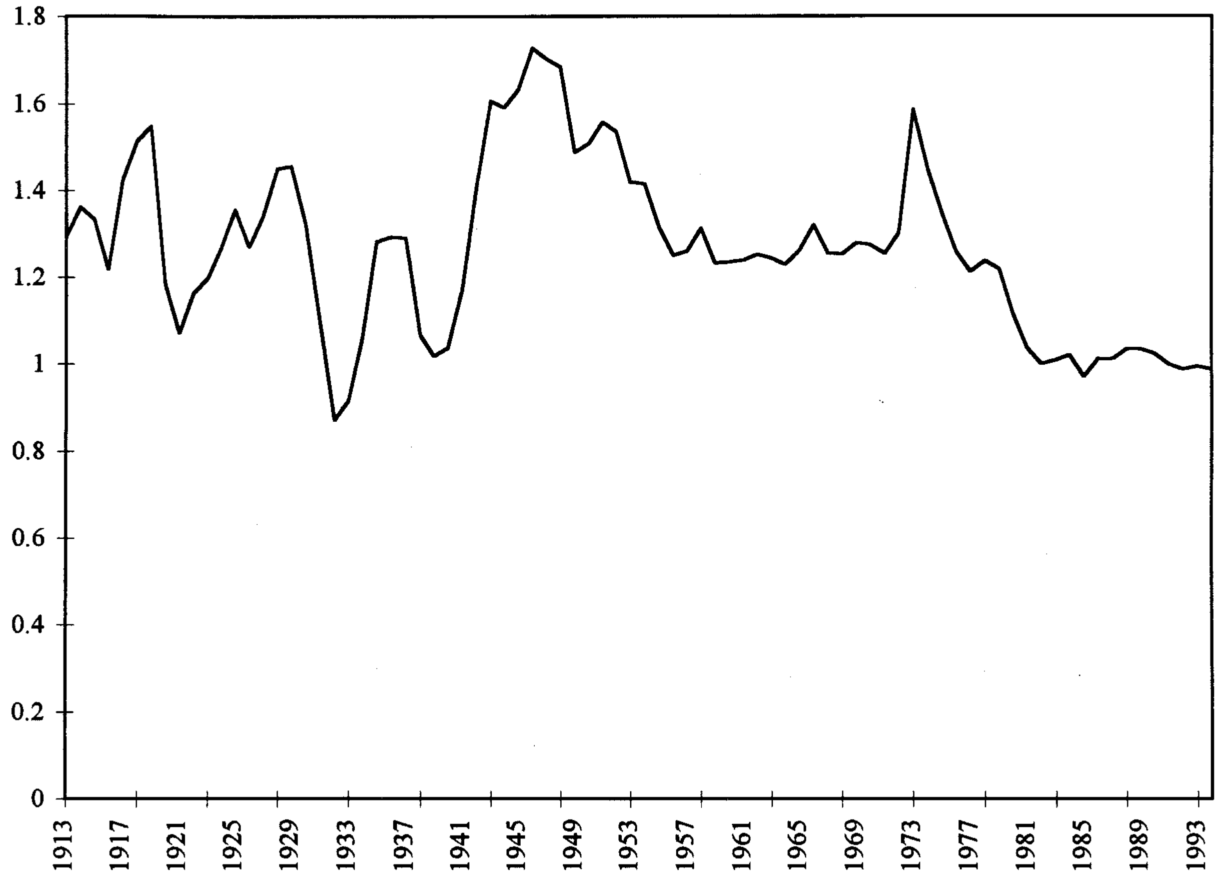


Figure 9
U.S. Relative Price of Agriculture Goods in Terms of Manufacturing Goods

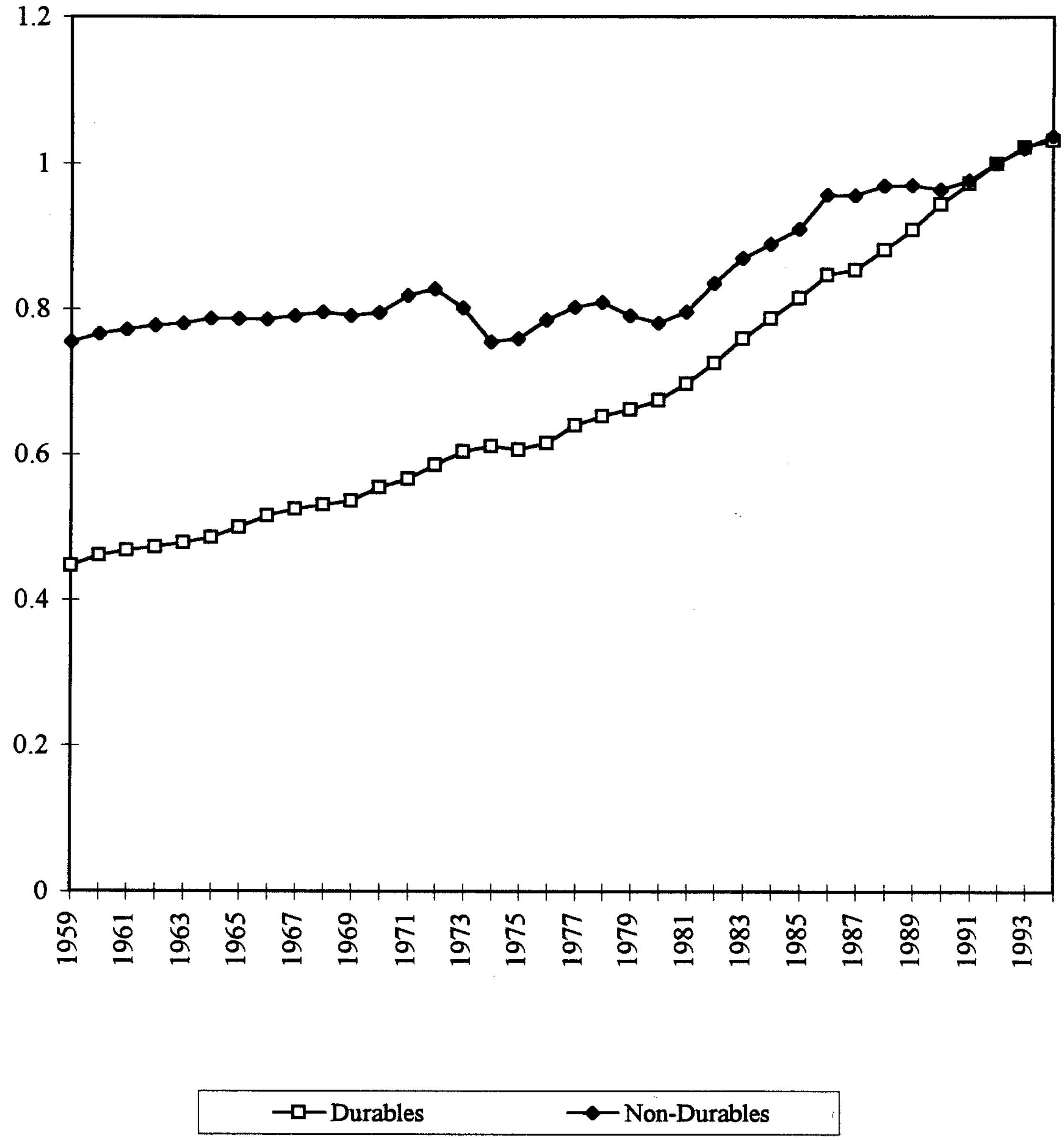


Source: Economic Report of the President, Historical Statistics of the United States from Colonial Times.

Figure 10
Canada: Relative Prices of Agricultural Goods in Terms of Manufacturing Goods

Source: Historical Statistics of Canada.

Figure 11
U.S. Relative Price of Services in Terms of Durables and Non Durables Manufacturing Goods



1992=1
Source: Survey of Current Business, January/February 1996.

Figure 12

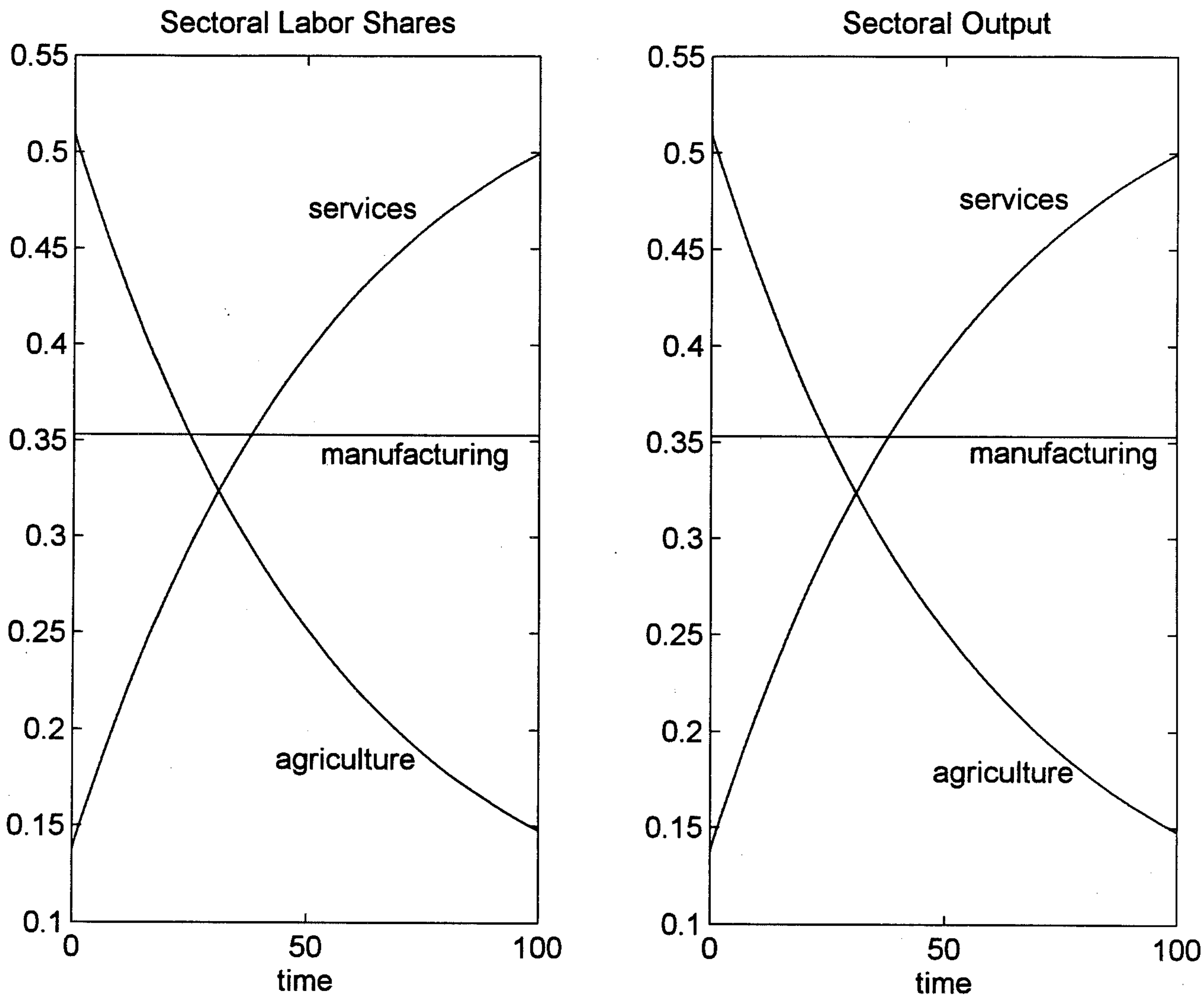
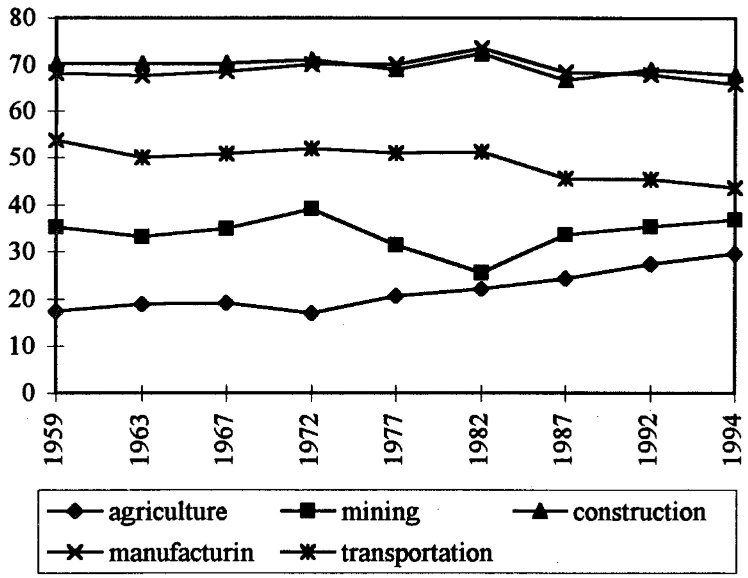
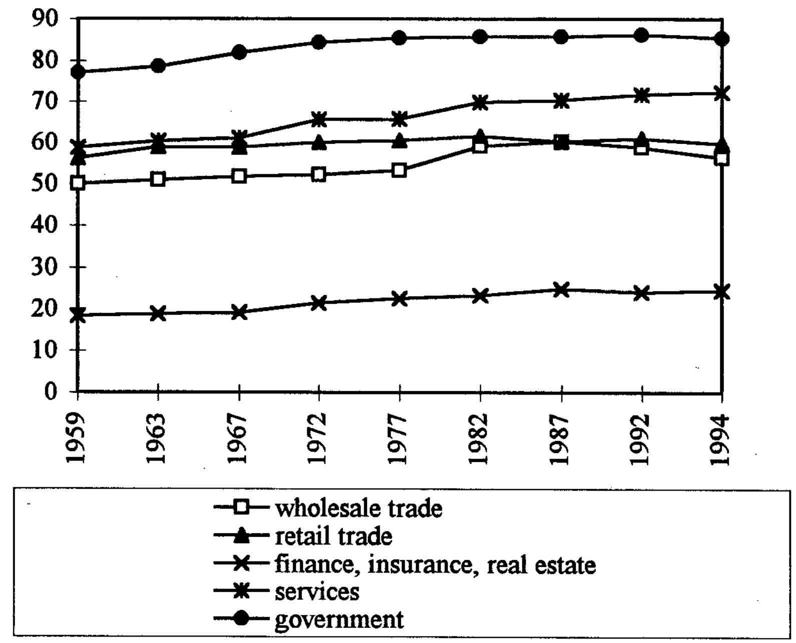

Sectoral Labor Shares
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0
50
100
time
services
manufacturing
agriculture
Sectoral Output
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0
50
100
time
services
manufacturing
agriculture

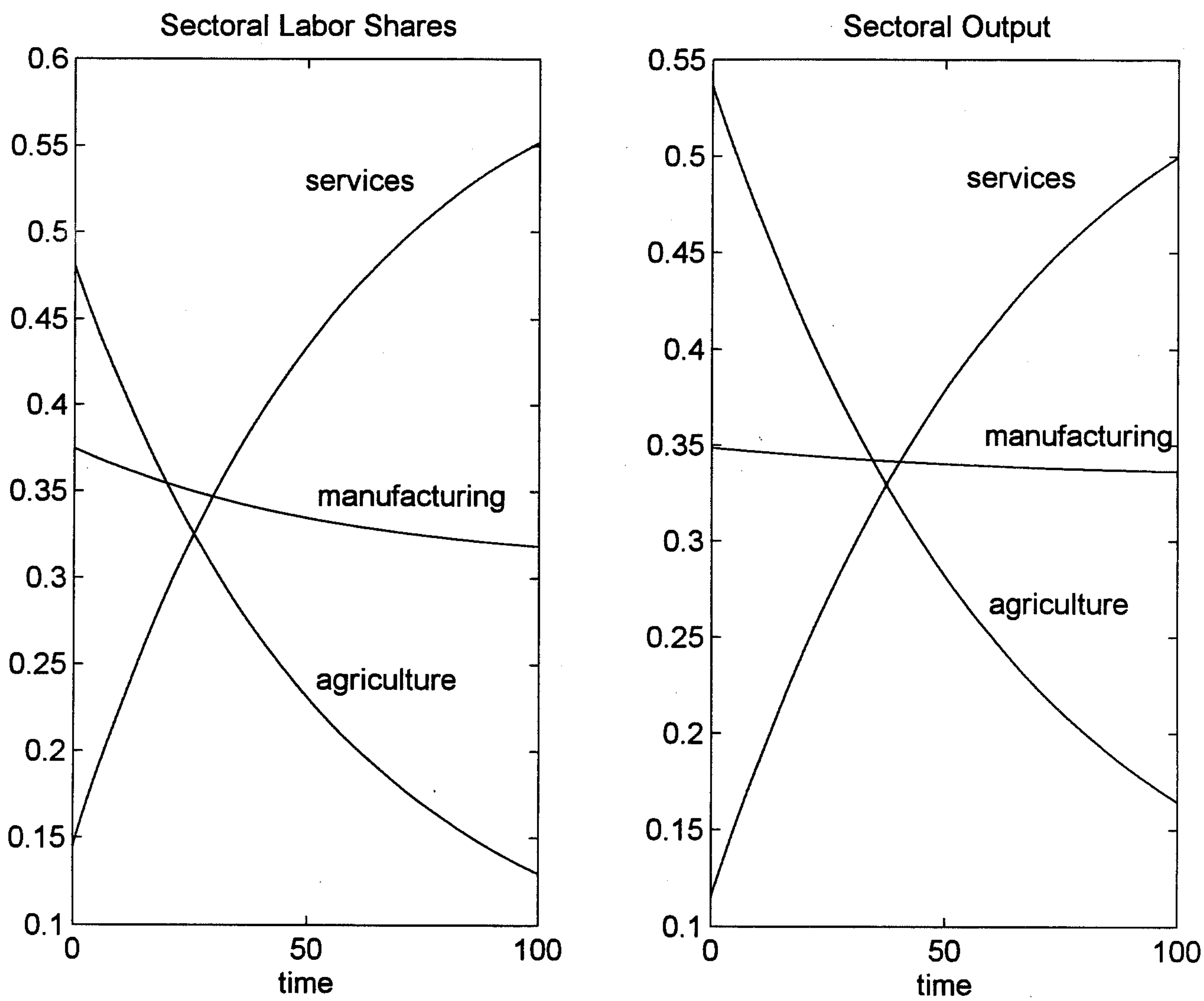
Figure 13
U.S. Sectoral Labor Income Shares





Source: Survey of Current Business, August 1996.

Figure 14



Sectoral Labor Shares
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0
50
100
time
services
manufacturing
agriculture
Sectoral Output
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0
50
100
time
services
manufacturing
agriculture