On the Axiomatic Method Part I: A User's Guide

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Abstract

This is Part I of a study of the axiomatic method and its recent applications to game theory and resource allocation. Part I is a user's guide. It describes the components of an axiomatic study, discusses the logical and conceptual independence of the axioms in a characterization, exposes mistakes that are often made in the formulation of axioms, and explain how each axiomatic study should be seen from the perspective of the axiomatic program. It closes with a schematic representation of this program. Part II discusses the scope of the axiomatic method and briefly presents a number of models where its application have been particularly successful, with emphasis on developments that have occured in the last few years.

Contents

1 Introduction

Until recently the axiomatic method[1] had been the primary method of investigation in a few branches of economics and game theory, such as abstract social choice, inequality measurement, and utility theory, but in the last few years its use has considerably expanded. This has certainly been the case for two important domains of game theory where it had been applied at the very beginning. One of them is bargaining theory, which is concerned with the selection of a utility allocation from some feasible set (see Thomson, 1996b, for a survey). The other is the theory of coalitional form games with transferable utility, which deals with the determination of players' rewards as a function of the profitability of the arrangements that they can make in groups (see Peleg, 1988, for a detailed treatment). More remarkably, several models for which the axiomatic method has proved extremely fruitful have recently been identified. Axiomatic studies of these models have shed additional light on well-known solutions, and sometimes led to the discovery of new solutions. The models concern the subjects listed below. In each case, I give a few representative references; general presentations can be found in Moulin (1988, 1995), Young (1994), Fleurbaey (1996), Roemer (1996), and Thomson (1996a,c).

(1) Apportionment: how should representatives in congress be allocated to states as a function of the states' populations, when proportionality is desired but exact proportionality is not possible? (See Balinski and Young, 1982, for a comprehensive treatment.)

(2) Bankruptcy and taxation: how should the net worth of a bankrupt firm be divided among its creditors? When money has to be raised to cover the cost of a public project, what fraction of his income should each

[1] A point of language needs to be clarified at the outset so as to delimit the scope of this essay. The axiomatic method has been used at different levels of formal analysis. I will not discuss its role in ancient mathematics (Euclidean geometry) and modern mathematics (*e.g.* the construction of number systems). Debreu's (1959) subtitle to the *Theory of value*, "An axiomatic analysis of economic equilibrium," reflects his objective of giving equilibrium analysis solid mathematical foundations, and to develop a theory whose internal coherence could be evaluated independently of the (economic) interpretation given to the variables. At a second level, we find the axiomatic foundations of utility theory and the von Neumann-Morgenstern axioms. I will not discuss these two levels, limiting myself to a third level, which concerns the search for *solutions* to classes of *problems* (formal definitions of these terms appear below).

taxpayer be assessed? (O'Neill, 1982; Aumann and Maschler, 1985; Chun, 1988; Dagan, 1996; see Thomson, 1995b, for a survey.)

(3) Quasi-linear social choice problems: given a finite set of public projects, and assuming that utility can be freely transferred among agents, which project should be chosen and what share of the cost (or monetary compensation) should each agent be charged (or receive)? (Moulin, 1985a,b; Chun, 1986.)

(4) Fair allocation in economic contexts: the general question is whether efficiency can be reconciled with equity, but equity is a multifaceted concept, and a myriad of specific issues can be raised. Many have now been resolved for a wide range of models. (See the surveys by Moulin, 1995; Thomson, 1996a,c; and Moulin and Thomson, 1996; Kolm, 1997.)

(5) Cost allocation: given a list of quantities demanded by a set of agents for a service or a good, and given the cost of producing the service or the good at various levels, how should the cost of satisfying aggregate demand be divided among the agents? (Tauman, 1988; Moulin, 1993b; Moulin and Shenker, 1991, 1992, 1994; Kolpin, 1994, 1996; Aadland and Kolpin, 1996.)

(6) Coalitional form games without transferable utility: given a set of feasible utility vectors for each group of agents, or "coalition", how should agents' payoffs be chosen? (Aumann, 1985; Hart, 1985; Peleg, 1985; see Peleg, 1988, for a survey.)

(7) Matching: given two groups of agents, each agent in a group being equipped with a preference relation over the members of the other group, how should they be paired? This problem and variants had been the object of a number of strategic analyses (Roth and Sotomayor, 1990), but their axiomatic analysis has recently expanded in a number of new directions (Sasaki and Toda, 1992; Sasaki, 1995; Kara and Sönmez, 1996, 1997; Toda, 1991, 1995, 1996; Sönmez, 1995, 1996).

(8) Measurement of the freedom of choice: given two sets of possible choices, when can one say that one set offers greater freedom of choice than the other? This literature, initiated by Pattanaik and Xu (1990),

is a very recent entry into the field but it is expanding fast (Bossert, Pattanaik, and Xu, 1994; Bossert, 1997; Klemisch-Ahlert, 1993; Kranich, 1996, 1997; Kranich and Ok, 1994; Puppe, 1995).

(9) Equal opportunities: given a group of agents with different talents or handicaps, how should resources be distributed among them? Here, the litterature is also very new (Bossert, 1994; Fleurbaey, 1994, 1995; Iturbe and Nieto, 1996; Maniquet, 1994; Bossert, Fleurbaey, and van de Gaer, 1996).

It may be timely to look at these various developments in a unified way and to review the methodology on which they are based. I have two main goals.

1. My first goal is the pedagogical one of explaining how an axiomatic study should be conducted and the axiomatic program envisioned.

2. My second goal is to give some idea of the recent progress that has been permitted by the use of the axiomatic method, in particular with regards to concretely specified models of resource allocation. For that reason, many of the examples that I take to illustrate points of pedagogy belong to this area. I certainly do not attempt to give a complete presentation of the axiomatic literature, and in particular, I take almost no example from the considerable theory of Arrovian social choice. On this subject, a number of other works are available (Sen, 1970; Kelly, 1978; Fishburn, 1987).

This study has grown much beyond what I had planned, and I have divided it into two parts.

A reader's guide. Each section opens with a compact statement of the point I am developing. This statement is usually italicized. I continue with examples illustrating the point, in one or several indented paragraphs. In addition to the theory of resource allocation, these examples are most often taken from the theory of cooperative games. Part I continues as follows:

- Section 2: I introduce the basic notions of a problem and a solution.
- Section 3: I describe the components of an axiomatic study, its starting point, its goals, and the sort of results that we should expect from it.

- Section 4: I discuss the issue of independence of the axioms in a characterization, by which I mean both their logical independence but also their conceptual independence.

- Section 5: I present two typical errors made in the formulation of axioms.

- Section 6: I widen the scope of the discussion and explain how each axiomatic study should be seen within the framework of what I call the axiomatic program. I describe the goals of this program.

- Section 7: I give a schematic representation of the scope of the axiomatic program.

In Part II, I discuss the scope of the axiomatic method and its recent applications to game theory and resource allocation. Alter a short introduction (Section 1), it continues as follows:

- Section 2: I present the alternatives to the axiomatic method, and discuss their connections to it.

- Section 3: I respond to a number of criticisms that have been raised against the axiomatic method.

- Section 4: I present and criticize the view that the scope of the axiomatic method is limited to abstract models, and to cooperative situations.

- Section 5: I discuss the relevance of the axiomatic method to the study of resource allocation. I introduce the distinction between abstract and concrete models and discuss the limitations and the merits of abstract models.

- Section 6: I discuss the relevance of the axiomatic method to the study of strategic interaction. I advocate a wider use of the axiomatic method in the study of conflict situations. I point out that the opposition that is often made between the axiomatic and the strategic approaches in game theory is conceptually flawed. Finally, I argue in favor of an integrated approach in which the axiomatic method is given the place it deserves.

2 Basic set-up: problems and solutions

Before defining the axiomatic method, I introduce the basic concepts of "problems" and "solutions", and the terminology that I will use.

2.1 Problems

Any investigation starts with the specification of a domain of ***problems*** with which it is concerned. A problem is given by specifying data pertaining to the alternatives available, and data pertaining to the agents (players, consumers, firms, generations ...). Usually included are the preferences of the agents over the alternatives.

Problems can be described in varying degrees of detail. To illustrate the wide range of possibilities, note that an "Arrovian" social choice problem (Arrow, 1963; Sen, 1970) simply consists of a (usually unstructured) set of feasible alternatives, together with the preferences of the agents over this set. Bargaining problems and coalitional form games consist only of sets of attainable utility vectors. For normal form games, a set of actions is specified for each agent, along with the utility vector associated with each profile of actions. For extensive form games, sequences of actions are given together with the utility vector associated with each profile of sequences of actions. These models are already more concrete, as they show how utilities result from individual choices, even more so of course when a description of the sequential structure of the actions is included. For allocation problems in economic environments, the precise physical structure of the alternatives is described. These problems stand at the opposite end of the spectrum from abstract social choice problems.

In what follows, I frequently take these concrete models as illustrations and I assume some familiarity with the basic definitions, and if not with all of the axioms that have been considered in their study, with at least the general principles underlying the central axioms, and with the main solutions. The appendix contains short descriptions of the models.

2.2 Solutions

Given a domain of problems, $\mathcal{D}$, a ***solution***[2] ***on*** $\mathcal{D}$ is a correspondence that associates with every $D \in \mathcal{D}$ a non-empty[3] set of alternatives in the feasible set of D.[4] My generic notation is F for solutions, X for the universal space of alternatives, and $X(D)$ for the feasible set of D. Altogether, a solution is therefore a correspondence $F: \mathcal{D} \to X$ such that $\emptyset \neq F(D) \subseteq X(D)$. The aim of the investigation is to identify "good" solutions, good in the sense that they provide either an accurate description of the way problems are resolved in the real world, or the recommendation that an impartial arbitrator could make.

Solutions are allowed to be multivalued in some models, and required to be *singlevalued* in others. Whether the objective is descriptive or prescriptive, *singlevaluedness* is of course desirable: a solution that makes precise predictions or recommendations is more likely to be useful. However, *singlevaluedness* is often a very strong requirement and for many models, the search has been for multivalued solutions.

> Bargaining theory is an example of a domain where *singlevaluedness* has been imposed in almost all cases.
>
> In the theory of coalitional form games with transferable utility, a number of *singlevalued* solutions exist but several important ones are multivalued. When utility is not transferable, *singlevaluedness* is very demanding.
>
> In the study of resource allocation, multivaluedness is usually permitted. Here too, *singlevaluedness* would be an unreasonably strong requirement. In some special cases, (examples are bankrupcty and taxation problems, and one-dimensional models with single-peaked preferences, both in the private good case and in the public good case), it is met by a number of interesting solutions.

[2] A variety of other terms are used, such as "rule", "mechanism", "solution function", "solution concept", and "correspondence".

[3] The non-emptiness requirement is not universally imposed. This issue of whether it should be is discussed later.

[4] Note that I do not consider here the problem of deriving a ranking of the set of feasible alternatives, the central objective of the Arrovian social choice literature.

3 The components of an axiomatic study

An axiomatic study often begins by noting that for a given domain of problems several intuitively appealing solutions exist, and that some means should be found of distinguishing between them. Alternatively, it may start with the observation that there appears to be only one natural candidate solution, and be motivated by the desire to find out whether other solutions may be available after all. Yet, for other classes of problems, no well-behaved solution is known, and the axiomatic approach is a good way of finally uncovering at least one such solution, or identifying how close solutions can get to meeting various criteria of good behavior.

Here are the components of an axiomatic study.

1. It begins with the explicit specification of a domain of problems, and the formulation of a list of desirable properties of solutions for that domain.

2. It ends with (as complete as possible) descriptions of the families of solutions satisfying various combinations of the properties.

It should also include

3. An analysis of the logical relations between the properties;

4. A discussion of whether plausible respecifications of the domain would affect the conclusions, and if so, how;

5. A discussion of the implications of substituting for the properties natural variants.

Studying the logical relations between the axioms is an effective way to evaluate their relative power. *Understanding the implications of alternative specifications of the domain* is important too since it is frequently the case that other choices could have been made that are just as natural. The robustness of our conclusions with respect to these choices should be tested. *Formulating and exploring variants of the axioms* is equally useful as it is not rare that the general ideas that constitute our point of departure can be given slightly different and almost equally appealing mathematical forms. We need to know the extent to which our conclusions are sensitive to choices between these various forms, choices whose conceptual significance may be limited.

An axiomatic study often results in *characterization theorems*. They are theorems identifying a particular solution or perhaps a family of solutions, as the only solution or family of solutions, satisfying a given list of axioms. A characterization is the most useful if it offers an explicit description of the solution(s); in the case of a family, a formula specifying it as a function of some parameter belonging to a space of small mathematical complexity (say a finite dimensional Euclidean space) is of greatest value.[5,6] The format of a characterization is as follows:

Theorem 1 (Characterization Theorem): *A solution $F: \mathcal{D} \to X$ satisfies axioms $A_1, \ldots, A_k$ if and only if it is solution F^* (alternatively, if and only if it belongs to the family $\mathcal{F}^*$.)*

An axiomatic study may also produce *impossibility theorems*, stating the incompatibility of a certain list of axioms on a certain domain.

3.1 The objective of an axiomatic study should not in general be the characterization of a particular solution

In the previous section, I stated that the objective of an axiomatic study should be to understand and to describe as completely as possible the implications of lists of properties of interest. Instead, authors often start with a sentence such as: "Our objective is to characterize the following solution: ..." Apart from two classes of exceptions discussed below, I do not consider this to be a legitimate goal.[7] Whatever reasons we have of being interested in a

[5]Of course, it is not up to the investigator whether such a formula exists.

[6]An analogy with particle physics may not be totally out of place. There, the search is for the minimal list of elementary particles in terms of which all other particles can be described. These elementary constituents are the "atoms" of the theory. Similarly, an axiomatic characterization can be seen as the "decomposition" of a solution into elementary properties. One important difference though is that a given solution can sometimes be characterized in several alternative ways.

[7]What motivates the analysis should not in principle affect the analysis itself, but in fact it often does, and a number of errors commonly made can be traced to this unjustified objective.

particular solution, and some of them may be quite justified, does not usually make a characterization of the solution a valid objective.

A first reason for such an interest is that the definition of the solution may be intuitively appealing. But this does not suffice to justify the exclusive focus on the solution because there may be other solutions whose definitions are intuitively appealing. How then are we to decide among them?

Another reason may be that the solution seems to give the right answers in particular situations about which, once again, intuition appears to be a reliable guide. But here too, other solutions may be equally successful for these examples. Moreover, for us to infer from the examples that good behavior is to be expected from the solution in general, they should be representative of sufficiently wide classes of situations. This suggests that the class of situations that each example illustrates be formally identified, that the requirement on a solution that it behave in the intuitively desirable way for that class be formulated as an axiom, and that the implications of this axiom be investigated. I will discuss this program in detail below.

3.2 The objective of characterizing a particular solution is legitimate in some situations

A first exception to the principle stated above, namely that the objective of an axiomatic study should not be the characterization of a particular solution, is when the solution happens to be widely used in practice. A second exception is when the solution has played an important role in theoretical literature. We may be able to discover through an axiomatization why the solution has come up in the real world or in theoretical studies.

1. Important examples of this kind can be found in the contexts of resource allocation and social choice.

> A primary one is the Walrasian solution. It is quite remarkable that the Walrasian solution has guided production and allocation decisions in so many different historical contexts, and very natural to infer that it must have special properties that no other solution satisfies: identifying the properties characterizing the Walrasian solution becomes a legitimate exercise.
>
> In the last two decades, some answers have been found to this question. Indeed, although its informational merits had been noted and given intuitive descriptions for a number of years, it is only relatively recently

that precise notions of informational efficiency have been formulated, and characterizations of the solution on the basis of these properties developed: under certain assumptions, it is "best" from that viewpoint (Hurwicz, 1977); under related assumptions, it is also "uniquely best" (Jordan, 1982).[8]

Majority rule and the Borda rule are examples of voting rules that are frequently applied, and again, it is proper to ask: What are the properties that these solutions must enjoy, and others not, that have led to such wide use? Here too, characterizations due to May (1952), Young (1974), Ching (1995) and others, have thrown considerable light on the issue.

2. Examples in the second category are formulas or algorithms that are sometimes suggested. We are often drawn to "simple" or "elegant" formulas, or formulas that can be given a simple interpretation. Similarly, certain algorithms or procedures may appeal to our intuition. I find it quite natural and justified to be curious about whether the intellectual appeal of a formula or algorithm is due to their embodying properties of general interest.

A solution for coalition form games with transferable utility defined by means of an attractive algorithm is the nucleolus (Schmeidler, 1969; Kohlberg, 1971). It is mainly this intuitive appeal that had made this solution a frequent point of reference in the game theory literature and it was natural to wonder whether a formal justification for it could be found. Such a justification, based on an idea of *consistency*, was indeed eventually discovered for a variant known as the prenucleolus (Sobolev, 1975; see below for a discussion).

[8]Clearly, if the objective is to understand what features of the Walrasian solution has made it an almost universal means of exchanging goods, this search for an axiomatization should proceed under an additional constraint, namely that the axioms be pertinent to the "spontaneous" development of institutions. In that respect, I believe that explanations based on considerations of informational simplicity are the most likely to be the "right" ones, whereas I very much doubt that the variable population considerations such as *consistency* that recently have led to the Walrasian solution have much relevance (more on this later). Of course, this does not mean that wanting to figure out the implications of *consistency* is not worthwhile, and it is of great interest that the Walrasian solution should have emerged from such considerations as well. To summarize, I would say that wondering whether certain properties of informational simplicity characterize the Walrasian solution is legitimate, but it is the characterization of the class of solutions satisfying *consistency* that we should be after, whether or not the Walrasian solution belongs to it, and not the characterization of this particular solution on the basis of such a condition.

But note that whether the goal is to understand why a solution is used in practice or where its intellectual appeal resides, if characterizations are possible, it is the properties on which they are based that should take center stage in further research on the subject.

> To pursue our last example, the focus of the literature that followed Sobolev's work has indeed been on identifying the implications of various *consistency* notions.

When we need to simply understand, perhaps not to characterize, a particular solution, because the solution has already merited our attention by enjoying some central properties and we would like to know more about it, I claim that the axiomatic method can be of great help, and I propose a protocol for its use in Section 5.2.

3.3 The characterization of a unique solution is not necessarily more useful than the characterization of a family of solutions

A characterization theorem has the merit of completely describing the implications of a list of properties, and that is why we should be striving for such results. *Although many authors prefer a characterization of a single solution, presumably because the class of problems under study has then been given a unique resolution, I will also challenge this view and say that such a characterization is actually not as valuable as the characterization of a family of solutions.*

Indeed experience tells us that, more often than we would like, impossibilities are precipitated by relatively short lists of properties. Typically, if we have shown that a certain list of properties are satisfied by all the members of a family of solutions, we will be eager to impose additional requirements. Some of them may be met by several members of the family, and our next task will be to find out exactly which they are. Starting with the property that we consider the most important, we should then identify the subfamily satisfying it. If this subfamily still contains more than one element, we should bring to bear the property that we consider to be the second most important ..., and we might very well proceed until a single solution remains.

More likely however, since we rarely have in mind a strict priority of properties, the analysis will branch off in several directions, depending on the order in which we impose the additional properties, each branch possibly ending with the characterization of a unique solution. This sort of tree structure of our findings is typical of an axiomatic study. Certainly, at a stage when several solutions are still acceptable, it is natural to want to know if they should really be thought of as equivalent, or whether they can be distinguished on the basis of additional properties of interest. Then, the objective of characterizing the various solutions "from each other" becomes legitimate.

We will probably want to conclude an axiomatic study with characterizations of *particular solutions*, because such theorems indicate that we have then reached the boundary of the feasible. However, the number of these individual characterizations, and therefore the scope of the study will be all the greater if our first findings are characterizations of *families of solutions*, that is, if we are successful in describing the implications of lists of properties that indeed are not strong enough to force uniqueness.

3.4 For practical reasons, once the objectives of an axiomatic study have been stated, the analysis itself may have to begin from solutions

Although properties come first conceptually, it is certainly useful from a practical viewpoint, and in some cases very useful, to have at our disposal several examples of solutions when starting an axiomatic study. In fact, we are more likely to achieve our goal if we have available a wide repertory of them. The examples can be used in testing conjectures concerning the compatibility of axioms and the independence of axioms in characterizations, an issue discussed next.[9]

4 Independence of axioms in characterizations

Here, I develop the view that the study of the independence of the axioms in a characterization should be part and parcel of the analysis. By the term independence, we usually understand "logical" independence, but I also

[9] It is actually unusual for a new solution to emerge for the first time in an axiomatic study. For most domains, the solutions that have been found the most valuable had been given intuitive definitions first and axiomatic justifications have been found later.

discuss what can be called the "conceptual" independence of the axioms. I first define these terms. Next, I argue that although axioms should be logically independent and express conceptually distinct ideas, they should be compatible in their spirit. Finally, I clarify a logical issue concerning the way in which a characterization is affected by expanding the domain of problems under consideration.

4.1 In a characterization theorem, the axioms should be logically independent

Recall the "if and only if" format of a characterization theorem. The issue of independence pertains to the statement: "If a solution satisfies a certain list of axioms, it is solution F^*." This is the "uniqueness part", the other direction being the "existence part". The axioms are *independent* if by deleting any one of them, it is not true that the solution F^* remains the only admissible one. Verifying existence is usually easier, principally because the work can be divided into separate steps, one for each of the axioms, whereas the uniqueness part has to do with the way the axioms interact.

4.1.1 A first reason to establish independence is to ensure that our results are stated in the most general form

The obvious argument in favor of independence concerns the generality of our conclusions: *if one of the axioms is redundant, we widen the scope of the result by deleting it.*

The interest of many researchers in characterizations lies in the mathematical appeal of results "packaged" as "if and only if" theorems. However, we often know more than what such a theorem says. In the course of analysis, we may have discovered that if some of the axioms were weakened in certain ways, the solution that is characterized would remain the only acceptable one (in other words, we know more than what the uniqueness part says). We may also have learned that the solution actually satisfies stronger versions of some of the axioms. Consequently, the "if and only if" format is a little dangerous: it conceals some of the information that we have. In particular, it may result in a uniqueness part in which the axioms are not independent.

If we have shown that a certain axiom can be deleted from the uniqueness part, we should write the characterization without it, but remark separately that the solution does satisfy it. If uniqueness does not hold without the axiom but does with a weaker but natural version of it,[10] we should use that version in the characterization but point out that the solution happens to satisfy the stronger version. If the solution satisfies much stronger versions of the axioms than the ones used in the uniqueness part, we should probably not present our findings as an "if and only if" theorem.

4.1.2 A second, practical, reason to establish independence of the axioms is to discover more general results

A *practical reason* for checking independence has to do with research strategy: it is a way of exploring the "neighborhood" of the characterization. The better we know this neighborhood, the more confident we will be about the correctness of our results. This exploration may also help us discover other techniques of proof for the characterization, or simplifications of the proof that we have.

4.1.3 How to establish logical independence

In order to establish the independence of A_1, say, from the other axioms, it suffices to exhibit one solution different from F^* and satisfying $A_2, \ldots, A_k$, but not A_1. However, we should not be satisfied with just one or any example of a solution, for several reasons.

1. First, *the examples should be as "natural" as possible*; ideally, they should be solutions that we might have been tempted to use on *a priori* grounds, such as solutions that we know enjoy other properties of interest, or solutions that have been used in the literature. Establishing independence in this way will provide a direct explanation of why these potentially worthwhile solutions are disqualified given our objectives.

> In the context of bargaining theory, in order to prove that *contraction independence* is independent of *Pareto-optimality*, *symmetry*, and *scale invariance*, four axioms that characterize the Nash solution, it is best to

[10] I write natural because it is often possible to carry out proofs with weaker but artificial conditions. In proofs, we only apply the axioms to some selected situations. The weaker conditions obtained by limiting the scope of the original ones to these situations will certainly suffice for the uniqueness proof but using them will not necessarily give a "better" theorem.

bring up the Kalai-Smorodinsky solution, which many studies have shown is a major competitor to the Nash solution, instead of obscure solutions.

2. *Second, to be really useful, the examples may very well have to satisfy properties that do not appear in the list* $A_2, \ldots, A_k$. A basic property may be implied by some list of axioms that is being studied, but not by the shorter list obtained by dropping one of them. Then, the independence of this axiom from the other should also be investigated under the additional assumption that the solution satisfies this basic property.

For instance, in many models, *continuity* is implied by lists of axioms that we consider, and therefore we do not impose the property separately. If the list $A_1, A_2, \ldots, A_k$ does not include *continuity*, the independence of A_1 from $A_2, \ldots, A_k$ can be established by exhibiting a solution that may well violate this property. However, *continuity* being in most cases very natural, we will want to know then whether A_1 is independent from $A_2, \ldots, A_k$ *together* with *continuity*. If not, A_1 can be replaced by *continuity* in the characterization, and this might be a better result (of course we should not forget to note then that the solution satisfies A_1, and also to state the characterization with A_1).

3. Finally, *we should look for as wide a class of counterexamples as possible.* Indeed, we might be able in the process to identify all of the solutions satisfying $A_2, \ldots, A_k$. This is not necessarily the result that we will write up though, since the proof of the stronger theorem will probably be more complex. If we judge that the cost of the additional technical developments is too high in relation to the added generality of the theorem, we should retain the simpler and less general result, but inform our readers of what we know, in a remark, a footnote, or an appendix, with a degree of detail that depends of course on our intended audience. From this characterization of the class of solutions satisfying $A_2, \ldots, A_k$, it will typically be easy to deduce how the class would be further restricted by adding either A_1, or one of several conditions that are reasonable alternatives to A_1.[11] Section 3.3 elaborates on this point.

[11] It is not entirely true that given any two lists of axioms related by inclusion, characterizing the implications of the shorter list is necessarily more difficult. For instance, the classes of solutions satisfying only *Pareto-optimality*, or only *symmetry*, are of course very simple to describe. It is probably more accurate to say that up to a point, the difficulty increases. Then, it starts decreasing. I am not making a formal point here, but this state-

In the context of bargaining theory, *symmetry* can be shown to be independent of the other three conditions that we listed earlier as characterizing the Nash solution, by simply producing the solution defined by maximizing the product of player 1's utility and the square of player 2's utility. However, the whole class of solutions satisfying these three conditions can essentially be obtained by noting that maximizing *any* product of utilities would also work, and it is much more informative to exhibit this class.[12]

4.2 In a characterization, the axioms should express conceptually distinct ideas

Although in a given characterization, several axioms may be motivated by the same general principle (such as a principle of fairness, or a principle of incentive-compatibility), *each axiom should preferably embody only one specific aspect of the general idea.*

I write "preferably" because, like most of the other rules formulated here, this recommendation should not be followed too rigidly. I now give three reasons for that.

1. *A first reason for a given axiom to incorporate distinct conceptual considerations is when it has a simple and direct procedural interpretation.*

In bargaining theory, the axiom of *midpoint domination*, which says that the solution outcome should dominate the average of the agents' most preferred alternatives, is an illustration. It does embody partial notions of *efficiency* (since the outcome should be sufficiently close to the boundary of the problem for this domination to be possible), *symmetry* (the point that is to be dominated is defined in a symmetric way), and *scale invariance* (the point is defined in a scale invariant way). However, it implies none of these three axioms.[13] Moreover, it is descriptive of an

ment describes fairly accurately most situations with which I have some familiarity. A main reason is that the basic axioms that we tend to impose first are one-problem axioms whose implications are usually much easier to determine than those of "multi-problem" axioms. The distinction is discussed in detail below.

[12]The qualification "essentially" is due to the fact that when violations of *symmetry* are extreme, certain dictatorial solutions and lexicographic extensions of them are also admissible.

[13]A very simple characterization of the Nash solution can be obtained by means of this axiom and Nash's *contraction independence* (Moulin, 1988).

intuitively appealing scheme that agents often use: the midpoint corresponds to the vector of utility levels that they reach when they randomize with equal probabilities between their preferred outcomes.

A closely related example, taken from the theory of coalitional form games with transferable utility, is the requirement that for the two-person case, the solution coincides with the so-called "standard solution" (Hart and Mas-Colell, 1989), the solution that picks the alternative at which the surplus above the individual rationality utility levels is split equally. Again, this requirement embodies partial notions of *efficiency* and *symmetry*, but it does so in a way that is very intuitive. It too corresponds to the flipping of the coin to which agents often resort in practice.

I will also note two difficulties — and these are the other two reasons to which I alluded above — in following the recommendation not to incorporate in an axiom distinct conceptual considerations, which should warn us against being too dogmatic in putting it in practice:

2. *Our judgment whether a given axiom does mix ideas that would better be kept separate may well depend on the perspective taken.*

In the theory of coalitional form games with transferable utility, and for the fixed population models in which it is typically used, the core can certainly be taken as a primitive notion. However, when the perspective enlarges so as to permit variations in populations, and axioms are introduced in order to relate the recommendations made by solutions in response to such variations, the core can be decomposed in terms of *individual rationality* and *consistency* (Peleg, 1985, 1986).

In the context of resource allocation, the notion of an envy-free allocation is another example that is intuitively appealing from a normative perspective, and it is difficult to conceive of more basic ones from which it could be derived. Yet, when the perspective shifts from uniquely normative considerations and strategic concerns are addressed in addition, no-envy can be derived under very mild domain assumptions from the much more elementary fairness condition of *equal treatment of equals* and the implementability condition of *Maskin-monotonicity* (Moulin, 1993a; Fleurbaey and Maniquet, 1997).[14] For an example taken from the theory of non-cooperative games, to which I return below, Nash equilibrium can

[14]This is not an exact decomposition, since these two axioms together only imply no-envy; they are not equivalent to it.

Young, P., "An axiomatization of Borda's rule", *Journal of Economic Theory 9* (1974), 43-52.

be decomposed — this decomposition is exact — in terms of *individual rationality*, *consistency*, and *converse consistency* (Peleg and Tijs, 1996).

3. The final reason is that *in the process of gaining a deeper understanding of a subject, our judgment about possible formal decompositions of an axiom into more elementary ones may change.* As we discover links between notions that we previously perceived as distinct, the way in which we partition and structure the "conceptual field" into individual conditions sometimes evolves.

On a variety of domains, *monotonicity* and *consistency* conditions are traditionally thought of as being unrelated, and they are stated separately. However, in some situations such as the allocation of private goods, they can actually be understood as "conditional" versions of a general "replacement principle", a strong requirement of solidarity, which says that a change in the environment in which agents find themselves should affect all of their welfares in the same direction. It pertains to situations in which agents are not "responsible" for the change when it is socially undesirable, nor deserve any "credit" for it when it is socially desirable. If the principle is applied to the departure of some of the agents, the issue is whether they leave empty-handed or with their components of what the solution has assigned to them. When imposed together with *efficiency*, we therefore obtain either a monotonicity condition or a consistency condition. (This point is developed in Thomson, 1995a.)

Similarly, we could argue that for the problem of fair division, the standard forms of the monotonicity conditions such as *resource-monotonicity*, which states that an increase in the social endowment should benefit everyone, or *population-monotonicity*, which states that an increase in the population, resources being kept fixed, should penalize every agent initially present, make sense only in the presence of *efficiency*. Since *efficiency* will indeed typically be required, the demand that all "relevant" agents be affected in the same direction if the parameter (resources or population) increases or decreases — this too is a requirement of solidarity — may be judged more natural (Thomson, 1995a).

Finally, in private ownership economies, an axiom such as *individual-endowment monotonicity*, which states that if an agent's endowment increases, he should not be made worse off, can be interpreted from the normative viewpoint, as reflecting the desire that the agent should benefit from resources on which we feel that he has legitimate rights, as he may have obtained them through an inheritance or thanks to his hard work for instance. Alternatively, it may be seen from the strategic viewpoint, as providing the agent the incentive never to destroy the resources he controls, as this would result in a socially inefficient outcome.

4.3 In a characterization, the axioms should be conceptually compatible

Although it is important that axioms be logically independent and that they express distinct ideas, it is equally important that they be conceptually compatible: the intuition underlying the formulation of one axiom should not be violated by the others. This point seems clear enough but nevertheless deserves to be made.

I will give an example from the theory of bargaining that has to do with the joint use of *continuity* and *consistency*. The most commonly used topological notion (Hausdorff topology) in that theory ignores subproblems involving subsets of the players. On the other hand, *consistency* is motivated by the desire to link recommendations across cardinalities, and certain subproblems appear explicitly in its statement. When this condition is imposed, I claim that it is therefore natural to use a continuity notion based on a topology that recognizes the importance of subproblems too. (Such a topology is used in Lensberg, 1985, and Thomson, 1985.)

The position could be adopted that in the formulation of each axiom we should take into account the essential ideas underlying the others. I illustrate the position with several examples, and for the reader who thinks that it may go a little too far — note that indeed its implementation creates a tension with the objective expressed in the previous subsection — I propose a less radical choice.

The first example again has to do with *efficiency* and *symmetry*, two properties that have been imposed together in a wide range of studies. In this application, an extreme form of the position stated in the previous paragraph is that if *efficiency* is imposed, the axiom of *symmetry* should be written so as to apply to problems from which it is only required that their Pareto-optimal boundary be symmetric (as opposed to problems that are fully symmetric). Such a formulation reflects a strong view that *efficiency* should be given precedence. For another illustration of this viewpoint in the context of Arrovian social choice in economic environments, see Donaldson and Weymark (1988). A somewhat more flexible formulation is to require that two problems with the same Pareto set be solved at the same point[15] and to keep the other axioms including *symmetry* in their usual forms.

To take another example, if *individual rationality* is one of the requirements, it may make sense in the formulation of monotonicity conditions to focus on the subset of the feasible set at which the individual rationality conditions are met. Here too, I would suggest instead that an axiom of *independence of non-individually rational alternatives* be used in conjunction with the others — such an axiom has indeed appeared in the literature (Peters, 1986).

4.4 Evaluating characterizations by the number of axioms on which they are based

Here, *I challenge the opinion sometimes heard that a characterization of a solution or a family of solutions that makes use of "few" axioms is superior to a characterization involving "many" axioms.* Before evaluating the validity of this position however, a "counting problem" needs to be confronted.

1. First, some requirements may be incorporated in the definition of what is meant by the term solution, instead of being imposed separately as axioms on solutions. If we believe that certain requirements are minimal, "non-negotiable" requirements, whereas our position concerning the others is more flexible, this way of proceeding may seem justified.

> A central example here is *non-emptiness*: some authors require solutions to associate with each admissible problem at least one feasible outcome (as I have done above), whereas others state *non-emptiness* as an axiom. Other conditions that are often taken as part of the definition of a solution are *efficiency* and *symmetry*.

The choice to write a given condition as a separate axiom may depend on how restrictive the condition is for the domain under consideration.

[15] Such a condition could be called *independence of non-Pareto optimal alternatives*.

For bargaining problems, existence is almost never an issue, whereas for coalitional form games without transferable utility, it often is. It is therefore safe to incorporate *non-emptiness* in the definition of a solution to the bargaining problem, and prudent to impose it as an axiom in the study of coalition form games.

However, I believe that even for requirements that we will certainly want to impose, the analysis always benefits from including a discussion of the extra freedom gained by deleting or weakening them, and for that reason, it is best to have them listed as separate axioms.

2. A second reason for the counting problem mentioned above is that it is of course always technically feasible to combine several axioms into one. By so doing, we decrease the number of axioms but not the demands on the solutions. I argued earlier that axioms should embody conceptually distinct desiderata, and this difficulty should in principle not occur, but practice is sometimes a different matter. I gave several reasons why in the previous section.

This counting problem being clarified, and contrary to the view stated above, *my position here is that the characterization of a solution or family of solutions making use of a large number of axioms must be seen as good news, provided, once again, that they are logically independent and express conceptually distinct ideas, as they should.* This is because, for the class of problems under study, a solution or family of solutions exists that is well-behaved from a variety of perspectives.[16] The benefit of such theorems is even larger when axioms are only greater when, more modestly, axioms only specify plausible desiderata.

On the other hand, and to emphasize a position that I expressed earlier, *we should in general be striving for theorems describing the implications of few properties together.* These are better theorems since the implications of additional properties will typically be easily obtained from them as corollaries. In order to take advantage of such theorems, we should of course thoroughly explore the derivation of these possible corollaries. This argument will take its full force below when I discuss the importance of seeing each axiomatic study from the perspective of what I refer to as "the axiomatic program".

[16]I find the argument that a characterization based on fewer axioms is more "elegant" to have no relevance to the program with which I am concerned here.

4.5 A logical issue: how enlarging or restricting the domain affects a characterization

It is important to understand how a characterization is affected by enlarging or restricting the domain of problems under consideration. Here, I attempt to clarify some common misconceptions about this issue. In particular, it is often said that the characterization of a given solution on a larger domain is a weaker theorem. To what extent is this a valid statement?

The first point I would like to make is that it is not technically meaningful to speak of the same solution as having been characterized on two different domains. Formally, a solution is a triple consisting of a domain, a range, and "arrows" from every point in the domain to the range. By changing the domain, we change the solution and therefore we cannot characterize a given solution on two distinct domains. A reason for much of the confusion here is that we often keep the same name for the mapping when we change the domain. There is of course a good reason for this: it is that in most cases a single formula, or a single algorithm, or a single set of equilibrium equations ... produces the two solutions.

> In the theory of resource allocation, we use the same phrase of "Walrasian solution" to designate the solution that selects the Walrasian allocations whether or not preferences are strictly monotonic or strictly convex .... It is certainly meaningful to apply the "Walrasian definition", or the "Walrasian formula", to these various domains.

A circumstance in which there is not much danger of thinking of a formula or algorithm as defining the "same" solution on various domains is when we mainly care about "one-problem" properties of solutions. However, as soon as properties involving comparisons of problems are brought in (axioms involving pairs, or triples, or sequences of problems), we risk making logical errors by not keeping in mind that applying the same definition on two different domains gives two different solutions.

To gain further understanding of the issue, think of a solution constructed by "combining" existing solutions as follows: arbitrarily divide the domain into two subdomains, and apply one or the other of two arbitrarily chosen solutions, depending upon which of the subdomains the problem to be solved belongs to.

For instance, on the domain of private good economies, consider the solution that selects the Walrasian allocations when all agents have Cobb-Douglas preferences, and the core otherwise.

We tend to immediately reject such hybrid solutions, but why? Is it because we feel that they are unlikely to meet any criteria of good behavior? Perhaps, but whether this is true really depends on which criteria we have in mind. If we truly cared only about one-problem properties for instance and these properties happened to be met by each of the component solutions, there would be nothing wrong with the hybrid solution, except perhaps for the inconvenience of having to check which of the two cases applies. We suspect however that for many additional criteria of good behavior, the hybrid solution would be disqualified. The axiomatic method can help us formally identify what these criteria are.

As a further illustration of the difficulty of deciding what is a legitimate solution, I will consider classes of problems involving variable populations, each economy being obtained by first drawing a finite group of agents from an infinite population of "potential" agents. A solution defined on such a domain associates with each group of agents and each specification of the data describing them (such as their preferences, their endowments, their production skills ...), a set of allocations. Consider now a solution constructed by switching back and forth between several known solutions according to how many agents are involved. Again, our first reaction, when confronted with such a solution, is to reject it as "artificial". In the paragraphs to follow, I will try to find out whether and to what extent this view is valid.

An example for private good economies is the solution obtained by selecting the Walrasian allocations when the number of agents is even and the core when it is odd. An objection to this solution might well be raised on the grounds that it is "unnatural" to go back and forth between Walrasian notions and core notions: we should make up our mind and pick Walrasian allocations for all cardinalities or the core for all cardinalities. This seems convincing enough but what are the formal arguments to support such an objection? In what sense does the Walrasian definition for even numbers "go together" or "fit" with the Walrasian definition for odd numbers, or the core for even numbers fit with the core for odd numbers?

In general, what is wrong with going back and forth between different existing notions in defining solutions? A possible answer is that our choice then cannot be described in terms of a single and simple formula. However, I believe that "compactness" of a definition is not much of an argument in its favor. To begin with, switching back and forth between two notions that are usually discussed separately is not necessarily a major technical complication. Second, and more importantly, arguments of simplicity of definitions should not take precedence over substantive considerations such as efficiency, fairness, monotonicity, consistency ... Such arguments are of course not completely irrelevant because solutions passing the single-and-simple-formula test are more likely to satisfy the various invariance or independence properties that have played an important role in axiomatic analysis. But if that is the underlying reason, these properties should be formally identified and incorporated in the analysis.

Moreover, the single-and-simple-formula test is not in general well defined because on a certain domain a given solution may be described in several distinct ways, each of which suggesting a different extension to larger domains. For solutions defined on classes of problems of arbitrary cardinalities, this difficulty often occurs because solutions that are in general distinct may coincide for the two-person case.

> To illustrate this point in the context of resource allocation, consider on the one hand the solution that selects the core for all economies, and on the other hand the solution that selects the individually rational and efficient allocations for all economies. These two solutions happen to coincide in the two-agent case, so how is one to say that the extension of what we choose for two-person economies to economies with more agents should be the core or the individual rationality and Pareto solution?

How to extend a certain definition from the two-person case to the general case is an issue that game theorists have also had to confront on many occasions. Similar issues have been how to pass from classes of bargaining problems to classes of coalition form games, or from classes of coalitional form games with transferable utility to classes of games without transferable utility.

> Extending the standard bargaining solutions to classes of coalition form games has motivated many studies. Extending the Shapley value

(1953) from coalitional form games with transferable utility to the non-transferable utility case has also been a central issue in the literature. In addition to Shapley 1969's proposal, we now know of several solutions to games without transferable utility that coincide with his 1953 value when restricted to the transferable utility case.

A second argument in favor of using solutions defined by means of a single-and-simple formula is the claim that whatever considerations would lead us to choosing a certain definition to solve problems of a given cardinality should have led us to choosing the same definition to solve problems of any other cardinality.

I agree with this view but only in so far as we do make the effort of uncovering what these considerations might be. This is precisely the role of axiomatic analysis to help us in this task, as they are certainly not given to us when we are presented with the definitions. To the extent that a characterization of a solution holds independently of the number of agents, and we do have many theorems of this kind, we may have a reason not to switch formulas as we move across the domain. However, a number of approaches can also be taken that explicitly address the issue of how components of solutions should be linked across cardinalities. *Consistency* or *population monotonicity* are two such principles that have provided arguments in favor of using the same definition across cardinalities. But note that *consistency* would not eliminate the solution that selects the core from equal division for two-person economies and the Walrasian allocations from equal division for economies of greater cardinalities. Yet, it would eliminate the solution that selects the core from equal division for all cardinalities, a solution that certainly passes the single-and-simple-formula test. I argued earlier that this test is not always well-defined nor necessary; this example shows that it is not sufficient either.

Let us now return to the issue of how the choice of domains affects the generality of characterizations. On the one hand it is sometimes claimed that the result pertaining to the larger domain is stronger. The opposite view, that by enlarging the domain, we facilitate and therefore weaken the uniqueness part of a characterization is also often heard. The argument here is that since there are "more" situations to which the axioms apply, we give them greater power.

To better evaluate these views I will rewrite the Characterization Theorem in the form of two separate lemmas. In their statements, I refer to an extension of a solution F^* defined on the domain $\mathcal{D}$ to the superdomain $\mathcal{D}'$ as F'^* (in practice, the same names might be used to designate both solutions):

Lemma 1 *If a solution $F: \mathcal{D} \to X$ satisfies axioms $A_1 - A_k$, then it is F^*.*

Lemma 2 *The solution $F^*: \mathcal{D} \to X$ satisfies axioms $A_1 - A_k$.*

Suppose that instead we have established the following two lemmas pertaining to a superdomain D' of D and a solution F'^* defined on D' and whose restriction to D is F^*:

Lemma 3 *If a solution $F: \mathcal{D}' \to X$ satisfies axioms $A_1 - A_k$, then it is F'^*.*

Lemma 4 *The solution $F'^*: \mathcal{D}' \to X$ satisfies axioms $A_1 - A_k$.*

Although it is clear that Lemma 4 is stronger than Lemma 2 — and to that extent the view that enlarging the domain provides a stronger result has some validity— there is in fact no logical relation between Lemmas 1 and 3. Indeed, in the proof of Lemma 3, it could very well be the case that in order to conclude that the solution coincides with F on $\mathcal{D}$, we use (and need) the fact that it satisfies the axioms on $\mathcal{D}' \backslash \mathcal{D}$. This is a sense in which working on the larger domain weakens the uniqueness lemma. On the other hand, precisely because the conclusion of Lemma 3 holds on a wider domain than that of Lemma 1, the two results are in fact not comparable.[17]

If the uniqueness result obtained on the larger domain is not logically weaker than its counterpart for the smaller domain, it may of course be more vulnerable to criticism: by working on a larger domain, we increase the chance that situations exist for which the axioms are not as convincing.[18]

5 Common mistakes in the formulation of axioms

Here, I discuss two mistakes commonly made in the formulation of axioms: tailoring axioms to a particular solution and losing sight of the fact that

[17]There could be several solutions satisfying the axioms on the larger domain which all coincide on the smaller domain.

[18]Although we should not expect of any axiom that it be equally appealing in all situations in which it applies, it is important however that the proof not rely precisely on applications to situations where the axiom is less desirable, a situation that I have unfortunately observed.

priority should be given to their economic meaning.

5.1 Axioms tailored to a particular solution and lacking general appeal

A frequent and unfortunate consequence of wanting to arrive at a particular solution, a goal whose legitimacy I questioned above, is *the formulation of axioms tailored to that solution and lacking general appeal.* (For a discussion of this point in the context of the search for inequality indices, see Foster, 1994.) Conceivably, by proceeding in this way, we could uncover properties of independent interest, but I have rarely witnessed this outcome. The most usual one is a characterization that simply amounts to restating the definition of the solution in a slightly different form. Of course, having at our disposal several equivalent definitions of a given solution may be useful. However, the axiom being typically satisfied only by the solution that the investigator is out to characterize (this is often the tell-tale sign[19]), the result does not come as a surprise.[20]

5.2 Technical axioms

Avoiding technical axioms is generally desirable since what motivates our work are economically meaningful objectives, not mathematical ones. Unfortunately, this is not always completely feasible: sometimes we are able to determine the implications of a condition of primary interest to us only in the presence of one or several auxiliary conditions of mainly technical interest. Note however that frequently an axiom appears technical at first, but when we look into it a little more closely, we discover that it does have economic content.

In the study of bargaining problems and coalitional form games without transferable utility, smoothness of boundaries, which is one of the

[19]We should not necessarily worry about this however. For instance, the fact that the Shapley value is essentially the only solution to games in coalition form to have a potential (Hart and Mas-Colell, 1989) does not make this characterization a less valuable result. Considerations of potential are so far removed from any previous consideration that had been brought to bear in the study of these games, and the proof so unlike any previous one, that the result is indeed very illuminating.

[20]One could argue that no result that is fully understood is a surprise, but clearly there are degrees to which the conclusion can be guessed from the hypotheses.

restrictions imposed on problems in the formulation of a number of axioms, is often thought of as a technical detail, but in fact it has economic significance. Indeed the rates at which utility can be transferred between players are meaningful information, and the fact that when moving along the boundary of a feasible set, one may suddenly face a change in these rates is quite relevant when selecting a payoff vector.

Perhaps an even more striking example is *continuity*. It is now well understood that in intertemporal models, the topologies on which such notions are based can be interpreted in terms of the agents' impatience, a central economic concept (on this point, see Bewley, 1972, and Brown and Lewis, 1981).

6 Axiomatic studies and the axiomatic "program"

We should not make too much of an axiomatic study in isolation and of the fact that a particular solution has come out as the best behaved from a certain viewpoint. By changing perspectives, some other solution might very well emerge.

6.1 The axiomatic program

That different studies may lead to different solutions has been seen as a difficulty with the axiomatic method, but the opposite would be surprising. In fact, the possibility that recommendations conflict should probably be expected, and it should be confronted. Each axiomatic study should be evaluated in the light of other studies, that is, it should be seen within the wider context of the *axiomatic program.*

The objective of the axiomatic program is to give as detailed as possible a description of the implications of properties of interest, singly or in combinations, and in particular to trace out the boundary that separates combinations of properties that are compatible from combinations of properties that are not.

Characterization theorems are landmarks on the boundary. One additional property is either redundant, or it takes us into the realm of the infeasible.

6.2 Establishing priorities between axioms

When different solutions result from different axiomatic considerations, the axiomatic program is essentially silent on which axiom to emphasize, and therefore on which solution to recommend. Deciding which axioms should be given priority is up to the "consumer" of the theory. No metatheory exists to help us. I will only state the obvious here, and observe that *since many of the critical axioms that are commonly imposed pertain to changes in the parameters entering the description of the problems, the relevance of these changes should be a primary consideration.*

> In stable economic environments, resources are fixed and in the short run, so are populations. Then, "variable resource" and "variable population" axioms are not very relevant. On the other hand, if frequent shocks occur in supplies, variable resource axioms may be important; since in the long run, population is more likely to vary than in the short run, variable population axioms could be considered.
>
> In teams, we do not have to worry about agents' misrepresenting the information they hold privately, but in more competitive situations, "implementability" requirements may be needed.

6.3 Formulating discrete weakenings of axioms

When an axiom of interest is shown to be incompatible with other important axioms, discrete weakenings of it can sometimes be identified and studied.

> For the problem of fair division in private good economies, the requirement that no agent receives a consumption bundle that dominates commodity by commodity that of any other agent — this condition is known as *no-domination* — is one such example, as a weakening of no-envy.

These weaker versions of the properties that were our starting point may of course not be as universally applicable.

> *No-domination*, as a weakening of no-envy, is meaningful only in situations where the space of alternatives is endowed with an order structure and preferences are monotonic with respect to that order (this is why it is indeed a weakening of no-envy), whereas no-envy is a meaningful condition even when no such structure is present.

6.4 Formulating parameterizations of axioms

Moreover, when a basic axiom is found not to be compatible with others, it is sometimes possible to formulate parameterized versions of it, with the parameter indicating the partial "degree" to which the axiom is satisfied. Then, we can attempt to identify the range of values of the parameter for which compatibility holds.

An illustration of this approach can be found in a study of the problem of fair division due to Moulin and Thomson (1988). There, the *equal division lower bound* (an allocation meets this bound if every agent weakly prefers his alloted bundle to an equal share of the social endowment) is shown to be incompatible with *efficiency* and *resource-monotonicity*. When the *equal division lower bound* is not imposed, a possibility was known to exist, so that the question was open where the line between possibilities and impossibilities had to be drawn. To answer it, Moulin and Thomson introduce a parameter in the interval $[0, 1]$ that turns the discrete requirement that the *equal division lower bound* be met into a continuum of "graduated" conditions of increasing restrictiveness: when the parameter is 0, the condition is vacuously satisfied and when it is 1, the condition is the *equal division lower bound* itself. The result is that for all positive values of the parameter, that is, no matter how much one weakens the *equal division lower bound*, the incompatibility with *efficiency* and *resource-monotonicity* persists. Thanks to the parameterization, the possibility can be shown to be the rare case, and the impossibility the norm.

6.5 Establishing functional relations between parameterized axioms

It is possible to go further however. *When several properties are given parameterized forms, it becomes in principle possible to describe the tradeoffs between them by means of a functional relation between the parameters. Then the identification of this relation becomes a natural next step in our research program.* A concern for several properties that are incompatible when imposed in full can be partially accommodated by an appropriate selection of the parameters. Instead of having to give up one or the other, we can decide on the importance we would like to give to each and choose the parameters accordingly.

In a series of papers, Campbell and Kelly (see for instance Campbell and Kelly, 1993, 1994a,b), have very completely described tradeoffs between efficiency and equity in the context of abstract social choice, in terms of proportions of profiles for which difficulties occur.

An example for resource allocation is given in Thomson (1987) where a functional relation is established between a parameter measuring the extent to which a certain distributional requirement is met and another parameter measuring the extent to which *resource-monotonicity* is satisfied.

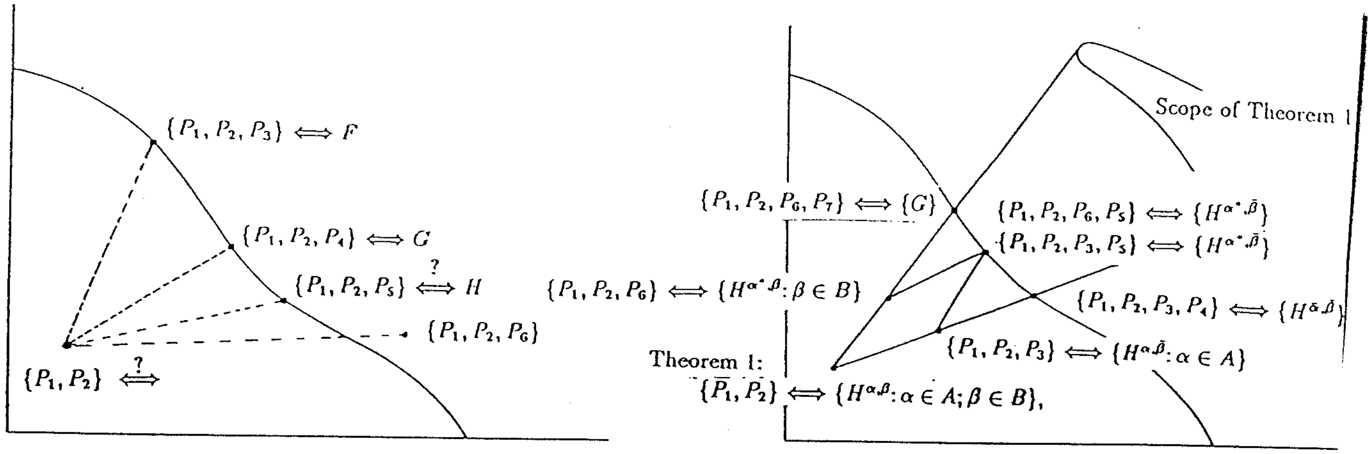
7 A schematic representation of the objectives of the axiomatic program

Figures 1 and 2, which give schematic representations of the objectives of the axiomatic program, summarize a number of the ideas discussed so far.

Each point in the plane is interpreted as a combination of properties. The downward sloping line is the boundary between combinations of properties that are compatible and combinations that are not. Think of the North-Easterly direction as indicating lists of increasing lengths. Close to the origin are short lists that are likely to be satisfied by large classes of solutions. As we progress in a North-Easterly direction, fewer and fewer solutions are acceptable. Eventually, we reach the boundary and the realm of the infeasible. Our goal is to trace out with as much detail as possible this boundary, and for combinations of properties that are compatible, to give complete descriptions of the class of solution(s) satisfying them all. To illustrate my notation, a characterization theorem identifying a family of solutions $\{H^\alpha : \alpha \in A\}$ as being the only solutions satisfying axioms P_1 and P_2 is written as "$\{P_1, P_2\} \Longleftrightarrow \{H^\alpha : \alpha \in A\}$".

1. Tradeoffs between properties (Figure 1a). A typical tradeoff betweeen two properties is illustrated by the points $\{P_1, P_2, P_3\} \Longleftrightarrow F$ and $\{P_1, P_2, P_4\} \Longleftrightarrow G$. They both lie on the boundary and therefore represent combinations of properties that can be met together but in a unique way, by solutions F and G respectively. In the presence of P_1 and P_2, only one of P_3 or P_4 can be met.

We may not have a good understanding of the implications of P_1 and P_2 together, as indicated by the point marked "$\{P_1, P_2\} \overset{?}{\Longleftrightarrow}$", but a theorem

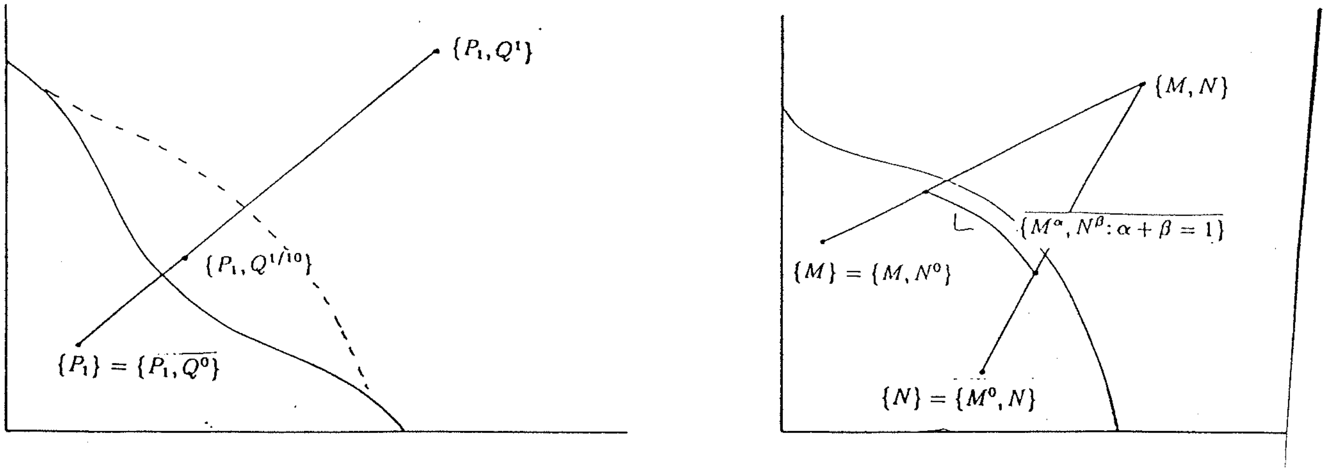


(a) (b)

Figure 1: The objectives of the axiomatic program. (a) An illustration of the trade-offs between properties P_3 and P_4. In the presence of P_1 and P_2, we cannot have both. (b) The scope of a theorem identifying a list of properties that do not force uniqueness, such as the pair $\{P_5, P_6\}$, is illustrated by the various corollaries derived from it by imposing additional properties. By adding P_3, we obtain a one-parameter family, and by adding P_4, only one member of the family remains acceptable. Alternatively, we could add P_5 and then P_6 ...

spelling out the implications of these properties would be very desirable. Most likely, the characterizations of F and G would be obtained as simple corollaries. Also, the implications of adding alternative properties such as P_5 might be easily obtained (perhaps to give another point of the boundary), and the fact that some other properties, such as P_6, are incompatible with P_1 and P_2 may also come out. This possibility is developed in the next paragraph. I have indicated these potential implications as dotted lines.

2. The scope of a theorem establishing the characterization of a family of solutions (Figure 1b). Suppose that we have shown that the solutions satisfying P_1 and P_2 constitute a two-parameter family, a result represented by the point marked "Theorem 1: $\{P_1, P_2\} \iff \{H^{\alpha,\beta}: \alpha \in A; \beta \in B\}$." Such a theorem is very useful because from it, we can often quite easily determine the implications of additional properties. By adding P_3, we reach a smaller family $\{H^{\alpha,\bar{\beta}}: \alpha \in A\}$, and then by adding either P_4 or P_5, we reach the boundary, at the points $H^{\bar{\alpha},\bar{\beta}}$ and $H^{\alpha^*,\bar{\beta}}$ respectively. Alternatively, starting from $\{P_1, P_2\}$, we could have added P_6 first, to obtain the family $\{H^{\alpha^*,\beta}: \beta \in B\}$, and then added P_5 (which perhaps would have taken us back to $H^{\alpha^*,\bar{\beta}}$), and so on. All of these corollaries indicate the "scope" of Theorem 1, which is symbolically indicated by the cone C whose vertex is the point labelled Theorem 1. The cone spans a whole section of the feasible region and of the boundary. It is not too far-fetched to think of it as a cone of light emanating from Theorem 1, its source.



(a) (b)

Figure 2: The objectives of the axiomatic program. (a) The parameterization of a property may allow us to determine the partial extent to which the property can be satisfied. (b) When several properties are parameterized, the trade-offs between them can sometimes be given the form of a functional relation.

3. Getting close to the boundary (Figure 2a). Suppose that we have established that P_1 can be met but the pair $\{P_1, Q\}$ cannot, so that the boundary passes between the points $\{P_1\}$ and $\{P_1, Q\}$. This raises the question of where exactly it lies. Does it pass "close" to $\{P_1\}$ (the solid line) or "close" to $\{P_1, Q\}$ (the dashed line)? Properties are discrete concepts and the question does not seem very meaningful. Yet, it is often possible to formulate parameterized versions of them, with the parameters indicating the partial extent to which they can be satisfied. Suppose that indeed we have a family $\{Q^\lambda : \lambda \in [0,1]\}$ of graduated conditions of increasing strength such that Q^0 is vacuously satisfied and $Q^1 = Q$. In the figure, we have schematically indicated that only a "small amount" of Q is incompatible with P_1.

4. Identifying a functional relation between parameterized axioms permitting to approach the boundary (Figure 2b.) When each of two properties is feasible but their combination is not, we can sometimes establish trade-offs between partial, parameterized versions of the properties. The two properties M and N have been parameterized as $\{M^\alpha : \alpha \in [0,1]\}$ and $\{N^\beta : \beta \in [0,1]\}$. For any pair of values of α and β such that $\alpha + \beta \leq 1$, the properties M^α and N^β are compatible. This is indicated by the curvilinear segment L, which represents pairs of values of the two parameters permitting compatibility.

8 Conclusion

In Part I of this two-part study, I have attempted to provide a user's guide. In Part II, I will discuss the alternatives to the axiomatic method, and give a bird's eye of the current state of the literature.

9 Appendix

This appendix contains short descriptions of the various models most often used as illustrations in the main body of the paper.

(1) A **bargaining problem** is a pair (B, d) of a non-empty, convex and compact subset of $\mathbb{R}^n_+$ and a point d in B. The set B is interpreted as a set of utility vectors attainable by the n agents if they reach a consensus on it, and d is interpreted as the alternative that will occur if they fail to reach any compromise. Let $\mathcal{B}^n$ denote the class of all such problems.

(2) A **transferable utility game in coalitional form** is a vector v in $\mathbb{R}^{2^n-1}$. The coordinates of v are indexed by the non-empty subsets of the set of players. A coordinate is interpreted as the amount of "collective utility" that the members of the corresponding coalition can obtain. Let $\mathcal{U}^n$ denote the class of these problems.

(3) A **normal form game** is a pair (S, h) where $S = S_1 \times \ldots \times S_n$ and $h: S \rightarrow \mathbb{R}^n$. For each player i, S_i is a set of actions that he may take, and the function h gives the payoffs received by all the players for each profile of actions. Let $\mathcal{G}^n$ denote the class of all such games.

(4) An **extensive form game** is a tree T, where each non-terminal node bears as label an element of $\{1, \ldots, n\}$, and each terminal node bears as label a point in $\mathbb{R}^n$. As compared to the previous class of games, a sequential structure is added to the set of actions, and the nodes indicate times at which agents choose actions. Let $\mathcal{E}^n$ denote the class of all such trees.

(5) An **exchange economy** is a list $(R_1, \ldots, R_n, \omega_1, \ldots, \omega_n)$ where each R_i is a continuous and monotonic preference relation defined on $\mathbb{R}^\ell_+$, and $\omega_i \in \mathbb{R}^\ell_+$ is agent i's endowment. The integer ℓ is the number of commodities. Let $\mathcal{H}^n$ denote the class of all such economies.

(6) An **economy with single-peaked preferences** is a list $(R_1, \ldots, R_n, \Omega)$ where R_i is a single-peaked preference relation defined over the non-negative reals. The number Ω gives the amount of a non-disposable good to be divided among the n agents. Let $\mathcal{S}^n$ denote the class of all such economies.

10 References

Aadland, D. and V. Kolpin, "Shared irrigation costs: an empirical and axiomatic analysis", University of Oregon mimeo, 1996.

Arrow, K., *Social Choice and Individual Values*, New York, Wiley, 1963.

Aumann, R., "An axiomatization of the non-transferable utility value", *Econometrica 53* (1985), 599-612

—— and M. Maschler, "Game theoretic analysis of a bankruptcy problem from the Talmud", *Journal of Economic Theory 36* (1985), 195-213.

Balinski, M. and P. Young, *Fair Representation*, Yale University Press, 1982.

Bewley, T., "Existence of equilibria in economies with infinitely many commodities", *Journal of Economic Theory 27* (1972), 514- 540.

Bossert, W., "Opportunity sets and individual well-being", *Social Choice and Welfare* (1997).

——, M. Fleurbaey and D. van de Gaer, "On second best compensation", mimeo 1996.

——, P. Pattanaik, and Y. Xu, "Ranking opportunity sets: an axiomatic approach", *Journal of Economic Theory 63* (1994), 326- 345.

Brown, D. and T. Lewis, "Myopic economic agents", *Econometrica 49* (1981), 359-368.

Campbell, D. and J. Kelly, "t or $t-1$, That is the trade-off", *Econometrica 61* (1993), 1355-1365.

—— and ——, "Trade-off theory", *American Economic Review 84* (1994a), 422-426.

—— and ——, "Lebesgue measure and social choice trade-off", *Economic Theory* (1994b), 445-459.

Ching, S., "What are the rationales for majority decisions? A characterization of pairwise majority rule", City University of Hong Kong mimeo, 1995.

Chun, Y., "The solidarity axiom for quasi-linear social choice problems", *Social Choice and Welfare 3* (1986), 297-310.

——, "The proportional solution for rights problems", *Mathematical Social Sciences 15* (1988), 231-246.

Dagan, N., "New characterizations of old bankruptcy rules", *Social Choice and Welfare 13* (1996), 51-59.

Debreu, G., *Theory of Value*, 1959, Yale University Press.

Donaldson, D. and J. Weymark, "Social Choice in Economic Environments", *Journal of Economic Theory 46* (1988), 291-308.

Fishburn, P.C., *Interprofile Conditions and Impossibility*, Harwood Academic Publishers, New York: 1987.

Fleurbaey, M., "On fair compensation", *Theory and Decision 36* (1994), 277-307.

——, *Théories Economiques de la Justice*, Economica, Paris, 1996.

—— and F. Maniquet, "Implementability and horizontal equity imply no-envy", University of Cergy-Pontoise mimeo, 1997, *Econometrica*, forthcoming.

Foster, J., "Normative measurement: is theory relevant?", *American Economic Review 84* (1994), 365-370.

Hart, S., "An axiomatization of Harsanyi's nontransferable utility solution", *Econometrica 53* (1985), 1295-1313.

—— and A. Mas-Colell, "Potential, value and consistency", *Econometrica 57* (1989), 589-614.

—— and ——, "Bargaining and value", *Econometrica 64* (1996), 357-380.

Hurwicz, L., "On the dimensional requirements of informationally decentralized Pareto-satisfactory processes", in *Studies in Resource Allocation Processes*, (K. Arrow and L. Hurwicz, eds), Cambridge University Press, 1977, 413-424.

Iturbe, I. and J. Nieto, "On fair allocations and monetary compensations", *Economic Theory 7* (1996), 125-138.

Jordan, J., "The competitive allocation process is informationally efficient uniquely", *Journal of Economic Theory 27* (1982), 1-18.

Kara, T. and T. Sönmez, "Nash implementation of matching rules", *Journal of Economic Theory 68* (1996), 425-439.

—— and ——, "Implementation of college admission rules", *Economic Theory*, 9 (1997), 197-218.

Kelly, J.S, *Arrow Impossibility Theorems*, New York, Academic Press, 1978.

Klemisch-Ahlert, M., "Freedom of choice: a comparison of different rankings of opportunity sets", *Social Choice and Welfare 10* (1993), 189-207.

Kohlberg, E. "On the nucleolus of a characterization function game", *SIAM Journal of Applied Mathematics 20* (1971), 62-66.

Kolm, S-C., *Modern Theories of Justice*, MIT Press, 1997.

Kolpin, V., "Coalition equitable cost sharing of multi-service facilities", University of Oregon mimeo, 1994.

——, "Multi-product serial cost sharing: an incompatibility with the additivity axiom", *Journal of Economic Theory 69* (1996), 227-233.

Kranich, L., "Equitable opportunities: an axiomatic approach", *Journal of Economic Theory 71*, (1996), 131-147.

—, "Equitable opportunities in economic environments", *Social Choice and Welfare 14* (1997), 57-64.

—— and E. Ok, "The measurement of opportunity inequality: a cardinality-based approach", *Social Choice and Welfare*, forthcoming, 1994.

Lensberg, T., "Stability, collective choice and separable welfare", Ph.D. Dissertation, Norwegian School of Economics and Business Administration", Bergen, Norway, 1985.

Maniquet, F., "Fair allocation with unequal production skills: the equal right approach to compensation", mimeo, 1994.

Maskin, E., "Nash equilibrium and welfare optimality", MIT mimeo, 1977.

——, "The theory of implementation in Nash equilibrium: a survey", in *Social Goals and Social Organization: Essays in Memory of Elisha Pazner*, (L. Hurwicz, D. Schmeidler, and H. Sonnenschein, eds), Cambridge University Press, 1985, 173-204.

May, K.O., "A set of independent necessary and sufficient conditions for simple majority rule", *Econometrica 20* (1952), 680-684.

McKelvey, R.D., "Intransitivity in multidimensional voting models and some implications for agenda control", *Journal of Economic Theory 12* (1976), 472-482.

Moulin, H., "The separability axiom and equal sharing methods," *Journal of Economic Theory 36* (1985a), 120-148.

——, "Egalitarianism and utilitarianism in quasi linear bargaining", *Econometrica 53* (1985b), 49-67.

——, *Axioms of Cooperative Decision Making*, Cambridge University Press, Cambridge, 1988.

——, "On the fair and coalitionproof allocation of private goods", in *Frontiers of Game Theory*, (K. Binmore, A. Kirman, and P. Tani, eds), MIT Press, 1993a, 151-163.

——, "Cost sharing under increasing returns", Duke University mimeo, 1993b.

——, *Cooperative Microeconomics; a Game-Theoretic Introduction*, Princeton University Press, 1995.

—— and S. Shenker, "A non-cooperative and fair mechanism for cooperative production", Duke University mimeo, 1991.

—— and ——, "Serial cost sharing", *Econometrica 60* (1992), 1009-1037.

—— and ——, “Average cost pricing versus serial cost sharing: an axiomatic comparison”, *Journal of Economic Theory 64* (1994), 178-207.

—— and W. Thomson, “Can everyone benefit from growth? Two difficulties”, *Journal of Mathematical Economics 17* (1988), 339-345.

—— and ——, “Axiomatic analysis of resource allocation”, in *Social Choice Reexamined*, (K. Arrow, A. Sen, and K. Suzumura, eds), Cambridge University Press, 1997, 101-120.

O'Neill, B., “A problem of rights arbitration from the Talmud”, *Mathematical Social Sciences 2* (1982), 345-371.

Pattanaik, P. and Y. Xu. “On ranking opportunity sets in terms of freedom of choice”, *Recherches Economiques de Louvain 56* (1990), 383-390.

Peleg, B., “An axiomatization of the core of cooperative games without side-payments”, *Journal of Mathematical Economics 14* (1985), 203-214.

——, “On the reduced game property and its converse”, *International Journal of Game Theory 15* (1986), 187-200.

——, “Lectures on cooperative games”, Hebrew University of Jerusalem, 1988.

—— and S. Tijs, “The consistency principle for games in strategic form”, *International Journal of Game Theory 25* (1996), 13-34.

Peters, H., “Characterizations of bargaining solutions by means of their status quo sets”, University of Limburg, Maastricht, The Netherlands, (1986).

Puppe, C., “Freedom of choice and rational decisions”, *Social Choice and Welfare 2* (1995), 137-153.

Roemer, J., *Theories of Distributive Justice*, Harvard University Press, Cambridge, 1996.

Roth, A.E. and M. Sotomayor, *Two-sided Matching*, Cambridge University Press, 1990.

Sasaki, H., “Consistency and monotonicity in assignment problems”, *International Journal of Game Theory 24* (1995), 373-397.

—— and M. Toda, “Consistency and the characterization of the core of two-sided matching problems”, *Journal of Economic Theory 56* (1992), 218-227.

Schmeidler, D., “The nucleolus of a characteristic function form game”, *SIAM Journal of Applied Mathematics 17* (1969), 1163-1170.

Sen, A., *Collective Choice and Social Welfare*, Holden Day, San Francisco, 1970.

Shapley, L., "A value for n-person games", in *Contributions to the theory of games* (H.W. Kuhn and N.W. Tucker, eds), Princeton University Press (1953), 307-317.

——, "Utility comparisons and the theory of games", in *La Décision*, (G. Guilbaud, ed), Editions du C.N.R.S., Paris, 1969,

Sobolev, A.I., "The characterization of optimality principles in cooperative games by functional equations," *Mathematical Methods in the Social Sciences 6* (1975), 150-165 (in Russian).

Sönmez, T., "Strategy-proofness and singleton cores in generalized matching problems", University of Michigan, 1995.

——, "Manipulation of matching rules by prearrangement", University of Michigan mimeo, (1996).

Tauman, Y., "The Aumann-Shapley prices: a survey", in *The Shapley value: Essays in Honor of Lloyd Shapley*, (A.E. Roth, ed), Cambridge University Press, 1988, 279-304.

Thomson, W., "A note on the Nash solution", University of Rochester mimeo, 1985.

——, "Is growth compatible with justice: another impossibility", University of Rochester mimeo, 1987.

——, "Population-monotonic allocation rules", Chapter 4 in *Social Choice, Welfare, and Ethics* (W. Barnett, H. Moulin, M. Salles and N. Schofield, eds), Cambridge University Press, Cambridge, 1995a, 79-124.

——, "The axiomatic analysis of bankruptcy and taxation problems: a survey", University of Rochester, 1995b.

——, "Fair allocation rules", University of Rochester mimeo, 1996a.

——, *Bargaining Theory: the Axiomatic Approach*, Academic Press, forthcoming, 1996b.

——, *The Theory of Fair Allocation*, University of Rochester mimeo, 1996c.

Toda, M., "On the fair allocations in the two-sided matching problems", Tokyo-Keizai University Discussion Paper, 1991.

——, "Characterizations of the core of two-sided matching problems which allow self-matchings", Tokyo-Keizai University Discussion Paper, 1995.

——, "Monotonicity properties and characterization of the core of matching problems", Tokyo Keizai University mimeo, 1996.

Yannelis, N., "A remark on the cardinal value equivalence theorem", mimeo, 1982.

Young, P., "An axiomatization of Borda's rule", *Journal of Economic Theory* *9* (1974), 43-52.