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1 Introduction

The importance of borrowing constraints for firm growth and survival has long been emphasized. Such constraints may arise in connection to the financing of investment opportunities faced by entrepreneurs or temporary liquidity needs, such as those required to survive a recession. Indeed, it has been argued that lower survival rates and more dramatic employment adjustments experienced by younger firms, could be attributed, to a large extent, to credit limits. On the theory side, several authors have suggested that borrowing constraints may arise as part of a constrained-efficient contract when borrowers face limited liability and repayment of debt cannot be perfectly enforced. The purpose of this paper is to develop a general model based on these two premises, characterize the constrained-efficient lending contracts and study their implications for firm survival, growth, equity share and dividend distribution policies.

The model is as follows. At time zero the borrower *(firm)* has a project which requires a fixed initial investment. The initial investment is financed by a lender *(the bank)*, with unlimited resources. Once the project is in operation, an advancement of working capital is required each period, which is also provided by the bank. Revenues increase with the amount of working capital advanced and are also affected by revenue shocks, which follow a Markov process. At any point in time, the project may be discontinued. Both, the borrower and lender, are risk neutral and discount future flows at the same rate. Information is shared symmetrically between the firm and the bank.

In the first part of the paper, we consider the case where the bank can commit to a long term contract contingent on all past shocks and actions. The lending contract specifies for each period, and as a function of all past shocks and actions *i)* whether the firm should exit or remain in operation; *ii)* the amount of working capital advanced and *iii)* a repayment schedule, which is restricted by the revenues collected by the firm. However, each period the firm may choose to default on the contract, obtaining an outside value which depends on its current revenue shock and the amount of working capital borrowed, and is known by the lender. We characterize the optimal lending contracts subject to the no-default constraint by the firm. Borrowing limits arise as part of the optimal contract because of this constraint: if enforcement was not a problem then in every period the firm would get from the bank all the working capital needed to maximize profits.

In the optimal long term contract, the borrowing constraints become less severe over time and disappear after a finite number of periods. This is consistent with the empirical studies discussed below, which suggest that young and small firms are more likely to face borrowing constraints. The relaxation of the borrowing constraints has a simple interpretation for the case of deterministic revenues. The contract gives the firm a certain value, equal to the discounted flow of its share of profits. It is precisely this value that holds the firm from defaulting. Since incentives for default increase with the amount of working capital advanced, the higher is the share of value of the firm, the more capital can thus be advanced. As the outstanding value of the firm grows over time (at the interest rate), the capital advanced increases towards the unconstrained level. In this way, the increased share of value of the firm provides the *bonding* necessary to raise increasing quantities of working capital.

In the presence of stochastic revenues, the growth in the equity share of the firm is contingent on the realizations of revenue shocks. In particular, the optimal contract must trade off the rate at which this share is increased depending on the future state of the revenue shock, subject to the constraint that the expected value of the equity share of the firm for the following period cannot exceed the current value plus the interest. The optimal contract has thus implications on the growth of firms and exit decisions, which are discussed in the paper. As in the deterministic case, at certain point the efficient frontier is reached, when the firm's share is sufficiently large to secure the optimal level of working capital for any possible contingency, so the unconstrained efficient outcome is obtained. Before that period is reached, conditional on a given revenue shock, the borrowing constraints are relaxed over time and the probabilities of survival increase. In this way, our model provides an added rationale for the increasing survival rates and size of firms, as a function of age, evidenced in the data.

The optimal contract has implications on earnings and the residual value of the firm. After the efficient frontier is reached, dividends are distributed contingent on the particular realization of the revenue shock. Under some regularity conditions, we show that no dividends are distributed in low revenue states, and that dividends increase with the level of earnings, for those states where they are distributed. In addition, according to numerical results obtained, at the efficient frontier the optimal contract can be approximated by a capital structure that specifies a constant marginal increase in equity per each extra dollar worth of the total value of the firm (i.e. debt plus equity).

In the setup described thus far, the lender has the ability to commit to a long term contract but the borrower does not. The second part of the paper considers the case in which lenders cannot commit to a long term contract either. As a consequence, contracts can be renegotiated every period. We study the case in which the lender has all the power in this bargaining problem, offering each period a new continuation contract to the borrower, which must respect all the constraints of the full commitment model. As before, the borrower may choose to default after the working capital is advanced. This defines a dynamic game between the borrower and the lender, where the latter is a Stackleberg leader in each period. We first provide conditions under which the optimal long term contract with one sided commitment considered above can be supported as a subgame perfect equilibrium of this dynamic game. As usual, this solution involves non-stationary trigger strategies. We then provide a full characterization of the unique stationary time-consistent equilibrium of this game. Compared to the first best -which is also the limiting behavior of the optimal long term contract with one sided commitment- capital advancements are below the optimal values and survival rates are also lower.

Our theory of debt relates more closely to that of Hart and Moore (1994). In their setup there is no uncertainty, and projects last for a known, fixed, duration, at which time they become worthless. Projects require an initial investment, generate a flow of returns thereafter and are worth a certain liquidation value to the lender if the entrepreneur defaults. The lending contract specifies an initial loan followed by a sequence of debt payments. As in our model, these debt payments are subject to a cash-flow constraint for the borrower. In addition, the threat of repudiation sets also a lower bound on the present value obtained by the entrepreneur, which in this setup is equivalent to an upper bound on the value of debt. In a repudiation-proof contract projects that are initiated are not terminated prior to their maturity. Inefficiency arises if the constraints on payments are such that the borrower is not able to commit to paying back the loan. In such case, the project is not started.

In our setup, it is also optimal for the project to be developed since it has a positive expected return. However, as in Hart and Moore (1994) and Fernández and Rosenthal (1990), default constraints put a limit on repayment schedules and in some cases make it infeasible for the borrower to credibly commit to paying back the loans received. In such cases, the project will not be undertaken unless the borrower contributes a certain fraction of the initial investment. Bulow and Rogoff (1989b) consider the general question of feasibility of borrowing and lending in the presence of default constraints. Their results imply that if upon defaulting, the borrower cannot be excluded from saving at the market interest rate and has access to actuarially fair insurance, there is no financially feasible contract. Furthermore, in any feasible contract, total debt is limited by the costs borne by the borrower upon default. As shown in the paper, their results apply to our setup. We extend their analysis, by characterizing the borrowing and lending contracts when they are feasible.

Our model extends Hart and Moore in three directions. First, we consider a stochastic environment. More importantly, revenues are not exogenously given: the amount of working capital advanced each period provides an added margin. As a consequence, in our setup a repudiation-proof contract has implications for firm growth and survival. Third, we allow for the borrower-lender relationship to be infinite lived. Financial constraints give rise to three types of inefficiencies: i) projects may not be financially feasible, as in Hart and Moore; ii) firms may be credit constrained and produce below the optimal level; iii) projects may be terminated too soon.

Imperfect enforcement is a source of contractual incompleteness that gives rise to a hold-up problem. An obvious way of dealing with this problem is through bonding. If a borrower has no collateral or there is limited liability, this may not resolve completely the enforcement problem. In our model, the hold-up problem is gradually resolved over time as the borrower builds up this bond by increasing its claims to future profits. A similar situation arises in the context of repeated insurance contracts when agents cannot commit not to take outside offers in the future. Harris and Hölmstrom (1982) use this mechanism to explain an increasing wage profile, when the ability of workers becomes known over time. Another example is Phelan (1995), that considers a repeated moral hazard model where agents can recontract with outside principals, generating increasing profiles of consumption.

There is an extensive literature on optimal debt financing with incomplete contracts.[1]

[1]Aghion and Bolton's (1992) seminal paper develops a theory of capital structure based on wealth

We briefly refer to the work that is most related to ours. A dynamic model of borrowing and lending with no default constraints was first introduced by Eaton and Gersovitz (1981), in the context of international lending. Kehoe and Levine (1993) present a general equilibrium theory under no default (participation) constraints. In their model economy information is also public knowledge. Therefore, all contracts have to obey a participation constraint, giving rise to endogenous debt limits. Also related is the work by Marcet and Marimon (1992). They analyze the effects of international financing on growth of an economy in environments of limited commitment and enforcement. Their simulations suggest that economic growth can be substantially influenced by the presence of limited enforcement.

Bulow and Rogoff (1989a) study a model of international lending with imperfect commitment, where the lender can punish the borrower by means of direct sanctions and contracts can be renegotiated. More closely related to our paper is Fernández and Rosenthal (1990), who study the problem of renegotiation in three variants of a strategic model of debt renegotiation. All payoffs are deterministic. As in our model, there is no external mechanism that commits the borrower to repay the debt. The present value of the debt and the structure of repayments are thus restricted by a no default constraint. Their model focuses on the renegotiation and repayment of a given stock of debt, so all financial flows go from the borrower to the lender. The borrower accumulates capital by reinvesting revenues collected, net of repayments. In their renegotiation game, the lender can propose each period an alternative repayment schedule. However, the borrower has always the option of rejecting this offer and sticking to the status quo contract. For some special cases, the equilibrium repayment schedule that they derive coincides with our optimal long term contract with one sided commitment.

The empirical importance of borrowing constraints has been suggested by several authors. Gertler and Gilchrist (1994) favor the explanation that small manufacturing firms are subject to liquidity constraints to the fact that these firms respond more to a tightening of monetary policy than do larger manufacturing firms. Fazzari, Hubbard and Peterson (1988) provide further evidence on the relevance of liquidity constraints across firms and as an explanation for the dynamic behavior of aggregate investment. Firm survival and growth have been widely documented in recent empirical work.[2]According to these studies: i) survival rates increase dramatically with the age of plants; ii) the rate of job creation decreases with the age of plants iii) gross job destruction decreases with age and iv) the variance of job creation decreases monotonically with age. Our theory implies, at a very general level, the first two regularities. The latter properties of firm growth depend more specifically on special characteristics of the model.

constraints on the part of the entrepreneur and on the inability of the parties to write contingent contracts. For an excellent survey of the literature, see Oliver Hart (1995.)

[2]For evidence on job creation and destruction for US manufactures see Davis, Haltiwanger and Schuh (1996) and Dunne Roberts, and Samuelson (1989a, 1989b), Evans (1987) and Hall (1987) provide evidence on growth properties of firms also by age. For evidence on entry and exit, see Dunne, Roberts and Samuelson (1989a, 1989b.)

The paper is organized as follows. Section 2 introduces the model. In sections 3 and 4 we characterize the optimal contract. Section 5 considers the feasibility of lending contracts. Section 6 describes the optimal contract under two extreme assumptions on the distribution of shocks: no uncertainty and *iid* shocks. In section 7 we solve for the contract in a model with limited commitment on both sides. Section 8 concludes.

2 The Model

Time is discrete and the time horizon is infinite. At time zero the borrower (firm) has a project which requires a fixed initial investment $I_0 \geq 0$. Thereafter, this project gives a random stream of revenues $R(s,k)$ each period, where $s \in S \subset \Re$ is a revenue shock and k the amount of working capital advanced in the period. The revenue shock s follows a Markov process with conditional cumulative distribution function $F(s_{t+1}, s_t)$ and an initial distribution $G(s_0)$.

The timing of events within a period is as follows. First the shock s is observed. After observing this shock, the firm can either be liquidated , at a value $L(s)$, or continue in operation. If the firm continues in operation, inputs are purchased, requiring an advancement of working capital k. At the end of the period, sales take place and revenues $R(s,k)$ are collected. These revenues are an increasing function of both, s and k.

We assume that the firm has no capital, and thus it requires a lender to finance the initial investment and the advancements of working capital every period. The firm is thus restricted at all times to a nonnegative cash flow. (This assumption can be easily relaxed to a lower bound.) We assume that both, the firm and the lender, observe the revenue shocks s_t, so there is no asymmetric information. Both, the firm and lender, discount flows using the same discount factor $\delta < 1$. Lenders can commit to long term contracts with the firm, as described below. However, contracts have limited enforceability as the firm can always choose to default. Limited commitment on both sides is discussed in section 7.

The long term contract specifies a contingent policy of working capital advancements from the lender to the firm that take place at the beginning of each period, and a payment from the firm to the lender at the end of the period. Letting τ denote this payment, $R(s,k) - \tau$ are the dividends collected by the firm. We assume, without loss of generality, that dividends are fully distributed. In consequence, the nonnegativity constraint on cash-flows of the firm requires that $\tau \leq R(s,k)$.

As in Hart and Moore (1994), we assume that only the borrower has the ability to run the firm. If the match is ended either voluntarily or not, the residual value for the borrower is given by a function $V^0(s,k)$, which is discussed in more detail below. The lender controls firm growth by choosing k_t and has priority over current earnings in deciding τ_t. This is the common approach in models of debt-financing.

The capital advancement and payments at any time t, are contingent on the history $h_t = \{k_{r-1}, \tau_{r-1}, s_r\}_{r=1}^{t}$ of previous transfers and all shocks, including s_t. Let H be the set of all possible histories.

Definition 1 *A feasible contract is a mapping* $\mathcal{C} : H \rightarrow \Re^2_+$ *such that for all* $h_t \in H$ *and* $(k_t, \tau_t) = \mathcal{C}(h_t)$, $\tau_t \leq R(s_t, k_t)$.

Notice that we include the limited liability (nonnegative cash-flow) constraint as part of our definition.

2.1 Contracts with perfect enforceability

In the absence of enforcement problems, the lender and the firm can commit to the above contract without any additional constraints. Since flows are discounted at the same rate, the optimal contract will maximize total expected discounted profits for the match.

Let $\pi(s) = \max_k R(s,k) - k$ denote the profit function. The following assumption guarantees a solution to the profit maximization problem.

Assumption 1 *The function* R *is continuous. For each* s, $R(s,k) - k$ *is quasiconcave in* k *and has a maximum. Furthermore, there exists some* $b < \infty$ *such that for all* s *and* k, $-b \leq R(s,k) - k \leq b$.

We will not require that profits be nonnegative, since we are interested in the possibility of firm exit.

The total surplus of the match $\tilde{W}(s)$ satisfies the following dynamic programming equation:

$$\tilde{W}(s) = \max\left(L(s), \pi(s) + \delta \int \tilde{W}(s') F(ds', s)\right). \tag{1}$$

If $\tilde{W}(s) = L(s)$ for all $s \in S$, the firm would not be viable and would be immediately closed. Let $\tilde{S} = \left\{s : \tilde{W}(s) > L(s)\right\}$, the set of states at which the firm would continue in the industry, the *survival set.*

Assumption 2 *The survival set* $\tilde{S}$ *is nonempty.*

2.2 Contracts with limited enforceability

We now consider the case where the firm can default on the contract. The firm is free to default before or after production has taken place. If the firm chooses to default it will do so prior to making any payments to the lender. We assume that this exit decision is irreversible, so the firm obtains a total value from defaulting equal to $V^0(s,k)$. This function summarizes the value of the outside investment opportunities faced by the firm and is common knowledge to both parties. For example, if the borrower can collect revenues and disappear, without being able to reestablish itself as a new firm, then $V^0(s,k) = R(s,k)$. If the firm can continue operations but is excluded from borrowing, saving and insurance (as in Manuelli (1985) and Marcet and Marimon (1992)), then $V^0(s,k)$ is the value obtained by the firm through optimal self-financing. Alternatively, a firm may be excluded from borrowing but not from saving or purchasing insurance, as in Bulow and Rogoff (1989b). $V^0(s,k)$ will be the value function thus obtained. Another example is obtained if the firm can establish a new contract with another bank, after paying a cost for breach of contract.

If at the beginning of some period the bank decides to liquidate the firm, then the latter obtains a value $V^0(s,0)$. This value represents an inalienable component of the firm's capital, as in Hart and Moore (1994.) The difference $L(s) - V^0(s,0)$ thus represents the component of the liquidation value that can be appropriated by the bank. We make the following assumptions on V^0.

Assumption 3 *The function V^0 has the following properties:*

(a) $V^0(s,k) \geq 0$.

(b) $V^0(s,k) - k \leq \tilde{W}(s)$.

(c) V^0 *is a continuous function.*

(d) V^0 *is nondecreasing in both arguments.*

Part (a) is in line with our limited liability assumption. Part (b) says that the net surplus obtained through an involuntary separation cannot exceed the value of the match. In particular, if it were optimal to liquidate the firm for some state s, then $L(s) \geq V^0(s,0)$.

After any history h_t, the contract specifies current and future contingent advances and payments which give the firm a discounted net cash flow V_t and the lender an expected discounted benefit B_t. Letting $V_{t+1}(s')$ denote the continuation value at the beginning of period $t+1$ after history $h_{t+1} = (h_t, k_t, \tau_t, s')$, the firm will choose not to default in period t provided that

$$V^0(s,k) \leq R(s,k) - \tau + \delta \int V_{t+1}(s')F(ds',s). \tag{2}$$

Since the lender can always include in the contract a recommendation to exit, we will require that the participation constraint (2) be satisfied at all times.

Notice that the values $V_{t+1}(s')$ provide a summary statistic for the future contract and together with the pair (k,τ) are sufficient to verify this non-default or participation constraint. Using V_t as a state variable, following Spear and Srivastava (1987), the contract can be specified in recursive form. Every period, given initial values $V_t = \bar{V}$ and $s_t = s$ and assuming exit is not recommended, the contract specifies a pair (k,τ) and continuation values $V(s')$. In practice, lender loans carry covenants that establish specific bonding activities by the firm, such as how much and what type of information should the firm report to the lender, restrictions in the investment and dividend policies, and in the financing policy. Note that, in fact, because we are assuming that there is no information asymmetry, we are giving the lender all the discretionary power in specifying $V(s')$ instead of a fixed valued for all possible s'.

Since the current value for the firm is the sum of the cash-flows generated this period plus the discounted value of the stream of future cash-flows, the following condition must be satisfied:

$$\bar{V} = R(s,k) - \tau + \delta \int V(s')\, F(ds', s). \tag{3}$$

The continuation values $V(s')$ must be supported by an enforceable continuation contract. By assumption 3, any positive continuation value $V(s') \geq V^0(s', 0)$ is feasible, since it can be obtained by giving the firm a transfer $t \leq V(s')$ such that

$$\max\left\{V^0(s', k) + t - k | 0 \leq k \leq t\right\} = V(s')$$

and liquidating the firm (or committing to no future advancements). Any continuation value $V(s') < V^0(s', 0)$ is not feasible. Hence, a value $V(s')$ can be supported by an enforceable contract if, and only if, $V(s') \geq V^0(s', 0)$.

Transfers are limited by the amount of earnings generated in the current period or the participation constraint, whichever is smaller.

$$\tau \leq \min\left\{R(s,k), R(s,k) - V^0(s,k) + \delta \int V(s')\, F(ds', s)\right\}. \tag{4}$$

Whenever $\tau < R(s,k)$ the lender is allowing the firm to distribute dividends.

We now write the optimal contract in recursive form. Let $B\left(\bar{V}, s\right)$ denote the maximum expected discounted cash flow that the lender can obtain, by choosing an enforceable contract that gives a current expected value $\bar{V}$ to the firm when the current revenue shock is s. This function satisfies the following dynamic programming equation:

$$\begin{aligned} B(\bar{V}, s) \quad = \quad & \max\Big\{L(s) - \bar{V}; \\ & \max_{k,\tau,V(s')\geq V^0(s',0)} -k + \tau + \delta \int B(V(s'), s')\, F(ds'; s)\Big\} \\ & \text{subject to constraints (3) and (4).} \end{aligned} \tag{5}$$

We assume that the lender has access to perfect capital markets as $\tau - k$ can be negative in some periods.

Exit occurs whenever the value of the contract $B(\bar{V}, s) = L(s) - \bar{V}$.

3 The Optimal Contract

To solve for the optimal contract, consider first the inner maximization problem in (5):

$$\max_{k,\tau,V(s')\geq V^0(s',0)} \left[-k + \tau + \delta \int B(V(s'), s')\, F(ds', s)\right] \tag{6}$$

subject to (3) and (4).

Using equation (3) to eliminate τ, (6) can be rewritten as follows:

$$\max_{k,V(s')\geq V^0(s',0)} \left[-k + R(s,k) - \bar{V} + \delta \int \left(B\left(V\left(s'\right), s'\right) + V\left(s'\right)\right) F\left(ds'; s\right)\right] \tag{7}$$

subject to:

$$\delta \int V\left(s'\right) F\left(ds', s\right) \leq \bar{V} \tag{8}$$

$$V^0(s,k) \leq \bar{V}. \tag{9}$$

Notice that the term $B\left(V\left(s'\right), s'\right) + V(s')$ under the integral in (7), is the total surplus of the match. Since maximizing the lender's surplus given V gives the same solution as maximizing total surplus given V, the optimal program can be reformulated as follows. Letting $W\left(V, s\right) = B(V,s) + V$, adding $\bar{V}$ to equation (5) and using (7), the optimal program solves:

$$\begin{aligned} W\left(\bar{V}, s\right) &= \max\left\{L\left(s\right),\right. \\ &\left.\max_{k,V(s')\geq V^0(s',0)} \left[R(s,k) - k + \delta \int W\left(V\left(s'\right), s'\right) F\left(ds'; s\right)\right]\right\} \\ &\text{subject to (8) and (9).} \end{aligned} \tag{10}$$

Standard results in dynamic programming imply that there is a unique solution $W\left(\cdot\right)$ to this problem.

It is interesting to compare this optimal program with (1), the one obtained for the case of perfect enforcement. Notice that if the no default constraint (9) were never binding, then k would be chosen so that $R\left(s,k\right) - k = \pi\left(s\right)$ and the solution to (10) would give $W(\bar{V}, s) = \tilde{W}\left(s\right)$.

Let $K\left(s\right) = \inf\left\{k : R\left(s,k\right) - k = \pi\left(s\right)\right\}$, and define $V^u\left(s\right) = V^0\left(s, K\left(s\right)\right)$. This is the smallest continuation value for the firm that, once reached, is compatible with static profit maximization. If $\bar{V} < V^u\left(s\right)$, current profit maximization cannot be enforced.

The following lemma is a direct consequence of Assumptions 1 and 3.

Lemma 1 *If $\bar{V} < V^u\left(s\right)$, then k will be chosen so that $V^0\left(s,k\right) = \bar{V}$. In this case, $R\left(s,k\right) - k < \pi\left(s\right)$. If $\bar{V} \geq V^u\left(s\right)$ then k will be chosen so that $R\left(s,k\right) - k = \pi\left(s\right)$.*

A direct consequence of this lemma, is that the static problem of determining k can be separated from the dynamic choice of $V\left(s'\right)$. In particular, defining the indirect profit function

$$\begin{aligned} \Pi\left(\bar{V}, s\right) = \quad & \max R\left(s,k\right) - k \\ & \text{subject to } V^0\left(s,k\right) \leq \bar{V}, \end{aligned}$$

the optimal contract (10) can be written

$$\begin{aligned} W\left(\bar{V}, s\right) &= \max\{L(s), \\ &\max_{V(s') \geq V^0(s',0)} \left[\Pi\left(\bar{V}, s\right) + \delta \int W\left(V(s'), s'\right) F(ds'; s)\right]\right\} \\ &\text{subject to } \delta \int V(s') F(ds', s) \leq \bar{V} \end{aligned} \tag{11}$$

The following Proposition gives properties of the indirect profit function $\Pi\left(\bar{V}, s\right)$ which are used below.

Proposition 1 *(i)* Π *is continuous, uniformly bounded, increasing in* s*, strictly increasing in* $\bar{V}$ *for* $\bar{V} < V^u(s)$ *and* $\Pi\left(\bar{V}, s\right) = \pi(s)$ *for* $\bar{V} \geq V^u(s)$.

(ii) If R *and* V^0 *are continuously differentiable and* $V_2^0(s,k) > 0$*, then* $\Pi_1\left(\bar{V}, s\right) = \frac{(R_2(s,k)-1)}{V_2^0(s,k)}$.

(iii) If R *and* V^0 *are twice continuously differentiable and*

$$R_{22}(s,k) V^0(s,k) - V_{22}^0(s,k)(R_2(s,k) - 1) \leq 0 \ \ (< 0)$$

for $k < K(s)$*, then* $\Pi_{11}\left(\bar{V}, s\right) \leq 0 \ (< 0)$ *for* $\bar{V} < V^u(s)$.

(iv) If R *and* V^0 *are twice continuously differentiable and*

$$R_{21}(s,k) V_2^0(s,k) - R_2(s,k) V_{21}^0(s,k) \geq -V_{21}^0(s,k) \ \ \left(> V_{21}^0(s,k)\right)$$

for all $k < K(s)$*, then* $\Pi_{21}\left(\bar{V}, s\right) \geq 0 \ (> 0)$ *for* $\bar{V} < V^u(s)$.

Proof. (i) follows is an immediate consequence of the maximum theorem and the properties given for R and V^0 in assumptions 1 and 3; (ii) is a direct application of the implicit function theorem; (iii) and (iv) follow from differentiating the expression given in part (ii). ■

Loosely speaking, condition (iii) says that if the curvature of the profit function with respect to k is greater than the curvature of the V^0 function, then the indirect profit function will be concave in $\bar{V}$. Condition (iv) says that if the *relative* degree of supermodularity of function R is greater than that of function V^0, then the indirect profit function will be supermodular. In particular, if $V^0(s,k) = R(s,k)$, then Π will be supermodular.

Assumption 4 *There exists* $M < \infty$*, such that* $V^u(s) \leq M$*, for every* s.

From part (i) of the above Proposition it follows that if the current promised utility to the firm $\bar{V}$ is sufficiently high, the period return will be identical to the one obtained in the unconstrained problem. However, this may not guarantee that the current total value $W\left(\bar{V}, s\right) = \tilde{W}(s)$, since the contract must also guarantee that the enforcement constraint will not bind in any future period. For example, if $V^u(s) < \bar{V} < \delta \int_{s' \in \tilde{S}} V^u(s') F(ds', s)$, then it must be the case that $V(s') < V^u(s')$ with positive probability on a subset of the survival set $\tilde{S}$, and thus next period's unconstrained profit maximum cannot be guaranteed. However, if $\bar{V}$ is high enough, it may be possible to guarantee that the enforcement constraint will not bind in any future period and thus the unconstrained optimal solution will be attained.

Let $V^n(s)$ be the minimum level of current initial value for the firm that is needed to guarantee that the enforcement constraint will not bind for at least n periods, including the current one. Then V^n can be defined recursively by

$$V^n(s) = \max\left(V^u(s), \delta \int_{s' \in \tilde{S}} V^{n-1}(s') F(ds', s)\right) \tag{12}$$

with $V^0(s') = 0$.

Let

$$\tilde{V}(s) = \lim_{n \to \infty} V^n(s).$$

Since $V^n(s)$ is an increasing sequence, which by Assumption 4 is uniformly bounded , this limit exists. As we will show below, starting from a state s, the unconstrained optimum will be achieved if, and only if, $\bar{V} \geq \tilde{V}(s)$. Before, we provide a simple characterization of $\tilde{V}(s)$.

Proposition 2 *The enforcement bounds $\tilde{V}(s) = \lim_{n\to\infty} V^n(s)$ are the unique solution to the following dynamic programming equation:*

$$\tilde{V}(s) = \max\left(V^u(s), \delta \int_{s' \in \tilde{S}} \tilde{V}(s') F(ds', s)\right). \tag{13}$$

Proof. Using Lebesgue's dominated convergence theorem, it follows immediately that these bounds are a solution to (13). This solution is unique, as Blackwell's sufficient conditions can be immediately verified. ■

The following Proposition provides a comparison between the constrained and unconstrained problems.

Proposition 3 *Let s be in the survival set $\tilde{S}$. Then:*

1. *$W\left(\bar{V}, s\right)$ is weakly increasing in $\bar{V}$.*

2. *For all $\bar{V} \geq \tilde{V}(s)$, $\tilde{W}(s) = W\left(\bar{V}, s\right)$.*

3. *For all $\bar{V} < \tilde{V}(s)$, $\tilde{W}(s) > W\left(\bar{V}, s\right)$.*

Proof. The weak monotonicity of $W(.,s)$ follows immediately from the monotonicity of $\Pi(.,s)$ applying standard dynamic programming arguments. We now establish 2. First notice that if $\bar{V} = \tilde{V}(s)$, then it is possible to choose as continuation values $V(s') = \tilde{V}(s')$. We will show that the dynamic programming equation given by problem (11) preserves the property defined by 2. Suppose then that $W(V,s) = \tilde{W}(s)$ for $V \geq \tilde{V}(s)$. Then, letting $V(s') = \tilde{V}(s')$, it follows immediately from (11) that

$$\begin{aligned} W\left(\bar{V}, s\right) &= \pi(s) + \delta \int_{s' \in \tilde{S}} \tilde{W}(V(s'), s') F(ds', s) \\ &= \pi(s) + \delta \int \tilde{W}(s') F(ds', s) = \tilde{W}(s). \end{aligned}$$

Since the set of functions $W(\cdot)$ such that $W\left(\bar{V}, s\right) = \tilde{W}(s)$ for $s \in \tilde{S}$ is closed in the norm topology, the unique solution to the dynamic programming equation (11) also satisfies this property.

We now establish 3. Note that $\tilde{W}(s) > L(s)$, as $s \in \tilde{S}$. Assume that $W\left(\bar{V}, s\right) > L(s)$, otherwise the result is trivial. If $\bar{V} < \tilde{V}(s)$, then there exists some n such that $\bar{V} < V^n(s)$. We now prove by induction that this implies $W\left(\bar{V}, s\right) < \tilde{W}(s)$. The result is immediate for $n = 1$, for in this case $\bar{V} < V^u(s)$ and letting $V(s')$ be the optimal continuation values starting from $\left(\bar{V}, s\right)$,

$$\begin{aligned} W\left(\bar{V}, s\right) &< \pi(s) + \delta \int W(V'(s), s) F(ds', s) \\ &\leq \pi(s) + \delta \int \tilde{W}(s) F(ds', s) = \tilde{W}(s). \end{aligned}$$

To continue with the induction argument, suppose that $\bar{V} < V^{n-1}(s)$ implies that $W\left(\bar{V}, s\right) < \tilde{W}(s)$ for all s. Then if $\bar{V} < V^n(s)$, either $\bar{V} < V^u(s)$ or $\bar{V} < \delta \int_{s' \in \tilde{S}} V^{n-1}(s') F(ds', s)$. In the first case, using the same argument as for the case $n = 1$, it follows immediately that $W\left(\bar{V}, s\right) < \tilde{W}(s)$. In the latter case, $V(s') < V^{n-1}(s')$ for a subset of $\tilde{S}$ with positive measure. Using the induction hypothesis, it follows immediately that $W\left(\bar{V}, s\right) < \pi(s) + \delta \int_{s' \in \tilde{S}} \tilde{W}(s') F(ds', s) = \tilde{W}(s)$. ■

Proposition 3 can be used to analyze the optimal dividend distribution policy. Since the borrower and lender are risk neutral and discount flows at the same rate, dividend distribution would seem to be indeterminate, up to the given expected present values for both parties. However, delaying dividend distribution allows for a faster growth of the borrower's share of value V_t, reducing the incentives for default and thus relaxing the borrowing constraint. Thus, whenever possible, it is optimal to choose continuation values $V(s') = \tilde{V}(s')$. If this is not feasible for all states, i.e. if $\bar{V} < \delta \int \tilde{V}(s') F(ds', s)$, then monotonicity of the value function implies that it is optimal to choose continuation values $V(s')$ such that $\bar{V} = \delta \int V(s') F(ds', s)$, distributing no dividends in the current period. Thus, a dividend distribution policy that sets dividends

$$d_t = \max\left(0, V_t - \delta \int \tilde{V}(s_{t+1}) F(ds_{t+1}, s_t)\right) \tag{14}$$

is optimal. Is this the only optimal policy? If $V_t > \delta \int \tilde{V}(s_{t+1}) F(ds_{t+1}, s_t)$, any policy that sets $V(s_{t+1}) \geq \tilde{V}(s_t)$ is also optimal.[3] Under some conditions listed below, the policy described by (14) distributes dividends at the earliest optimal time, i.e. whenever $V_t > \delta \int \tilde{V}(s_{t+1}) F(ds_{t+1}, s_t)$. We will call such policy a *dividend priority* policy.

Definition 2 *A* dividend priority *policy is a selection $d\left(\bar{V}, s\right)$ that picks among the set of solutions to (11) the one that minimizes $\int V(s') F(ds', s)$.*

The existence of such policy follows from the continuity and boundedness of the value function (see Lemma 2 in the Appendix.) If $V_t < \int \tilde{V}(s_{t+1}) F(ds_{t+1}, s_t)$ and W is strictly increasing in V for $V < \tilde{V}(s)$, then the optimal policy requires that $d_t = 0$. However, if W fails to be strictly increasing, then it may be optimal to set $d_t > 0$. This possibility is illustrated in the following example.

Example 1 *Suppose $S = \{s_0, s_1, s_2\}$, s_1 and s_2 are absorbing states and $P(s_2|s_0) = P(s_1|s_0) = \frac{1}{2}$. Let s_0 be a transitory state, with no production possibilities. Suppose $R(s_2, k) = \min\left(\gamma k, \gamma \bar{k}\right)$ for some $\bar{k} > 0$, $\gamma > 1$, $L(s) = 0$ for all s and that $V^0(s, k) = R(s, k)$. Suppose that $\pi(s_1) = \epsilon$, but there is a large fixed cost and profit margins are so small, that exit in the absorbing state s_1 is optimal whenever $\bar{V} < V^*$. Finally assume that $V^* > 2V_1/\delta > 2V_2/\delta > \gamma\bar{k}$. Then starting at state s_0 with value V_1 or V_2, it is always optimal to set $V(s_1) = 0$, exiting the next period if state s_1 is verified and set $V(s_2) = \gamma\bar{k}$. Since s_2 is an absorbing state, $\tilde{V}(s_2) = V^u(s_2) = \gamma\bar{k}$. It follows that $W(V_1, s_0) = W(V_2, s_0)$, though both V_1 and V_2 are strictly below $\delta V^*/2 \leq \tilde{V}(s_0)$. Consequently, a dividend priority policy would prescribe distributing some dividends in the first period.*

If $W(V, s)$ were increasing in V for $V < \tilde{V}(s)$, no dividends would be distributed when $\bar{V} < \delta \int \tilde{V}(s') F(ds', s)$, so the optimal dividend priority policy is defined by (14). The following proposition gives two conditions that guarantee this strong monotonicity. The first one, implies that it is never optimal to exit, which excludes the above example. The second condition is the concavity of the value function. This assumption is not problematic in the absence of exit. The possibility of exit introduces a nonconcavity in the optimization problem. However, as we show later, allowing for randomizations on the exit decisions, the value function is concave.

Proposition 4 *Suppose either* i) $R(s, k) - k \geq L(s)$ *for all k and s, or* ii) $W(V, s)$ *is concave in V for all s. Then if $W\left(\bar{V}, s\right) > L(s)$ and $\bar{V} < \tilde{V}(s)$ it follows that $W(V, s) < W\left(\bar{V}, s\right)$ for all $V < \bar{V}$.*

[3] An upper bound on V is necessary to avoid a policy were dividends are rolled over indefinitely. Notice also that if firms were less patient than the bank, there would be a unique solution. The solution given by equation (14) can be interpreted as the unique solution corresponding to the limit as the discount factor of the firm converges (from below) to the discount factor of the bank.

Proof. We first prove that condition *(i)* is sufficient. The proof will proceed by induction on n. We know that the function $W(V,s)$ is nondecreasing in V. Let $V < \bar{V}$ and without loss of generality suppose that $W(V,s) > L(s)$. Since $V < \tilde{V}(s)$, there exists $n \geq 1$ such that $V < V^n(s)$. If $n = 1$, $V < V^u(s)$, and the result follows immediately from the quasiconcavity of the profit function given by Assumption 1. So suppose the Proposition is true for V^{n-1} and $V^u(s) \leq V < \bar{V} < V^n(s)$. Let $V_1(s)$ be the optimal continuation value policy for (s,V). Since $V^u(s) \leq V < V^n(s)$, it follows that $V < \delta \int V^{n-1}(s') F(ds',s)$ and thus $V_1(s') < V^{n-1}(s')$ on a positive probability set $S_1 \subset S$. Since $\bar{V} > V$, there exists $\epsilon > 0$ such that $\epsilon \int_{S_1} F(ds',s) = \bar{V} - V$, and $V_1(.)$ is feasible for the pair $(\bar{V}, s)$. Consequently,

$$
\begin{aligned}
W\left(\bar{V}, s\right) &\geq \pi(s) + \delta \int_{S_1^c} W(V_1(s'), s') F(ds', s) + \delta \int_{S_1} W(V_1(s') + \epsilon, s') F(ds', s) \\
&> \pi(s) + \delta \int W(V_1(s'), s') F(ds', s) = W(V, s),
\end{aligned}
$$

where the last inequality follows from the induction hypothesis.

We now show that *(ii)* is also a sufficient condition. If $W(V,s) = W\left(\bar{V}, s\right)$ for some $V < \bar{V} < \tilde{V}(s)$, then by concavity of the value function it follows that $W\left(\bar{V}, s\right) = W\left(\tilde{V}(s), s\right)$, contradicting Proposition 3. ■

Corollary 1 *Under the assumptions of the previous proposition, if* $\bar{V} < \delta \int \tilde{V}(s') F(ds', s)$ *the optimal contract requires that* $\bar{V} = \delta \int V(s') F(ds', s)$, *so no dividends are distributed, and the optimal dividend priority is given by (14).*

For future reference, we will call the condition $R(s,k) - k \geq L(s)$ the *no exit assumption.*

Proposition 3 establishes that the enforcement constraints will reduce total surplus until the boundaries are reached, i.e. $\bar{V} \geq \tilde{V}(s)$. The policy described by (14) can be used to provide a bound on the time necessary to reach this boundary. Let V_0 be the initial value for the borrower in the contract and let V_t denote the value at time t following the optimal policy given by (14) and let T denote the random variable indicating the first time when $V_t \geq \tilde{V}(s_t)$. The following proposition gives a bound on T.

Proposition 5 *The stopping time* $T \leq \log_\delta (V_0/M) + 1$, *where* M *is the bound given by Assumption 4.*

Proof. The dividend policy given by (14) implies that $EV_{t+1} = V_t/\delta$ until the boundaries are crossed for all states and also implies that $V_t \leq \tilde{V}(s_t)$. Let Ω_t denote the set of sample paths ω starting from (V_0, s_0) such that $V_t < \tilde{V}(s_t)$ and no exit occurs. By the law of iterated expectation, $E_{\Omega_t} V_t \geq \left(E_{\Omega_{t-1}} V_{t-1}\right)/\delta$ and thus $E_{\Omega_t} V_t \geq V_0/\delta^t$. In particular, letting t_0 be the smallest integer such that $t \geq \log_\delta (V_0/M)$ it follows that $E_{\Omega_{t_0}} V_{t_0} \geq M$.

But since $V_t \leq \tilde{V}(s_t) \leq M$ for all t, and $V_t(\omega) < M$ for all $\omega \in \Omega_t$ it follows that the set of sample paths in Ω_{t_0} has measure zero. ■

We now turn to the exit rule when the no exit assumption is not verified. It is optimal for the firm to remain active, provided that

$$\begin{aligned} 0 \ < \ & \max_{k,V(.)} \quad R(s,k) - k + \delta \int W(V(s')) F(ds', s) \\ & \text{subject to (8) and (9).} \end{aligned}$$

Let $S^*\left(\bar{V}\right)$ be the set of initial values s satisfying this inequality. The following Proposition gives some properties of the optimal exit policy.

Proposition 6 *The optimal exit rule satisfies the following properties: 1)* $\bar{V}_2 \geq \bar{V}_1$ *implies that* $S^*\left(\bar{V}_2\right) \supset S^*\left(\bar{V}_1\right)$. *2)* $S^*\left(\bar{V}\right) \subset \tilde{S}$. *3) If* $s \in \tilde{S}$, *then* $s \in S^*\left(\tilde{V}(s)\right)$.

Proof. These three properties follow from Proposition 3. ■

The next Proposition summarizes the properties of the optimal contract derived thus far.

Proposition 7 *Suppose Assumptions 1-4 are satisfied. Let the current state be s and the continuation value for the borrower* $\bar{V}$.

1. The advancement $k = \inf\left\{k | V^0(s,k) = \min\left(V^u(s), \bar{V}\right)\right\}$. *The advancement will be below the optimal level iff* $\bar{V} < V^u(s)$.

2. If $\bar{V} \geq \delta \int_{\tilde{S}} \tilde{V}(s') F(ds', s)$ *and* $s \in \tilde{S}$, *then it is optimal to set* $V(s') = \tilde{V}(s')$ *and distribute dividends* $\bar{V} - \delta \int_{\tilde{S}} \tilde{V}(s') F(ds', s)$.

3. The boundaries $\tilde{V}(s_t)$ *will be reached almost surely no more than* $T = \log_\delta\left(\bar{V}/M\right) + 1$ *periods.*

4. If, in addition, either $R(s,k) - k \geq L(s)$ *for all* s, k *or the value function* W *is concave in* V, *and provided that* $\bar{V} < \delta \int \tilde{V}(s') F(ds', s)$, *then the optimal continuation policy* $V(s')$ *must satisfy* $\bar{V} = \delta \int V(s') F(ds', s)$, *so no dividends are distributed. In this case, the optimal dividend priority policy satisfies*

$$d\left(\bar{V}, s\right) = \max\left(0, \bar{V} - \int \tilde{V}(s') F(ds', s)\right).$$

5. If $s \notin S^*\left(\bar{V}\right)$ *the firm exits and the lender pays to the firm* $\bar{V} - V^0(s, 0)$ *and keeps* $L(s) - \bar{V}$.

In the optimal long term contract, the borrowing constraints become less severe over time and disappear after a finite number of periods. The relaxation of the borrowing constraints has a simple interpretation for the case of deterministic revenues. The contract gives the firm a certain value, equal to the discounted value of its share of profits. This value, i.e. the anticipation of the firm's share of future profits, is precisely what holds the firm from defaulting. Since incentives for default increase with the amount of working capital advanced, the higher is the share of value of the firm, the more capital can thus be advanced. As the outstanding value of the firm grows over time (at the interest rate), the capital advanced increases towards the unconstrained level. In this way, the increased share of value of the firm provides the *bonding* necessary to raise increasing quantities of working capital.

From the previous analysis, it follows that the optimal contract has implications for the evolution of the shares of value of the borrower and lender over time, i.e. an optimal capital structure. Throughout the period where dividends are not distributed and no exit occurs, the value for the lender evolves as follows:

$$B\left(\bar{V}, s\right) = \Pi\left(\bar{V}, s\right) + \delta \int B\left(V\left(s'\right), s'\right) F\left(ds', s\right)$$

Letting Π_t denote total profits, W_t total value and B_t the value for the lender at time t, and letting $b_t = B_t / W_t$ be the corresponding share of value, it follows that

$$b_t = \frac{\Pi_t}{W_t} + \delta \frac{E\ b_{t+1} W_{t+1}}{E W_{t+1}} \cdot \frac{E W_{t+1}}{W_t}.$$

Since $W_t = \Pi_t + \delta E W_{t+1}$, it follows that

$$b_t - \frac{E\ b_{t+1} W_{t+1}}{E W_{t+1}} \overset{s}{=} \Pi_t. \tag{15}$$

Since $\frac{E\ b_{t+1} W_{t+1}}{E W_{t+1}}$ is the expected future share of the lender (weighted by total value), equation (15) has the following implications for the evolution of the share of the lender.

Proposition 8 *The share of value of the lender will decrease (increase) in expectation if total profits in the current period are greater (less) than zero.*

It is important to emphasize, that this proposition refers to the lender's share of total value, rather than the lender's value B_t. As we show in section 6.2, in absence of revenue shocks it can be established that the latter will also decrease over time. [4]

As in the case of deterministic revenues, when revenues are stochastic and $\bar{V} < \delta \int \tilde{V}\left(s'\right) F\left(ds', s\right)$, no dividends are distributed so the expected value of the borrower grows at the fastest possible rate. However, in contrast to the deterministic case, a non-trivial choice of continuation values must be made, trading off borrowing constraints along different future paths. As we show below, a sharp characterization of this optimal policy can be obtained when the value function is concave. As mentioned earlier, if randomizations are introduced the value function will be concave. However, we provide here conditions that guarantee concavity in the absence of randomizations.

[4] This need not happen when revenues are stochastic, as obtained in numerical results. Inded, when the process governing revenue shocks is mean reverting, these numerical results suggest that for low values of s_t, $EB_{t+1} > B_t$, while for high values of s_t the opposite inequality holds.

Proposition 9 *Suppose the no exit assumption is satisfied and* $\Pi\left(\bar{V}, s\right)$ *is concave in* $\bar{V}$ *for each* s. *Then the total value function* $W\left(\bar{V}, s\right)$ *will be concave in* $\bar{V}$ *for all* s. *Furthermore, if* $\Pi\left(\bar{V}, s\right)$ *is strictly concave for* $\bar{V} < V^u(s)$, *then* $W(\cdot, s)$ *is strictly concave in* $\bar{V}$ *whenever* $\bar{V} < \tilde{V}(s)$.

Proof. Concavity of $W(\cdot, s)$ follows from the concavity of $\Pi\left(\bar{V}, s\right)$ by theorem 9.8 in Stokey and Lucas (1989). Now suppose that $V < \tilde{V}(s)$. Then there exists an n such that $V < V^n(s)$. We will show by induction on n that this implies that $W(\cdot, s)$ is strictly concave in a neighborhood of V. For $n = 1$, $V < V^u(s)$. Since in this region the return function Π is strictly concave, then the value function $W(\cdot, s)$ will also be strictly concave. Suppose now that the result is true for all s and $V < V^{n-1}(s)$ and that $V^u(s) \leq V < V^n(s)$. Letting $V(s')$ be the optimal continuation values, then $V(s') < V^{n-1}(s')$ on a subset of S with positive measure given s. Now, strict concavity follows by the induction hypothesis. ■

Recall that Proposition 1 gives conditions that guarantee concavity and strict concavity of Π.

Under the assumptions of Proposition 9, the value function is concave in $\bar{V}$ and thus almost everywhere differentiable. We now use these facts to derive some further properties of the optimal policy function. For $\bar{V} < \delta \int \tilde{V}(s') F(ds', s)$, the constraint in (11) is binding, and it is easy to see that at an interior solution, $V(s')$ will be chosen so that $W_1(V(s'), s')$ is equalized for all s'. Applying the envelope theorem,

$$W_1\left(\bar{V}, s\right) = \Pi_1\left(\bar{V}, s\right) + W_1(V(s'), s'). \tag{16}$$

This equation must hold for all s' and in particular for $s' = s$. For $\bar{V} < V^u(s)$, $\Pi_1(V, s) > 0$ and consequently, $W_1\left(\bar{V}, s\right) > W_1(V(s), s)$, which implies that $V(s) > V$. In contrast, if $V^u(s) \leq \bar{V} < \tilde{V}(s)$, then $\Pi_1(V, s) = 0$ and $W_1\left(\bar{V}, s\right) = W_1(V(s), s)$, which under strict concavity of the profit function implies, by Proposition 9, that $V(s) = \bar{V}$.

Proposition 10 *Suppose* $W(\cdot, s)$ *is strictly concave in* $\bar{V}$ *whenever* $\bar{V} < \tilde{V}(s)$. *At an interior solution, the following properties of the optimal continuation value policy will be satisfied:*

1. *The continuation values* $V(s')$ *starting from any state* $\left(s, \bar{V}\right)$, *with* $\bar{V} < \delta \int \tilde{V}(s') F(ds', s)$ *are increasing in* $\bar{V}$ *and constant and equal to* $\tilde{V}(s')$, *otherwise.*

2. *If the initial value* $\bar{V} < V^u(s)$, *then the continuation value for the same state* $V(s) > \bar{V}$.

3. *If* $V^u(s) \leq \bar{V} \leq \tilde{V}(s)$, *then the corresponding continuation value* $V(s) = \bar{V}$.

Proof. For part 1, take $\delta \int \tilde{V}(s') F(ds', s) > \bar{V}_2 > \bar{V}_1$ and assume, by way of contradiction, that $W_1(V_2(s'), s') > W_1(V_1(s'), s')$ for some s', where $V_i(s')$ are the optimal continuation values starting from $(\bar{V}_i, s)$. Since derivatives are equalized in this range of values for $\bar{V}_i$, this inequality must hold for all s'. Therefore, $\bar{V}_2 = \delta \int V_2(s') F(ds', s) \leq \delta \int V_1(s') F(ds', s) = \bar{V}_1$, since by concavity of $W(\cdot, s')$, $V_2(s') < V_1(s')$. For values of $V \geq \delta \int \tilde{V}(s') F(ds', s)$, the constraint on the continuation values is not binding and it is feasible to choose $\tilde{V}(s')$. The proofs of parts 2 and 3 follow immediately from equation (16), as detailed above. ■

Notice that Proposition 10 implies that a firm will increase its asset position for the same state next period if, and only if, it is unable to raise the profit maximizing capital in the current period. Though this may at first sight look like a myopic policy, it is indeed dynamically optimal.

Furthermore, equation (16) implies that the derivative $W_1(V_t, s) \geq W_1(V_{t+1}, s')$ for all s and s' in the respective surviving sets, with strict inequality iff $V_t < V^u(s)$. By repeated application of this inequality, this implies that for fixed s, V_t is an increasing sequence. This implies that profits will not decrease over time for any given state. Furthermore, if a firm does not exit at time t in state s_t, then it won't exit anytime in the future if it returns to the same state. Thus, conditional on the revenue shock, firm survival increases over time.

Corollary 2 *Conditional on the revenue state of the firm, profits and survival probability increase with age.*

We now consider the question of how the continuation values $V(s')$ vary with s'. We will show that if $\Pi(\bar{V}, s)$ is supermodular and concave in $\bar{V}$, then $V(s')$ will be nondecreasing in s'. This result and others below require the following regularity condition, which is assumed for the rest of the paper.

Assumption 5 *F is jointly continuous and $F(s', \cdot)$ is nonincreasing.*

The first requirement is a standard continuity requirement. The second condition implies that the conditional probability distribution increases with the current shock in the first order stochastic dominance sense. This is a very standard assumption, which is satisfied by many stochastic processes, in particular by AR1 processes with persistence less than one.

Proposition 11 *At an interior solution:*

1. *If $W(\bar{V}, s)$ is strictly concave in $\bar{V}$ for $V < \tilde{V}(s)$ and $\Pi(\bar{V}, s)$ is supermodular and concave, then $W(\bar{V}, s)$ will be supermodular.*

2. *If $W(\bar{V}, s)$ is strictly concave in $\bar{V}$ for $\bar{V} < \tilde{V}(s)$ and it is supermodular, then any optimal continuation policy $V(s')$ must be nondecreasing. Furthermore, if $s_2 > s_1$ and $V^i(s')$ denotes the optimal policy function starting from $(\bar{V}, s_i)$, then $V^2(s') \leq V^1(s')$.*

Proof. We first prove 2. Consider the problem

$$g(\bar{V},s) = \max \delta \int W\left(V\left(s'\right),s'\right)F\left(ds',s\right) \qquad \text{subject to } \delta \int V\left(s'\right)F\left(ds',s\right) \leq \bar{V}. \tag{17}$$

Let $s_2 > s_1$. We now show that for any pair $\left(\bar{V}, s\right)$ the solution to (17) $V\left(s_2\right) \geq V\left(s_1\right)$. Since W is concave in $\bar{V}$, it is almost everywhere differentiable. The first order conditions of problem (17) imply that $W_1\left(V\left(s_2\right),s_2\right) = W_1\left(V\left(s_1\right),s_1\right)$ at all points where the solution is interior. Otherwise, by strict concavity and supermodularity in (V,s), it follows immediately that

$$\begin{aligned} W_1\left(V\left(s_1\right),s_1\right) &< W_1\left(V\left(s_2\right),s_1\right) \\ &\leq W_1\left(V\left(s_2\right),s_2\right). \end{aligned}$$

Since $V\left(s'\right)$ is an increasing function, $\int V\left(s'\right)F\left(ds',s_2\right) \geq \int V\left(s'\right)F\left(ds',s_1\right)$. Letting $V^i\left(s'\right)$ obtained from initial values $\left(V,s_i\right)$, it follows that $V^2\left(s'\right) \leq V^1\left(s'\right)$ for some set of positive measure. But then, by the strict concavity of W, this inequality must hold for all s'.

We now prove part 1 We will show that the Bellman equation maps the set of supermodular functions into itself. So suppose that we start with a function W that is supermodular and, as assumed, strictly concave for $V < \tilde{V}\left(s'\right)$. It follows immediately that the function g will be strictly concave in $\bar{V}$, for all $\bar{V} < \tilde{V}\left(s\right)$. Using the envelope theorem, it follows that $g_1\left(\bar{V},s_i\right) = W_1\left(V^i\left(s'\right),s'\right)$, where V^i is the optimal solution starting from s_i. Let $s_2 > s_1$ and assume that $\bar{V} < \tilde{V}\left(s_1\right)$. By part 2 it follows that $V^1\left(s'\right) \geq V^2\left(s'\right)$, so $g_1\left(\bar{V},s_1\right) \leq g_1\left(\bar{V},s_2\right)$, and thus g is supermodular. The function W satisfies the following Bellman equation

$$W\left(\bar{V},s\right) = \max\left\{L\left(s\right), \Pi\left(\bar{V},s\right) + g\left(\bar{V},s\right)\right\}$$

where function g is understood to depend on W as defined in (17). Since Π is assumed to be supermodular and g was proved to be so, it follows that the function W is also supermodular. ■

4 Additional Properties

This section provides additional properties of the optimal contract and gives some extensions of the basic theory. Section 4.1 considers in more detail the properties of our selected dividend priority policy once the boundaries have been reached. Section 4.2 considers exit behavior and extends the model by allowing for randomizations. Section 4.3 discusses some implications of the optimal contract on the growth of firms.

4.1 Earnings and Residual Value of the Firm

We consider here the dividend distribution once the region of no default is achieved, i.e. for any state $\left(s, \bar{V}\right)$ such that $\bar{V} = \tilde{V}(s)$. Recall the dynamic programming equation that these boundaries satisfy:

$$\tilde{V}(s) = \max\left(V^u(s), \delta \int_{\tilde{S}} \tilde{V}(s') F(ds', s)\right). \tag{18}$$

It is simple to observe, that the left hand term of the maximization must bind for certain values of s. In the absence of persistence, where $F(s', s) = G(s')$ for all s, then it must be the case that $\tilde{V}(s) = V^u(s)$ for all $s \geq s^*$, for some $s^* \in S$, as shown in Section 6.3. The following Proposition gives conditions under which such cutoff rule will hold more generally.

Proposition 12 *Suppose that $\delta \int_{\tilde{S}} V^u(s') F(ds', s) - V^u(s)$ is decreasing in s. Then $V^u(s) - \delta \int_{\tilde{S}} \tilde{V}(s') F(ds', s)$ is increasing in s, and there exists an s^* such that $\tilde{V}(s) = V^u(s)$ for all $s \geq s^*$ and $\tilde{V}(s) = \delta \int_{\tilde{S}} \tilde{V}(s') F(ds', s)$ for all $s < s^*$.*

Proof. Letting $Z(s) = \tilde{V}(s) - V^u(s)$, equation (18) can be rewritten as

$$\begin{aligned} Z(s) &= \max\left(0, \delta \int_{\tilde{S}} Z(s') F(ds', s) + \right. \\ &\qquad \left. \delta \int_{\tilde{S}} V^u(s') F(ds', s) - V^u(s)\right). \end{aligned} \tag{19}$$

This equation also defines a dynamic programming problem. Using the assumption above, it is easy to verify that the corresponding Bellman equation maps the set of nonincreasing functions into itself, so Z must be a nonincreasing function. This implies that $\tilde{V}(s) - V^u(s)$ is nonincreasing. Since it cannot be the case that $\tilde{V}(s) = \delta \int_{\tilde{S}} \tilde{V}(s') F(ds', s)$ for all s, because $\delta < 1$, $s^* \geq \inf S$, exists, thus proving the Proposition. ■

Remark 1 *When $V^0(s, k) = R(s, k)$, a sufficient condition for the extra assumption given in the Proposition, that $\int_{\tilde{S}} V^u(s') F(ds', s) - V^u(s)$ is decreasing in s, is to assume that for a competitive firm in a homogenous good industry $\int_{\tilde{S}} y(s') F(ds', s) - y(s)$ is decreasing, where $y(s)$ is the optimal output of the firm in state s. This implies that the increments to output are decreasing with the current level of output. Subtracting the unconditional mean of $y(s)$, this assumption is equivalent to mean reversion of the output process.*

Consider now the dividend process $d(s) = \max\left\{0, \tilde{V}(s) - \delta \int_{\tilde{S}} \tilde{V}(s) F(ds', s).\right\}$ Under the assumptions of Proposition 12, $d(s) = 0$ for $s < s^*$ and $d(s) = V^u(s) - \delta \int_{\tilde{S}} \tilde{V}(s') F(ds', s)$ for $s \geq s^*$. The latter is increasing in s as follows from the proof of Proposition 12. Hence, distributed dividends are zero up to s^* and increase with revenues thereafter.

The total value of the firm is $\tilde{W}(s)$, of which in the limit the borrower must hold a fraction equal to at least $\tilde{V}(s)/\tilde{W}(s)$. Retained earnings, as a fraction of the value of the borrower, are then $d(s)/\tilde{V}(s)$, which is equal to zero for $s < s^*$ and equal to $1 - \delta \int_{\tilde{S}} \tilde{V}(s') F(ds', s) / V^u(s)$ otherwise.

4.2 Exit, Randomizations and Concavity

One of the potentially interesting implications of this model, are its predictions about firm survival. Most industries exhibit very large turnover of firms, predominantly for small and young firms. Survival rates are lower for smaller and for younger firms. Proposition 6 implies that the exit set is weakly decreasing in V. Since in the early years the value V is smaller, this suggest that the model will imply that, conditional on s, exit rates will be higher for younger firms.

The possibility of exit introduces a potential non concavity in the total surplus function $W(\cdot, s)$. In that case, randomizations could improve the contract. In particular, randomizations on the exit decision for some values of $\left(\bar{V}, s\right)$ are to be expected. The next Proposition shows that these randomizations are also sufficient.

Suppose that at the beginning of the period, after observing a shock s and when the initial value is $\bar{V}$, a randomization is used and define the new value function W^r by:

$$\begin{aligned} W^r\left(\bar{V}, s\right) &= \max_{\lambda \in [0,1], V_2 \geq 0, V_1 \geq V^0(s,0)} \lambda W(V_2, s) + (1-\lambda) L(s) \\ & \quad \text{subject to } \lambda V_2 + (1-\lambda) V_1 = \bar{V}, \end{aligned}$$

where the function W is defined as in equation (11), but replacing W^r for W under the integral sign. The following Proposition provides conditions under which this function is concave, and thus no further randomizations are required.

Proposition 13 *Suppose $\Pi(V, s)$ is concave. Then, the value function $W^r(\cdot, s)$ is concave for all $s \in S$.*

Proof. The proof follows the lines of the one given for Proposition 9. ■

If shocks are correlated over time, so assumption 5 holds, one may expect that the exit sets will be characterized by thresholds, conditional on V, such exit occurs if and only if shocks are below this threshold. The following Proposition provides such characterization.

Proposition 14 *Suppose $\Pi(V, s)$ is concave in V. Then $W\left(\bar{V}, \cdot\right)$ is weakly increasing in s. Furthermore, if $L(s)$ is constant, there exist weakly decreasing functions $e\left(\bar{V}\right)$, such that a firm will exit if and only if its current state $s < e\left(\bar{V}\right)$.*

Proof. In Proposition 4 we show that W is weakly increasing in V. We prove now that $W(V,\cdot)$ is also weakly increasing in s. The proof uses some advanced concepts in probability theory.

Assume first that we start with a weakly increasing function W on the right hand side of the Bellman equation. Suppose $s_2 \geq s_1$ and let $V^i(s)$ be the optimal continuation values from (V, s_i). Let $\mu_i, i = 1,2$ denote the probability distribution corresponding to $F(\cdot, s_i)$. By assumption 5, μ_2 stochastically dominates μ_1, so there exists a transition function $P(s,s')$ with support on the set $\{(s,s') \in S \times S : s' \geq s\}$ such that $\mu_2 = P\mu_1$. Let v_1 be a probability measure on $S \times \Re_+$ with support on the graph of this function $V^1(\cdot)$ such that $\nu_1(ds, V_1(ds)) = \mu_1(ds)$. It follows by construction that $\int W(V_1(s), s) F(ds, s_1) = \int W(V,s)\nu_1(dV, ds)$. Define a measure ν_2 on $S \times \Re_+$ by lifting ν_1 with P, i.e. for any rectangle set $A \times B$ let $\nu_2(A \times B) = \int P(s,A)\nu_1(ds,B)$. It is easy to verify that since ν_1 has first marginal μ_1 and $\mu_2 = P\mu_1$, then ν_2 will have first marginal μ_2. By the Radon-Nikodym theorem, there exists a transition function $Q : S \times \Re_+ \rightarrow [0,1]$ such that $\nu_2 = Q\mu_2$. $Q(s,.)$ is a probability measure on $\Re_+$ for each s, which may be interpreted as a randomized strategy on continuation values V, given that state s. Allowing for the randomization given in Proposition 13, if necessary, W is concave in V. Consequently, the optimal continuation policy $V^2(s)$, given (s_2, V) must give a payoff no lower than the one obtained from this randomized strategy, i.e.

$$\begin{aligned} \int W\left(V^2(s), s\right) F(ds, s_2) &\geq \int W(V,s) Q(s,V) F(ds, s_1) \\ &= \int W(V,s)\nu_2(dV, ds) \end{aligned} \tag{20}$$

Finally, comparing ν_2 and ν_1,

$$\begin{aligned} \int W(V,s)\nu_2(dV, ds) &= \int W(V,s) P(s', s)\nu_1(dV, ds') \\ &\geq \int W(V,s')\nu_1(dV, ds'), \end{aligned} \tag{21}$$

where the last inequality follows from the fact that $P(s',\cdot)$ has support on values $s \geq s'$ and the induction assumption that W is weakly increasing in s. Combining equations (20) and (21), it follows that the Bellman equation maps the set of nondecreasing functions into itself, and thus W is nondecreasing.

Now let $e\left(\bar{V}\right) = \sup\left\{s | \Pi\left(\bar{V}, s\right) + g\left(\bar{V}, s\right) < 0\right\}$, where function g is as defined in the proof of Proposition 11, and if this set is empty set it equal to $\inf S$. It is immediate to check that this function is weakly decreasing and gives the exit thresholds. ■

Remark 2 *The condition that $L(s)$ is constant can be replaced by $\Pi(V,s) - L(s) + \delta \int L(s') F(ds', s)$ is nondecreasing in s. Details are available from the authors by request.*

Together with Proposition 10, the above Proposition implies that if $s_t \in S^*(V_t)$, then for exit to occur after time t, a state $s < s_t$ must be reached. Thus, according to our theory, limited enforcement contributes to a positive relationship between firm survival and age.

4.3 Growth of firms

How does the optimal contract affect the growth properties of a constrained firm? While the firm is in a set where $\bar{V} < V^u(s)$, firms will grow with age. Once a region is reached where $\bar{V} = \tilde{V}(s)$, age will have no further growth effect. This implies that younger firms will, in average, grow faster. This result does not require mean reversion of productivity shocks or learning, as in models of pure stochastic evolution, like Hopenhayn (1992a) and Jovanovic (1982).

Consider now two firms with the same initial value $\bar{V}$ but two different levels of shock $s_2 > s_1$. Suppose that both firms reach in the following period the same state s. Obviously the one in the lower state will grow faster. Moreover, under the assumptions of Proposition 11, and as a consequence of part 2 of the Proposition, the continuation policies $V^1(s) \geq V^2(s)$. This further enhances the difference in growth rates by part 1 of Proposition 10.

5 Equilibrium contracts with competing lenders

Suppose the firm is free, at the beginning only, to engage in one long term contract with any lender in a competitive market. This should obviously drive the share of the surplus of the lender to zero. Suppose the initial shock of the firm s_1 is picked randomly from distribution G. The firm contracts with the lender prior to the realization of this shock. If V_0 is the initial value given by the contract to the firm, then $V(s)$ will be chosen so that

$$\begin{aligned} W_0(V_0) &= \max_{V(s) \geq V^0(s,0)} \delta \int W(V(s), s)\, G(ds) \\ & \quad \text{subject to } \delta \int V(s)\, G(ds) \geq V_0. \end{aligned}$$

$W_0(V_0) - V_0 - I_0$ is the initial value to the lender, net of the initial setup cost I_0. In equilibrium, V_0 is the supremum of the set of initial values that give nonnegative value to the lender. Notice that, in particular, this implies that V_0 must be in a region where $W_0(V_0) - V_0$ is decreasing. Furthermore, if the value function W is concave, then W_0 will also be a concave function of V_0 and thus $W_0'(V_0) = W_1(V(s), s) < 1$. Additionally, letting V_t denote the value after t periods for a given path of shocks, this also implies that $W_1(V_t, s) \leq W_1(V_0, s)$, so there will be no mutual gains from renegotiating the contract in the future.

If the set $\{V_0 | \; W_0(V_0) - V_0 - I_0 \geq 0\}$ is empty, then there is no feasible lending contract. In this case, no contract is possible unless the firm has funds to contribute to this initial investment. More specifically, letting $\bar{B} = \max_{V_0}(W_0(V_0) - V_0)$, the firm must contribute at least $I_0 - \bar{B}$. Notice that in this case a weaker enforcement structure, defined by higher values $V^0(s,k)$, will reduce the total surplus of the project and thus require a higher initial investment by the firm.

Bulow and Rogoff (1989b) consider the feasibility of long term lending contracts between a lending and a borrowing country with one sided commitment. Their results apply to our setup, so we will restate their main proposition in terms of our notation. Suppose that if the borrower defaults, it cannot continue operating in the industry, and is excluded from further borrowing, but may save or buy insurance. In particular, the firm may write a contract where it pays currently A_t in exchange for contingent claims $a(s')$ in the following period such that the following two conditions are met:

$$\begin{aligned} \int a(s') F(ds', s) &= A_t/\delta, \\ a(s') &\geq 0, \end{aligned}$$

where A_t is limited by the stock of liquid assets the firm has at time t. Bulow and Rogoff call this a *cash-in-advance contract.* Suppose additionally that if a firm defaults, it must pay a penalty that has a present value of $C(s,k)$. In terms of our model, $V^0(s,k)$ corresponds to the value the firm gets from following a cash in advance contract starting with a shock s and initial capital k, net of the cost $C(s,k)$. The following proposition gives a tight upper bound to the value of the lender in any feasible contract.

Proposition 15 *(Bulow and Rogoff, Theorem 2.) In a feasible contract,* $B\left(\bar{V}, s\right) - C(s,k) \leq 0.$

In particular, if $C(s,k) = 0$, i.e. there is no penalty for default, and after defaulting firms have access to cash in advance contracts, there is no possibility of borrowing and lending.

Fernández and Rosenthal (1990) consider a model of debt repayment where the borrower obtains a reward after all debt has been repaid. Debt renegotiation is modelled as a dynamic game. There is an initial level of debt forgiveness f which leaves the borrower indifferent between following the optimal repayment strategy or choosing autarky, conditional on no future reductions in the outstanding debt. All subgame perfect equilibria are payoff-equivalent to one where the lender reduces the initial debt precisely by this amount f and the borrower repays the remaining debt optimally.

A parallel to their model can be established by letting the value of default $V^0(s,k)$ equal the value of autarky net of the discounted value of the future reward of full repayment. In our model, the implicit level of debt is the value $B(V_t, s_t)$. Debt forgiveness occurs whenever $B(V_{t+1}, s_{t+1}) < (B(V_t, s_t) - \Pi(s_t, V_t) + d_t)/\delta$. Since $EB(V_{t+1}, s_t) = (B(V_t, s_t) - \Pi(s_t, V_t) + d_t)/\delta$, periods of debt forgiveness -as well as debt appreciation-would occur with positive probability, unless the distribution for future shocks is degenerate. A closer comparison can be obtained if we consider the special case where the shocks obtained initially with distribution $G(s_0)$ are permanent. Letting I_0 represent the initial value of debt, as above, $I_0 = \int B(V_0(s_0), s_0) G(ds_0)$, where $B(V_0(s_0), s_0) = W(V_0(s_0), s_0) - V_0(s_0)$; debt forgiveness will occur for all initial states where $I_0 > B(V_0(s_0), s_0)$.

6 Special cases

In this section we work out the properties of the optimal debt contract for three special cases. We consider first a case which is closely connected to Hart and Moore, where the value of default only depends on the revenue shock. The case where there is no uncertainty is considered next. Finally, we analyze the where revenue shocks are i.i.d.

6.1 Value of default independent of capital advancement

Suppose the value of default depends only on the revenue shock s and not on the capital advanced. Denote this value by $V^0(s)$. It follows immediately that the optimal capital advancement will be supplied each period, so $\Pi\left(\bar{V}, s\right) = \pi(s)$ and $V^u(s) = V^0(s)$. Using equation (13) it follows that

$$\tilde{V}(s) = \max\left\{V^0(s), \delta \int \tilde{V}(s') F(ds', s)\right\}. \tag{22}$$

These are now the lowest values needed for each state to guarantee that the contract is not terminated too soon. If $\bar{V}_0 < \int \tilde{V}(s) G(ds)$, there is positive probability that a point will be reached where the value granted to the firm $V_t < V^0(s_t)$, so the project will be inefficiently terminated. Notice also that, according to our previous results, dividends will not be distributed until the efficient boundary given by (22) is reached.

Hart and Moore (1994) consider a case where the threat of repudiation by the entrepreneur is used to determine, via bargaining, its continuation value. Though their model is deterministic, it can be easily extended to our setup. In their model, the outside value for the entrepreneur is zero, so the liquidation value for the bank is $L(s)$. Letting $\tilde{W}(s)$ denote the optimal value for the project at state s, and assuming -as in their model- that as a result of bargaining the lender gets $B(s) = \max\left\{\frac{1}{2}\tilde{W}(s), L(s)\right\}$, the value for the borrower is $V^0(s) = \tilde{W}(s) - B(s) = \min\{\frac{1}{2}\tilde{W}(s), \tilde{W}(s) - L(s)\}$. Assuming that $\tilde{W}(s) > L(s)$, the project will not be terminated. Since it is not feasible to grant the borrower less than $V^0(s)$, the initial value to the borrower V_0 must be greater or equal to $\int \tilde{V}(s) G(ds)$. If the total ex-ante value of the project $W_0 = \int \tilde{W}(s) G(ds) < V_0 + I_0$, the lender cannot break even in the contract unless the borrower contributes the amount $W_0 - I_0 - V_0$ to the initial investment. The dividend priority policy implied by (22) is

$$d(s) = \max\left\{0, V^0(s) - \delta \int \tilde{V}(s') F(ds', s)\right\}$$

and the corresponding net cash-flow to the bank is

$$p(s) = \pi(s) - d(s) = \min\left\{\pi(s), \pi(s) - \delta \int \tilde{V}(s') F(ds', s)\right\}. \tag{23}$$

This corresponds to the slowest repayment schedule, as defined by Hart and Moore (1994).[5]

[5]It is instructive to show how Hart and Moore's model can be accomodated in our setup. In their

6.2 No Uncertainty

When there is no uncertainty the optimal policy rule for the lender wishing to maximize the total surplus of the match is to let the borrower's equity grow at the highest possible rate, which is exactly the rate of interest when no dividends are distributed. The unconstrained maximum value of the match is thus reached in the least feasible number of periods. Letting $\bar{V}_t$ be the continuation value for the firm at time t, this optimal policy implies that $\bar{V}_{t+1} = \min\left(\bar{V}_t/\delta, V^u\right)$ Once the enforcement bound $\tilde{V} = V^u$ is reached, the firm gets a constant dividends equal to $(1-\delta)V^u$.

Assuming that the indirect profit function $\Pi(V)$ is concave, the value function will also be concave. In this case, a renegotiation free contract requires that $W'(V_t) < 1$ for all t. Hence, the value $B_t = W(V_t) - V_t$ for the lender will decrease over time. Since $B_t = \Pi(V_t) + \delta B_{t+1}$, this implies that $\Pi(V_t) > (1-\delta)B_t$. In particular, at time 0, $\Pi(V_0) > (1-\delta)B_0 \geq 0$, with strict inequality if the lender contributes to an initial investment $I_0 > 0$. Since V_t increases over time, $\Pi(V_t)$ will be positive every period, so the cash flow of the contract will be positive every period. As a consequence, the lending contract can be decomposed in two parts, a long term debt used to finance the initial investment I_0 and short term advancements of working capital every period k_t. Each period the lender pays back the working capital advanced and contributes the quantity $\Pi(V_t)$ to the repayment of the long term debt. Once the boundary V^u has been reached, the borrower keeps $(1-\delta)V^u$ as dividend and pays the constant amount $\Pi(V^u) - (1-\delta)V^u$ as interest on the outstanding long term loan.

Under some conditions, the positive cash-flow allows the borrower to self-finance the working capital without the need of short term advancements. In particular, if $R(k_t) - \delta k_{t+1} \geq 0$, the optimal contract can be implemented by giving the borrower an initial long term loan $B_0 = I_0 + k_0$. The borrower makes repayments $R(k_t) - \delta k_{t+1}$ in period t, saving the quantity δk_{t+1} at the gross interest rate $1/\delta$, thus securing k_{t+1} for the following period. Once the enforcement bound is reached, the borrower pays every period to the lender $R(k^*) - \delta k^* - (1-\delta)V^u$, where k^* is the profit maximizing amount of capital. Furthermore, it can be easily established that the payments to the bank, $R(k_t) - \delta k_{t+1}$ will be decreasing (increasing) if $R'(k_t)$ is smaller (larger) than $V^{0\prime}(k_t)/V^{0\prime}(k_{t+1})$. As in the contracts studied by Fernández and Rosenthal (1990) and Hart and Moore (1994), in this case, the enforcement problem translates into a constraint on the initial loan and a particular timing of the repayment of this loan.[6]

As an example, consider the case where $V^0(k) = R(k)$, i.e. where the borrower can

model, the project's profits follow a deterministic path, and become zero after some finite time T, when the project ends. Letting $S = \{1, 2, ..., T\}$, the initial distribution $G(1) = 1$, the transition function $F(t+1,t) - F(t,t) = 1$ for $t < T$ and $F(T,T) - F(T-1,T) = 1$,defining $\pi(T) = 0$ and the residual value at time T, $L(T) = 0$, their model is mapped into our structure. The payment schedule $p(t)$ derived from (23) corresponds exactly to their slowest repayment schedule.

[6]The firm's choice of reinvestment of earnings parallels the capital accumulation decision in Fernández and Rosenthal (1990). Forgiveness would occur if $I_0 > W(V_0) - V_0$. This could happen if, as noted in section 5 there are some initial shocks to the value of the project.

run away with the revenues of the firm. Since $0 \leq R(k_{t+1}) - k_{t+1}$, and $\delta V^0(k_{t+1}) = V^0(k_t) = R(k_t)$, it follows that $0 \leq R(k_t) - \delta k_{t+1}$, so the optimal contract can be implemented with a long term loan. Furthermore, if R is concave, and given that prior to reaching the enforcement bounds $R'(k_t) > 1$, then $V^{0\prime}(k_t) / V^{0\prime}(k_{t+1}) = R'(k_t) / R'(k_{t+1}) < R'(k_t)$, implying that the repayment of the long term debt increases over time.

Starting from an initial value $\bar{V}_0$, the number of periods T needed to reach this enforcement bound is the smallest integer greater than $\log_\delta \left(\bar{V}_0 / V^u\right)$. Once this maturity stage is reached the total surplus of the contract is $\tilde{W} = \pi / (1 - \delta)$, and growth stops. At this point the lender gets $\tilde{W} - V^u$, and the firm gets the remaining, V^u. Since T is finite, we can recover the value function by working backwards from $\tilde{V}$. Clearly,

$$W\left(\bar{V}\right) = \sum_{t=0}^{T-1} \delta^t \Pi(V_t) + \delta^T \tilde{W}$$

with $V_t = \delta^{-t} \bar{V}$. The initial value of $\bar{V}$ will have to be such that $W\left(\bar{V}\right) - \bar{V} > 0$, and that

$$W_1\left(\bar{V}\right) = \sum_{t=0}^{T-1} \Pi_1(V_t) \leq 1,$$

otherwise the contract would not be profitable for the lender, or there would be gains from renegotiating the contract. The welfare loss due to the lack of enforcement is given by:

$$DWL = \sum_{t=0}^{T} \delta^t \left(\pi - \Pi\left(\bar{V}_0 / \delta^t\right)\right).$$

6.3 The *iid* Case

When shocks are i.i.d., equity will also grow on average at the rate of interest until the efficient boundary is reached. However, there is a nontrivial choice in terms of how equity grows at different states. Letting F denote the distribution for revenue shocks, the continuation values $V(s')$ satisfy the equation

$$\int V(s') F(ds') = \bar{V} / \delta.$$

The values $V(s')$ grow over time until a state is reached where $V(s') = \tilde{V}(s')$. Letting $\tilde{V} = \int_{s^e} \tilde{V}(s) F(ds)$, where s^e is the exit threshold, it follows that

$$\tilde{V}(s) = \max\left(V^u(s), \delta \tilde{V}\right).$$

Assuming that $V^u(s)$ is an increasing function, there exists some value s^* such that $\tilde{V}(s) = V^u(s)$ for $s \geq s^*$ and $\tilde{V}(s) = \delta \tilde{V}$ otherwise. This implies that $\delta \tilde{V} = V^u(s^*)$ and thus the following equation gives the unique solution to s^*

$$V^u(s^*) = \delta \left\{ \int_{s^*} V^u(s) F(ds) + \left(F(s^*) - F(s^e)\right) V^u(s^*) \right\}$$

Rearranging terms we get

$$V^u(s^*)(1-\delta(1-F(s^e)))=\delta\int_{s^*}(V^u(s)-V^u(s^*))F(ds).$$

Recall that no dividends will be paid when $s \leq s^*$. For higher shocks, dividends will be given by $d(s)=V^u(s)-V^u(s^*)$. Some illustrative comparative statics can be derived from this equation. Consider the case where firms face a fixed cost which is independent of their revenue shock. Then an increase in this fixed cost would obviously increase the exit point s^e and thus decrease s^*. Notice that this implies an increase in the region over which the firm distributes dividends and also an increase in the dividends distributed. At the boundaries, the equity share of the firm would also be larger. Consider a change in δ for the case when $F(s^e)=0$, i.e. when there is no exit. A decrease in δ will also lead to a reduction in s^*, and thus higher distributed dividends and equity share of the firm at the boundaries.

We now examine some properties of the growth process for firms prior to reaching the boundaries. Suppose the value for the firm at some initial period is $\bar{V}$. Consider two different revenue shocks $s_i > s_j$ for the following period and the corresponding optimal continuation values V_i and V_j. Assuming that $\Pi(V,s)$ is supermodular and concave, by Proposition 11, $V_i \geq V_j$. Optimality requires that

$$W_1(V_i,s_i)=W_1(V_j,s_j).$$

For any shock s_k, the envelope condition implies that:

$$\begin{aligned}W_1(V_i,s_i) &= \Pi_1(V_i,s_i)+W_1(V_{ik},s_k)\\ &= \Pi_1(V_j,s_j)+W_1(V_{jk},s_k)\\ &= W_1(V_j,s_j),\end{aligned}$$

for optimal continuation values V_{ik} and V_{jk} starting from V_i and V_k, respectively. Given the i.i.d. assumption, it is easy to check that $V_{ik} \geq V_{jk}$. By concavity, it follows that $W_1(V_{jk},s_k) \geq W_1(V_{ik},s_k)$ and consequently $\Pi_1(V_i,s_i) \geq \Pi_1(V_j,s_j)$. This implies that, at the margin, starting from a point where $\bar{V} < \delta\tilde{V}$, a firm will be more liquidity constrained in the following period for higher revenue shocks than for lower ones.

7 Incomplete Contracts and Bargaining

The previous analysis assumes that the lender can commit to the contract. Incomplete contracts have been emphasized in much of the current literature on theory of the firm. How is the optimal contract modified if the lender cannot commit to a long term contract and each period the surplus is distributed through a bargaining process? We will consider here the extreme case where the lender has all the bargaining power and each period makes a take-it-or-leave-it offer to the firm.

This problem defines a dynamic game between the firm and the lender. Each period, the lender is a Stackleberg leader and proposes a take-it-or-leave-it continuation contract to the firm. If the firm accepts, then the lender provides the corresponding capital k. After production takes place, the firm decides how much of total revenues it transfers to the lender.

With no restrictions on the strategy space, the equilibrium set could be quite large and difficult to characterize. However, we will show that if the value of default is sufficiently large, the optimal long term contract derived above can be supported as a subgame perfect equilibrium. Assume that $V^0(s,k) \geq R(s,k) - k$. These conditions imply that there is no capital advancement k and net transfer τ such that the lender breaks even in the current period and the firm obtains dividends in excess of the value of default. We also assume that $L(s) = V(s,0) = 0$. The following subgame perfect equilibrium strategies give zero value to both players: the lender always proposes a contract with zero value to the firm and sets $k = 0$; the firm always chooses to default. As usual, this bad equilibrium can be used to support some cooperation. The following Proposition gives conditions under which the optimal contract with one sided commitment can be also supported.

Proposition 16 *Suppose that* $V^0(s,k) \geq R(s,k) - k$, $W\left(\tilde{V}(s), s\right) - \tilde{V}(s) \geq 0$ *for all* $s \in \tilde{S}$ *and* $W(\cdot, s)$ *is concave. Then there exists a subgame perfect equilibrium of the dynamic game which supports the optimal contract with one-sided commitment.*

Proof. The assumptions imply that $W\left(\bar{V}, s\right) - \bar{V} \geq 0$ for all $\bar{V} \leq \tilde{V}(s)$, since we started the contract with $W_1(V,s) \leq 1$. Thus, this contract guarantees the lender non-negative surplus. Following any deviation from this contract by the lender or the firm, let both players use the strategies that give an equilibrium with zero value. It is easy to verify that these are equilibrium strategies for the subgame starting after any history as they provide no incentives for deviating from the original contract at any decision node. ■

Remark 3 *We have assumed that the value for both players is zero after separation, which is used to get a threat point with zero value. A more general -but messier- result can be given for the case in which threat points may be positive and dependent on* s, *by relating the assumptions in the proposition directly to the corresponding values at the threat points.*

We now provide results when strategies are restricted to be functions of the state s only. First notice that if the bank cannot commit to exclude the firm when it defaults, in a Markov equilibria where strategies cannot depend on the default history, the firm would always choose to default, and no contract is possible. We thus restrict our analysis to the case where such minimal commitment is possible. Let $B(s)$ denote the expected discounted value to the lender at state s and $V(s)$ the corresponding value to the firm.

Each period the lender chooses k and τ to maximize current profits $\tau - k$, subject to the no-default and non-negative cash flow constraints for the firm. Then $B(s)$ satisfies the following dynamic programming equation:

$$\begin{aligned} B(s) &= \max_{\tau,k} \tau - k + \delta \int B(s') F(ds', s) \qquad (24) \\ \text{subject to } \tau &\leq \min\left(R(s,k), R(s,k) - V^0(s,k) + \delta \int V(s') F(ds', s)\right). \end{aligned}$$

where the constraints are identical to the ones derived before. We will refer to the solution of the above problem as the *optimal time consistent policy* for the lender (see Chari and Kehoe (1993)). Letting $\kappa(s)$ and $\tau(s)$ denote the optimal solutions, $V(s)$ satisfies the functional equation:

$$V(s) = R(s, \kappa(s)) - \tau(s) + \delta \int V(s') F(ds', s).$$

We now provide a complete characterization of the solution. There are three possible cases, which are considered in turn.

Case 1. The first part of the constraint in equation (24) does not bind at the optimum. Then it follows that

$$\kappa(s) = k_1(s) = \arg\max_{k \geq 0} R(s,k) - V^0(s,k) - k, \qquad (25)$$

$$V^0(s, k_1(s)) > \delta \int V(s') F(ds', s) \qquad (26)$$

$$V(s) = V^0(s, \kappa(s)). \qquad (27)$$

Case 2. The first constraint binds at the optimum but the second one does not. In this case, $\kappa(s) = K(s)$, the solution to the static profit maximization problem, and

$$V^0(s, K(s)) < \delta \int V(s') F(ds', s). \qquad (28)$$

Case 3. Both constraints bind at the optimum and thus $\kappa(s)$ satisfies

$$V^0(s, \kappa(s)) = \delta \int V(s') F(ds', s).$$

Note also that in both, cases 2 and 3,

$$V(s) = \delta \int V(s') F(ds', s).$$

When will each of these cases hold? If condition (28) is satisfied, then the first best can be attained and all profits captured by the bank, and thus Case 2 will prevail. If condition (28) is not satisfied, but condition (26) holds, then the solution will fall in Case

1. If neither of these conditions is verified, then Case 3 will prevail. Finally, note that if $k_1(s) \leq K(s)$ -which would occur, for example, when the objective in (25) is concave-, then Case 1 will hold if and only if condition (26) is satisfied. Furthermore,

$$V(s) = \max\left(V^0(s, k_1(s)), \delta \int V(s') F(ds', s)\right). \tag{29}$$

We have thus proved the following Proposition.

Proposition 17 *Suppose that $B(s) \geq 0$, and that $k_1(s) \leq K(s)$. Then the optimal time consistent policy for the lender is to offer the firm the equity value $V(s)$ as defined in (29) and current dividends*

$$d(s) = \max\left(0, V^0(s, k_1(s)) - \delta \int V(s') F(ds', s)\right).$$

Since $k_1(s) \leq K(s)$, and $V^0(s,k)$ is increasing in k, the value of the firm in a contract where strategies can only depend on the current shock of the firm, $V(s)$, is smaller, for each s, than the corresponding level at the enforcement bounds, $\tilde{V}(s)$. The restriction to Markov-perfect equilibria thus reduces the share of the project collected by the firm, at least in the long term.

Some special cases of interest are now examined. First, suppose that the firm can choose to default on the contract by "running away " with the revenues $R(s,k)$. In this case, $V^0(s,k) = R(s,k)$. It is easy to see that $k_1(s) = 0$, and thus Case 1 could never apply. But then the unique solution to functional equation (29) is $V(s) = 0$, and thus no contract with positive value is feasible.

Consider now the case studied in section 6.1, where V^0 is independent of k. It is immediate to verify that equation (29) is identical to (22), so the boundaries are exactly the same as those derived under one sided commitment.

Finally, consider the special case where the firm gets a constant value θ when it defaults, i.e. where $V^0(s,k) = \theta$. It immediately follows that $k_1(s) = K(s)$. We now show that there is a feasible contract that gives the firm a value θ and produces every period the profit maximizing output. This contract is obviously optimal for the bank. Letting $V(s) = \theta$, then $V^0(s, K(s)) = \theta > \delta\theta$, and thus case 1 is always verified. The firm receives every period a dividend $(1-\delta)\theta$ and thus has no incentive to default. The bank, in turn, cannot offer a lower dividend in any period.

Fernández and Rosenthal (1990), also analyze the question of debt renegotiation in a repeated game. In their renegotiation game, the lender can propose each period an alternative repayment schedule. However, the borrower has always the option of rejecting this offer and sticking to the status quo contract. This status quo contract is replaced by the one proposed by the lender only with the consent of the borrower. In the special case considered in section 6.2, where payoffs are deterministic and firms can self-finance from retained earnings, our contract design problem becomes a debt repayment problem. The equilibrium repayment schedule that Fernández and Rosenthal derive in their paper, coincides with our optimal long term contract with one sided commitment. Though they do not explicitly assume that the lender has such commitment, the ability of the borrower to invoke the current status quo contract provides the necessary commitment device.

8 Final remarks

In this paper we have developed a general model of borrowing constraints based on the idea of limited enforcement. In our model, borrowing constraints arise as part of the optimal borrowing and lending contract. Our model extends previous theories of borrowing and lending, such as Hart and Moore (1994), allowing for uncertainty and dynamic effects of the resulting credit constraints. In particular, the model has implications for firm growth and survival, implying that younger firms tend to grow faster and have lower survival rates. Both properties are consistent with empirical regularities.

We have kept our analysis at a fairly abstract level in order to describe the general properties of a class of models of borrowing and lending based on the idea of limited commitment and limited liability. We certainly believe that our structure is very flexible and could be used to develop more specialized models. To illustrate this point, we have considered several special cases -including Hart and Moore's setup.

The paper also highlights the value of long term borrowing/lending relationships. In absence of commitment, we have shown that history dependent strategies in such dynamic relationship can be used to support higher lending. Thus, the paper provides a framework to asses the value of these long term relationships. Kiyotaki and Moore (1996), study the macroeconomic implications of limited borrowing in a model where firms' collateral is subject to endogenous fluctuations. In the presence of sunk costs of investment or when some assets -such as human capital- are not fully expropriable, loans cannot be fully collateralized with assets. In such cases, long term contracts of the sort described here may provide an added value. The macroeconomic implications of such contracts can be an interesting question.

There are several interesting possible extensions of the theory. One of them, is to allow for capital accumulation. As in Hart and Moore (1994), in our model investment only occurs at time zero. All lending activities that follow are restricted to the provision of working capital. Allowing for capital accumulation, or other sources of investments such as R&D, would significantly enhance the model. This could be done in our framework, for example, by allowing the conditional distribution of random shocks for the following period dependent on the amount of working capital advanced. We have not explored in this paper the general equilibrium implications of the type of borrowing constraints considered. This is obviously another interesting direction for further research.

In this paper we treat the value of default as exogenous, with fairly general properties to accommodate most existing models. In an equilibrium framework, the value of default will not be exogenous and should in turn be influenced by the optimal contract. There are many alternative ways of *closing* a model of the sort developed here, some of which have been already explored in the literature (see, for instance, Ghosh and Ray, 1996 and 1997.)

As an example, consider a situation where, upon defaulting, borrowers can enter into new long term contracts with other lenders, at some additional cost. An equilibrium contract would require taking into account this endogeneity. This is obviously a very interesting area of research. Though our paper does not address this equilibrium problem, the general results obtained here could still prove very useful in developing that research program.

There are obviously alternative ways of modeling borrowing constraints which are worth exploring. In particular, informational asymmetries, absent in this model, are an obvious alternative source of frictions to generate credit constraints. Diamond (1990, 1991a, 1991b) considers the relationship between the unobservable quality of the borrower's project and the maturity/seniority structure of debt. Green (1987) analyzes debt contracts in a repeated environment when agents have unobserved endowments. Marcet and Marimon (1992) and Atkeson (1991) study the effect of moral hazard on growth in the context of international lending. Developing models of firm growth and survival based on such foundations is also an important direction of research in this area. Recent work by Gomes (1997) suggests another promising line of research. Gomes models borrowing constraints by introducing a parametrized cost of funds function, with an increasing marginal cost, in an industry equilibrium model. This is obviously an alternative to deriving the borrowing constraints from fundamentals, as pursued in this paper. It is however an interesting complementary direction of research, which can shed light on the type of borrowing constraints that seem more plausible candidates to explain the observed facts.

A Additional Proofs

Lemma 2 *There exists a dividend priority policy.*

Proof. Given an initial vector $\left(\bar{V}, s\right)$, let

$$v_0 = \inf\left\{\int V_n\left(s'\right) F\left(ds', s\right) | V_n\left(s'\right) \text{ is a solution to 11}\right\}.$$

Without loss of generality, assume $v_0 < \bar{V}$. Let $V_1 \neq V_2$ be two optimal solutions such that $\int V_i\left(s'\right) F\left(ds', s\right) < \bar{V}$, $i = 1, 2$. Suppose there exists a set A of positive probability such that for all $s' \epsilon A$, $W\left(V_1\left(s'\right), s'\right) < W\left(V_2\left(s'\right), s'\right)$. This implies that for all $\epsilon < \int_A F\left(ds', s\right)$ there exists a subset $A_\epsilon \subset A$ with $0 < \int_A F\left(ds', s\right) < \epsilon$ such that

$$\int_{A_\epsilon} W\left(V_1\left(s'\right) F\left(ds', s\right)\right) < \int_{A_\epsilon} W\left(V_2\left(s'\right) F\left(ds', s\right)\right)$$

Pick ϵ such that $\epsilon \left\|V_1 - \tilde{V}\right\| < \bar{V} - \int V_1\left(s'\right) F\left(ds', s\right)$. Consider a function $V\left(s'\right) = V_1\left(s\right)$ on $S \backslash A_\epsilon$ and $V\left(s'\right) = \min\left(\tilde{V}\left(s'\right), V_2\left(s'\right)\right)$ on A_ϵ. It is straightforward to check that $V\left(s'\right)$ is a feasible policy from $\bar{V}, s$ and that it gives a higher expected value for W than V_1, thus contradicting the assumption that the latter is an optimal solution. This implies that the set of states $s' \in S$ such that $W\left(V_2\left(s'\right), s'\right) > W\left(V_1\left(s'\right), s'\right)$ has zero conditional probability given s. Applying a symmetric argument, it follows that $W\left(V_1\left(s'\right), s'\right) = W\left(V_2\left(s'\right), s'\right)$ almost surely.

Now choose sequences V_n of optimal solutions such that $\int V_n\left(s'\right) F\left(ds'\right) \to v_0$. Let $V\left(s'\right) = \inf_n V_n\left(s'\right)$. Wince $W\left(V, s\right)$ is continuous in V, by Lebesgue's dominated convergence theorem it follows that V is also an optimal solution, and $\int V\left(s'\right) F\left(ds', s\right) = v_0$. ■

References

[1] Aghion, Philippe and Patrick Bolton (1992), "An Incomplete Contracts Approach to Financial Contracting," *Review of Economic Studies*, 59, 473-494.

[2] Atkeson, Andrew (1991) "International Lending with MOral Hazard anr Risk of Repudiation." *Econometrica,* 59, pp. 1069-1090.

[3] Bulow, Jeremy and Kenneth Rogoff (1989a), "A Constant Recontracting Model of Sovereign Debt," *Journal of Political Economy*, 97 (1), 155-178.

[4] Bulow, Jeremy and Kenneth Rogoff (1989b), "Sovereign Debt: Is to Forgive to Forget?," *American Economic Review*, 79:43-50.

[5] Chari, V. V., and Patrick Kehoe (1993), "Sustainable Plans and Debt," *Journal of Economic Theory*, 61, 230-261.

[6] Davis, Steven J., John C. Haltiwanger and Scott Schuh (1996), *Job Creation and Destruction,* MIT Press, Cambridge, Mass.

[7] Diamond, D. (1990), "Seniority and Maturity Structure of Bank Lonas and Publicly Traded Debt, " University of Chicago working paper, 1990.

[8] ——— (1991a), "Monitoring and Repudiation, The Choice between Bank Lonas and Directly Placed Debt," *Journal of Political Economy,* **99** , 689-721.

[9] ——— (1991b), "Debt Maturity Structure and Liquidity Risk," *Quarterly Journal of Economics,* 106, 709-37.

[10] Dunne, Timothy, Mark Roberts, and Larry Samuelson (1989a) "The Growth and Failure of U.S. Manufacturing Plants, " *Quarterly Journal of Economics,* 104, no. 4:671-98.

[11] —————. (1989b) "Plant Turnover and Gross Employment Flows in the U.S. Manufacturing Sector." *Journal of Labor Economics* 7, no.1: 48-71.

[12] Eaton, J. and M. Gersovitz (1981), "Debt with Potential Repudiation: Theoretical and Empirical Analysis," *Review of Economic Studies* **48,** 289-309.

[13] Evans, David. (1987) "The Relationship between Firm Growth, Size and Age: Estimates for 100 Manufacturing Industries." *Journal of Industrial Economics* 35, no.4: 567-81.

[14] Fazzari, Steven M., R. Glenn Hubbard and Bruce C. Petersen (1988), "Financing Constraints and Corporate Investment," *Brookings Papers on Economic Activity*, 1, 141-206.

[15] Fernández, R. and R.W. Rosenthal (1990). "Strategic Models of Sovereign-Debt Renegotiations," *Review of Economic Studies* 57:331-350.

[16] Gertler, Mark and Simon Gilchrist (1994), "Monetary Policy, Business Cycles, and the Behavior of Small Manufacturing Firms," *Quarterly Journal of Economics*, 109 (2), 309-340.

[17] Ghosh, P. and D. Ray (1996) "Cooperation in Community Interaction Without Information Flows," *Review of Economic Studies,* **63**. 491-519.

[18] ——— (1997) "Cooperation and Trust Among Anonymous Agents: Application to Informal Credit," unpublished manuscript, Department of Economics, Boston University.

[19] Green, E. (1987) "Lending and the Smoothing of Uninsurable Income," in *Contractual Arrangements for Intertemporal Trade,* University of Minnesota Press, Minneapolis, Minnnesota.

[20] Gomes, Joao (1997) "Financing Investment," unpublished manuscript, University of Rochester.

[21] Harris, M., and B. Hölmostrom (1982), "A Theory of Wage Dynamics," *Review of Economic Studies,* 56, 315-34.

[22] Kehoe, Timothy J. and David K. Levine (1993), "Debt-Constrained Asset Markets," *Review of Economic Studies*, 60, 865-888.

[23] Hall, Bronwyn. 1987. "The Relationship between Firm Size and Firm Growth in the U.S. Manufacturing Sector." *Journal of Industrial Economics* 35, no. 4: 583-606.

[24] Hart, Oliver (1995), "Firms, Contracts and Financial Structure," Clarendon Lectures in Economics, Clarendon Press - Oxford.

[25] Hart, Oliver and John Moore (1994), "A Theory of Debt Based on the Inalienability of Human Capital," *Quarterly Journal of Economics*, November, 841-879.

[26] Hopenhayn, Hugo A. (1992a), "Entry, Exit, and Firm Dynamics in Long Run Equilibrium," *Econometrica*, 60 (5), 1127-1150.

[27] Hopenhayn, Hugo A. (1992b), "Exit, selection, and the value of firms," *Journal of Economic Dynamics and Control* 16, 621-653.

[28] Jovanovic, Boyan (1982), "Selection and Evolution of Industry," *Econometrica*, 50 (3), 649-670.

[29] Kiyotaki, N. and J. Moore (1996), "Credit Cycles," *Journal of Political Economy,* forthcoming.

[30] Manuelli, Rody (1985), "A General Equilibrium Model of International Credit Markets." Ph.D. thesis, University of Minnesota.

[31] Marcet, Albert and Ramon Marimon (1992), "Communication, Commitment, and Growth," *Journal of Economic Theory*, 58, 219-249.

[32] Marsh, Terry A., and Robert C. Merton (1987), "Dividend Behavior for the Aggregate Stock Market," *Journal of Business* 60, 1:1-40.

[33] Phelan, Chris (1995) "Repeated Moral Hazard and One-Sided Commitment," *Journal of Economic Theory*, 66 (2), 488-506.

[34] Shiller, R. J. (1983), "Do stock prices move to much to be justified by subsequent changes in dividends? Reply," *American Economic Review* 73, 1:236-237.

[35] Spear, E. and S. Srivastava (1987), "On Repeated Moral Hazard With Discounting," *Review of Economic Studies*, 54, 599-617.

[36] Stokey, Nancy L. and Robert E. Lucas, Jr. with Edward C. Prescott (1989), *Recursive Methods in Economic Dynamics*, Harvard University Press.