Turnpikes

McKenzie, Lionel W.

Working Paper No. 447
January 1998

Turnpikes

Lionel W. McKenzie

Rochester Center for Economic Research
Working Paper No. 447

January 1998

TURNPIKES

Lionel W. McKenzie

Richard T. Ely Lecture

Presented at the
Annual Meetings of the
American Economic Association
Chicago, IL
January 3, 1998

TURNPIKES*

Lionel W. McKenzie

I will sketch the history of the so-called turnpike theorems and describe some interesting recent developments regarding them. I will also discuss the attempt to apply the ideas of turnpike theorems in the literature of optimal capital accumulation to the theory of competitive equilibrium over time. Finally I will make some remarks on the relation of this literature to some recent developments in the theory of economic growth which are often referred to comprehensively as the New Growth Theory. Of course the New Growth Theory like the Old Growth Theory is not concerned directly with optimal capital accumulation, but with the actual course of events in markets, which indeed need not be perfectly competitive. I do not pretend for a moment to have a comprehensive mastery of this enormous literature. Especially I must apologize to those many able economists, beginning with Brock and Mirman (1972), who have developed turnpike theorems under uncertainty. Time does not permit my discussing their work, nor does my command of the literature. Finally, I have tried to pitch my lecture at a level that does not require a great deal of prior knowledge of the subject.

1 Origins

I do claim to have one qualification for delivering the Ely lecture. I learned my first economics from Ely's textbook in a small junior college in middle Georgia, named appropriately Middle Georgia College. I should add that the summer after sitting in the principles class I did consult an even earlier and more highly revered source, The Wealth of Nations, read from the Harvard Classics. Somehow this volume prevailed over The Origin of Species from the same collection and led me to economics rather than biology, although my scholarship to Duke was obtained with the aid of my biology professor. He never responded to my letter informing him of my apostasy.

There are two principal sources of the modern theory of optimal capital accumulation as well as modern growth theory. They were very nearly simultaneous. The earlier to be published was Frank Ramsey's Mathematical Theory of Saving, Economic Journal, 1928, one of Ramsey's three great contributions to economics, this one on the suggestion of Keynes. The other is the equally famous paper by John von Neumann which was first delivered to a mathematics seminar in Princeton in 1932. This was one of his two great contributions to economics. It was also given in 1936 to a mathematics colloquium in Vienna led by Karl Menger, spelled with a K, the son of Carl Menger, the economist, spelled with a C. I heard the paper in an economics seminar in Princeton around 1940. I can provide partial confirmation of Morgenstern's remark that no one in the audience understood a word of it. It was translated and published in the Review of Economic Studies in 1945 under the title A Model of General Economic Equilibrium. It is rather intriguing that both Ramsey and von Neumann were mathematicians of the first rank. I do not believe the loss of any other five papers from the economics literature would have had a greater impact on the development of economic theory.

Ramsey assumes there is one good which serves both for capital accumulation and for consumption. The good is produced by capital and labor. In addition there is a social utility function with this good and labor as arguments. An essential assumption for his principal result is that the economy can achieve, or at least asymptotically approach, satiation, either in production or utility, a condition he called Bliss. Then without assuming that future utility is discounted he derives a rule for capital accumulation which realizes the maximum sum of utility over time. In those days in Cambridge, England, discounting future utility over generations was not favored. Ramsey's famous criterion was, and I quote, "*[the] rate of saving multiplied by [the] marginal utility of consumption should always equal bliss minus [the] actual rate of utility enjoyed.*" Keynes gave him a nice intuitive way of seeing that this criterion is correct. However the rule which he also derived and which remains central to the modern optimal growth theory is that in the absence of discounting the marginal utility of consumption should fall at a proportionate rate given by the rate of interest. In perfectly competitive markets this is the marginal product of a unit of capital.

The contribution of von Neumann was along very different lines. Ramsey uses what today is called a macro model. von Neumann uses a disaggregated general equilibrium model, whose production sector is an activities model. It has the peculiarity that there is no explicit recognition of labor inputs. Capital goods produce capital goods. Moreover there is no utility function in his model. He sought to show that there is an economic equilibrium in the sense that prices exist which will allow activities in use to cover costs while no activity offers a positive profit. Moreover, the activities in use can expand the stock of capital goods at a maximal feasible rate. He proved that the rate of expansion of the capital stock and the rate of decline in prices will be equal in this equilibrium. He also generalized a famous fixed point theorem due to Brouwer which later played a critical role in proving the existence of a competitive equilibrium.

When von Neumann presented this result in Cambridge, Massachusetts, he asserted that maximization of an objective function had no part in his theory. Paul Samuelson who was in the audience rose to challenge this statement asserting that maximization would enter once disequilibria were considered. von Neumann offered to bet him a cigar that this was wrong. Amazingly Samuelson did not accept the wager. Nonetheless he feels that should they meet at St. Peter's gate he should ask von Neumann for a cigar. It was in the context of a von Neumann model, in a publication of the Rand Corporation, that Samuelson (1949) first described the idea of the turnpike.

2 The Samuelson Turnpike

The subsequent history of models of optimal growth has featured an interplay of these two foundations. That is, the Ramsey objective of maximizing a utility sum over time has been introduced into the disaggregated model of von Neumann, and the von Neumann production sector featuring numerous activities has been introduced into the Ramsey model. It is the accomplishment of the von Neumann model to describe the conditions for an equilibrium of production over time.

Samuelson conjectured the existence of a turnpike in a von Neumann model where the objective to be maximized is the size of the terminal capital stocks in certain assigned ratios. The turnpike was to be the path of most rapid balanced growth of the capital stock, the von Neumann equilibrium. Therefore, it seems appropriate to refer to the turnpike in the von Neumann model as the Samuelson turnpike. Later Dorfman, Samuelson, and Solow (1958) presented a proof of the Samuelson turnpike conjecture for a model with two capital goods. However, the first rigorous proof was found by Roy Radner (1961). His model allowed any number of capital goods. He also introduced the most useful method of proof for turnpike theorems, which I have called the value loss method.

The value loss method exploits the fact that, compared with the capital stocks of paths on the turnpike, the capital stocks of paths off the turnpike lose value at the equilibrium prices that support the turnpike. Since it is possible to use the turnpike for most of the time in an alternative path, the losses that the optimal path can suffer are limited. This limits the time the optimal path can spend outside a neighborhood of the turnpike, just as for your trips by car through the countryside. Radner showed that any sufficiently long optimal path in an irreducible model will spend most of the time in a small angular neighborhood of the ray on which von Neumann equilibrium paths lie. He assumed that the production set, which is a convex cone, is strictly convex near the von Neumann ray except for constant returns to scale. This theorem has the weakness that strict convexity is not consistent with the neoclassical production model, for which different industries have independent production processes. The difficulty is that the social production set is convex but not strictly convex under neoclassical assumptions. However the argument can be adapted to prove convergence to a flat piece of the production set on which the von Neumann equilibrium path lies, which I call the von Neumann facet. More of that later. Other arguments on some additional assumptions imply that the optimal path will converge further to the von Neumann equilibrium itself.

Morishima (1961) and I (1963) independently, using approaches different from Radner's, proved a turnpike theorem for a von Neumann model of Leontief type with circulating capital and variable coefficients. As a von Neumann model there are no explicit labor inputs. This model has a neoclassical production sector. My approach was to use a theorem of Solow and Samuelson (1953) which implies that the equilibrium prices converge to the turnpike prices over time. This causes the production coefficients to converge to the coefficients that produce the turnpike. Then tracing the production path backwards in time I show that the optimal path must stay near the turnpike most of the time if the period of accumulation is long enough. It has recently been shown by Michael Kaganovich (forthcoming) that it is possible to introduce a utility function into this model and prove a turnpike theorem where the objective is to maximize a discounted utility sum over the infinite future. More about this later.

3 The Ramsey Turnpike

The first moves beyond the theorem of Ramsey with its assumption of one sector and the state of Bliss as the turnpike was made independently by David Cass (1966) and Tjalling Koopmans (1965). In the modern literature the state of the economy where population is allowed to grow and saturation has been reached in the sense of a maximum of sustainable per capita utility is sometimes referred to as a Golden Age. If discounting is allowed as well, an optimal path of accumulation with constant per capita capital stock is called a modified Golden Age. Cass and Koopmans gave rigorous proofs of convergence of optimal paths to a modified Golden Age where utility is discounted and population is growing. Koopmans used a social utility function defined on per capita consumption plus possible additional discounting, while Cass equivalently assumed that the discount rate on social utility exceeds the rate of population growth. Ramsey had introduced generalizations in these same directions but his arguments were not understood and probably not complete. The theorems of Cass and Koopmans were subsequently generalized by James Mirrlees (1967) to allow technical progress as well.

Some essays in the New Growth Theory take their departure from a one sector model of this type, that is, the Ramsey model into which population growth, technical progress, and discounting have been introduced with the rate of discounting of social utility exceeding the sum of the rate of population growth and the rate at which individual utility increases because of technical progress. In such a model the New Growth theorists seek to prove turnpike theorems in the sense of convergence of the capital stock to the capital stock of a modified Golden Age. This literature is comprehensively surveyed in the recent book *Economic Growth* (1995) by Robert Barro and Xavier Sala-i-Martin.

One should note that the meaning of turnpike in the one sector model of Ramsey is just the level of capital accumulation reached at a saturation point, either a Golden Age, or a modified Golden Age. On the other hand, in the von Neumann model the turnpike is given by the combination of capital goods that attains the fastest growth rate. When this idea is adapted to a Ramsey model where there are resources like labor and land that cause production to be bounded, the comparable task is to discover the combination of capital goods that supports production in the state of saturation. This does not exclude sustained growth so long as it is made possible by exogenous factors like population growth and technical progress. When there is more than one capital good the combination of capital goods must be found which can play the role of the turnpike, that is, to which optimal paths converge.

The first proof of a turnpike for a Ramsey model with more than one sector was found by Hiroshi Atsumi (1965) in a model with one capital good and one consumer good and no discounting. He assumed like Koopmans and Cass an expanding population and he used the Ramsey objective based on undiscounted per capita utility. He introduced the overtaking criterion so this objective would be meaningful over an infinite future. He also relates his argument to the competitive market and derives the optimal savings ratio in the sense of Ramsey. His argument is a value loss argument and he derives a generalization of the lemma which was the basis of Radner's proof of the Samuelson turnpike theorem. However Atsumi's theorem does not achieve full generality since he uses only one capital good.

The turnpike theorem for any number of capital goods and consumers was made in a model with a finite horizon by me (1968) and with an infinite horizon by David Gale (1967). Gale assumed strict concavity of social utility, at least at the Golden Age, while I considered the case where the concavity was not strict. Utility was not explicitly discounted but for an expanding population an interpretation of our models in terms of per capita utility would implicitly amount to discounting social utility at the rate of population growth. Of course the general model should allow for greater levels of discounting. However it took a surprisingly long time to prove turnpike theorems with greater levels of discounting. The extension to a multi-sector model with discounting exceeding the rate of population growth was made independently by Cass and Shell (1976) and José Scheinkman (1976). They proved the convergence of optimal paths to the modified Golden Rule path. Their theorems, like the theorem of Gale, require strict concavity of the reduced utility function. The reduced utility is the maximum social utility achievable in a single period, given the initial and terminal capital stocks. Concavity of the reduced utility function is implied by strict concavity of the social utility function, defined on consumption, and strict convexity of the production possibility set, except for constant returns to scale. The theorems are stated in terms of discount factors, net of population growth, which are close enough to one, that is, net discount rates sufficiently close to zero, so that the optimal path from certain initial capital stocks converge to an optimal balanced path. The optimal balanced path itself converges to the unique optimal balanced path for the undiscounted case as the discount rate goes to zero. Thus one way of interpreting their theorems is that the optimal paths are continuous in the discount factor as the discount factor approaches one

4 Neighborhood Turnpikes and von Neumann Facets.

Given strictly concave utility and strictly convex production sets convergence to an optimal balanced path, or a modified Golden Age, may require that discount rates be very close to zero, or equivalently discount factors close to one. However, I have shown that this requirement may be relaxed if convergence to a modified Golden Age is replaced by convergence to a neighborhood of a modified Golden Age. Then the discount rate must be closer to zero the smaller the neighborhood chosen. The modified Golden Ages need not be unique but they will all lie in similar neighborhoods of each other. The differentiability requirements on the utility and production functions are weaker in my theorem. Later theorists have found that all kinds of complex patterns of optimal paths are possible even when differentiability is assumed. For example, periodic paths or even chaotic paths may occur. Such cases have been extensively investigated and described by Boldrin, Mitra, Nishimura, Yano and many others. Much of the literature is surveyed by Boldrin and Woodford (1990). However these complex patterns must lie within the neighborhoods employed by the neighborhood turnpike theorems, when the reduced utility function is strictly concave at the modified Golden Age. These neighborhoods close down on the modified Golden Age as the discount rate approaches zero.

The added generality I will now describe when strict convexity and concavity assumptions are relaxed cannot be too easy to grasp since Tjalling Koopmans told us at Stanford in 1965 that he did not understand it. However, Roy Radner was present and said that he did understand it, which somewhat relieved my mind. I first introduced (1963) this generalization with reference to Radner's theorem on the Samuelson turnpike where Radner assumed strict convexity of the production possibility set at the von Neumann ray except for constant returns to scale. However, I will confine our attention here to the Ramsey model. In the absence of strict concavity of the reduced utility function the prices that support the modified Golden Age may also support a convex set of input-output combinations which surround the modified Golden Age, perhaps including many which are not balanced. In this case what the value loss argument produces is not convergence to the modified Golden Age, but to the set of input-output combinations for capital stocks which are supported by the same prices that support the modified Golden Age. I call this convex set of input-output combinations a von Neumann facet in the Ramsey model. The convergence theorem requires the supporting price vector to be unique and the value losses off the facets to be uniformly bounded from zero (1983). The theorem leaves open how the optimal path will behave within the neighborhood, even when the von Neumann facet is trivial and contains only the modified Golden Age. Complex patterns may also arise on the von Neumann facet, when it is not trivial. In this case they may be cyclic but not chaotic. On the other hand it is also possible that the difference equations which govern paths which lie on the facet will be such that convergence to a small neighborhood of the Golden Age must also occur for a small enough discount rate. These considerations are rather too complicated to describe more exactly in this lecture. Unfortunately, however, the secondary sources are very inadequate on neighborhood convergence and on von Neumann facets as well.

The neoclassical model always has nontrivial von Neumann facets. Given the prices of inputs and outputs in each production process the most profitable combination of inputs and outputs will be chosen, and with perfect competition the profit will be zero. Through the replication of the elementary production units, assuming they are small, the level at which these combinations are realized may be varied nearly continuously. The consequence is that a great variety of total inputs and outputs are consistent with given prices. Moreover, although the dimension of this variation of inputs and outputs is reduced if the variation in activity levels is constrained by side conditions, it is not eliminated so long as the side conditions are fewer in number than the number of processes. For example, the total supplies of the primary resources may be side constraints. These relations have been discussed in a model without joint production by Harutaki Takahashi (1985).

In the model with bounded paths and differentiability of social utility near the von Neumann facet Makoto Yano (1997) has recently proved a dual turnpike theorem, which is stated in terms of prices rather than goods, on assumptions significantly weaker than those used for the primal theorem, which is stated in terms of goods. My theorem requires uniformity in value losses off the von Neumann facets for an interval of discount rates bounded by zero. His theorem does not require this uniformity. The dual turnpike theorem asserts that the prices that support the optimal path will converge to a neighborhood of the unique price vector that supports a modified Golden Age. In a competitive economy where perfect foresight paths are optimal paths his theorem implies that fiscal policies that transfer income from some consumers to others will not affect the spending of these consumers to any significant extent if the transfers are temporary. This extends results of Milton Friedman based on the permanent income hypothesis from a partial equilibrium framework to a general equilibrium framework.

5 Unbalanced Growth with Bounded Paths

Almost all the attention to asymptotic convergence has been concentrated on convergence to balanced paths although it is not clear that optimal balanced paths will exist. This type of path is virtually impossible to believe in, if the model is disaggregated beyond the division into human capital and physical capital, and new goods and new methods of production appear from time to time. However the fundamental convergence results in models in which paths are bounded, after allowing for population growth and exogenous technical progress, do not depend on the presence of an optimal stationary path or the absence of changes in technology and taste. The basic result is that optimal paths from different initial stocks have a tendency to converge whatever their shapes may be. I made this point in articles published in 1974 and 1976. It was also emphasized in my chapter in the Handbook of Mathematical Economics (1986). If the discount factor is equal to one, that is to say, future utilities are treated on a par with the utility of the current period as Ramsey would prefer, and the reduced utility function is strictly concave, it is a general phenomenon that optimal paths from different starting points converge in some sense in models with bounded growth if the paths are not isolated, in other words, if the optimal path from one starting point can be reached from the other starting point. Of course, if the optimal path from the second starting point can be reached only with great difficulty from the first starting point, the convergence may be slow.

There is a simple argument which does not use value losses that proves this result. We are free to normalize utility so that the utility along one optimal path is zero in every period. Also assume that the starting point of this path is interior to the set of capital stocks whose utility sums are bounded above minus infinity after the normalization is made. Suppose the optimal path from the second starting point stays away from the optimal path from the first starting point by at least a small distance for an indefinite number of periods. Now consider a path lying halfway between these paths. By convexity of the production set it is feasible. If this convexity is uniform, which is not unreasonable if the path is bounded, the midpath will enjoy a utility that exceeds the average utility of the two paths by more than some positive amount in every period when they are apart by more than a given distance. If there are an indefinite number of such periods, and no discounting, the total utility along the mid path will exceed the average of the total utilities along the two original paths by an unbounded amount. In other words the utility gain is infinite over the infinite path. I cannot give a complete argument since I do not wish to use even simple algebra in this lecture, but it is easily shown that this leads to a contradiction. The contradiction can only be escaped if the paths converge.

If the discount factor is less than one the obvious difficulty arises that the utility gains of the midpath will not be unbounded, but have a finite sum. However an argument adapted from Truman Bewley (1982) still allows a neighborhood theorem to be proved. Consider the gains of the midpath over the average of the two optimal paths from an arbitrary time from which the path is observed. It can be shown, if the discount factor is close enough to one, that the sum of future gains decreases over time by at least a certain amount if the path from the second starting point remains outside a given neighborhood of the path from the first starting point. This argument as the earlier one depends on uniform concavity of the reduced utility function, at least in the neighborhood of one of the paths. Since the sum of gains cannot become negative, we reach a contradiction unless the paths converge. This theorem can be given a generalization to facets if strict concavity fails to hold.

6 The Old Growth Theory

The work that I have been reviewing should be called the Old Optimal Growth theory. Robert Solow (1956) and Trevor Swan (1956) introduced what is properly called the Old Growth Theory. They are concerned with competitive equilibria. The crucial assumption in Solow's model is that saving by consumers is fixed at a percentage of net output or net income. Production is guided by current prices as in a Walrasian model with stationary expectations. With these behavioral assumptions and strongly diminishing returns to capital, given the labor supply, Solow shows that the competitive equilibrium over time will approach a stationary value, that is, a state in which saving is just adequate to meet the demand for capital arising from the expanding population. If population is not growing and saving is a proportion of net output, capital will continue to grow. On the other hand, it seems unlikely that people would continue to accumulate capital after additions to capital no longer increase, or may even reduce, output. Thus it is necessary to introduce some dependence of the saving rate on the level of accumulated capital. This dependence presumably requires some attention to utility considerations and some foresight. If there is exogenous technical progress which takes the form of the increasing productivity of labor the analysis may be carried through using units of effective labor rather than units of labor. The Old Growth Theory was introduced to refute the theories of Harrod and Domar which used fixed coefficients of production and implied that equilibria would always be unstable. It accomplished this purpose by bringing economic considerations into the choice of inputs for production, but it left consumption levels still divorced from economizing choice over time.

Solow has a very clear presentation of the view of the old growth theorists toward the positive and normative theories in his Radcliffe lectures (1970). There is no reluctance to discuss the question of what saving rate would be optimal and the analysis is conducted in the same style that Ramsey used. However the normative theory is placed squarely in the realm of policy and there is no suggestion that the competitive market by itself will achieve a normative result. On the other hand it is suggested that a loose approximation to the optimal policy may be good enough.

There is a model of population growth which bears a suggestive similarity to the von Neumann model and the Old Growth Theory in its positive form. The Old Growth Theory assumes a rate of saving and an aggregate initial stock of capital and explores the implications in a model of capital accumulation. Turnpike theorems in the von Neumann model assume an initial stock of capital goods of arbitrary composition and follow the evolution of the stock where all output in excess of worker's subsistence is reinvested and production processes satisfy no profit conditions. The demographic growth theory assumes birth rates and death rates by cohort and an initial composition of the population and traces the subsequent development of the population by size and age distribution which these assumptions imply. If the birth and death rates are assumed to be constant, the composition of the population is shown to converge to a constant composition which is independent of the initial composition, and the rate of growth of the population becomes constant as well. This development does not correspond exactly to either of their economic models but there are analogies to both the von Neumann and the Old Growth Theory. However, the demographic analogy goes further. If the birth and death rates are assumed to change over time, it still holds true that the composition of the population by age is asymptotically independent of the initial composition of the population. We found for the Ramsey model of bounded growth that the path itself was asymptotically independent of the initial capital stocks. However convergence in the demographic model will be like convergence in the von Neumann model in that the size of the later populations will be proportionate to the initial population for a given initial composition. The growth rates will converge in both cases. This demographic theory was first conjectured by Ansley Coale and successfully used by him to make population projections. The theorems were proved by Ansley's student Alvaro Lopez (1961).

We may observe that the demographic theory is subject to the same exception that has been taken to the Old Growth Theory. That is, the decisions of the population about the size of families are not explained by utility maximizing choices made in face of anticipated incomes and death rates but assumed to remain at certain levels, probably those realized in the past. There have been efforts by economists, in particular by Gary Becker (1990), to introduce economizing into demographic theory. There is the same dispute among demographers as among economists on whether these moves toward optimizing models have been useful.

7 Optimal Growth and Competitive Equilibrium

So far we have discussed turnpikes chiefly in terms of optimal growth, which is a normative theory. However, as you know, there have been applications made of the optimal growth theory to the market economy. Robert Becker (1982) showed how the the Ramsey description of optimal growth could be placed in correspondence with Irving Fisher's theory of markets over time. Truman Bewley (1982) showed, if perfect competition and perfect foresight are assumed, using the methods that Negishi applied to proving the existence of competitive equilibrium, that paths over time are optimal paths for a social welfare function which is the sum of individual utilities, weighted in a certain manner. The weights are the reciprocals of the marginal utilities of money for the individual consumers, that is, of whatever is used to state prices. These do not represent moral values. The theory is positive not normative. The consumers make decisions that allow for bequests, so that their decisions represent consumption plans into the indefinite future.

A further step is to show that competitive paths over the infinite horizon will have a turnpike property as a consequence of being optimal paths given the appropriate welfare function. However some caution is needed when a competitive path is characterized as a turnpike. The social utility function which is maximized along the dynamic equilibrium path depends on the initial conditions. This is because the weights given consumers in the equilibrium are dependent on the income distribution and therefore depend on the distribution of ownership in the initial stocks and on the level of these stocks. However, Makoto Yano (1985) shows that the effect of these initial conditions tend to disappear as the discount rate on utility approaches zero. Then we obtain a turnpike theorem in the original sense.

Most of the applications of turnpike theory to competitive equilibrium have been made under quite restrictive assumptions. First it is assumed that individual utility over time can be represented by utility functions which are separable and additive over time and discounted at a constant rate. In a period model with long periods the assumption of additivity and separability may not be very demanding though certainly not without objection. However, it is much less acceptable to assume that all consumers have the same discount rates on utility. If discount rates are allowed to be endogenous and dependent on levels of wealth the assumption of equal discounts rates may be somewhat more acceptable. This move was introduced by Uzawa (1969) for an aggregative model. More recently recursive utility functions for individual consumers have been introduced by Lucas and Stokey (1984) who prove turnpike theorems when, loosely speaking, the discount rates on utility increase with increasing wealth. This discourages runaway saving by rich people. Recursive utility is treated in considerable detail by Becker and Boyd in their book *Capital Theory, Equilibrium Analysis and Recursive Utility*, published this year.

8 The New Growth Theory

At this point we are left with two quite different turnpike theories. One theory is concerned with economies that expand indefinitely at rates determined endogenously. When the objective is to maximize terminal stocks the economy eventually expands at the maximal possible rate. The other theory is concerned with economies that may expand indefinitely but at rates determined outside the economic model perhaps by the rate of growth of population. Moreover by adding the population growth rate to the discount rate and using this as an expanded discount rate, the economy can be viewed as moving to a saturation level for discounted utility, a modified Ramsey's Bliss. The question then arises whether, apart from population growth, the economy can grow indefinitely from endogenous factors in the manner depicted by the von Neumann growth model. And if this may occur does this economy have turnpike properties as the earlier ones do.

The first paper in the optimal growth literature in which the rate of technical progress was made endogenous seems to be that by Uzawa (1965) in a model with one produced good serving both as capital good and as consumption good. He assumed that technical progress depends on the portion of labor devoted to the educational sector which leads to increased labor productivity in the form of Harrod neutral technical progress. His paper has a normative slant. This type of model was developed further by Lucas (1988) twenty three years later with more positive intentions. The interest of Uzawa's model was somewhat reduced by his assumption that utility is proportional to consumption. Lucas uses a constant elasticity consumption function and a Cobb-Douglas production function with Harrod neutral technical progress which he describes as resulting from accumulating human capital. Also he adds a term to represent external effects from the accumulation of human capital in the economy. As a consequence of this factor competitive equilibria do not realize Pareto optima. On the other hand it should be noted that sustained growth in his model does not depend on the presence of the external effects.

On the other hand somewhat earlier Paul Romer (1986) used external economies in the production of goods arising from the spread of innovations to make possible indefinite expansion. This occurs in the context of competitive equilibrium but because of the external economies the equilibrium is not Pareto optimal. In a later model proposed by Romer (1990) knowledge is produced by research using human capital as an input to design new intermediate products. The quantity of human capital is constant and separate from the labor supply. The human capital is divided between research and production.

The papers of Romer and Lucas began an avalanche of papers in this style, which continues unabated. Uzawa proved convergence of the optimal path to a balanced path, that is, a turnpike. The problem is simplified by the fact that only one good is present so that it is not necessary to choose the combination of capital goods that appear on the optimal balanced path. Romer and Lucas were primarily interested in the path under free enterprise and they conjectured convergence to balanced paths in these economies but did not pursue the question.

We should note that the New Growth Theory is distinguished by two principal characteristics. There is continuing growth at endogenously determined rates, and the growth path is treated as a general equilibrium of a competitive market, but not necessarily perfectly competitive market. The use of general equilibria of imperfectly competitive markets in growth theory was pioneered by Paul Romer (1986).

Subsequently there have been many efforts toward proving turnpike theories in models of the New Growth Theory sometimes with two capital goods denoted physical capital and human capital. As in the cases of Uzawa, Lucas, and Romer these models have allowed for unbounded growth even without population growth. If Harrod neutral technical progress is possible in the production sector from devoting a part of labor to the education sector, the possibility of continued growth is clear. Of course it is just the same if human capital can be produced by the use of human capital alone. Indeed it is also possible if the production of human capital requires both goods and human capital so long as neither goods production nor the production of human capital is constrained by a given labor supply. The conditions that are necessary for such continued growth were recently described by Jim Dolmas (1996). What is needed is that some subset of capital goods can be produced out of themselves. This subset of capital goods is analogous to the set of basic goods used by by Piero Sraffa in his book *Production of Commodities by Means of Commodities* (1960).

If such a subset of basic goods exists unlimited expansion in the supply of other goods can occur even if their production does involve fixed factors provided there is sufficient substitutability between the basic goods and the fixed factors. For example Cobb-Douglas production functions suffice. Let labor and capital be the two factors in a production function for a consumption good and assume that constant returns prevail, but allow capital to reproduce itself. Then the rate at which the production of the consumption good increases along a balanced path is equal to the product of the share of capital in the output of the consumption good times the own rate of return of capital, which is determined in the industry producing the capital good. If the share of capital in the consumption goods industry is constant, the rate of expansion in the supply of the consumption good is constant though less than the rate of expansion of the supply of capital. The return to capital per unit in terms of consumption goods approaches zero although it does not reach zero. If the utility function depends only on consumption and has a constant elasticity of substitution, there will be an optimal balanced path which is optimal from the appropriate initial stocks provided the discount rate is sufficiently high, so the utility sum is finite. The conditions for the existence of an optimal balanced path are expounded in the general case by Dolmas (1996). They were first stated for a one sector model in an early paper by Brock and Gale (1969).

9 Turnpikes and Endogenous Growth

As I mentioned earlier Michael Kaganovich (forthcoming) has shown that my method of proving the Samuelson turnpike theorem in a model of production without joint production can be extended to a Ramsey turnpike theorem where the objective is the sum of discounted future utility. The Ramsey objective first was introduced into a Leontief model with fixed coefficients by Swapan Dasgupta and Tapan Mitra (1988). Kaganovitch introduced variable coefficients along the lines of my proof of the Samuelson turnpike theorem. But he goes further to prove that the linearity of the model only need hold asymptotically.

One of the models of the New Growth Theory is a special case of Kaganovich's model. It is a feature of the von Neumann model that growth is sustained and the rate of growth is endogenous to the model. Since sustained growth is the principal distinguishing feature of the New Growth Theory it is not surprising that the model of von Neumann and the Samuelson turnpike theorem should prove relevant. However, the economic significance of von Neumann's model has been questioned from the fact that it does not recognize the supply of labor as a constraint on production. The New Growth Theory in one of its incarnations tries to avoid this criticism by replacing raw labor by human capital which is assumed to be reproducible without limit out of human capital and perhaps physical capital. The model in the New Growth Theory has the utility of consumption as the objective, but this is not a barrier to balanced growth if the utility function is homogeneous of some degree, a usual assumption the New Growth theorists make.

Kaganovich shows that constant returns in production need only hold asymptotically, that is, in the long run. Asymptotically the dynamics of the Kaganovich model is the same as the dynamics of the Dasgupta-Mitra model. The growth rate of consumption and capital stocks is determined in just the same way as in the simple Ak model of Sergio Rebelo (1991) with the von Neumann growth rate replacing the coefficient A. The condition that characterizes optimal balanced growth may be found by differentiating the simple Euler equations, based on the reduced utility function in discrete time. No fancy mathematics from the calculus of variations is needed, much less an appeal to Pontryagin. To determine the growth rate of an optimal path, first, take the product of the von Neumann maximum rate of growth times the discount factor. Then raise the product to a power equal to the elasticity of intertemporal substitution. Asymptotically this is the growth rate of the optimal path. In addition, relative prices and interest rates are asymptotically independent of preferences, thus of discount rates on future utility. In view of these facts it is clear that asymptotically the composition of consumption and the ratios of inputs into productive processes are constant. These results are reminiscent of the arguments of Piero Sraffa, as one might expect when human capital is treated as an output, since these models really do represent production of commodities by means of commodities.

Other convergence theorems have been proved in the New Growth literature in special models of endogenous growth. These use either one or two sector models. Some of the models have had competitive equilibria which are Pareto optimal. Others have been competitive but feature monopolistic competition or external economies where the competitive equilibrium is not Pareto optimal. In the case of competitive equilibrium which is Pareto Optimal the convergence is to a balanced path of optimal capital accumulation which is optimal, given the welfare function determined by the market equilibrium. However the proofs of these results have involved features special to the dimensions of the models. Moreover the linearization of the equations governing the evolution of the optimal paths have mostly led to matrices which are quite singular. For example, one of the capital goods has usually been produced without involving the other capital good, directly or indirectly. That is, human capital or education is produced without the introduction of physical capital. On the other hand, if the matrices which give the local approximation for the Euler equations for discrete time are assumed to be nonsingular the balanced growth paths of the endogenous growth models with any number of capital goods are easily proved to be locally stable when the discount factors are near enough to one.

So far as I know, interesting conditions that guarantee global stability for the case of any number of capital goods have not been found for the case of joint production. Of course, the no joint production model seems to include all the models that have been described in the New Growth Theory literature, in which the labor supply is not treated as a primary resource. If leisure is introduced into the utility function, I would say this condition is violated. So it is not surprising that multiple balanced paths arise, some of which are unstable. There is an excellent account of this literature in the January 1997 issue of *The Journal of Economic Dynamics and Control.* To have a proof of the turnpike in a truly general convex model it is necessary to eliminate the no joint production condition. Then the theorem would match Radner's proof of a Samuelson turnpike for the von Neumann model. Such a theorem seems to me to be within reach.

10 The Question of Foresight

A weakness of the turnpike theory In the setting of competitive equilibrium is the requirement of long term foresight by the economic agents. It was first shown by Edmond Malinvaud (1953) that efficiency in production over time was not guaranteed by period by period maximization of profits since capital overaccumulation could occur. More recently Frank Hahn (1966) in particular has emphasized that satisfying the conditions of market equilibrium in the short run is no guarantee that the economy is on an optimal path. It is well known that the prices that guide the choices of the economic agents must be such that a transversality condition at infinity is satisfied. However, it was suggested by David Gale in his classic paper (1967, p. 2) that continual revisions of the plan might be able to overcome the deficiencies of long range foresight. Quoting him, "to describe the situation figuratively, one is guiding a ship on a long journey by keeping it lined up with a point on the horizon even though one knows that long before that point is reached the weather will change (but in an unpredictable way) and it will be necessary to pick a new course with a new reference point, again on the horizon rather than just a short distance ahead."

Attempts have been made to prove theorems where short term foresight is sufficient. In the context of planning for optimal growth this program was begun by Goldman (1968) who showed that planning for a finite program which was revised every period would converge to the optimal program in a one sector Ramsey style neoclassical model if the planning period was chosen sufficiently long. He required that the terminal capital stock at the end of the planning period be at least as large as the capital stock at the time of planning. This result has been generalized to multisector models since then.

These attempts to reduce the demands on foresight do not deal with the questions of technical progress and the introduction of new goods. A recent unpublished and, indeed, unfinished, paper by Boldrin and Levine (1997) entitled "Innovation Growth and Cycles in General Equilibrium" approaches these problems by way of a model with an infinite number of goods and activities. Only a finite set of goods and activities are present at any one time. Anticipating today's talk by Alan Greenspan their utility functions are defined in terms of characteristics, rather than goods. They remark, "...convergence to any balanced growth path is, at best partial and temporary as new feasible activities are implemented when they become profitable and others are disbanded when not profitable anymore." Their arguments are interesting and provocative, but it is not possible for me to discuss them today.

In the end the growth models like all economic models are guides to the kinds of things that may happen. They cannot predict with the accuracy expected of the natural sciences. Of course the refined accuracy of prediction of the natural sciences is realized principally in controlled situations in the laboratory or in industry, with the perhaps singular exception of the movements of the celestial bodies. In the context of demographic theory Nathan Keyfitz (1977, p. 86) remarks, "An exposition of the mathematics of population is not more directly concerned with prediction of future changes than a book on hydrodynamics is concerned with the prediction of floods. The most one can hope is that theoretical formulations will give the practitioners who do the predicting some help in thinking about their problem." I suppose we should not expect the position of the theoretical economist to be stronger than that of the theoretical demographer.

References

Atsumi, Hiroshi. "Neoclassical Growth and the Efficient Program of Capital Accumulation". *Review of Economic Studies*, 1965, 32(2), pp. 127-136.

Barro, Robert and Xavier Sala-i-Martin. *Economic Growth.* New York: McGraw-Hill, 1995.

Becker, Gary S.; Murphy, Kevin M. and Tamura, Robert. "Human Capital, Fertility, and Economic Growth." *Journal of Political Economy*, 1990, 98(5), part II, S12-S37.

Becker, Robert A. "The Equivalence of a Fisher Competitive Equilibrium and a Perfect Foresight Competitive Equilibrium in a Multi-Sectoral Model of Capital Accumulation." *International Economic Review*, 1982, 23(1), pp. 19-34.

Becker, Robert A. and Boyd, John H. III. *Capital Theory, Equilibrium Analysis and Recursive Utility.* Malden, Massachusetts: Blackwell, 1997.

Bewley, Truman. "An Integration of Equilibrium Theory and Turnpike Theory." *Journal of Mathematical Economics*, 1982, 10(2/3), pp. 514-540.

Boldrin, Michele and Levine, David. "Innovation Growth and Cycles in General Equilibrium". Very preliminary and incomplete.

Boldrin, Michele and Woodford, Michael. "Equilibrium in Models Displaying Fluctuations and Chaos: A Survey." *Journal of Monetary Economics*, 1990, 25, pp. 189-222.

Brock, William A. and Gale, David. Optimal Growth under Factor Augmenting Progress." *Journal of Economic Theory*, 1969, 1(3), pp. 229-243.

Brock, William A. and Mirman, Leonard J. "Optimal Economic Growth and Uncertainty: the Discounted Case." *Journal of Economic Theory*, 1972, 4, pp. 479-513.

Cass, David. "Optimum Growth in an Aggregative Model of Capital Accumulation: a Turnpike Theorem." *Econometrica*, 1966, 34(4), pp. 833-50.

Cass, David and Shell, Carl. "The Structure and Stability of Competitive Dynamical Systems." *Review of Economic Theory*, 1976, 12(1), pp. 1-10.

Dasgupta, Swapan and Mitra, Tapan. "Intertemporal Optimality in a Closed Linear Model of Production." *Journal of Economic Theory*, 1988, 45(2), pp. 288-315.

Dolmas, Jim. "Endogenous Growth in the Model with Overlapping Generations." *International Economic Review*, 1996, 37(2), pp. 403-421.

Dorfman, Robert; Samuelson, Paul A. and Solow, Robert. *Linear Programming and Economic Analysis*, New York: McGraw-Hill, 1958.

Gale, David. "On Optimal Development in a Multi-Sector Economy." *Review of Economic Studies*, 1967, 34(1), pp. 1-18.

Goldman, S. M. "Optimal Growth and Continual Planning Revision." *Review of Economic Studies*, 1968, 35(102), pp. 145-154.

Hahn, Frank. "Equilibrium Dynamics with Heterogeneous Capital Goods." *Quarterly Journal of Economics*, 1966, 80(4), pp. 633-646.

Kaganovich, Michael. "Sustained Endogenous Growth with Decreasing Returns and Heterogeneous Capital." *Journal of Economic Dynamics and Control*, forthcoming.

Keyfitz, Nathan. *Introduction to the Mathematics of Population with Revisions.* Reading, Massachusetts: Addison-Wesley, 1977.

Koopmans, Tjalling C. "The Concept of Optimal Economic Growth." *The Econometric Approach to Development Planning*, Pontificae Academiae Scientiarum Scripta Varia No. 28. Amsterdam: North Holland, 1965.

Lopez, Alvaro. *Problems in Stable Population Theory.* Princeton, New Jersey: Office of Population Research, 1961.

Lucas, Robert E., Jr. "On the Mechanics of Economic Development". *Journal of Monetary Economics*, 1988, 22(1), pp 3-42.

Lucas, Robert E., Jr. and Stokey, Nancy L. "Optimal Growth with Many Consumers." *Journal of Economic Theory*, 1984, 32, pp. 139-171.

Malinvaud, Edmond. "Capital Accumulation and Efficient Allocation of Resources." *Econometrica*, 1953, 21(2), pp. 233-268.

Mckenzie, Lionel W. "Turnpike Theorem of Morishima". *Review of Economic Studies*, 1963, 30(2), pp. 169-176.

McKenzie, Lionel W. "Turnpike Theorems for a Generalized Leontief Model." *Econometrica*, 1963, 31(1-2), pp.165-180.

McKenzie, Lionel W. "Accumulation Programs of Maximum Utility and the von Neumann Facet." In J. N. Wolfe, ed., *Value, Capital, and Growth.* Edinburgh: Edinburgh University Press, 1968.

McKenzie, Lionel W. "Turnpike Theorems with Technology and Welfare Function Variable." In J. Los and M. Los, ed., *Mathematical Models in Economics*: New York, American Elsevier, 1974.

McKenzie, Lionel W. "Turnpike Theory." *Econometrica*, 1976, 44(5), pp. 841-865.

McKenzie, Lionel W. "Turnpike Theory, Discounted Utility, and the von Neumann Facet." *Journal of Economic Theory*, 1983(2), 30, pp. 330-352.

McKenzie, Lionel W. "Optimal Economic Growth, Turnpike Theorems and Comparative Dynamics". In *K. J. Arrow and Michael D. Intriligator, eds., Handbook of Mathematical Economics, vol.III*: Amsterdam, North Holland, 1986.

Mirrlees, James A. "Optimum Growth When Technology Is Changing". *Review of Economic Studies*, 1967, 34(1), pp.95-124.

Morishima, Michio. "Proof of a Turnpike Theorem: the No Joint Production Case." *Review of Economic Studies*, February, 1961, 28.

Radner, Roy. "Paths of Economic Growth that Are Optimal with regard Only to Final States." *Review of Economic Studies*, 1961, 28(1), pp. 98-104.

Ramsey, Frank P. "A Mathematical Theory of Saving." *Economic Journal*, 1928, 38(152), pp. 543-559.

Rebelo, Sergio. "Long-Run Policy Analysis and Long-Run Growth." *Journal of Political Economy*, 1991, 99(3), pp. 500-521.

Romer, Paul M. "Increasing Returns and Long-Run Growth." *Journal of Political Economy*, 1986, 94(5), 1002-1037.

Romer, Paul M. "Endogenous Technological Change." *Journal of Political Economy*, 1990, 98(5), S71-S102.

Samuelson, Paul A. *Market Mechanisms and Maximization, Part III.* Santa Monica, California: Rand Corporation, 1949.

Scheinkman, José A. "On Optimal Steady States of n-sector Growth Models when Utility Is Discounted." *Journal of Economic Theory*, 1976, 12(1), pp. 11-30.

Solow, Robert M. "A Contribution to the Theory of Economic Growth." *Quarterly Journal of Economics*, 1956, 70(1), pp. 65-94.

Solow, Robert M. *Growth Theory, An Exposition.* Oxford, The Clarendon Press, 1970.

Solow, Robert and Samuelson, Paul A. "Balanced Growth under Constant Returns to Scale." *Econometrica*, 1953, 21(3), pp. 412-424.

Sraffa, Piero. *Production of Commodities by Means of Commodities: Prelude to a Critique of Economic Theory.* Cambridge: Cambridge University Press, 1960.

Swan, Trevor W. "Economic Growth and Capital Accumulation." *Economic Record* 32 (November), 334-361.

Takahashi, Harutaka. *Characterization of Optimal Programs in Infinite Horizon Economies.* PhD Thesis, University of Rochester, 1985.

Uzawa, Hirofumi. "Optimal Change in an Aggregative Model of Economic Growth," *International Economic Review*, 1965, 6 (1), pp. 18-31.

Uzawa, Hirofumi. "Time Preference and the Penrose Effect in a Two-Class Model of Economic Growth." *Journal of Political Economy*, 1969, 77, pp. 628-652.

von Neumann, John. "Über ein Ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsätzes." *Ergebnisse eines Mathematischen Kolloquim*, 1937, 8, 73-83. Translated in "A Model of General Equilibrium", *Review of Economic Studies*,1945, 32, pp. 85-104.

Yano, Makoto. "The Turnpike of Dynamic General Equilibrium Paths and Its Insensitivity to Initial Conditions." *Journal of Mathematical Economics*, 1984, 13(3), pp. 235-254.

Yano, Makoto. "On the Dual Stability of a von Neumann Facet and the Inefficacy of Temporary Fiscal Policy." *Econometrica*, forthcoming.

*This academic year marks the 40th anniversary of the founding of the Rochester Economics Department. Therefore, I wish to dedicate this paper, unworthy though it is, to the faculty and graduate students of the Rochester Department over those years. Many of their names will appear in my paper. Even more should appear were there more time to discuss their contributions.

University of Rochester

January 10, 1998