Consistency and Its Converse: An Introduction

Thomson, William

Working Paper No. 448
February 1998

Consistency and Its Converse: An Introduction

William Thomson

Rochester Center for Economic Research
Working Paper No. 448

February 1998

Consistency and its Converse: an Introduction

William Thomson*

February 4, 1998

*This paper was prepared for a NATO Advanced Research Institute entitled "Game Theory and Resource Allocation: the Axiomatic Approach", held in SUNY Stony Brook, July 21-31, 1997. I thank NATO for providing the finances that made it possible, and SUNY Stony Brook for assistance with the local arrangements. I also thank Bettina Klaus and Eiichi Miyagawa for helpful comments.

Abstract

This essay is an introduction to the recent literature on the "consistency principle" and its "converse". An allocation rule is consistent if for any problem in its domain of definition and any alternative that it selects for it, then for the associated "reduced problem" obtained by imagining the departure of any subgroup of the agents with their "components of the alternative" and reassessing the options open to the remaining agents, it chooses the restriction of the alternative to that subgroup. Converse consistency pertains to the opposite operation. It allows us to deduce that the rule chooses an alternative for some problem from the knowledge that for all two-person subgroups, it chooses its restriction to the subgroup for the associated reduced problem this subgroup faces.

We present two lemmas, the Elevator Lemma and the Bracing Lemma, involving these properties. These lemmas have been found useful in the analysis of a great variety of models. We also describe some of their applications. Finally, we illustrate the versatility of consistency and of its converse by means of a sample of characterizations based on these principles.

Contents

1 Introduction

This essay is an introduction to the recent literature on the "consistency principle" and its "converse". The principles pertain to the behavior of solutions whose domains of definition contain problems involving variable sets of agents and in particular sets of different sizes, and for which it is therefore meaningful to compare the choices they make for different populations. By contrast, most of the axiomatic literature has been written for a fixed set of agents. A solution is *consistent* if whenever it chooses a certain alternative for some problem, then for the "reduced problem" obtained by imagining the departure of some of the agents with their components of the alternative and reassessing the opportunities open to the remaining agents, it chooses the restriction of the alternative to this subgroup. *Converse consistency* has to do with the opposite operation. It allows us to deduce that the solution chooses a certain alternative for some problem from the knowledge that for all two-person subgroups, it chooses its restriction to the subgroup for the associated reduced problem this subgroup faces.

The principles, which were first investigated for abstract models of cooperative game theory, have recently been examined in the context of a great variety of concrete problems of resource allocation, and for many models, their implications are now quite well understood. We will survey some of these developments.

The paper is organized as follows. We first introduce the basic concepts of a problem and of a solution, and our two principles. Then, we state two lemmas that have been critical in proofs for a wide range of models. Finally, we describe several characterizations based on *consistency* and its *converse*.

In order to illustrate various points we make, we introduce a number of models in succession. Our objective is not a comprehensive account of what is known of *consistency* and of its *converse* for these models, but rather to give the flavor of the usefulness of the conditions in evaluating allocation rules, and in exposing the mechanics of proofs based on these principles. For a detailed survey of the vast literature devoted to the study of the two principles, see Thomson (1997).

2 Basic concepts: domains and solutions

A ***problem*** is given by a set of ***alternatives*** and a set of ***agents*** whose preferences are defined over this set or over "personal" components of it. The objective is to identify one or several feasible alternatives for each problem satisfying some regularity conditions. Depending upon the context, these alternatives are interpreted as the recommendations that an arbitrator, (alternatively a planner, a high level manager, a judge ...) could make, or as predictions of what the agents would choose if left to their own devices. Instead of handling each problem separately however, we will look for general methods of selecting alternatives for each admissible problem. A ***solution*** is a correspondence defined on some ***domain of problems*** that associates with each problem in the domain a non-empty subset of its feasible set.

A number of tests can be devised to evaluate how satisfactory a solution is. The test of consistency involves comparing the choices it makes for some problem involving some "initial" group of agents to the choices it makes for associated "reduced" problems involving subgroups.

We use the following notation throughout. There is an infinite set of "potential" agents indexed by the natural numbers, $\mathbb{N}$. For each group of agents N drawn from the family $\mathcal{N}$ of non-empty finite subsets of $\mathbb{N}$, there is a class of problems that N could face. Solutions are defined over the union of these classes as N varies in $\mathcal{N}$. Our generic notation is $\mathcal{D}^N$ for the class of problems that N could face and $\mathcal{D}$ for the union $\cup_{N\in\mathcal{N}}\mathcal{D}^N$. When an alternative is chosen by a solution φ for some problem D, we say that it is ***φ-optimal for D***. If a solution φ' only selects alternatives that are also selected by some solution φ, we say that ***φ' is a subsolution of φ***. We then write $\varphi' \subseteq \varphi$.

To see the need for testing how solutions behave when the population of agents varies, we introduce our first domain, which pertains to fair division.

Domain 1 *A **classical problem of fair division** (see Thomson, 1996, for a survey) is a pair (R, Ω) where $R = (R_i)_{i\in N}$ is a list of preference relations defined on the non-negative quadrant of the ℓ-dimensional commodity space, $\mathbb{R}^\ell_+$, and $\Omega \in \mathbb{R}^\ell_{++}$ is a social endowment. Preferences are continuous, increasing, and convex. The asymmetric part of R_i is denoted P_i and indifference is denoted I_i. A feasible allocation for (R, Ω) is a list $z \in \mathbb{R}^{\ell N}_+$ such* *that* $\sum z_i = \Omega$.[1]

Several of the solutions defined next will also play a role, when appropriately adapted, in the analysis of models introduced later, and we will not repeat the formal definitions then.

Examples of solutions for Domain 1 *The* ***Pareto solution,*** P*, selects the feasible allocations* z *for which there is no other feasible allocation* z' *such that for all* $i \in N$, $z'_i \; R_i \; z_i$*, strict preference holding for at least one* $i \in N$*. The* ***no-envy solution,*** F*, selects the feasible allocations* z *such that for all* $i, j \in N$, $z_i \; R_i \; z_j$*. The* ***equal division lower bound solution*** *selects the feasible allocations* z *such that for all* $i \in N$, $z_i \; R_i \; \Omega/|N|$*. The* ***Walrasian solution operated from equal division,*** W_{ed}*, selects the feasible allocations* z *for which there exists a price vector* $p \in \Delta^\ell$ *such that for all* $i \in N$, z_i *maximizes* R_i *in the budget set* $\{z'_i \in \mathbb{R}^\ell_+ : pz'_i \leq p\Omega/|N|\}$.[2] *The* ***egalitarian-equivalence solution*** *selects the feasible allocations* z *such that for some reference bundle* $z_0 \in \mathbb{R}^\ell_+$*, and for all* $i \in N$, $z_i \; I_i \; z_0$.

Figure 1a represents the solution that chooses the Walrasian allocations from equal division for all economies and Figure 1b the solution that chooses the envy-free allocations for all economies. When a solution is defined over a domain of problems of arbitrary cardinalities, there is no reason in principle why it could not choose allocations in completely different ways as the number of agents varies. Giving free rein to our imagination, let us consider for instance the solution that selects the efficient allocations for all two-person economies, the envy-free allocations for all three-person economies, and the Walrasian allocations from equal division for all economies involving more than three agents (Figure 1c)! Figures 1d-e represent two other solutions, both of which also seem quite hard to justify. These examples make it clear that some test is needed to relate the choices made by solutions for different sets of agents. *Consistency* is such a test.

[1]By $\mathbb{R}^N$ and $\mathbb{R}^{\ell N}$ we mean the cross-products of $|N|$ copies of $\mathbb{R}$ and $\mathbb{R}^\ell$ respectively, indexed by the members of N.

[2]The notation Δ^ℓ designates the unit simplex in the ℓ-dimensional Euclidean space $\mathbb{R}^\ell$.

⋮	⋮	⋮	⋮	⋮	⋮
General n	W_{ed}	F	W_{ed}	P	P
⋮	⋮	⋮	⋮	⋮	⋮
$n = 4$	W_{ed}	F	W_{ed}	P	W_{ed}
$n = 3$	W_{ed}	F	F	F	P
$n = 2$	W_{ed}	F	P	W_{ed}	F
	(a)	(b)	(c)	(d)	(e)

Figure 1: Solutions defined on a domain of problems of arbitrary cardinalities. Which one(s) of the following solutions is (are) *consistent*? (a) This solution selects the Walrasian allocations from equal division for all cardinalities. (b) This solution selects the envy-free allocations for all cardinalities. (c) This hybrid solution is more and more restrictive as the number of agents increases: it selects the efficient allocations for cardinality two, the envy-free allocations for cardinality three, and the Walrasian allocations from equal division for greater cardinalities. (d) By contrast, this solution is less and less restrictive as the number of agents increases. (e) This solution makes choices that do not seem to follow any particular pattern as the number of agents changes.

3 Consistency and its converse

In this section, we introduce the notion of a *consistent* solution and illustrate it by means of several examples. We also define the notion of a *conversely consistent* solution. We only give a few sample proofs that certain solutions are *consistent* or *conversely consistent*, or violate the properties. In most cases, these are simple exercises, which we suggest to the reader as a way of progressively strengthening his or her understanding of the conditions, as well as gaining familiarity with the models, not all of which are standard, that we will discuss. The figures should be seen as an integral part of our exposition, as their legends sometimes contain sketches of proofs.

3.1 Consistent allocation rules: the general definition

Very informally, a solution is *consistent* if there is never a need to revise an alternative it has chosen after some of the agents "have received their components of it" and left. At this point, the clause "have received their components of the alternative" is only meant to be suggestive and we will devote the next few pages to clarifying it.

Somewhat more precisely, let $N \in \mathcal{N}$ and D be a problem that N could face. Let φ be a solution and x one of the φ-optimal alternatives for D. Now, we imagine some of the agents leaving with their components of x and we reevaluate the situation from the perspective of the remaining agents. If N' is the subgroup of remaining agents, we denote $r^x_{N'}(D)$ the set of alternatives at which the agents who leave receive their components of x, and refer to it as the ***reduced problem of D with respect to N′ and x***.

Consistency: For all groups $N \in \mathcal{N}$, all problems $D \in \mathcal{D}^N$, all subgroups $N' \subset N$, and all φ-optimal alternatives of D, x, if the reduced problem of D with respect to N' and x, obtained from D by assigning to all agents in $N \backslash N'$ their components of x, belongs to $\mathcal{D}^{N'}$, then the restriction of x to N' is φ-optimal for it: $x_{N'} \in \varphi(r^x_{N'}(D))$.

The following variants of the basic idea have been explored in the literature. (i) First is the slightly weaker condition obtained by limiting attention to subgroups of *two* remaining agents, a variant called ***bilateral consistency***. (ii) Apart from size, it is sometimes natural to impose other restrictions on the subgroups. The class they constitute may be endowed with some particular structure so as to reflect relevant aspects of social organization, such as communication networks, trade groups, family relations .... (iii) A third variant is obtained, for *single-valued* solutions and in models equipped with a convex structure, by instead of asking that the restriction of x to each subgroup be chosen for the reduced problem associated with x this subgroup faces, requiring that for each agent, his component of x coincides with the *average* of what he would receive in the reduced problems associated with x and all the proper subgroups of N to which he belongs.

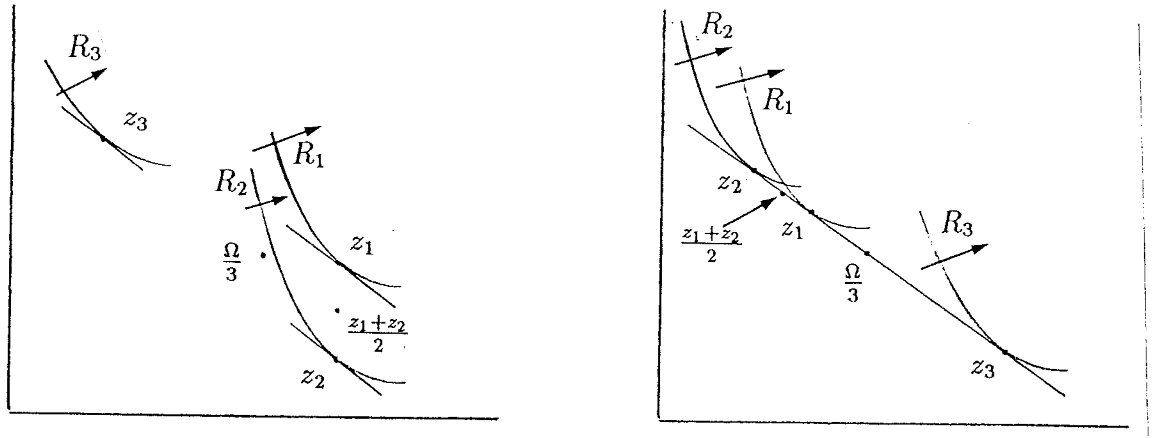
We will now illustrate the various choices that we have in defining reduced problems by considering several applications.

3.1.1 The reduction operation for models formulated in commodity space

We start with the allocation of privately appropriable goods. There, the natural "separability" of the allocation space suggests a very simple way of defining a reduced economy. When some agents leave, they take along the bundles intended for them, so that the set of options available to the remaining agents is simply the set of lists of bundles obtained by distributing among them the resources that are left; this is of course the sum of the bundles that were intended for them in the first place. Specifically, given $e = (R, \Omega)$, $N' \subset N$, and $z \in \varphi(e)$, the reduced economy of e with respect to N' and z is the pair $(R_{N'}, \Omega - \sum_{N \setminus N'} z_i)$, or equivalently $(R_{N'}, \sum_{N'} z_i)$.

It is easy to see that the Pareto solution is *consistent* since, if no Pareto-improving reallocation of goods can be achieved by the group N, then of course no subgroup $N' \subset N$ can achieve any Pareto-improving reallocation of the resources it has received. The no-envy solution is *consistent* too: if an agent does not want to exchange bundles with anyone in the initial group N, then *a fortiori*, he does not want to exchange bundles with anyone in any subgroup $N' \subset N$.

On the other hand, the equal division lower bound solution is not *consistent* (Figure 2a). It is tempting to say that this is because in a reduced economy, the point of equal division is typically not the same as what it is in the original economy, so that if an agent finds his component of some allocation at least as desirable as equal division, there is no reason why this should still be the case in an associated reduced economy. But that is not the whole story, because the Walrasian solution operated from equal division also depends on what equal division is, and it is *consistent* (Figure 2b). The reason



(a) (b)

Figure 2: Examples of consistent solutions for the problem of fair division. (a) The equal division lower bound and Pareto solution is not *consistent*: z meets the bound in the three-person economy represented here, but since $\frac{z_1+z_2}{2}\ P_2\ z_2$, its restriction (z_1, z_2) to the group $\{1, 2\}$ does not meet the bound in $(R_1, R_2, z_1 + z_2)$. (b) The Walrasian solution operated from equal division is *consistent*: here, z is Walrasian from equal division in the three-person economy; after agent 3 leaves with his bundle z_3, the resources that remain available to agents 1 and 2 are $z_1 + z_2$, and if each of them is endowed with $\frac{z_1+z_2}{2}$, equilibrium is indeed achieved by quoting the same prices, the corresponding allocation being (z_1, z_2).

is that for that solution the points of equal division of the reduced economies associated with a chosen allocation are related in a very special way: they all have the same value at the initial equilibrium prices. Consequently, in any of these reduced economies, if the same prices are quoted, the resulting budget sets are the same as in the initial economy; this preserves the maximizing bundles of the members of the subgroup of remaining agents, which in turn guarantees equality of demand and supply in the reduced economy.

3.1.2 The reduction operation for models formulated in utility space

For the model of bargaining presented next, feasible sets are given in utility space.

Domain 2 *A* ***bargaining problem*** *is a convex, compact, and comprehensive*[3] *subset of* $\mathbb{R}^N_+$, T, *containing at least one strictly positive point.*

Examples of solutions for Domain 2 *The* ***Nash solution*** *(Nash, 1950) selects the point of* T *that maximizes the product of utilities. The* ***egalitarian solution*** *(Kalai, 1977) selects the maximal point of* T *of equal*

[3]By comprehensiveness we mean that if $x \in \mathbb{R}^N_+$ is feasible, then so is any other vector y such that $0 \leq y \leq x$.

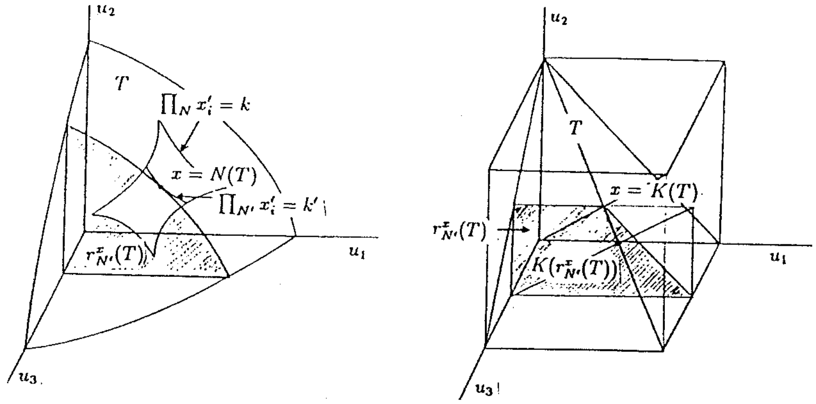


Figure 3: Bargaining problem. (a) The Nash solution is *consistent*: if x achieves maximal product of utilities in T, then (x_1, x_2) achieves maximal product of utilities in the section of T by the plane parallel to the $\{1, 2\}$-coordinate subspace passing through x. (b) The Kalai-Smorodinsky solution is not *consistent*: it selects x for T but it does not select (x_1, x_2) for the section of T by the plane parallel to the $\{1, 2\}$-coordinate subspace passing through x.

utilities. The ***Kalai-Smorodinsky solution*** *(Kalai and Smorodinsky, 1975) selects the maximal point of T proportional to the ideal point of T, the point whose i-th coordinate is the maximal feasible utility agent i can achieve in T.*

Here, agents "leave" with utility levels, and it is most natural to define the set of options open to the remaining agents as the subset of the initial problem consisting of all the vectors at which the departing agents receive their promised payoffs. Given a problem T involving the initial group N, the reduced problem of T with respect to $N' \subset N$ and x is therefore defined as $\{x' \in \mathbb{R}^{N'}$: for some $y \in T, y_{N \setminus N'} = x_{N \setminus N'}$, and $y_{N'} = x'\}$. Geometrically, this is the section of T by a plane parallel to the N'-coordinate subspace through x.

The Nash solution is *consistent* but the Kalai-Smorodinsky solution is not (Figure 3). The egalitarian solution is *consistent* on the subdomain of problems on which it selects Pareto-optimal outcomes.

Note that when a feasible set is obtained as the image in utility space of the set of allocations obtainable by distributing a fixed bundle of goods, if the departing agents were to leave with physical amounts of goods giving them their agreed-upon utilities, the set of options available to the remaining agents would typically be a subset of the feasible set of the reduced game as we have just defined it (except in the one-good case).

3.1.3 The reduction operation when the departing agents remain "available"

In other models, it is important to think of the agents who leave as remaining "available". This is illustrated by the next model.

Domain 3 *A (transferable utility)* ***coalitional game*** *is a vector* $v \in \mathbb{R}^{2^{|N|-1}}$*, with coordinates indexed by the non-empty subgroups of* N*, called coalitions, each coordinate being interpreted as what the corresponding coalition can achieve; this amount is called the worth of the coalition.*

Examples of solutions for Domain 3 *The* ***core*** *(Gillies, 1959) selects the payoff vectors* x *such that* $\sum_N x_i = v(N)$ *and for all* $S \subset N$, $\sum_S x_i \geq v(S)$*. The* ***Shapley value*** *(Shapley, 1953) selects the payoff vector whose* i*-th coordinate is equal to* $\sum_{S:i\in S} k_S(v(S) - v(S\backslash i))$ *for certain combinatorial coefficients* k_S*. Now, define the dissatisfaction of coalition* S *at the payoff vector* x *to be the difference* $v(S) - \sum_S x_i$*; then, the* ***prenucleolus*** *(Schmeidler, 1969) selects the feasible payoff vector at which the dissatisfactions of coalitions are minimized in a lexicographic way, starting with the most dissatisfied coalition.*[4]

Here, we have several ways of defining the set of options available to the remaining agents. Given a coalition $S \subset N'$, a first possibility is to calculate, as for bargaining problems, what S can achieve by getting together with the departing agents and giving them their agreed-upon payoffs. This yields the difference $v(S \cup N\backslash N') - \sum_{N\backslash N'} x_i$ as the worth of S. Since the complement of N' is involved in the reduced game, we name the resulting condition "complement consistency" (Moulin, 1988).

Alternatively, we can let S choose which ones of the departing agents to "cooperate with". By getting together with $S' \subset N\backslash N'$, the worth $v(S\cup S')$ is generated, but since the members of S' have to be paid $\sum_{S'} x_i$, what remains for S is the difference $v(S \cup S') - \sum_{S'} x_i$. Here, the worth of S in the reduced game is defined to be the *maximal* such difference when S' ranges over the subsets of $N\backslash N'$. This definition being based on a maximization exercise, we name it "max-consistency" (Davis and Maschler, 1965).

[4]This means that it selects the payoff vector at which the dissatisfaction of the most dissatisfied coalition is minimal if there is a unique such vector. Otherwise, among these minimizers, it picks the vector at which the dissatisfaction of the second most dissatisfied coalition is minimal if there is a unique such vector; otherwise ...

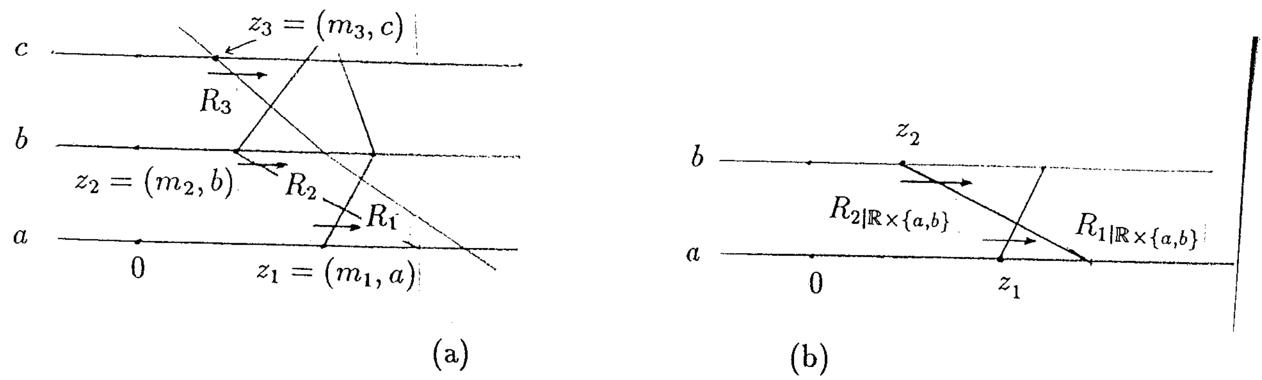


Figure 4: Reducing a consumption space and a preference relation. (a) We start from a three-person economy in which each of the three agents has preferences defined over the product of $\mathbb{R}$ with the set consisting of the three objects available, $\{a, b, c\}$. The chosen allocation is z. (b) Agent 3 leaves with his bundle z_3, which contains object c. In the reduced economy associated with the group $\{1, 2\}$ and z, the preferences of agents 1 and 2 are restricted to the product of $\mathbb{R}$ with the set consisting of the remaining objects, $\{a, b\}$.

It turns out that the core satisfies both definitions and that the Shapley value satisfies neither. The prenucleolus only satisfies the second one.

3.1.4 When the reduction operation suggests a reduction of consumption spaces

In any model it is possible and sometimes appealing to require that solutions only depend on the restrictions of preferences to the set of bundles that are actually feasible.[5] When some of the agents leave with certain resources, the set of bundles achievable by any one of the remaining agents will of course get smaller. For some models, consumption spaces are "decomposable" in a way that makes requiring this kind of independence even more tempting. Then, we will say that we "reduce" consumption spaces and preference relations. An illustration is provided by the next domain. Another example is matching (Domain 6 defined below).

Domain 4 *An allocation problem with indivisible goods when monetary compensations are possible (Svensson, 1983) is a list (M, A, R) where $M \in \mathbb{R}$ is some amount of an infinitely divisible good called "money", A is a finite set of "objects" drawn from some infinite list $\mathcal{A}$, and $R = (R_i)_{i \in N}$ is a list of preference relations defined over the product $\mathbb{R} \times A$. Preferences are continuous and strictly monotonic with respect to money. We assume $|N| = |A|$. A feasible allocation is a pair (m, σ) where $m \in \mathbb{R}^N$ is a list of*

[5] A number of interesting solutions do not satisfy this requirement however, the Walrasian solution being an example.

monetary amounts satisfying $\sum_N m_i = M$, and σ is a bijection from N to A indicating which object each agent receives.

This model is illustrated in Figure 4a for a three-person example. With each of the objects is associated an axis along which the amount of money that will go with it is measured, thereby defining a bundle that will be assigned to one of the agents. The broken lines connect bundles that are indifferent to each other. Note that given that no sign constraint is imposed on the consumption of money, any bundle containing an existing object is feasible for any of the agents.

Examples of solutions for Domain 4 *The* ***Pareto, no-envy,*** *and* ***egalitarian-equivalence*** *solutions are all still meaningful here.*[6]

Whether or not we reduce consumption spaces and preference relations affects which solutions are *consistent*. For instance, if we do, the egalitarian-equivalence solution violates the property; indeed, an agent could leave with the object appearing in the reference bundle associated with an egalitarian-equivalent allocation. But if we do not, the solution is *consistent*, just as it is on the classical domain (Domain 1). For the no-envy solution however, it does not matter which specification is adopted; it is *consistent* either way.

3.1.5 Closedness of domains under the reduction operation

According to our definition of *consistency*, nothing is required of the solution if the reduced problem does not belong to the domain. Alternatively, we may *require* the reduced problem to be in the domain.

How strong this additional requirement is depends on the domain of problems under investigation. There are domains such that, for any feasible outcome, the natural way to define the reduction produces a problem that is admissible. We then say that the "domain is closed under the reduction operation". An example here is the classical domain (Domain 1): if (R, Ω) is admissible and z is a feasible allocation, then $(R_{N'}, \sum_{N'} z_i)$ is admissible too. On the other hand, suppose that instead of thinking of the departing agents leaving with their components of z, we had imagined them leaving with the understanding that whatever allocation is eventually chosen should give them the welfare levels they experience at z. Then the reduced problem

[6]For this model, the no-envy solution is a subsolution of the Pareto solution (Svensson, 1983).

of (R, Ω) with respect to N' and z would be the set of lists $(z'_i)_{i \in N'} \in \mathbb{R}_+^{\ell N'}$ such that for some list $(z'_i)_{i \in N \setminus N'} \in \mathbb{R}_+^{\ell N \setminus N'}$, we have (i) $\sum_N z'_j = \Omega$ and (ii) for all $i \in N \setminus N'$, $z'_i \; I_i \; z_i$ (as discussed in Subsection 3.1.2). Such a reduced economy could not be described as a pair $(R_{N'}, \Omega')$ for some $\Omega' \in \mathbb{R}_+^{\ell}$, and closedness would fail.

In some cases, the reduction operation yields a problem that may not satisfy all of the assumptions imposed on the elements of the domain. For example, consider standard production economies. There, what first comes to mind in defining the production set of a reduced economy is to translate the production set of the initial economy by the vector of goods taken with them by the departing agents. (Some of the coordinates of this vector, the ones corresponding to labor inputs, have negative coordinates.) But it is unlikely that this resulting production set will satisfy the same regularity conditions, such as "no free lunch", "increasing returns to scale", ..., that may have been imposed on the initial set.

In other cases, the reduced problem is admissible only if the outcome is chosen in certain ways. If this happens for the outcomes chosen by a particular solution, we say that "the domain is closed under the reduction operation for the solution". This is illustrated by our next domain.

Domain 5 *A* ***bankuptcy problem*** *(O'Neill, 1982; Aumann and Maschler, 1985) is a list* $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$ *such that* $\sum c_i \geq E$. *The number* c_i *is the claim of agent* i *on the net worth* E *of a bankrupt firm. A feasible allocation for* (c, E) *is a list* $x \in \mathbb{R}_+^N$ *such that* $\sum x_i = E$.

Examples of solutions for Domain 5 *The* ***proportional solution*** *selects awards proportional to claims. The* ***constrained equal awards solution*** *selects the feasible award vector such that, for some* $\lambda \in \mathbb{R}_+$, *each claimant* $i \in N$ *receives* $\min\{c_i, \lambda\}$. *The* ***constrained equal losses solution*** *selects the feasible award vector such that, for some* λ, *each claimant* $i \in N$ *receives* $\max\{c_i - \lambda, 0\}$. *The* ***random arrival rule*** *selects the average of the vectors of awards obtained as follows: for each possible order in which agents could arrive, give to each of them the minimum of his claim and whatever remains. The average is calculated under the assumption that all orders of arrival are equally likely.*

Here, the most natural way of defining the reduced problem of (c, E) with respect to $N' \subset N$ and a feasible allocation x is $(c_{N'}, E - \sum_{N \setminus N'} x_i)$, or equivalently, $(c_{N'}, \sum_{N'} x_i)$. It is easy to see that the proportional solution is *consistent*, and that so are the constrained equal awards and constrained equal losses solutions. On the other hand, almost any example reveals that the random arrival solution is not.

Now, note that in general in a reduced problem we may not have $\sum_{N'} c_i \geq \sum_{N'} x_i$. However it makes sense to require of a solution φ that if $x = \varphi(c, E)$, then for all $i \in N$, $x_i \leq c_i$. If this property of ***claims boundedness*** is satisfied, then for all $N' \subset N$, we have $\sum_{N'} c_i \geq \sum_{N'} x_i$. Therefore, we can say that the domain of bankruptcy problems is closed under the reduction operation for any solution satisfying *claims boundedness.*

A surplus sharing problem (Moulin, 1987) is defined like a bankruptcy problem except that the inequality $\sum c_i \leq E$ is imposed instead. The number c_i is interpreted as the investment made by agent $i \in N$ in a successful venture whose worth is E. A feasible allocation for (c, E) is a vector $x \in \mathbb{R}^N_+$ such that $\sum x_i = E$. Again, for a solution that only chooses award vectors x satisfying the natural requirement that for all $i \in N$, $x_i \geq c_i$, closedness of the domain holds.

Finally, consider the domain consisting of all bankruptcy *and* surplus-sharing problems; for a problem in this enlarged domain, no relation is imposed between $\sum c_i$ and E. Here, any reduced problem is admissible. Starting from a surplus-sharing problem, and given a feasible allocation for it, an associated reduced problem may be a bankruptcy problem, and conversely, but this does not create any difficulty since the solution is applicable in any case.

3.2 Constructing consistent solutions. Minimal consistent extensions. Maximal consistent subsolutions

Here, we identify several operations preserving *consistency.* These operations will permit us to construct new *consistent* solutions from solutions known to have the property.

We start with an informal observation: given an allocation chosen by a solution for some problem, if the solution is not very restrictive for subgroups of agents, there is a better chance that it will choose the restrictions of the allocation for the associated reduced economies. Therefore, a *consistent* solution is more and more "tapered" for problems involving more and more agents (Figure 5a). This observation should provide some intuition for the

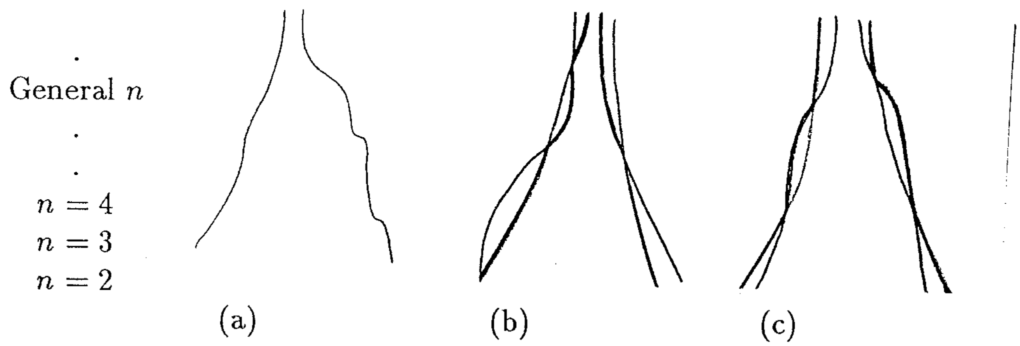


Figure 5: The "shape" of consistent solutions. (a) A *consistent* solution is more and more restrictive for groups of agents of greater and greater cardinalities. This results in a "tapered" shape. (b) If two solutions are *consistent*, then so is their intersection: the "inner lining" of the two tapered-shape solutions, which represents this intersection, is also tapered-shape. (c) If two solutions are *consistent*, so is their union: the "outer envelope" of the two solutions is tapered-shape.

claims made in Subsections 3.2.1-4 below.

3.2.1 Constructing new consistent solutions by intersecting consistent solutions

Consistency is preserved under intersections: given two *consistent* solutions, if their intersection (the "inner lining" of Figure 5b) is well-defined, that is, if it is non-empty for all admissible problems, then it is *consistent.* To illustrate, for the problem of fair division (Domain 1), both the Pareto solution and the no-envy solution are *consistent.* Therefore, their intersection, which under standard assumptions is well-defined, is *consistent* too.

In fact, *consistency* is preserved under *arbitrary* intersections, and this permits us to define a *consistent* approximation (from above) to a solution that may not be *consistent*, as follows. Let φ be a solution. Note that the solution that associates with each economy its whole feasible set is *consistent.* Therefore, the family of *consistent* solutions containing φ is non-empty. Let $\bar{\varphi}$ be the intersection of all of its members. Since they all contain φ, so does $\bar{\varphi}$. As we just argued, $\bar{\varphi}$ is also *consistent.* Obviously then, it is the smallest *consistent* solution to contain φ (Figure 6a). Formally, the ***minimal consistent extension***[7] of a solution φ is defined as $\bigcap_{\psi \in \Psi} \psi$, where $\Psi =$

[7]This notion, and the notion of the *maximal consistent subsolution* of a given solution,

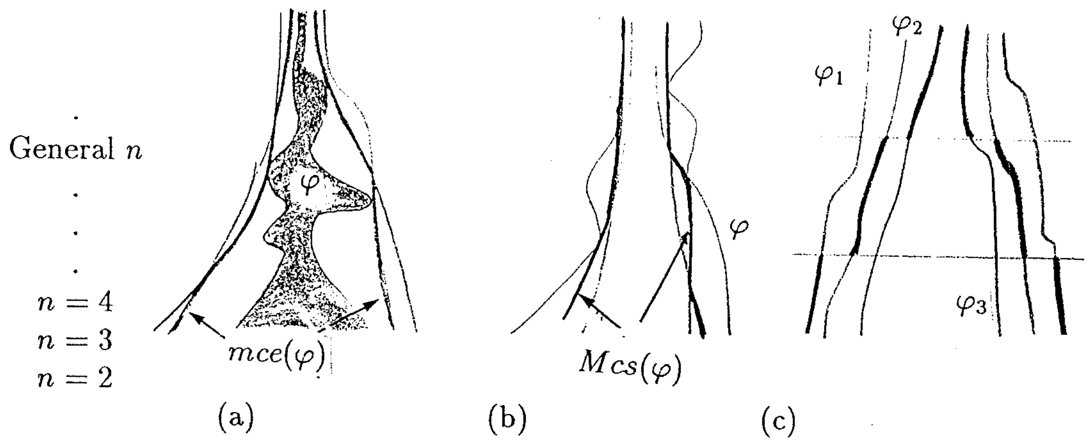


Figure 6: Constructing consistent solutions. In each of the three panels, the new solution is indicated by the thicker lines. (a) *Minimal consistent extension of* φ, $mce(\varphi)$. (b) *Maximal consistent subsolution of* φ, $Mcs(\varphi)$. (c) If several solutions related by inclusion are *consistent*, any solution obtained by successively switching to the less and less permissive ones as the cardinality of problems increases is also *consistent.*

$\{\psi \colon \psi \supseteq \varphi, \psi$ is *consistent*$\}$.

3.2.2 Constructing new consistent solutions by taking the union of consistent solutions

Similarly, *consistency* is preserved under arbitrary unions (the outer envelope of Figure 5c), so that if a solution is not *consistent* but has at least one *consistent* subsolution, it has a ***maximal consistent subsolution***, simply the union of all of its *consistent* subsolutions. Formally, the *maximal consistent subsolution* of a solution φ that contains at least one *consistent* solution is defined as $\bigcup_{\psi \in \Psi'} \psi$, where $\Psi' = \{\psi \colon \psi \subseteq \varphi, \psi$ is *consistent*$\}$.

3.2.3 Constructing new consistent solutions from consistent solutions ordered by inclusion

Let $(\varphi^\ell)_{\ell \in \{1,\ldots,k\}}$ be a list of *consistent* solutions related by inclusion, $\varphi^1 \subseteq \ldots \subseteq \varphi^k$, and $(n^\ell)_{\ell \in \{1,\ldots,k-1\}}$ a list of integers such that $n^1 < \ldots < n^{k-1}$.

presented next, are proposed and studied in Thomson (1994b). To say that the *minimal consistent extension* of a solution "approximates" it, as we did above, is not always justified however since a solution may differ considerably from its *minimal consistent extension*. Nevertheless, it represents the closest we can get to the solution so as to recover *consistency.*

Now, consider the solution that coincides with φ^1 for all problems of cardinalities no greater than n^1, with φ^2 for all problems of cardinalities between $n^1 + 1$ and n^2, ..., and with φ^k for all problems of cardinalities greater than n^{k-1}. This solution is clearly *consistent*. The examples represented in Figures 1c and 6c illustrate the operation.

3.2.4 Constructing new consistent solutions by partitioning the domain into subdomains each of which is closed under the reduction operation for a particular consistent solution

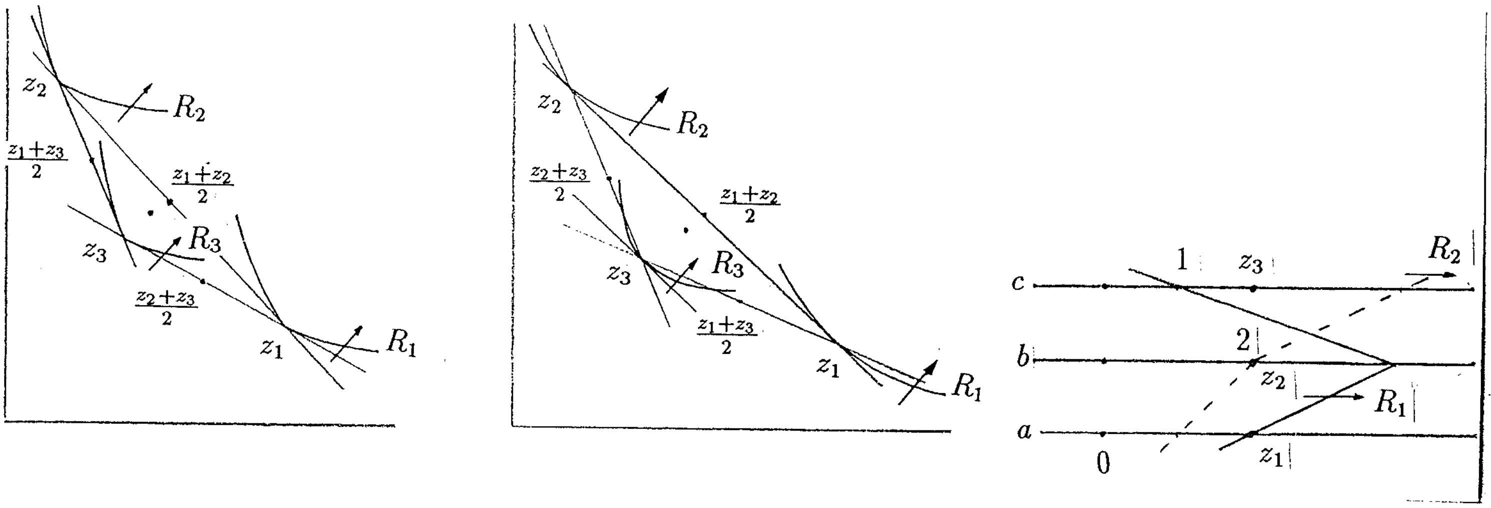
Let φ^1 and φ^2 be two *consistent* solutions whose common domain of definition $\mathcal{D}$ can be partitioned into two subdomains $\mathcal{D}^1$ and $\mathcal{D}^2$ such that (i) $\mathcal{D}^1$ is closed under the reduction operation for the solution φ^1, and (ii) $\mathcal{D}^2$ is closed under the reduction operation for the solution φ^2. Then, the solution that coincides with φ^1 on $\mathcal{D}^1$ and with φ^2 on $\mathcal{D}^2$ is *consistent.*

3.3 Conversely consistent allocation rules

Our second central property of a solution permits us to deduce that an alternative x is chosen for some problem by the solution if its restriction to each two-person group is chosen for the reduced problem associated with the subgroup and x.

Converse consistency: For all groups $N \in \mathcal{N}$, all problems $D \in \mathcal{D}^N$, and all feasible alternatives x of D, if for all subgroups N' of cardinality two, the restriction $x_{N'}$ of x to N' is φ-optimal for the reduced problem $r^x_{N'}(D)$ obtained from D by assigning to all agents in $N \backslash N'$ their components of x, then x is φ-optimal for D.

This property is certainly not as conceptually compelling as *consistency* but it is of great computational interest, as it permits us to determine whether an alternative would be chosen for a problem possibly involving a large number of agents from the knowledge that its restrictions to subgroups of two agents, for which calculations are generally much less involved, are chosen for the associated reduced problems. Of course, if there are many agents initially, there are many reduced problems for which this simpler calculation has to be carried out. Also, *converse consistency* does not help us in *dis-*



(a) (b) (c)

Figure 7: Converse consistency. (a) For the classical problem of fair division, the Walrasian solution operated from equal division is not *conversely consistent*: here, the restriction of z to each two-person group $\{i, j\}$ is Walrasian from equal division for the reduced economy (R_i, R_j, z_i+z_j), but z is not Walrasian from equal division for (R_1, R_2, R_3, Ω). Under smoothness of preferences, the property would hold however. (b) The equal division lower bound solution is not *conversely consistent* either, but here, no natural restriction on preferences can help recover the property. (c) For the problem of allocating indivisible goods when monetary compensations are possible, the Pareto solution is not *conversely consistent.* Suppose that $(2, a)$ I_1 $(4, b)$ I_1 $(1, c)$, and that the preferences of agents 2 and 3 are obtained by "rotation": $(2, b)$ I_2 $(4, c)$ I_2 $(1, a)$ and $(2, c)$ I_3 $(4, a)$ I_3 $(1, b)$. It is easy to see that no Pareto improving trade can take place in any of the three two-person reduced economies associated with the allocation z represented here, which we make feasible by choosing the amount of the divisible available to be 6. However, the allocation (z_3, z_1, z_2) Pareto-dominates z in the three-person economy.

covering φ-optimal alternatives, but simply in *checking* whether a proposed alternative is φ-optimal.[8]

A different formulation consists in writing the hypothesis for all reduced problems of cardinalities up to $|N|-1$, but it turns out that for many models, this amounts to the same thing.

For some models, the hypothesis for the two-agent case is actually no restriction on the solution. The condition should then be rewritten with the hypothesis stated for the smallest number of agents for which it does constitute a meaningful restriction. An example of such a model is matching (Domain 6 below).

It is useful to note that *converse consistency* too is preserved under intersections and unions, so that the *minimal conversely consistent extension* and *maximal conversely consistent subsolution* of a given solution can be defined analogously to the way we defined its *minimal consistent extension* and *maximal consistent subsolution.*

Here are a few examples of *conversely consistent* solutions: for the bargaining problem, the Nash solution provided preferences are smooth, and the egalitarian solution when it selects Pareto-optimal points; for the classical problem of fair division (Domain 1), provided preferences are smooth, the Pareto solution and the Walrasian solution operated from equal division are *conversely consistent.* Otherwise they are not (Figure 7a). The equal division lower bound solution is not (Figure 7b), and here, no natural restriction on preferences exist under which the property holds. In any context where it is meaningful, such as for various problems of fair allocation, including classical economies (Domain 1), or economies with indivisible goods when monetary compensations are possible (Domain 4), or economies with single-peaked preferences (Domain 8 defined below), the no-envy solution is *conversely consistent*, precisely because it is based on two-person tests. Whenever the notion of proportionality is well-defined, such as for bankruptcy (Domain 5) or fair allocation in economies with single-peaked preferences (Domain 8 defined below), the proportional solution is *conversely consistent.* For the allocation of indivisible goods when monetary compensations are possible (Domain 4), the Pareto solution is not *conversely consistent* (Figure 7c).

[8]Yet, the property suggests algorithms that sometimes converge to a φ-optimal alternative.

3.4 Logical relations between properties

Consistency and its *converse* are not in general logically related. Indeed, for bargaining problems, the Nash solution is *consistent* (Figure 3a) but not *conversely consistent.* The opposite holds for the egalitarian solution. Both are *single-valued*, so this example settles in the negative a conjecture that is often made, that this property helps relate *consistency* and its *converse.* However, for some models, interesting logical relations do hold or hold under minor additional conditions (Chun, 1997).

A property that has been discussed in connection with *consistency* and its *converse* is the following: a solution φ is ***flexible*** if, starting from a φ-optimal allocation, subgroups can redistribute between their members what they have jointly received, and provided they do that according to φ, the conjunction of these redistributions produces an allocation that is also φ-optimal for the initial economy (Balinsky and Young; 1982).

It is easy to see that a *single-valued* solution is *consistent* if and only if it is *flexible.*

3.5 Extensions of properties across cardinalities

It is often the case that if a certain property is imposed on a *consistent* solution for the two-person case, then the property is automatically "transferred" to the other cardinalities.

For instance, consider the requirement of ***equal treatment of equals***, which says that two agents with identical characteristics should be treated identically; for resource allocation problems, this means that they should receive bundles that are indifferent according to their common preferences. Let φ be a *consistent* solution. Let e be an economy in which two agents i and j have the same characteristics. Then, we claim that if x is φ-optimal for e, the two agents should receive indifferent bundles. Indeed, by *consistency* of φ, in the reduced economy of e with respect to $\{i, j\}$ and x, the two agents should still receive their components of x; since they have the same characteristics and φ satisfies *equal treatment of equals* in the two-person case, they should receive indifferent bundles.

Other properties can be extended in this way. Examples are continuity and certain invariance properties.

4 The Elevator Lemma

Consistency and its *converse* are versatile principles and they have been studied in models exhibiting a great diversity in their mathematical structures. An unfortunate consequence of this diversity is that few theorems are available that apply across all models. However, we can offer two extremely useful lemmas that are "model-free". We illustrate them in the context of several models. In Section 6, where we present a number of characterizations, we will see that much of their proofs consist in showing that the hypotheses of the Lemmas are satisfied.

4.1 Statement of the Elevator Lemma

The first lemma identifies conditions on two solutions guaranteeing that if an inclusion relation between them holds for the two-person case, then it also holds for the n-person case: if $\varphi \subseteq \varphi'$ for the two-person case, φ is *consistent*, and φ' is *conversely consistent*, then $\varphi \subseteq \varphi'$ for all cardinalities. Its proof consists in moving down from an arbitrary number of agents to two agents by means of the *consistency* of φ, and moving back up again by means of the *converse consistency* of φ'. Using the image of a building whose floors are indexed by the cardinalities of problems (Figure 8), we refer to this lemma as the "Elevator Lemma": *consistency* is the "Down" button and *converse consistency* the "Up" button. The building of Figure 8 has no first floor because for most models nothing is learned from how a solution behaves on the class of one-person problems. Therefore no generality is lost in excluding these degenerate problems from the domain.[9]

Lemma 1 *(The* "Elevator Lemma"*) Let φ and φ' be two solutions defined on a domain $\mathcal{D}$ that is closed under the reduction operation for the solution φ. If (i) on the subdomain of two-person problems, φ is a subsolution of φ', (ii) φ is* consistent*, and (iii) φ' is* conversely consistent*, then, φ is a subsolution of φ' on the entire domain $\mathcal{D}$.*

[9]An important exception is the domain of strategic games (Peleg and Tijs, 1996) where the objective is precisely to relate the way multi-person interactions are resolved from the knowledge of how one-person decision problems are solved. We will not discuss strategic games here, but only note that *consistency* has played an important role in linking their study to the study of cooperative games, as most clearly exemplified in Serrano's work. See for example, Serrano (1995).

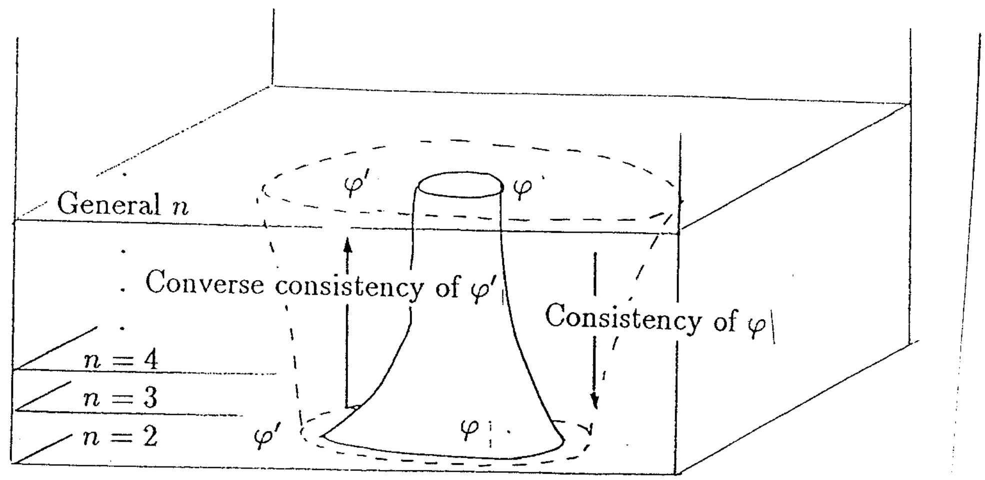


Figure 8: The Elevator Lemma. Classes of problems involving increasing number of agents are stacked up like the floors of a building. The Elevator lemma states that if a *consistent* solution φ is a subsolution of some *conversely consistent* solution φ' in the two-person case, then this inclusion holds in general.

Proof: Let $N \in \mathcal{N}$, $D \in \mathcal{D}^N$, and x be φ-optimal for D. We need to show that x is φ'-optimal for D. Since φ is *consistent*, then for all subgroups N' of N, the restriction $x_{N'}$ of x to N' is φ-optimal for the associated reduced problem $r^x_{N'}(D)$. This is true in particular for all $N' \subset N$ such that $|N'| = 2$. Since in the two-person case, φ is a subsolution of φ', then for all subgroups N' of N such that $|N'| = 2$, the restriction $x_{N'}$ of x to N' is φ'-optimal for $r^x_{N'}(D)$. This is exactly the statement that x satisfies the hypothesis of *converse consistency* for φ'. Since φ' is *conversely consistent*, x is φ'-optimal for D. □

Note that *bilateral consistency* would suffice in the Elevator Lemma. Also, *converse consistency* is not imposed on φ. We stated earlier that this property may not be as compelling as *consistency*, but many solutions do satisfy it and the Elevator Lemma shows how this fact can be profitably exploited.[10]

4.2 Applications

The Elevator Lemma is widely applicable because there are many models for which an inclusion relation holds between certain solutions for the two-person case. We give three examples:

[10]In Section 3.5, we explained how *consistency* helps extend properties from the two-person case to the general case. At the risk of seeing the cable of that elevator snapping under the increasing weight of our metaphor, we can say that *consistency* "lifts" the property from the second floor to the other floors.

1. For the problem of fair division (Domain 1 or Domain 8 defined below), in the two-person case, any allocation that meets the equal division lower bound is envy-free.

2. For coalitional games (Domain 3), requiring that a solution be a subsolution of the core is a very mild requirement in the two-person case; it reduces to the requirement that the chosen payoff vectors be "imputations", namely that they meet the individual rationality conditions and be efficient. (There are no coalitions of intermediate size then.)

3. For bankruptcy problems (Domain 5), a number of different ways of thinking about the problem give us the random arrival solution in the two-claimant case.

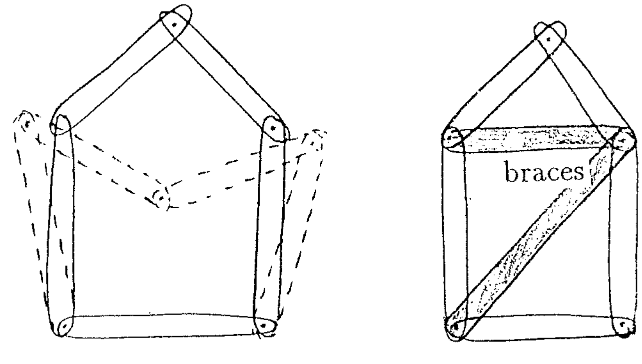
5 The Bracing Lemma

We now turn to the second lemma, which identifies conditions on two solutions only assumed to be related by inclusion, guaranteeing that in fact they coincide.

5.1 Statement of the Bracing Lemma

Let φ be a *consistent* solution and suppose that it is a subsolution of some solution $\bar{\varphi}$. Given a problem D and an alternative x that is $\bar{\varphi}$-optimal for D, note that in general there will be some freedom to move away from x without leaving the $\bar{\varphi}$-optimal set. However, suppose that additional agents can be introduced and D extended to the enlarged set of agents in such a way that (i) only one alternative is $\bar{\varphi}$-optimal for the augmented problem, (ii) the restriction of that alternative to the initial group is precisely x— we will say that this alternative is ***an augmentation of x***—, and (iii) the reduction of the augmented problem with respect to the initial group of agents and that augmented alternative is D. Now, since $\varphi \subseteq \bar{\varphi}$ and (i) holds, the augmented alternative is the only φ-optimal alternative for the augmented problem. Then, since φ is *consistent*, and since (ii) and (iii) hold, we conclude that x is φ-optimal for D.

We will illustrate the Bracing Lemma with another physical metaphor, that of a building. Consider the "house" of Figure 9a, put together by nailing boards together. This structure will not be very stable because the nails will serve as axes of rotation for the boards. Two of their infinitely many



(a) (b)

Figure 9: The Bracing Lemma. (a) The house on the left is not stable because the boards have several degrees of freedom. One of the possible configurations is indicated by the solid lines and another one by the dotted lines. (b) To stabilize it, we add two boards connecting two pairs of corners. Note that if only one of the two boards were added, the structure would not be stabilized. In fact, one board would not be sufficient no matter where it would be placed. On the other hand, there are other ways to position two new boards so as to obtain a stable structure. A third board would be redundant.

possible configurations are indicated. However, it is possible to eliminate the unwanted degrees of freedom by adding "braces", as shown in Figure 9b. (In the example, we have several choices of where to place braces but note that two of them are needed. If the structure to be stabilized were more complex, we could need more.) These braces are the additional agents of the lemma.

Lemma 2 *(The* "Bracing Lemma"*) Let φ be a consistent subsolution of some solution $\bar{\varphi}$. Suppose that $\bar{\varphi}$ is such that for all $N \in \mathcal{N}$, all $D \in \mathcal{D}^N$, and all $x \in \bar{\varphi}(D)$, there are $N' \supset N$, $D' \in \mathcal{D}^{N'}$, and x' in the feasible set of D', such that (i) x' is the only $\bar{\varphi}$-optimal alternative for D', (ii) the restriction of x' to N' is x, and (iii) the reduced problem of D' with respect to N' and x' is D. Then, $\varphi = \bar{\varphi}$.*

The proof follows directly from the statement of the lemma, and in fact we have essentially given it in the paragraph preceding it. Perhaps, it is not so much a bracing "lemma" as a bracing "construction".

Sometimes the bracing requires only one additional agent—an example is the allocation of indivisible goods when monetary compensations are possible (Domain 4)—, sometimes two are required, as for the allocation of *identical* indivisible goods, (a special case of Domain 4), and sometimes almost as many agents as are present initially are needed; an example here is the allocation of an infinitely divisible good when preferences are single-peaked (Domain 8 defined below). In some situations, instead of assuming the set of potential agents to be infinite, it is more natural to impose an upper bound on the number of agents; then, the Bracing Lemma will only apply to a restricted class of situations.

5.2 Applications and variants of the Bracing Lemma

In applications, the question is when the "extension to uniqueness" of the Bracing Lemma is possible, and this depends on the richness of the domain of problems over which the solution is defined. To return to our architectural metaphor, the bracing there is possible only if we have available a board that is long enough to be nailed diagonally.

5.2.1 When the bracing is not possible

The following example will make it obvious that an augmentation to uniqueness is not always possible. On the domain of exchange economies with continuous, strictly monotonic, and strictly convex preferences, consider the solution that associates with each economy its set of Pareto-optimal allocations. Starting from some economy and an arbitrary Pareto-optimal allocation for it, there is in general no way to introduce new agents and additional resources for them so that in the augmented economy, this augmented allocation is the unique Pareto-optimal allocation.[11] If this augmentation were possible, then by the Bracing Lemma, there would be no *consistent* subsolution of the Pareto solution, but we know this not to be true: the Walrasian solution operated from equal division is one (of course, we have other information about the structure of the set of Pareto-optimal allocations that confirms this).

Here is an example in the context of the problem of fair allocation for which the answer may not be so clear. It involves the no-envy solution (Figure 10). Let us say that the "envy constraints are met at an allocation for an individual" if he finds his assigned bundle at least as desirable as each of the bundles assigned to the other agents. If these constraints are all met strictly for all individuals, then reallocations will be possible within the envy-free set in every direction (except when the boundary would get in the way; Figure 10a). If some of them are met as indifferences, then such reallocations may still be feasible, but we will have to be much more careful (Figure 10b).

[11] It is of course easy to perform the extension in such a way that there is a Pareto-optimal allocation whose restriction to the initial set of agents is z. For instance, add an arbitrary agent and no new resources; give nothing to the new agent.

Figure 10c illustrates a natural way to go about augmenting a two-person economy with agent set $\{1, 2\}$ in an attempt to obtain a unique envy-free allocation in the augmented economy. Starting from z, imagine a new agent coming in, agent 3, and let z_3 designate a bundle intended for him. As we just noted, the augmented allocation (z, z_3) will have a chance to be the unique envy-free allocation in the augmented economy only if sufficiently many of the envy constraints are met as indifferences. Since we can choose z_3 as well as agent 3's preferences, we should probably make these choices so that he be indifferent between z_3 and as many as possible of the bundles received by the agents initially present. Also, the option of introducing more than one agent gives us a chance of increasing the proportion of the envy constraints that are met as indifferences, thereby getting us closer to our objective of creating a structure where all degrees of freedom are eliminated.

However, once again this construction will not work because here too, the Walrasian solution operated from equal division is a *consistent* subsolution of the no-envy solution. Nevertheless, this is essentially the right way to proceed so as to brace an allocation, and this approach will work for other models: for the allocation of indivisible goods when monetary compensations are possible for example (Domain 4), it does produce the desired bracing. We explain why in Subsection 5.2.4. (Figures 11 and 12).

5.2.2 A model in which the Bracing Lemma is directly applicable

An important example illustrating the usefulness of the Bracing Lemma is the domain of coalitional games, but we will not go into the details as it is unfortunately among the most difficult domains to work with.

5.2.3 Bracing requiring an augmentation of the consumption spaces

We saw earlier that for some models, the reduction operation is most naturally accompanied by a reduction of consumption spaces and a restriction to that reduced space of the preferences of the agents that stay. This means that conversely, an augmentation of a problem will have to involve an augmentation of the consumption spaces of the agents initially present, and an extension of their preferences to the augmented space. We have two examples to offer as illustrations of this operation, one pertaining to the allocation of indivisible goods (Domain 4) and the other pertaining to a certain class of

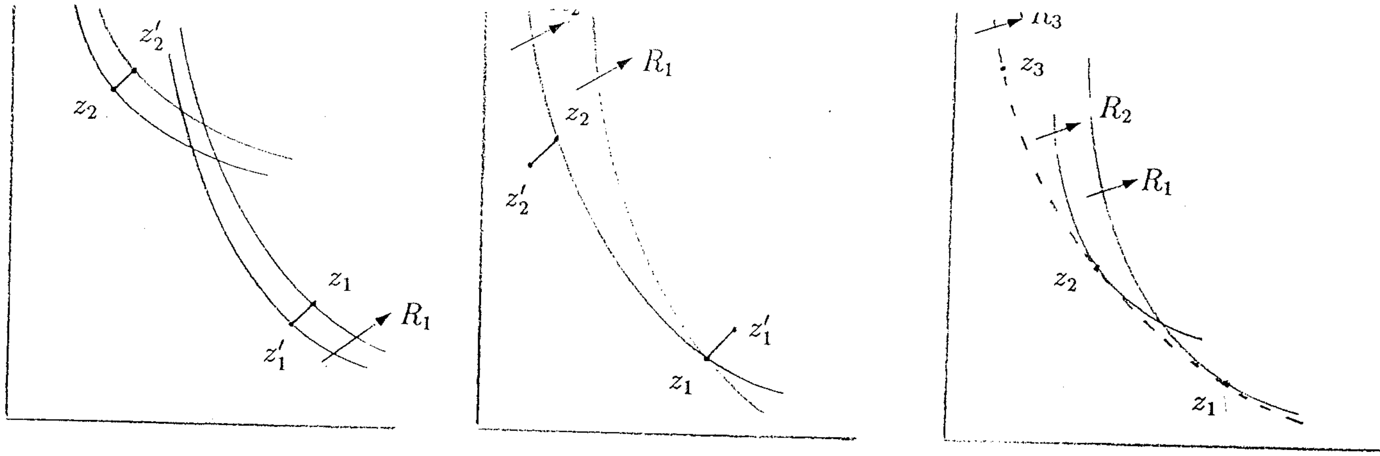


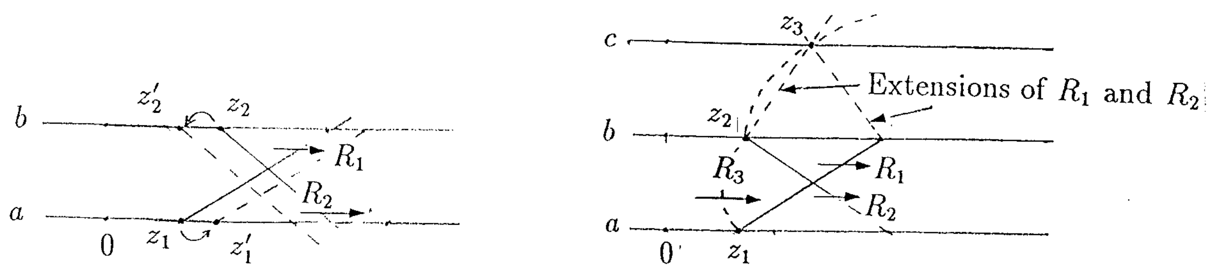
Figure 10: Illustration of the Bracing Lemma for the problem of fair division. (a) If all envy constraints are met strictly (as they are at z), reallocations are possible *in any direction* without envy being violated provided they are small enough (z' is just one example of an allocation that can be reached). (b) Here, one of the envy constraints is met at indifference and redistributions in certain directions would lead to a violation of envy (a move to z' would create envy). (c) This figure illustrates a natural attempt at augmenting an economy that admits many envy-free allocations so as to obtain a unique such allocation. This attempt will be unsucessful however. Indeed the Walrasian solution operated from equal division is a *consistent* subsolution of the no-envy solution. However, for the problem of allocating indivisible goods, we will see that the same idea will work.

matching problems known as marriage problems (Domain 6). For a discussion of the former model, we refer the reader to Subsection 5.2.4, where the example is used again to illustrate another issue.

Domain 6 *A* ***marriage problem*** *(Shapley and Shubik, 1972; Roth and Sotomayor, 1990) is defined by first partitioning the set of agents N into two groups, denoted M and W, and called "men" and "women"; then, specifying a list R of strict preference relations such that for each $i \in M$, R_i is defined over W, and for each $i \in W$, R_i is defined over M. A feasible allocation is a bijection, or "match", from the set of men to the set of women.*

Examples of solutions for Domain 6 *The* ***Pareto solution*** *is defined in the usual way. The* ***stable solution*** *selects the set of matches such that there is no pair of a man and a woman that prefer each other to their assigned mates.*[12] *The* ***man-optimal solution*** *selects the stable match that is best in*

[12]This is saying that the bijection is not "blocked" by a pair of a man and a woman; requiring that the no-blocking conditions be met for all groups, as in the usual definition of the core, is actually not more restrictive: the stable solution coincides with the core.



(a) (b)

Figure 11: The Bracing Lemma for the allocation of indivisible goods. (a) A two-person economy and an envy-free allocation for it in which the no-envy constraints are not met strictly: by redistributing money according to the arrows, we do not violate the no-envy constraints. (b) This freedom to move in the envy-free set is eliminated up to a neutral exchange by adding one object and some arbitrary amount of money, introducing a new agent (here, agent 3) and specifying his preferences so that he is indifferent between this bundle of additional resources (z_3 in the figure) and the components of z; finally, extending the preferences of the old agents so that they are indifferent between their old bundles and the new bundle.

*the set of all stable matches for all the men. The **woman-optimal** solution is defined in a symmetric way.*

It is easy to see that as usual, the Pareto solution is *consistent*. So is the stable solution. The man-optimal solution is not, and of course neither is the woman-optimal solution.

5.2.4 Bracing "up to neutral exchanges" or "up to indifferent exchanges"

For a number of economic domains and for certain solutions $\bar{\varphi}$ of interest, the $\bar{\varphi}$-optimal set is often not a singleton but the allocations it contains are Pareto-indifferent. A useful variant of the Bracing Lemma in such situations involves the requirement that the solution φ also satisfies ***Pareto-indifference***: if x and x' are feasible alternatives of D such that x is φ-optimal for D and x' is Pareto-indifferent to x, then x' should also be φ-optimal for D. This requirement, which seems innocuous enough, is not always met however.[13] Nevertheless, if imposed on φ and satisfied by $\bar{\varphi}$, the equality $\varphi = \bar{\varphi}$ is obtained by a slight modification of the proof of the Bracing Lemma.

[13] For instance, for the problem of fair division, the so-called no-envy solution violates it.

For the allocation of indivisible goods when monetary compensations are possible (Domain 4), the bracing is achievable only up to a "neutral exchange": the allocation z' is related to the allocation z by such an exchange if its components are obtained by reshuffling the components of z but all agents are indifferent between their old and new bundles (Figure 11).[14]

5.2.5 When the bracing is only possible for distinguished alternatives

In some cases, the bracing is not possible for all $\bar{\varphi}$-optimal alternatives but only for some distinguished ones. If these distinguished alternatives exist for all problems, they constitute a well-defined solution φ^*, and we will be able to conclude that any solution satisfying the required conditions has to contain φ^*:

Lemma 3 *(Variant of the Bracing Lemma). Let φ be a consistent subsolution of some solution $\bar{\varphi}$. Let φ^* be a subsolution of $\bar{\varphi}$. If the "extension to uniqueness" described in the Bracing Lemma is possible for all $x \in \varphi^*(D)$, then $\varphi \supseteq \varphi^*$. Therefore, if φ^* is consistent, it is the minimal consistent subsolution of φ.*

An illustration of Lemma 3 is provided by the allocation of a single indivisible object (a prize, say) when monetary compensations are possible, the equality between the numbers of objects and agents being reestablished by introducing "null objects". They correspond to not getting the "real" object (Figure 12). We use the notation ν for the null object. :

Domain 7 *A problem of* ***allocating a single indivisible good when monetary compensations are possible*** *is the simple version of the problem of allocating indivisible goods (Domain 4) in which there is only one indivisible real object. The domain also includes economies where only money is to be allocated.*

Examples of solutions for Domain 7 *The* ***winner's curse solution*** *(Tadenuma and Thomson, 1993) selects the envy-free allocation(s) at which the* ***winner****, the agent that is assigned the real object, is indifferent between his bundle and the common bundle of the losers.*

[14]Note that this is a special case of a Pareto-indifferent exchange.

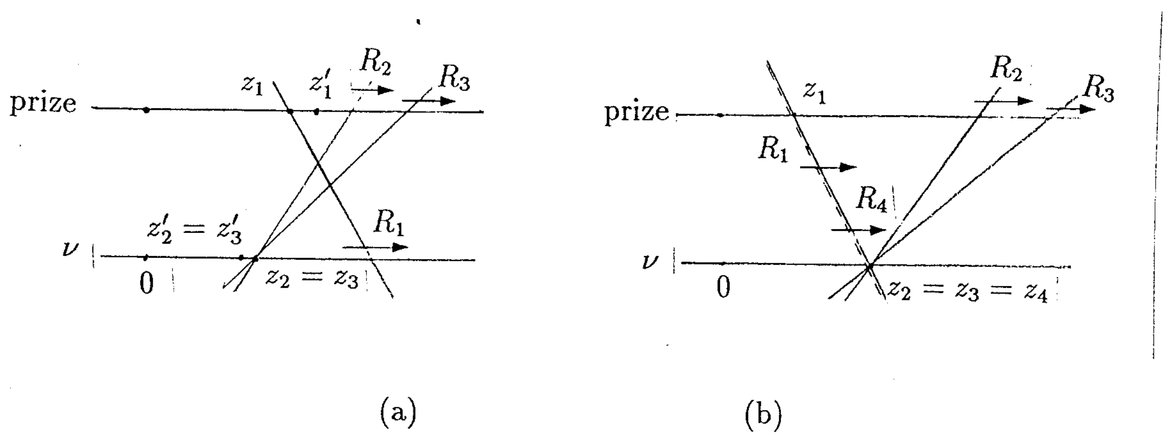


Figure 12: Lemma 3 applied to the allocation of a prize. Initially, there are three agents, agents 1, 2, and 3. Agent 1 is the "winner" of the object. The point z_1 denotes his bundle. Agents 2 and 3 are "losers". By no-envy, agent 1's indifference curve through z_1 passes above z_2, agents 2 and 3's bundles are the same (they consist of the same amount of money and the "null object", denoted ν), and their indifference curves through this common bundle passes above z_1. (a) The allocation z' represents another envy-free allocation, obtained from z by transfering money from the losers to the winner. The allocation z cannot be braced. (b) The allocation at which the winner is indifferent between his bundle and the common bundle of the losers can be braced. By introducing a new agent, agent 4, and specifying his preferences so that he is indifferent between z_1 and z_2, adding m_2 units of money, and giving him the bundle $z_4 = (m_2, \nu)$, the augmented allocation (z, z_4) is the only envy-free allocation up to a neutral exchange (between agent 1 and agent 4).

See the legend of Figure 12 for an explanation of how Lemma 3 applies, with φ^* being the winner's curse solution.

5.2.6 When the bracing is achieved approximately

In some situations, the bracing is achievable only approximately, but with a "tolerance" that can be made arbitrarily small. Then, another useful variant of the Bracing Lemma is obtained by imposing a continuity property on the solution. An example illustrating this possibility is the problem of fair rationing in the two-good case. Such a problem can be modelled as an economy with single-peaked preferences, defined as follows:

Domain 8 *An* ***allocation problem with single-peaked preferences*** *(Sprumont, 1991) is a pair* (R, Ω) *where* $R = (R_i)_{i \in N}$ *is a list of single-peaked preference relations defined on* $\mathbb{R}_+$*, and* $\Omega \in \mathbb{R}_+$ *is some amount of a social endowment of an infinitely divisible commodity. Single-peakedness of* R_i *means that* R_i *has a satiation amount, denoted* $p(R_i)$*, and for all* x_i, x_i' *such that* $x_i' < x_i \leq p(R_i)$ *or* $p(R_i) \leq x_i < x_i'$*, we have* $x_i \; P_i \; x_i'$*. A feasible allocation is a list* $x \in \mathbb{R}_+^N$ *such that* $\sum x_i = \Omega$*.*

Examples of solutions for Domain 8 *The* ***Pareto solution*** *is defined in the usual way. The* ***uniform rule*** *selects the allocation* x *such that if* $\sum p(R_i) \geq \Omega$*, then for all* $i \in N$*,* $x_i = \min\{p(R_i), \lambda\}$*, and if* $\sum p(R_i) \leq \Omega$*, then for all* $i \in N$*,* $x_i = \max\{p(R_i), \lambda\}$*, in each case* λ *being chosen so as to make* x *feasible (Figure 13a illustrates this definition).*

We refer to Figure 13 for a sketch of the analysis of the example. It pertains to a solution assumed to be a *consistent* subsolution of the no-envy and Pareto solution and it shows that for such a solution, if an allocation is chosen, it has to be sufficiently close to the uniform allocation. It represents a two-person economy for which $\sum p(R_i) \geq \Omega$, and whose uniform allocation is denoted x (there $\lambda = x_2$). The proof uses the obvious fact that at an efficient allocation, all agents receive at most their satiation amounts if $\sum p(R_i) \geq \Omega$, and all agents receive at least their satiation amounts if $\sum p(R_i) \leq \Omega$.

6 Characterizations: a sampler

In this subsection, we state a few results involving *consistency* and its *converse.* They constitute but a small fraction of the literature, but we have

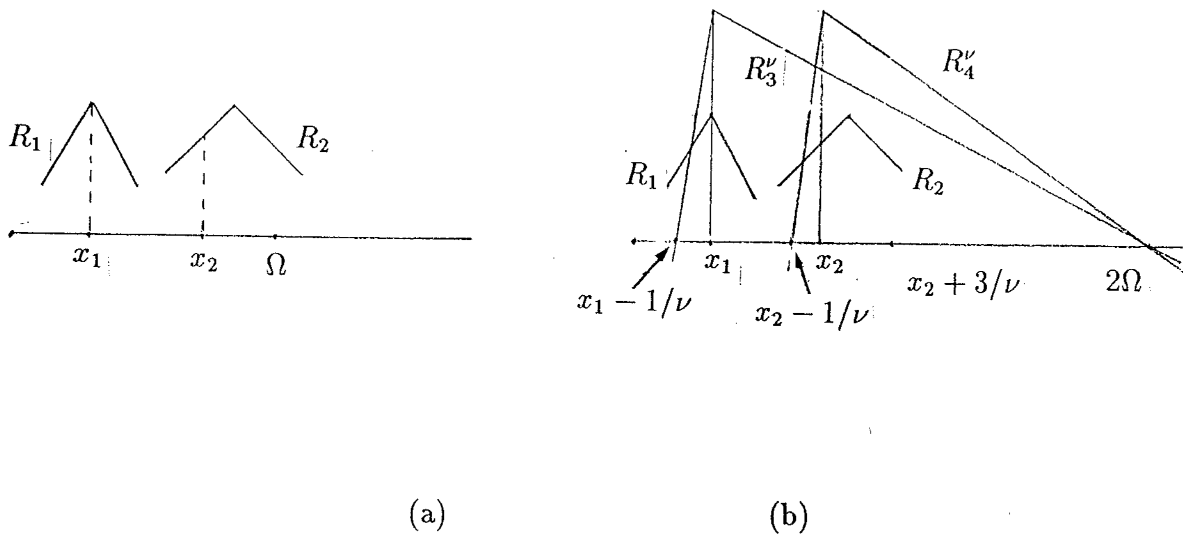


Figure 13: Approximate bracing. (a) We start with an economy e involving the group $\{1, 2\}$, for which $\sum p(R_i) \geq \Omega$, say, and identify its uniform allocation, $x = (x_1, x_2)$. (b) We introduce two new agents, agents 3 and 4. Given ν, an integer that will go to infinity, we specify their preferences R_3^ν and R_4^ν so that $p(R_3^\nu) = p(R_1)$, $p(R_4^\nu) = x_2$, $(x_1 - 1/\nu)\ I_3^\nu\ 2\Omega$, and $(x_2 - 1/\nu)\ I_4^\nu\ 2\Omega$. We double the social endowment. Let y^ν be an allocation chosen by a selection from the no-envy and Pareto solution for the augmented economy $(R_1, R_2, R_3^\nu, R_4^\nu, 2\Omega)$. By efficiency, for each $i \in \{1, \ldots, 4\}$, $y_i^\nu \leq p(R_i)$. Then by no-envy, $y_1^\nu = y_3^\nu$ and $y_2^\nu = y_4^\nu$. This implies that at least one of agents 2 and 4 consumes more than $p(R_3^\nu)$, and for agent 3 not to envy that agent, he should consume at least $p(R_3^\nu) - 1/\nu = p(R_1^\nu) - 1/\nu$. So, agent 1 should consume at least that amount. Similarly, agent 4 should consume at least $p(R_4) - 1/\nu = x_2 - 1/\nu$, and therefore agent 2 should consume at least that amount. We then deduce that agent 2 should not consume more than $p(R_2) + 3/\nu$. Altogether, y_1^ν belongs to the interval $[p(R_1) - 1/\nu, p(R_1)]$ and y_2^ν to the interval $[x_2 - 1/\nu, x_2 + 3/\nu]$, so that (y_1^ν, y_2^ν) converges to (x_1, x_2) as ν goes to infinity.

selected them so as to give a flavor of the range of existing applications of the principles, and whet the reader's appetite. Several of them make use of one or the other of the two Lemmas and variants. In some cases, very little work is required beyond showing that the hypotheses of the Lemmas are met.

6.1 Bargaining

For our first result, which pertains to bargaining (Domain 1), we will need the two basic properties of ***Pareto-optimality***, whose definition we will not repeat, and ***anonymity***, which says that the solution should be invariant under renamings of agents. We will also impose ***scale invariance***, which says that a rescaling, independent agent by agent, of their utilities, is accompanied by a similar rescaling of the outcome.

Theorem 1 (Lensberg, 1988) *The Nash solution is the only solution satisfying single-valuedness, Pareto-optimality, anonymity, scale invariance, and consistency.*

The proof, which is illustrated in Figure 14, involves an operation that can be described as an "augmentation to anonymity". Starting from an arbitrary problem S that may not have any particular symmetry, we augment it so as to obtain a problem T that is sufficiently symmetric so that we can deduce by *Pareto-optimality* and *anonymity* the point x that it should choose; moreover, the reduced problem of T with respect to the initial group of agents and x is S.

The proof works for any S whose boundary contains a "sufficiently long" (in relation to the common value of the coordinates of its Nash outcome) segment centered at its Nash outcome. Otherwise, the section of T through x contain $S' = S$ as a *strict* subset, and we cannot derive what we want about S. However, the desired conclusion can then be obtained by introducing more than one new agent and extending the replication, the number of new agents that are needed being all the greater the shorter this segment. A continuity argument is required for a problem that is strictly convex at its Nash outcome.

A very general result that does not involve the invariance or symmetry conditions is given by Lensberg (1987). He essentially obtains a characterization of the class of solutions obtained by maximizing a sum of concave functions of the agents' utilities.

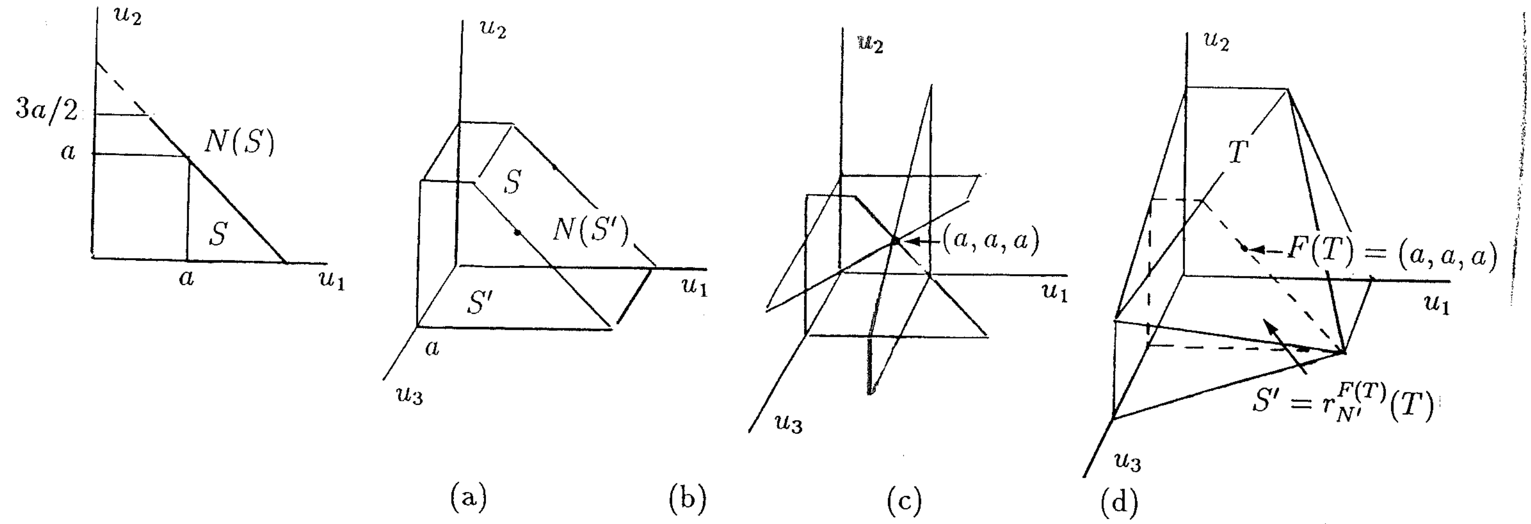


Figure 14: Characterization of the Nash solution. (a) We start with a two-person problem S involving agents 1 and 2. By *scale invariance*, we can assume that its Nash outcome has equal coordinates, (a, a). (b) We introduce a third agent, agent 3, and translate S along the third axis by the amount a. Let S' denote the result. (c) We replicate S' twice by having the roles played by agents 1 and 2 in S' be played by agents 2 and 3 respectively, and then by agents 3 and 1 respectively. (d) We construct the smallest convex and comprehensive problem, T, containing S' and its two replicas. By *Pareto-optimality* and *anonymity*, the point chosen for T is (a, a, a). The reduced problem of T with respect to $\{1, 2\}$ and (a, a, a) is S'. By *consistency*, the solution outcome of S' is (a, a). Since $S = S'$, we are done.

6.2 Coalitional games with transferable utility

The literature on *consistency* for coalitional games (Domain 3) is extensive, partly because, as we have already seen, reduced games can be defined in more than one way. We list three basic results involving the two notions of *consistency* introduced in Subsection 3.1.3. Theorems 2 and 3 involve ***individual rationality***, the requirement that each agent's payoff be at least as large as the agent's worth. Both of them rely on the Bracing Lemma.

Theorem 2 (Tadenuma, 1992) *On the domain of TU coalitional games whose core is non-empty, the core is the only solution satisfying individual rationality and complement consistency.*

The next result involves the condition of ***super-additivity***, which says that if x is chosen for some game v, and y is chosen for some game w, then $x + y$ is chosen for the sum game $v + w$.

Theorem 3 (Peleg, 1986) *On the domain of TU coalitional games whose core is non-empty, the core is the only solution satisfying individual rationality, super-additivity, and max-consistency.*

The next theorem involves ***homogeneity***, the requirement that if all utilities are multiplied by the same number, the chosen payoff vector should also be scaled by that number. Its proof is by means of an augmentation to anonymity analogous to that carried out in the proof of Theorem 1.

Theorem 4 (Sobolev, 1975) *The prenucleolus is the only solution satisfying single-valuedness, individual rationality, Pareto-optimality, anonymity, homogeneity, and max consistency.*

A third notion of *consistency*, in which the solution itself appears, was proposed by Hart and Mas-Colell (1989), and it essentially leads to a characterization of the Shapley value.

Counterparts of Theorems 2 and 3 for the non-transferable utility case are available (Tadenuma, 1992; Peleg, 1985). Numerous contributions have been made to the study of this class of problems. We note Dutta (1990) and Maschler and Owen (1989).

6.3 Fair division

To present the results of this section, which pertain to fair division in classical economies (Domain 1), we need two additional properties. ***Replication-invariance*** says that if an allocation is chosen for some economy, then for any integer k, the k-replica of the allocation is chosen for the k-replica of the economy. In this replica, each of the preference relations appearing in the list R is cloned k times and the social endowment is multiplied by k.

Theorem 5 (Thomson, 1988) *Suppose that preferences are smooth. If a subsolution of the equal division lower bound and Pareto solution satisfies replication invariance and consistency, then it is a subsolution of the Walrasian solution operated from equal division.*

The proof is sketched in the legend of Figure 15a where the replication operation is denoted with a star ($k * z$ is the k replica of z and $(k * R, k\Omega)$ is the k-replica of (R, Ω)), and φ is a solution assumed to satisfy the properties listed in the theorem. It involves a variant of the Elevator Lemma, in which the role of *converse consistency* is played by *replication invariance*, which is a (very) weak form of it.

We also have the following result, which involves ***anonymity***, the requirement that the chosen allocations be independent of the names of agents:

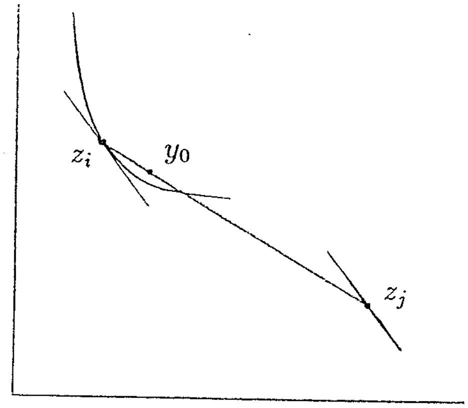


Figure 15: Characterization of the Walrasian solution operated from equal division. An economy and an efficient allocation for it, z, at which the implicit incomes of two agents, agents i and j, are not equal. By smoothness of preferences, there exist $k_i, k_j \in \mathbb{N}$ such that $y_0 \; P_i \; z_i$, where $y_0 \equiv \frac{k_i z_i + k_j z_j}{k_i + k_j}$. We replicate the economy $k \equiv \max\{k_i, k_j\}$ times. Now, if z is chosen for the initial economy, then, by *replication-invariance*, $k * z$ is chosen for $(k * R, k\Omega)$. Let N' be a subgroup consisting of k_i agents of type i and k_j agents of type j. By *consistency*, $(k_i * z_i, k_j * z_j)$ is chosen for $((R_\ell)_{\ell \in N'}, k_i z_i + k_j z_j)$. Since the solution is a subsolution of the equal division lower bound solution, we have $z_i \; R_i \; y_0$, in contradiction with the specification of y_0.

Theorem 6 (Thomson, 1994a) *Suppose that preferences are smooth. If a subsolution of the equal division lower bound and Pareto solution satisfies anonymity and converse consistency, then in the two-person case, it is a subsolution of the Walrasian solution operated from equal division. If for the two-person case, equality holds, then it is a subsolution of the Walrasian solution operated from equal division for all cardinalities.*

Other results have been obtained by Maniquet (1996) and Fleurbaey and Maniquet (1994). Roemer (1988) formulates and studies a notion of *consistency* where it is the number of goods that varies.

6.4 Bankruptcy

To state our next result, which pertains to bankruptcy (Domain 5), we need the concept of a ***parametric solution***. Consider a family of real-valued continuous, and nowhere decreasing functions $f(\bar{c}, .)$ defined on some interval $[a, b]$ and such that for all $\bar{c} \in \mathbb{R}_+$, $f(\bar{c}, a) = 0$ and $f(\bar{c}, b) = \bar{c}$. Now, given any (c, E) with agent set N, choose the vector $x \in \mathbb{R}^N_+$ such that $\sum_N x_i = E$ and for some λ and all $i \in N$, $f(c_i, \lambda) = x_i$: this is the choice made by the parametric solution associated with the family f. It is easy to see that the proportional, constrained equal awards, constrained equal losses, and Talmudic solutions (defined next), are parametric solutions. Figure 16 gives

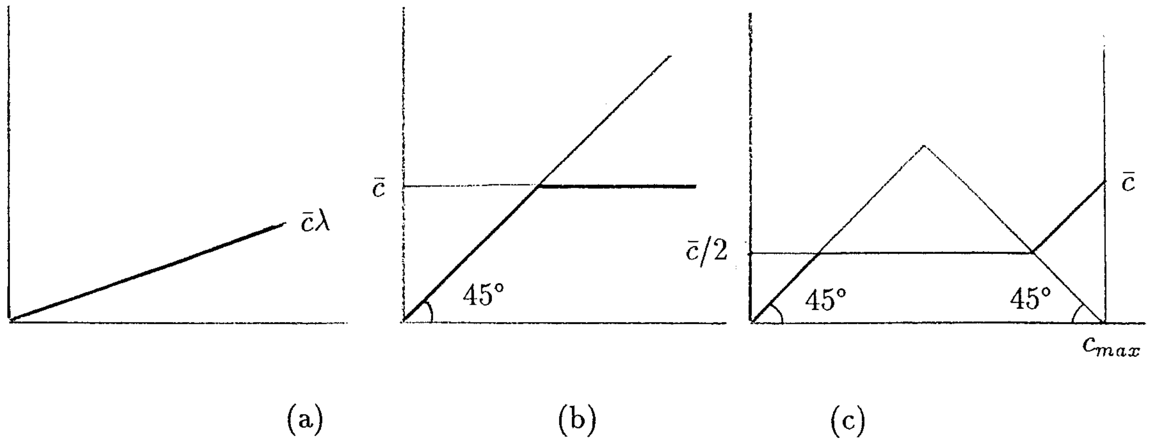


Figure 16: Parametric representations of three solutions. (a) The proportional solution. (b) The constrained equal awards solution. (c) The Talmudic solution when there is upper bound on claims, c_{max}.

parametric representations for three of them.

To define the ***Talmudic solution***, we distinguish two cases: if $\sum c_i/2 \geq E$, each claimant $i \in N$ receives what he would receive under an application of the constrained equal awards solution to the problem $(c/2, E)$; if $\sum c_i/2 \leq E$, each claimant $i \in N$ receives $c_i/2$ plus what he would receive under an application of the constrained equal losses solution to the problem $(c/2, E - \sum c_i/2)$.

Theorem 7 (Young, 1987) *A solution satisfies continuity, symmetry, and consistency if and only if it is a parametric solution.*

By imposing additional conditions on rules, interesting subfamilies of the parametric family can be identified (Young, 1988), including several solutions that have played a prominent role in the public finance literature. Another relevant contribution developing the notion of *average consistency* is Dagan and Volij (1997).

6.5 Allocation of indivisible goods when monetary compensations are possible

The two theorems below, which pertain to the allocation of indivisible goods when monetary compensations are possible (Domains 4 and 7), involve bracings up to neutral exchanges, as illustrated in Figure 12.

Theorem 8 (Tadenuma and Thomson, 1993) *In the one-object case, there is a smallest subsolution of the no-envy solution satisfying neutrality and consistency. It is the winner's curse solution.*

The bracing used in proving the next theorem is illustrated in Figure 11.

Theorem 9 (Tadenuma and Thomson, 1991) *In the multiple-object case, if a subsolution of the no-envy solution satisfies neutrality and consistency, then in fact, it is the no-envy solution.*

If several identical objects have to be allocated, any envy-free allocation can be braced by introducing two agents, specifying their preferences in such a way that they are indifferent betwen the two bundles initially received by the losers and the winners, and adding resources so that one of them can be given the winners' bundle and the other can be given the losers' bundle.

Bevia (1996) studied *consistency* in situations where each agent may receive more than one object.

6.6 Allocation with single-peaked preferences

For problems of fair division with single-peaked preferences (Domain 8), we have the following characterization:

Theorem 10 (Thomson, 1994a) *There is a smallest subsolution of the no-envy and Pareto solution (alternatively, of the equal division lower bound and Pareto solution) satisfying upper semi-continuity with respect to the social endowment and consistency. It is the uniform rule.*

The proof for a solution φ required to be a subsolution of the no-envy solution relies on the approximate bracing illustrated in Figure 13. It concludes as follows. Let $\Omega^\nu = \sum_{\{1,2\}} y_i^\nu$ and $e^\nu = (R_1, R_2, \Omega^\nu)$. By *consistency*, $y^\nu_{\{1,2\}} \in \varphi(e^\nu)$. Since $y^\nu_{\{1,2\}} \to x$ as $\nu \to \infty$, it follows that $\Omega^\nu \to \sum_{\{1,2\}} x_i = \Omega$. By *upper semi-continuity with respect to the social endowment*, $x \in \varphi(e)$.

When the search is for a subsolution of the equal division lower bound solution the conclusion is a direct consequence of the Elevator Lemma and the fact that for the two-person case, the no-envy solution is less restrictive.

A counterpart of Theorem 6 holds for this model, and its form is even a little simpler since the uniform rule is *single-valued.* A result related to Theorem 10 is due to Dagan (1996). Results on the dual case when preferences are single-troughed is given by Klaus (1997).

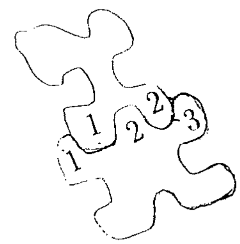


Figure 17: A logo for consistency. The two-person case, symbolized by a two-pronged piece of a jigsaw puzzle, "fits" just right with the three-person case, symbolized by the three-pronged piece.

6.7 Matching

For matching problems (Domain 6), we will impose *converse consistency*, but note that here, for this condition to make sense, we need to demand that its hypotheses holds for all problems involving two men and two women. The following theorem involves the Bracing Lemma.

Theorem 11 (Sasaki and Toda, 1992) *If a subsolution of the Pareto solution satisfies anonymity, consistency, and converse consistency, then it is the stable solution.*

A number of additional results have been obtained by Toda (1993), for a model in which remaining single is an option, and also for a different notion of a reduced problem.

7 Conclusion

Figure 17 is a logo illustrating the idea of *consistency.* It consists of two interlocking pieces of a jigsaw puzzle, one with two prongs and the other three prongs, symbolizing the two-person case and the three-person case respectively. The interlocking of the two pieces indicates the way the recommendations made by a *consistent* solution for one cardinality "fits" perfectly the recommendations it makes for other cardinalities.

A jigsaw puzzle can also help illustrate the notion of *converse consistency.* Say that two pieces of a puzzle are "correctly positioned" if when they are meant to interlock, you have indeed placed them in their interlocking position, and otherwise, you have kept them apart. Suppose now that you have correctly positionned any two pieces of the puzzle. Then, you have completely solved the puzzle. Correct positioning of pieces two-by-two guarantees correct positioning altogether. An implication of this property of puzzles is that several people can work independently on the same puzzle, one person doing the frame, the other the trees, a third the mountain range in the background ...

Of course, this depends on the puzzle having been well designed, that is, on the way in which the pieces have been cut. To see this, consider the case when the pieces are squares. Then, any two pieces, when placed adjacently to each other, would appear to be correctly positioned, but this positioning would not solve the puzzle. This shows that one could take it to be the definition of a good puzzle that it be *conversely consistent.*

8 References

Aumann, R. and M. Maschler, "Game theoretic analysis of a bankruptcy problem from the Talmud," *Journal of Economic Theory* 36 (1985), 195-213.

Balinsky, M. and P. Young, *Fair Representation*, Yale University Press, 1982.

Bevia, C., "Identical preferences lower bound solution and consistency in economies with indivisible goods", *Social Choice and Welfare* 13 (1996), 113-126.

Chun, Y., "Equivalence relations of axioms for bankruptcy problems", University of Rochester mimeo, November 1997.

Dagan, N., "A note on Thomson's characterizations of the uniform rule", *Journal of Economic Theory* 96 (1996), 255-261.

——, and O. Volij, "Bilateral comparisons and consistent fair division rules in the context of bankruptcy problems", *International Journal of Game Theory* 26 (1997), 11-25.

Davis, M. and M. Maschler, "The kernel of a cooperative game," *Naval Research Logistics Quarterly* 12 (1965), 223-259.

Dutta, B., "The egalitarian solution and reduced game properties in convex games," *International Journal of Game Theory* 19 (1990), 153-169.

Fleurbaey, M. and F. Maniquet, "Fair allocation with unequal production skills: the solidarity approach to compensation", Université de Cergy-Pontoise mimeo, December 1994.

Gillies, D.B., "Solutions to general non-zero sum games", in *Contributions to the Theory of Games IV* (Annals of Mathematics Studies 40), ed. by A.W. Tucker and R.D. Luce, Princeton: Princeton University Press, 1959, 595-614.

Hart, S. and A. Mas-Colell, "Potential, value and consistency", *Econometrica* 57 (1989), 589-614.

Kalai, E., "Proportional solution to bargaining problems: interpersonal utility comparisons," *Econometrica* 45 (1977), 1023-1030.

—— and M. Smorodinsky, "Other solutions to Nash's bargaining problem," *Econometrica* 43 (1975), 513-518.

Klaus, B., *Fair Allocation and Reallocation: an Axiomatic Study*, University of Maastricht, 1997.

Lensberg, T., "Stability and collective rationality," *Econometrica* 55 (1987), 935-961.

——, "Stability and the Nash solution," *Journal of Economic Theory* 45

(1988), 330-341.

Maniquet, F., "Horizontal equity and stability when the number of agents is variable in the fair division problem", *Economics Letters* 50 (1996), 85-90.

Maschler, M. and G. Owen, "The consistent Shapley-value for hyperplane games," *International Journal of Game Theory* 18 (1989), 390-407.

Moulin, H., "Equal or proportional division of a surplus, and other methods," *International Journal of Game Theory* 16 (1987), 161-186.

——, *Axioms of Cooperative Decision Making*, Cambridge University Press, 1988.

Nash, J., "The bargaining problem", *Econometrica* 18 (1950), 155-162.

O'Neill, B., "A problem of rights arbitration from the Talmud," *Mathematical Social Sciences* 2 (1982), 345-371.

Peleg, B., "An axiomatization of the core of cooperative games without side-payments," *Journal of Mathematical Economics* 14 (1985), 203-214.

——, "On the reduced game property and its converse," *International Journal of Game Theory* 15 (1986), 187-200. "A correction", *International Journal of Game Theory* 16 (1987).

—— and S. Tijs, "The consistency principle for games in strategic form", *International Journal of Game Theory*, 25 (1996), 13-34.

Roemer, J., "Axiomatic bargaining theory on economic environments," *Journal of Economic Theory* 45 (1988), 1-31.

Roth, A. and M. Sotomayor, *Two-sided Matching*, Cambridge University Press, 1990.

Sasaki, H. and M. Toda, "Consistency and characterization of the core of two-sided matching problems," *Journal of Economic Theory* 56 (1992), 218-227.

Schmeidler, D., "The nucleolus of a characteristic function form game", *SIAM Journal of Applied Mathematics* 17 (1969), 1163-1170.

Serrano, R., "Strategic bargaining, surplus sharing and the nucleolus", *Journal of Mathematical Economics* 24 (1995), 319- 329.

Shapley, L., "A value for n-person games," in *Contributions to the Theory of Games II* (Annals of Mathematics Studies 28), ed. by H.W. Kuhn and A.W. Tucker, Princeton: Princeton University Press, 1953, 307-317.

—— and M. Shubik, "The assignment game I: the core.", *International Journal of Game Theory* 1 (1972), 111-130.

Sobolev, A.I., "The characterization of optimality principles in cooperative games by functional equations," *Mathematical Methods in the Social*

Sciences 6 (1975), 150-165 (in Russian).

Sprumont, Y., "The division problem with single-peaked preferences", *Econometrica* 59 (1991), 509-519.

Svensson, L., "Large Indivisibilities: An Analysis with Respect to Price Equilibrium and Fairness," *Econometrica* 51 (1983), 939-954.

Tadenuma, K., "Reduced games, consistency, and the core", *International Journal of Game Theory* 20 (1992), 325-334.

—— and W. Thomson, "No-envy and consistency in economies with indivisible goods," *Econometrica* 59 (1991), 1755-1767.

—— and ——, "The fair allocation of an indivisible good when monetary compensations are possible," *Mathematical Social Sciences* 25 (1993), 117-132.

Thomson, W.,"A study of choice correspondences in economies with a variable number of agents," *Journal of Economic Theory* 46 (1988), 247-259.

——, "Consistent solutions to the problem of fair division when preferences are single-peaked," *Journal of Economic Theory* 63 (1994a), 219-245.

——, "Consistent extensions," *Mathematical Social Sciences* 28 (1994b), 35-49.

——, "Fair allocation rules", mimeo, 1996.

——, "Consistent allocation rules", 1997, forthcoming, in *Fundamentals of Pure and Applied Economics*, Harwood Academic Publishers.

Toda, M., "Characterizations of the core of two-sided matching problems which allow self-matchings", Tokyo Keizai University working paper, 1993.

Young, P., "On dividing an amount according to individual claims or liabilities," *Mathematics of Operations Research* 12 (1987), 398-414.

——, "Distributive justice in taxation", *Journal of Economic Theory*, 48 (1988), 321-335.