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Abstract

We propose a new rule to solve claims problems (O'Neill, 1982) and show that this rule is best in achieving certain objectives of equality. We present three theorems describing it as the most "egalitarian" among all rules satisfying two minor conditions. We refer to it as the "constrained egalitarian" rule. We show that it is consistent and give a parametric representation of it. We also define several other rules and relate all of them to the rules that have been most commonly discussed in the literature.

Key-words: Claims problems, constrained egalitarian rule, Talmud rule, consistency.
J.E.L classification numbers: D63, D70.

1 Introduction

A man dies leaving behind debts adding up to more than the value of his estate. How should the estate be divided between the agents holding claims against it? We propose a new rule to solve such "claims problems" (O'Neill, 1982) and show that this rule is best in achieving certain objectives of equality. We present three theorems describing it as the most "egalitarian" among all rules satisfying two minor conditions. Therefore, we refer to it as the "constrained egalitarian" rule.[1] We also define a rule that can be seen as its "dual"; there, the focus is on the losses claimants experience instead of on the awards made to them. Among all rules satisfying the two conditions alluded to above, this rule is the most "egalitarian" in terms of losses.

The constrained egalitarian rule is inspired in part by a rule that has played a central role in the study of an important class of rationing problems—the "uniform rule"—and specifically by recent characterizations of the uniform rule on the basis of egalitarian considerations (Schummer and Thomson, 1997).[2] Our strategy is to exploit formal analogies between claims problems and rationing problems, making accommodations for the critical distinctions that exist between the two classes of problems — in the context of claims problems, a number of properties have to be required of rules that are not pertinent to rationing.

We close with a discussion of a certain consistency property that the constrained egalitarian rule enjoys. Informally, a rule is "consistent" if the recommendation it makes is unaffected by the departure of some of the claimants with their awards. Somewhat more precisely, consider some problem and apply the rule to it. Then, let some of the claimants collect their awards and

[1] It should not to be confused with the "constrained equal awards" rule, defined later.

[2] The uniform rule has also been characterized on the basis of a variety of other considerations, including incentives (Sprumont, 1991), various notions of monotonicities (Thomson, 1994a, 1995), and consistency (Thomson, 1994b).

leave. Finally, consider the "reduced" problem obtained by reevaluating the situation from the viewpoint of the remaining claimants, and apply the rule to it. Consistency requires that the restriction to the subgroup of remaining claimants of the original recommendation be the recommendation the rule makes for the reduced problem. We show that the constrained egalitarian rule is consistent, a fact that allows us to describe it in a particularly convenient way, thanks to a representation theorem due to Young (1987).

In the course of our analysis, we present several rules that have played a central role in the literature, and we close by proposing several new ones that are related to these rules in simple ways.

In addition to introducing new rules to solve claims problems, we hope that our work will contribute to making the fundamental and very intuitively appealing idea of consistency known to a wider public than specialists in game theory and mechanism design. Although it is only recently that a systematic investigation of its implications has taken place (see Thomson, 1997, for a comprehensive survey), much is already known about the property and we feel that it is ready to be incorporated in the body of concepts that economists should routinely appeal to when evaluating resource allocation rules. Presenting the idea in the context of claims problems has an important advantage: this class of problems is among the simplest that one could possibly consider and yet it is surprisingly rich. A wealth of rules have been defined for it, and it can conveniently serve as a laboratory in which to introduce many of the concepts and techniques of the modern theory of mechanism design.

The paper is organized as follows. In Section 2, we introduce the concept of a claims problem and several well-known rules. In Section 3, we define the constrained egalitarian rule. In Section 4, we discuss its consistency and provide a parametric representation for it. In Section 5, we define our other proposals.

2 Claims problems and rationing problems

We first present the family of claims problems and a related family of rationing problems. In each case, $N = \{1, \ldots, n\}$ is the set of agents.

A ***claims problem*** (O'Neill, 1982)[3] is a vector $(c, E) = (c_1, \ldots, c_n, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$, where for each $i \in N$, c_i represents the ***claim*** of claimant i on an ***estate*** of value E.[4] The claims cannot be jointly met, that is, $\sum c_i \geq E$.[5] A ***feasible allocation for*** (c, E) is a vector $x \in \mathbb{R}_+^N$ such that $\sum x_i = E$ and for all $i \in N$, $x_i \leq c_i$. A ***rule*** is a function defined on the class $\mathcal{C}^N$ of all of these problems, which associates with each problem a unique feasible allocation. Our generic notation for a rule is F.

Numerical examples of claims problems, the so-called "contested garment" and "marriage contract" problems, appear in the Talmud and we return to them later to illustrate how our proposals differ from the resolutions discussed in the Talmud for these examples.[6]

This model has other applications. One of them is to bankruptcy: here the members of N are creditors, and for each $i \in N$, c_i is the claim of creditor i on the net worth E of a bankrupt firm. Also, consider the problem of allocating aid by an international agency. There, N is a set of countries in need of aid, with c_i representing the financial need of country i, and E is the budget of the agency. For a final example, think of N as the participants

[3]Using the language of game theory, the problems we consider here can be described as "transferable utility" claims problems. For a theory of "non-transferable utility" claims problems, see Chun and Thomson (1992).

[4]By $\mathbb{R}^N$ we mean the Cartesian product of $|N|$ copies of $\mathbb{R}$ indexed by the members of N.

[5]For convenience, we include the limit case when equality holds.

[6]These examples have been discussed extensively (O'Neill, 1982; Aumann and Maschler, 1985; Young, 1987; Curiel, Maschler, and Tijs, 1987; Chun, 1988, 1998; Serrano, 1993; Dagan and Volij, 1993; Dagan, 1996; Herrero, Maschler, and Villar, 1997; Benoît (1997); Herrero, 1998; Herrero and Villar, 1988; for a survey, see Thomson, 1996).

to a scientific meeting, with c_i representing the travel expenses incurred by participant i, and E being the conference budget. In any of these cases, the inequality $\sum c_i \geq E$ is quite descriptive of the real world.

A rule to solve claims problems, advocated in particular by Maimonides (see Aumann and Maschler, 1985), and known as the ***constrained equal awards rule, CEA***, consists in making awards as equal as possible, subject to the condition that no claimant receives more than his claim. For all $i \in N$,

$$CEA_i(c,E) = \min\{c_i, \lambda\}, \text{ where } \lambda \text{ is chosen so that } \sum \min\{c_i, \lambda\} = E.$$

A dual of this rule, the ***constrained equal losses rule, CEL***, makes awards such that the losses agents experience are as equal as possible, subject to the condition that no claimant receives less than zero. For all $i \in N$,

$$CEL_i(c,E) = \max\{0, c_i - \lambda\}, \text{ where } \lambda \text{ is chosen so that } \sum \max\{0, c_i - \lambda\} = E.$$

Another rule, introduced by Aumann and Maschler (1985), and which we refer to as the ***Talmud rule, T***, is obtained by "combining" the constrained equal awards and constrained equal losses rules (Figure 1). For all $i \in N$,

$$T_i(c,E) = CEA_i(c/2, \min\{\sum c_k/2, E\}) + CEL_i(c/2, \max\{E - \sum c_k/2, 0\}).$$

This rule can also be conveniently described as a function of the estate, assuming agents to be ordered by their claims, so that $c_1 \leq \cdots \leq c_n$. The first units of the estate are divided equally until all agents have received an amount equal to half of the smallest claim. Then, agent 1 does not receive anything for a while. Additional units are divided equally among the other agents until they have received an amount equal to half of the second smallest claim, and so on. This process goes on until each agent has received half of his claim, which is when the estate is worth $\sum c_i/2$. Each agent's award schedule, the function that gives his award as a function of the amount to divide, is then completed by symmetry with respect to the point $(\sum c_k/2, c_i/2)$; or equivalently, starting from an estate equal to $\sum c_i$, which allows each agent to receive his claim, we consider shortfalls of increasing sizes: the first units of such a shortfall are divided equally until each agent's loss is equal to half of the smallest claim. Then, agent 1's loss stops. Any additional shortfall is divided equally between the other agents until their common loss is equal to half of the second smallest claim ... This process goes on until each agent's loss is equal to half of his claim.

Next, we present the related problem of rationing. Rationing is necessary in resource allocation problems when prices are not allowed to adjust, or have not had the time to adjust, so as to achieve equality of demand and supply. In the two-good case, by restricting attention to the budget lines, we obtain the following reduced model, in which preferences have the well-known property of single-peakedness: a preference relation R_i defined on $\mathbb{R}_+$ with asymmetric part P_i is ***single-peaked*** if there is a number, which we denote by $p(R_i)$ and call the ***peak amount for*** $\boldsymbol{R_i}$, such that for all $x_i, x_i' \in \mathbb{R}_+$, if $x_i < x_i' \leq p(R_i)$ or $p(R_i) \leq x_i' < x_i$, then $x_i'\ P_i\ x_i$. A ***rationing problem*** (Sprumont, 1991) is a list $(R, M) = (R_1, \ldots, R_n, M)$ where the R_i's are single-peaked preference relations defined on $\mathbb{R}_+$,[7] and $M \in \mathbb{R}_+$ represents an amount to divide. A ***feasible allocation for*** $\boldsymbol{(R, M)}$ is a vector $x \in \mathbb{R}_+^N$ such that $\sum x_i = M$. A ***rule*** is a function defined on the class $\mathcal{E}^N$ of these problems that associates with each problem a unique feasible allocation. The most important rule in the literature devoted to the analysis of this class of problems is obtained by confronting all agents with the same "uniform" (upper or lower, as the case may be) bound, and letting them maximize their preferences subject to not exceeding, or not falling short of, that bound. It is known as the ***uniform rule,*** $\boldsymbol{U}$ (Benassy, 1982): for all $i \in N$, if $M \leq \sum p(R_k)$, then $U_i(R, M) = \min\{p(R_i), \lambda\}$, λ being chosen so

[7]Alternatively, we could require preferences to be defined over some interval $[0, M_0]$, or over the interval $[0, M]$, M being the amount to divide.

that $\sum x_i = M$; if $\sum p(R_k) \geq M$, then $U_i(R, M) = \max\{p(R_i), \lambda\}$, λ being chosen so that $\sum x_i = M$.

In order to make clear the connection between the uniform rule and the Talmud rule introduced earlier, it is convenient also to describe the uniform rule parametrically as a function of the amount to divide (Thomson, 1994a). Assume agents to be ordered by their peaks, so that $p(R_1) \leq \cdots \leq p(R_n)$. The first units of the commodity are divided equally until all agents have received an amount equal to agent 1's peak amount. Then, agent 1 stops receiving anything for a while. Additional units are divided equally among the other agents, until they have received an amount equal to agent 2's peak amount. Then, agent 2 also stops receiving anything for a while. This process goes on until each agent has received his peak amount. At that point, additional units go first to agent 1 until he receives an amount equal to agent 2's peak amount. Additional units are divided equally between agents 1 and 2 until each receives an amount equal to agent 3's peak amount. This process goes on until agents 1 to $n-1$ all receive amounts equal to agent n's peak amount. Additional units are divided equally among all agents.

The following distinctions between problems of fair division and claims problems should be noted. First, In a claims problem, the value of the estate is by definition less than the sum of the claims ($\sum c_i \geq E$). In a rationing problem, no comparable restriction on M in relation to $\sum p(R_i)$ was imposed. Also, in a rationing problem, no natural upper bound can be imposed on what agents receive, whereas in claims problems, it makes sense to require that no agent receives more than his claim. This is the property of ***claims boundedness***.

Second, it is desirable in claims problems that payments respect the ordering of claims. Agents differ only in their claims and this requirement of ***fair ranking*** is quite natural. The corresponding requirement for rationing problems, that the amounts awarded respect the ordering of peaks, is not compelling in general unless of course the rule depends only on peaks, (a property satisfied by the uniform rule.[8])

3 The constrained egalitarian rule

Motivated by recent characterizations of the uniform rule as the only rule to satisfy certain egalitarian objectives subject to efficiency (Schummer and Thomson, 1997), our objective here is to investigate the existence of rules for claims problems that would enjoy similar properties. A natural way to proceed is of course to try to adapt the uniform rule to the present situation. Given the various ways in which rationing problems and claims problems differ, some care should be exercised however. A good starting point is the Talmud rule, as it seems to correspond to the first half of the uniform rule (up to the point where each agent receives his peak). To obtain a rule for claims problems that reflects the totality of the uniform rule, we proceed as in the definition of the constrained equal awards rule but take from the Talmud rule its switchpoint of $E = \sum c_i/2$. Aumann and Maschler (1985) note numerous examples in the Talmud where the midpoint is viewed as an important special case.

The Talmud rule and the rule we obtain exhibit substantial differences. This is because, under the uniform rule, it is the agent with the smallest peak amount that receives the first additional units after the switchpoint is reached, whereas under the Talmud rule, it is the agent with the largest claim that does.

Also, for payments to respect the ordering of claims, we have to make sure that when agent i starts receiving additional units, he should not receive so much as to overtake agent $i+1$.

[8]This property is also satisfied by the proportional rule, for which payments are proportional to peaks.

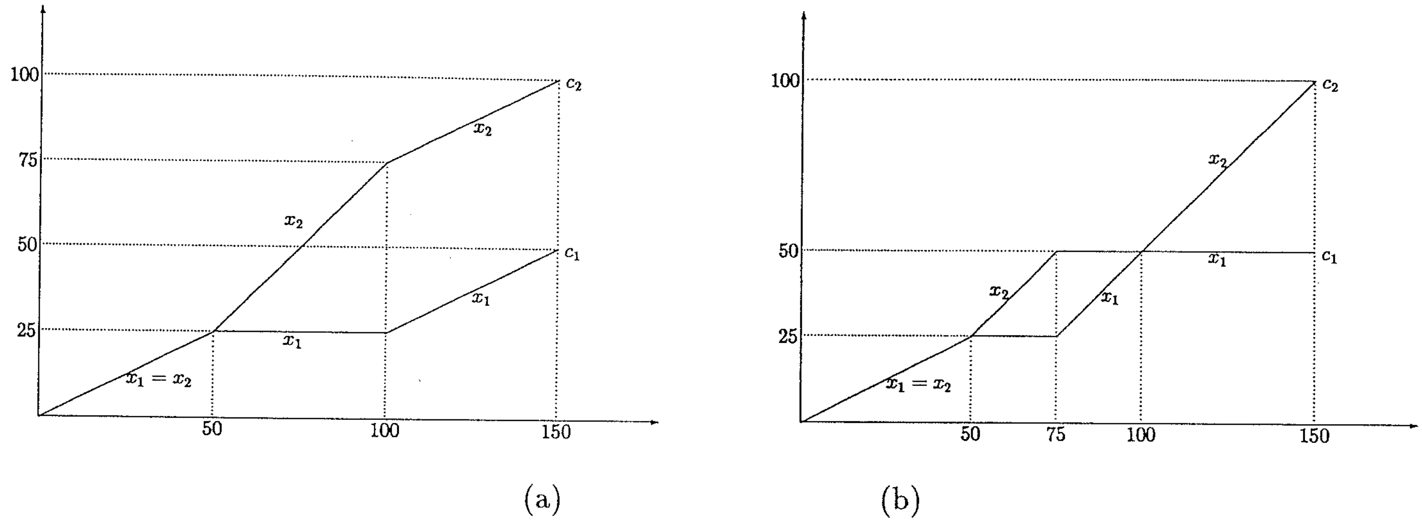


Figure 1: The Talmud rule and the constrained egalitarian rule illustrated for the contested garment problem: $c = (50, 100)$ (a) The Talmud rule. (b) The constrained egalitarian rule.

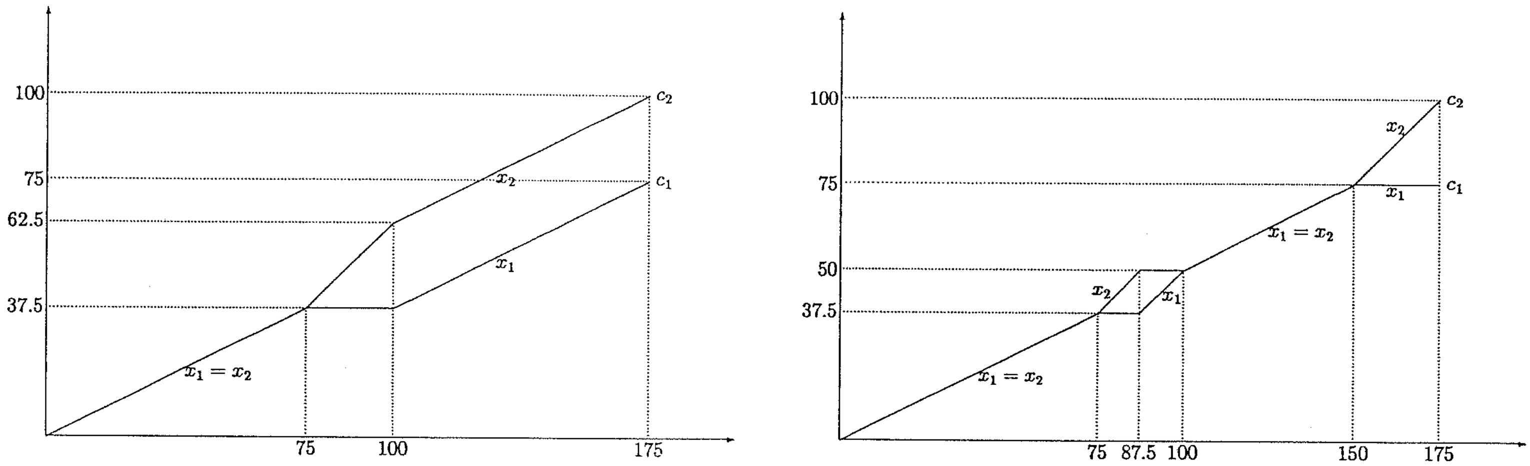
We are now in a position to propose an explicit description, still assuming claimants to be ordered by claims: $c_1 \leq \cdots \leq c_n$. For values of the estate up to $\sum c_i/2$, payments are computed as for the Talmud rule. At that point, any additional unit goes to agent 1 until he reaches his claim or half of the second smallest claim, whichever is smallest. If $c_1 \leq c_2/2$, he stops there. If $c_1 > c_2/2$, any additional unit is divided equally between agents 1 and 2 until agent 1 reaches his claim, in which case he drops out, or they reach $c_3/2$. In the first case, any additional unit is given entirely to agent 2 until he reaches his claim or $c_3/2$. In the second case, any additional unit is divided between the three agents until agent 1 reaches his claim or they reach $c_4/2 \cdots$. A compact formula for the rule is given by:

The constrained egalitarian rule, CE: For all $i \in N$,

$$CE_i(c, E) = \begin{cases} \min\{c_i/2, \lambda\} & \text{if } E \leq \sum c_k/2, \\ \max\{c_i/2, \min\{c_i, \lambda\}\} & \text{otherwise,} \end{cases}$$

where in each case, λ is chosen so as to achieve feasibility.

The constrained egalitarian rule is illustrated in Figures 1 to 3. Figure 1 pertains to the two-person ***contested garment problem*** discussed

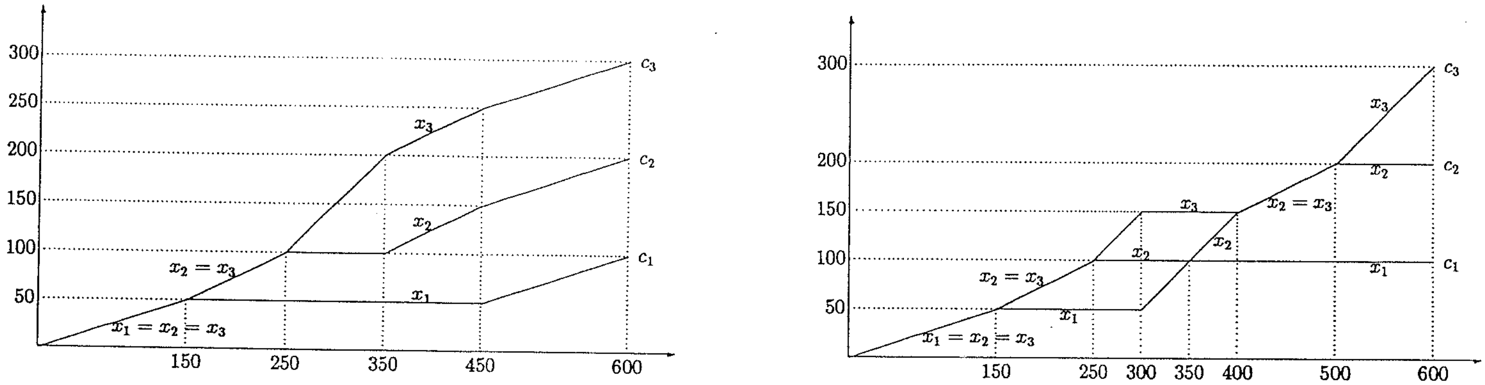


(a) (b)

Figure 2: The Talmud rule and the constrained egalitarian rule illustrated for another two-person problem: $c = (75, 100)$ (a) The Talmud rule. (b) The constrained egalitarian rule.

in the Talmud (Baba Metzia 2a), where $(c_1, c_2) = (50, 100)$. It represents the awards as a function of the estate for both the Talmud rule and the constrained egalitarian rule. The Talmud considers the case $E = 100$ and recommends $(25, 75)$. Figure 2 represents similar graphs for a two-person problem with $(c_1, c_2) = (75, 100)$, the values of the claims being chosen so as to help reveal the range of possible shapes for the schedules of awards for the constrained egalitarian rule. Figure 3 gives the corresponding graphs for the three-agent ***marriage contract problem*** of the Talmud (Kethubot 93a), where $(c_1, c_2, c_3) = (100, 200, 300)$; the Talmud considers the case $E = 100$ for which it recommends equal division, the case $E = 300$ for which it recommends proportional division, and the case $E = 200$ for which it recommends $(50, 75, 75)$. Note that the relationship between c_i and $c_{i+1}/2$ is crucial in determining the shape of the schedules for the constrained egalitarian rule.

The constrained egalitarian rule differs from ***Pineles' rule, P,*** (Aumann and Maschler, 1985). This rule also coincides with the Talmud rule up



(a) (b)

Figure 3: The Talmud rule and the constrained egalitarian rule illustrated for the marriage contract problem in the Talmud: $c = (100, 200, 300)$ (a) The Talmud rule. (b) The constrained egalitarian rule.

to $E = \sum c_k/2$, but for $E > \sum c_k/2$, it is obtained by simply "replicating" the payments given by the Talmud rule up to $E = \sum c_k/2$. For all $i \in N$,

$$P_i(c, E) = CEA_i(c/2, \min\{\sum c_k/2, E\}) + CEA_i(c/2, \max\{E - \sum c_k/2, 0\}).$$

For the contested garment problem, it recommends $(37.5, 62.5)$, whereas the constrained egalitarian rule recommends $(50, 50)$.

The constrained egalitarian rule also differs from all of the other rules discussed by O'Neill (1982).

4 Characterizations

In this section, we show that we have achieved our objective of contrained egalitarianism by presenting three characterizations of the constrained egalitarian rule. The first constraint is that if the estate increases, the amount received by each claimant should not decrease. This property is satisfied by all of the rules that have been discussed in the literature and it is very difficult to think of any reason why one may accept that it be violated.

Estate-monotonicity: For all (c, E), $(c', E') \in \mathcal{C}^N$, if $c = c'$ and $E \leq E'$, then $F(c, E) \leqq F(c', E')$.[9]

A property that has been considered in a number of studies expresses the idea that "gains and losses should be put on the same footing": a given problem can be seen from two perspectives, either as pertaining to the division of whatever amount is available as we have done so far, or as pertaining to the allocation of the losses that claimants have to incur. The requirement that what is available be distributed symmetrically to what is missing is known as ***self-duality.*** A number of rules are *self-dual,* including the proportional rule (according to which awards are proportional to claims) and the Talmud rule. The idea expressed in *self-duality* is very ancient. Aumann and Maschler (1985) cite numerous passages in the Talmud where it appears.

Note that *self-duality* implies that if the estate is equal to half of the sum of the claims, then every claimant should receive half of his claim. We will write this as a separate condition.

Midpoint property: For all $(c, E) \in \mathcal{C}^N$, if $E = \sum c_i/2$, then $F(c, E) = c/2$.

Focusing on this special case gives us a considerably weaker condition than *self-duality*; for each claims vector, it applies to *only one* value of the estate whereas *self-duality* relates the choice made for *any* value of the estate E to the symmetric value $\sum c_i - E$.

Second, the property is satisfied by interesting rules that are not *self-dual,* an example being Pineles' rule. Another example is the variant of the proportional rule obtained as follows (Curiel, Maschler, and Tijs, 1987): first truncate the claims by the estate and attribute to each claimant the minimum

[9] Vector inequalities: $x \geqq y$, $x \geq y$, $x > y$.

of the difference between the estate and the sum of the claims of the other agents and zero; adjust the claims down by these amounts, and in a second round, apply the proportional rule to the problem so redefined. The award to each claimant is the sum of the awards made in each of these two rounds.

Our results are that among rules satisfying *estate-monotonicity* and the *midpoint property*, the constrained egalitarian rule behaves in the most egalitarian way, as we intended. Their proofs are relegated to an appendix. The proofs of Theorems 1 and 2 below are obtained by a simple adaptation of the proofs of the characterizations of Schummer and Thomson (1997). The proof of Theorem 3 is a variant of the proof of Theorem 1.[10]

Theorem 1 *The difference between the largest amount received by any claimant and the smallest such amount is smaller at the award vector selected by the constrained egalitarian rule than at the award vector selected by any other rule satisfying* estate-monotonicity *and the* midpoint property.

Theorem 2 *The variance of the amounts received by all the claimants is smaller at the award vector selected by constrained egalitarian rule than at the award vector selected by any other rule satisfying* estate-monotonicity *and the* midpoint property.

Theorem 3 *The constrained egalitarian rule selects the award vector that maximizes the Lorenz ordering among all rules satisfying* estate-monotonicity *and the* midpoint property.

[10]It is worth noting that these characterizations of the uniform rule are obtained by considering one rationing problem at a time, whereas the characterizations we propose here are defined for rules defined on the whole domain of claims problems. The *midpoint condition* pertains to a small subdomain of the domain of claims problem, and it applies to each element of that subdomain separately, but *estate-monotonicity* relates the choices made for any two problems that differ only in the amount to divide.

We note that Aadland and Kolpin (1998) have characterized rules to solve the problem of sharing the cost of an irrigation system (this problem is mathematically equivalent to the problem of sharing the cost of an airport runway, as formulated by Littlechild and Owen, 1973), by means of a maximizing exercise of the kind defined in Theorem 1. One of the rules they characterize can be seen as a counterpart of the game-theoretical concept of Shapley value (Shapley, 1953). In the context of claims problem, the Shapley value corresponds to a rule that differs from the constrained egalitarian rule. It is called the ***random arrival rule.*** To define it, first order the claimants. Then, give to each claimant the minimum of his claim and the amount that is left when he arrives, full satisfaction having been given to the claimants preceding him in the ordering. Finally, give to each claimant the average over all possible orderings of the amount just calculated.[11]

5 Consistency

In this section, we turn to a property of allocation rules that has played an important role in recent developments. "Consistency" links the recommendations made by a rule as the population of claimants varies. So far, the set of claimants had been kept fixed, but we now think of it as a variable. Informally, the requirement is that there should never be any need to reconsider a recommendation once some of the claimants have left with their assigned amounts. Somewhat more formally, and given a claims problem $(c, E) \in \mathcal{C}^N$, let x be the recommendation made by a given rule for it. Now, let claimant $i \in N$ leave the scene with x_i and let us reevaluate the situation

[11]This process is indeed reminiscent of the way the rule introduced by Shapley for coalitional games can be defined, and it is the underlying reason for the formal correspondence between the random arrival rule and the Shapley value. This correspondence was brought to light by O'Neill (1982).

from the viewpoint of the remaining claimants. Their claims are unchanged; the leftover is $E - x_i$. The rule is ***consistent*** if for the "reduced" problem so defined it recommends the same awards for them as initially.[12]

Note that for this operation to be meaningful, the rule has to be defined for classes of problems of arbitrary cardinalities. Formally then, we have a universe of "potential agents" which we index by the natural numbers, $\mathbb{N}$. To define a problem, we first specify a non-empty finite subset of $\mathbb{N}$. Let $\mathcal{N}$ denote the class they constitute. Given $N \in \mathcal{N}$, let $\mathcal{C}^N$ denote the class of claims problems involving the group N. A ***rule*** is a mapping defined on the union of the $\mathcal{C}^N$ that associates with every $N \in \mathcal{N}$ and every $(c, E) \in \mathcal{C}^N$ a unique feasible allocation of (c, E). The rules discussed so far are adapted to deal with variable populations in a straightforward way, by applying them separately to each set of agents. For instance, what we now mean by the "Talmud rule" is the rule that selects for each group of agents and for each claims problem that this group could face, the awards vector recommended by the formula of Section 3 that we presented under that the Talmud rule. The general statement of the property is as follows:

Consistency: For all $N \in \mathcal{N}$, all $(c, E) \in \mathcal{C}^N$, and all $N' \subset N$, we have $x_{N'} = F(c_{N'}, \sum_{N'} x_i)$, where $x = F(c, E)$.

Many well-known rules are *consistent.* (An example of a rule that is not is the random arrival rule.) The constrained egalitarian rule is too.[13] It also satisfies ***continuity***, the requirement that small changes in the data should not cause large changes in the recommended vector of awards, and it is ***symmetric***, in that two agents with the same claims receive the same awards: if $c_i = c_j$, then $F_i(c, E) = F_j(c, E)$. Therefore, by Young's (1987)

[12] For a survey of the literature on the "consistency principle", see Thomson (1997).

[13] It is easy to see that the uniform rule, as a rule to rationing problems, is *consistent* (Thomson, 1994b).

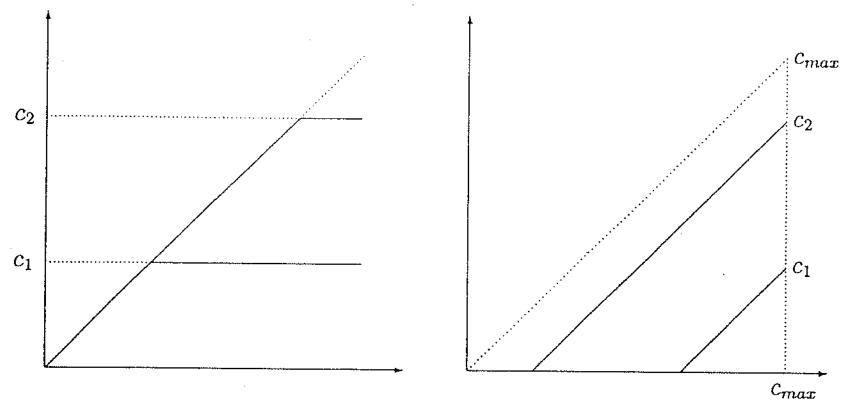
theorem, it has a ***parametric representation***: say that the rule F has such a representation if there is a family of continuous, nowhere-decreasing, and onto real-valued functions $f(c_0, \cdot) : \mathbb{R} \times [a,b] \to \mathbb{R}$, where $[a,b] \subseteq]-\infty, +\infty[$, indexed by the parameter $c_0 \in \mathbb{R}_+$, each c_0 being interpreted as the possible value of a claim, such that for all $N \in \mathcal{N}$ and all $(c, E) \in \mathcal{C}^N$, $F(c, E) = \{f(c_i, \lambda)\}_{i \in N}$, for $\lambda \in \mathbb{R}$ chosen so that $\sum_N f(c_i, \lambda) = E$.

Young gives a parametric representation of the Talmud rule. For the purposes of exposition, we will restrict our attention to claims problems where each claim is bounded by some fixed number c_{max}. Then, a very simple parametric representation—it is piecewise linear—of the Talmud rule is possible (Figure 5a). Postulating an upper bound on the claims, a simple parametric representation for the constrained egalitarian rule can also be given. The following theorem gives such a representation. We omit the proof, which is straightforward.

Theorem 4 *The constrained egalitarian rule is* consistent *and it admits the following parametric representation:*

$$f(c_i, \lambda) = \begin{cases} \lambda & \text{if } \lambda \in [0, c_i/2]; \\ c_i/2 & \text{if } \lambda \in [c_i/2, c_i/2 + c_{max}/2]; \\ \lambda - c_{max}/2 & \text{if } \lambda \in [c_i/2 + c_{max}/2, c_i + c_{max}/2]; \\ c_i & \text{if } \lambda \in [c_i + c_{max}/2, 3c_{max}/2]. \end{cases}$$

More illuminating than the formula is its graph. We will find it useful to introduce it in stages, by first giving the corresponding graphs for the constrained equal awards and constrained equal losses rules. This is because the Talmud rule as well as the constrained egalitarian rule are built up from these rules. The Talmud rule is obtained for values of the estate up to half the sum of the claims, by applying the constrained equal awards rule to the problem obtained by dividing the claims by two, and for greater values of the estate by applying the constrained equal losses rule. This switch between



(a) (b)

Figure 4: Parametric representations of the constrained equal awards and constrained equal losses rules. (a) The constrained equal awards rule. (b) The constrained equal losses rule.

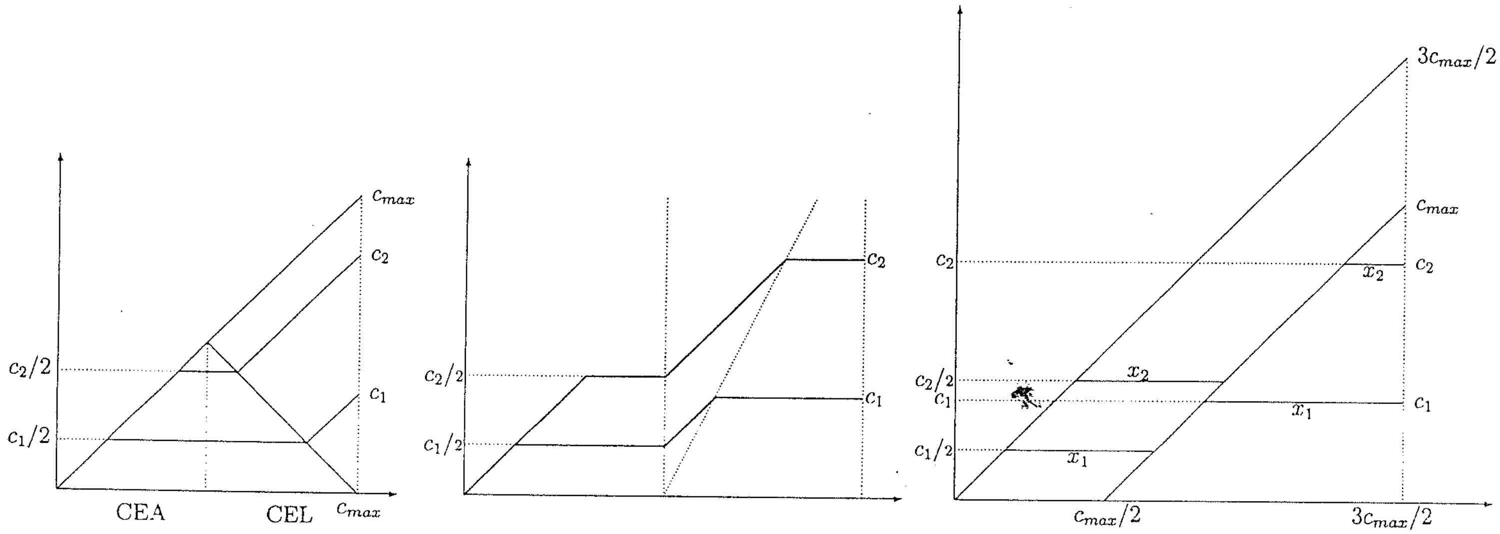
constrained equal awards and constrained equal losses guarantees *self-duality*, a property that the constrained egalitarian rule does not have, although by not insisting on it, we obtain egalitarianism in the various forms expressed in Theorems 1 to 3.

It should be noted that the *midpoint property* is satisfied in the two-person case by rules that are not *consistent*, but in the light of *continuity*, *symmetry*, and *consistency*, the *midpoint property* for two persons is inherited for the general case, as stated in the following lemma. Its proof, as well as the proofs of the other lemmas, are relegated to the appendix:

Lemma 1 *If a rule is* continuous, symmetric, consistent, *and satisfies the* midpoint property *in the two-person case, then it satisfies the* midpoint property *in general.*

A converse of *consistency* says that if a feasible award vector has the property that for any two-person subgroup of the agents, its restriction to that subgroup would be chosen by the rule for the associated reduced problem, then it is chosen by the rule for the problem.

Converse consistency: For all $N \in \mathcal{N}$, all $(c, E) \in \mathcal{C}^N$, and all $x \in \mathbb{R}^N$ such that $\sum x_i = E$, if for all $N' \subset N$ with $|N'| = 2$, we have $x_{N'} =$



(a) (b) (c)

Figure 5: Parametric representations of three rules. (a) The Talmud rule. (b) Pineles' rule. (c) The constrained egalitarian rule.

$F(c_{N'}, \sum_{N'} x_i)$, then $x = F(c, E)$.

Lemma 2 *If a rule is* conversely consistent *and satisfies the* midpoint property *for the two-person case, then it satisfies the* midpoint property *in general.*

The final lemma helps to relate these results:

Lemma 3 *If a rule is* estate-monotonic *and* consistent, *then it is* conversely consistent.

6 Three more rules

In this concluding section, we propose several additional rules. First is a rule that is symmetric to the Talmud rule in that the roles played by the constrained equal awards and constrained equal losses are reversed. For the ***reverse Talmud rule***, $\boldsymbol{RT}$, for all $i \in N$,

$$RT_i(c, E) = CEL_i(c/2, \min\{E, \sum c_k/2\}) + CEA_i(c/2, \max\{E - \sum c_k/2, 0\}).$$

This rule is self-dual, as is the Talmud rule.

Another rule is the ***dual of Pinelès' rule***, $\boldsymbol{P^*}$, which is obtained by applying the constrained equal losses twice. For all $i \in N$,

$$P_i^*(c, E) = CEL_i(c/2, \min\{E, \sum c_k/2\}) + CEL_i(c/2, \max\{E - \sum c_k/2, 0\}).$$

Finally, we have the dual of the constrained egalitarian rule, $\boldsymbol{CE^*}$. For all $i \in N$,

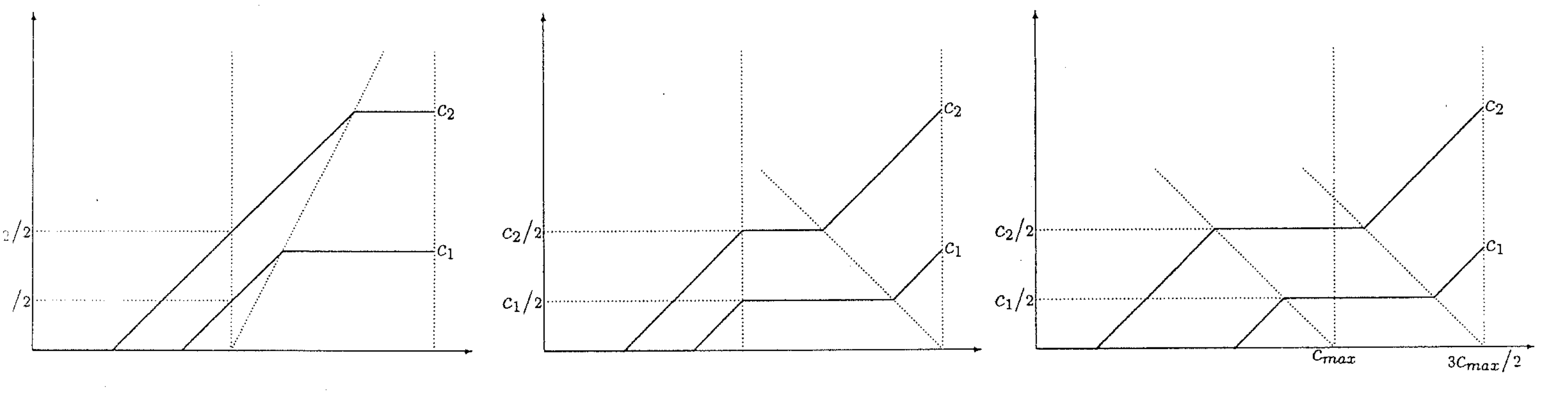
$$CE_i^*(c, E) = \begin{cases} \min\{c_i/2, \max\{0, c_i - \lambda\}\} & \text{if } E \leq \sum c_k/2, \\ \max\{c_i/2, c_i - \lambda\} & \text{otherwise,} \end{cases}$$

where in each case λ is chosen so as to achieve feasibility.

The three characterizations of the constrained egalitarian rule given in Section 5 can be converted into characterizations of its dual by focusing on losses agents experience instead of on what they receive. We dispense with formal statements of these results.

The three rules just defined are *consistent* too. This can be seen directly from the definitions. For the latter two, it also follows from the obvious fact that if a rule is *consistent*, so is its dual. Representations of the rules are given in Figure 6.

Pineles's rule is also *consistent* and therefore it too has a parametric representation. This rule can be obtained by imposing of ***composition from midpoint***, the requirement that that it should not make any difference whether an estate is divided in one step, or in two steps as follows: first, divide whichever amount is smaller, the estate or half the sum of the claims, and in



(a) (b) (c)

Figure 6: Parametric representations of three other rules. (a) The Reverse Talmud rule. (b) The dual of Pineles' rule. (c) The dual of the constrained egalitarian rule.

the latter case, allocate the remainder after dividing the claims by two.[14] Now it is clear that Pineles' rule is the only one to maximize the Lorenz ordering among all rules satisfying *estate-monotonicity*, the *midpoint property*, *fair ranking*, and *composition from midpoint*. Its dual can be characterized in a symmetric way, focusing on losses instead.

By way of summary, we plot in the two-person case, the schedule of awards for each of the main rules that we have discussed in this paper (Figures 7 and 8).

We close with a table gathering most of the rules that we have discussed. The second row lists their duals. Three of the rules coincide with their duals. All except for the proportional rule are defined by applying ideas of equal awards or equal losses. The Talmud and reverse Talmud rule, Pinele's rule and its dual, are obtained by implementing these ideas in succession, after dividing the claims by two, the switch occuring for a value of the estate equal to half of the sum of the claims. They differ in the order in which these ideas are applied. The constrained egalitarian rule and its dual are based on similar ideas of equality of awards or losses but they involve an additional constraint. In that sense, they are a little less radical in reaching for equality than the constrained equal awards and constrained equal losses rules, which take minimal account of differences in claims.

Rule	Pro	CEA	T	RT	Pin	CE
Dual of the rule	$Pro^*(= Pro)$	CEL	$T(= T^*)$	$RT(= RT^*)$	Pin^*	CE^*

Six rules and their duals. Key: Pro is the proportional rule; CEA, the constrained equal awards rule; T, the Talmud rule; RT, the "reverse Talmud" rule; Pin, Pineles' rule; CE, the constrained egalitarian rule; and CEL, the constrained equal losses rule.

[14]Formally, the condition is as follows: for all $N \in \mathcal{N}$ and all $(c, E) \in \mathcal{C}^N$, $F(c, E) = F(c, \min\{E, \frac{1}{2}\sum c_i\}) + F(\frac{1}{2}c, \max\{E - \frac{1}{2}\sum c_i, 0\})$.

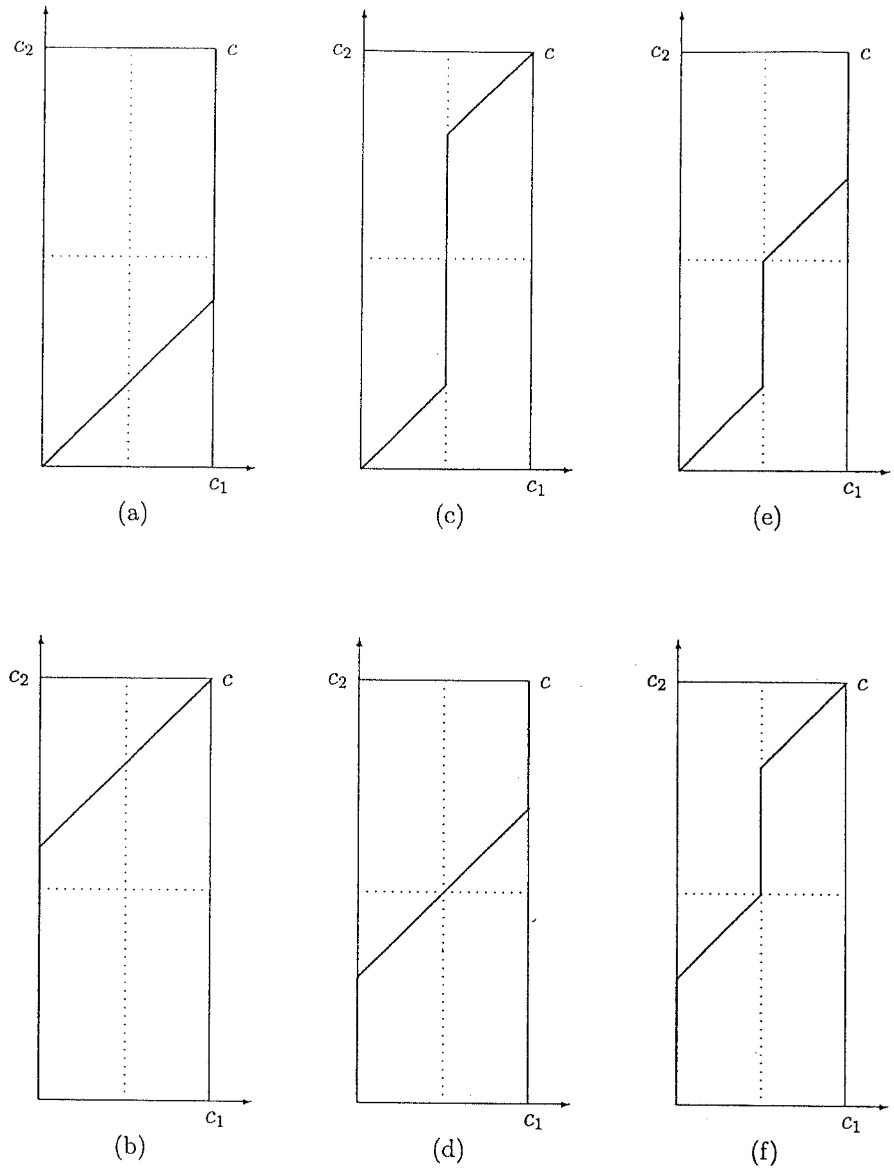


Figure 7: Schedules of awards for six rules. (a) The constrained equal awards rule. (b) The constrained equal losses rule. (c) The Talmud rule. (d) The reverse Talmud rule (e) Pineles' rule. (f) The dual of Pineles' rule.

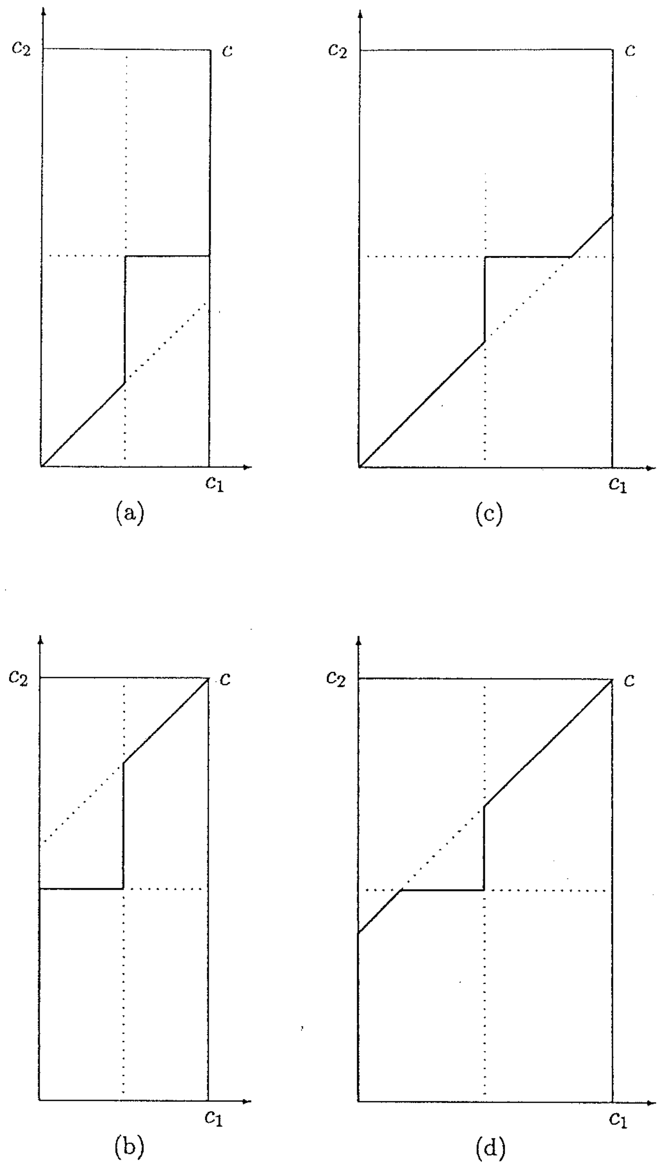


Figure 8: Schedules of awards for the constrained egalitarian rule and its dual for two choices of the claims vector. (a) The constrained egalitarian rule. (b) The dual of the constrained egalitarian rule. (c) and (d) The two rules for a different claims vector.

7 Conclusion

Interesting open questions remain. First is whether the constrained egalitarian rule as well as the other proposals made in this paper can be obtained by first associating with each claims problem a coalitional game, and then applying to this game one of the well-known solutions to coalitional games. As shown by Aumann and Maschler (1985), the Talmud rule can be so obtained.[15] So can the random arrival rule, which can be derived in a similar way from the Shapley value (O'Neill, 1982). For another contribution linking rules to abstract games (bargaining problems and coalitional games respectively) and rules to claims problems or problems of fair division, see Dagan and Volij (1993) and Otten, Peters, and Volij (1994).

Another open question is whether our new rules can be provided non-cooperative foundations. For the formulation and the study of non-cooperative games associated with claims problems, see Chun (1989), Serrano (1993) and Sonn (1992).

Finally is whether the new rules have axiomatic justifications.

We will leave these questions to future research.

[15] By applying the nucleolus or the kernel.

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Appendix

Proof: (of Lemma 1) Let F be a rule satisfying the hypotheses of the lemma. Then by Young (1987), it has a parametric representation $f: \mathbb{R}_+ \times [a,b] \to \mathbb{R}$. For each $c_0 \in \mathbb{R}_+$, and since $f(c_0, .)$ is continuous and nowhere decreasing, there is a (unique) closed interval $I_0 \subset [a,b]$ where it takes the value $c_0/2$. Since F satisfies the *midpoint property* in the two-person case, for each pair $c_0, c_0' \in \mathbb{R}_+$, the intervals I_0 and I_0' overlap. Now, given $N \in \mathcal{N}$ and $(c, E) \in \mathcal{B}^N$ with $E = \sum_N c_i$, we prove by induction on the number of possible values of the claims that the intervals $(I_i)_{i \in N}$ have at least one point in common. For $|N| = 2$, there is nothing to prove. Suppose the result holds up to $n-1$, and let $N = \{1, \ldots, n\}$. Let $I^* = \cap_{k=1}^{n-1} I_k \subset [a,b]$. By the induction hypothesis, $I^* \neq \emptyset$. Suppose by contradiction that $I^* \cap I_n = \emptyset$ and without loss of generality, that $I_n \subset [a, \underline{\lambda}[$, where $\underline{\lambda} = min\{\lambda \colon \lambda \in I^*\}$. For each $i < n$, let $\lambda_i \in I_i \cap I_n$, and $\lambda^* = \max_{i=1}^{n-1} \lambda_i$, and i^* be such that $\lambda_{i^*} = \lambda^*$. Since $\lambda_i \in I_i$ for all $i < n$, $\lambda^* \in I^*$. Since λ^* is obtained for the coalition $\{i^*, n\}$, $\lambda \in I_n$, a contradiction.

By choosing λ in $\cap_N I_i$, we obtain the vector of awards chosen by the rule, but then, every agent gets half of his claim. □

Proof: (of Lemma 2) Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{B}^N$, and $x = c/2$. By the *midpoint property* for the two-person case, for all $N' \subset N$ with $|N'| = 2$, $x_{N'} = F(c_{N'}, \sum_{N'} x_i)$. By *converse consistency*, $x = F(c, E)$. □

The proofs of Theorems 1 and 2 are obtained by a simple adaptation of the proofs of characterizations of the uniform rule appearing in Schummer and Thomson (1997). We include them for completeness.

Proof: (of Theorem 1) Let F be a rule satisfying *estate-monotonicity* and the *midpoint property*. Let $(c, E) \in \mathcal{C}^N$ be given. By *estate-monotonicity* and the *midpoint property*, either $F(c, E) \geqq \frac{c}{2}$ or $F(c, E) \leqq \frac{c}{2}$.

(i) Suppose that $E \leq \sum c_k/2$, so that $F(c,E) \leqq \frac{c}{2}$. Let $A(c,E) \equiv \{x \in \mathbb{R}^N_+ : 0 \leqq x \leqq \frac{c}{2} \text{ and } \sum x_i = E\}$. For all $x \in \mathbb{R}^N_+$, let $r(x) \equiv \max_i x_i - \min_i x_i$. Since $A(c,E)$ is compact and r is continuous, there exists $x \in \arg\min_{y \in A(c,E)} r(y)$. Suppose by contradiction that $x \neq CE(c,E)$.

Let $y^{max} = \max_j x_j$. Since $x \neq CE(c,E)$, there exists $i \in N$ such that $x_i < y^{max}$ and $x_i < c_i/2$. Let $\delta = \min\{\frac{c_i}{2}, y^{max}\} - x_i$. Let $N' = \{j \in N : x_j = y^{max}\}$.

Define $z \in A(c,E)$ as follows: $z_i = x_i + \frac{\delta}{2}$, for all $j \in N'$, $z_j = x_j - \frac{\delta}{2|N'|}$, and for all $k \notin N' \cup \{i\}$. $z_k = x_k$. It is straightforward to check that $z \in A(c,E)$ and $r(z) < r(x)$, contradicting our choice of x.

(ii) A similar argument can be developed for the case $E > \sum c_k/2$. □

Proof: (of Theorem 2): The first paragraph is identical to the first paragraph of the previous proof and we do not repeat it.

(i) Suppose that $E \leq \sum c_k/2$, so that $F(c,E) \leqq \frac{c}{2}$. Let $A(c,E) \equiv \{x \in \mathbb{R}^N_+ : 0 \leqq x \leqq \frac{c}{2} \text{ and } \sum x_i = E\}$. For all $x \in \mathbb{R}^N_+$, let $v(x) \equiv \frac{1}{n}\sum_N (x_i - M/n)^2$. Since $A(c,E)$ is compact and v is continuous, there exists $x \in \arg\min_{y \in A(c,E)} v(y)$. Suppose $x \neq CE(c,E)$. Then, there exist $i, j \in N$ such that $x_i < CE_i(c,E)$ and $x_j > CE_j(c,E)$. By the definition of the constrained egalitarian rule, *estate-monotonicity,* and the *midpoint property* imply that $x_i < CE_i(c,E) \leq CE_j(c,E) < x_j$.

Let $\delta = \min\{CE_i(c,E) - x_i, x_j - CE_j(c,E)\}$. Note that $\delta > 0$. Let $y \in A(c,E)$ be defined by $y_i = x_i + \delta$, $y_j = x_j - \delta$, and $y_k = x_k$ for all $k \notin N \backslash \{i,j\}$. Since $x \in A(c,E)$, $CE(c,E) \in A(c,E)$, $y_i \leq CE_i(c,E)$, $y_j \leq x_j$, and $\sum y_k = E$, $y \in A(c,E)$. Letting $m = M/n$, we have

$$n \cdot v(x) - n \cdot v(y) \quad = \quad (x_i - m)^2 + (x_j - m)^2 - (y_i - m)^2 - (y_j - m)^2$$

$$
\begin{aligned}
&= \{(x_i - m)^2 - (y_i - m)^2\} + \{(x_j - m)^2 + (y_j - m)^2\} \\
&= (x_i - 2m + y_i)(x_i - y_i) + (x_j - 2m + y_j)(x_j - y_j) \\
&= (2x_i + \delta - 2m)(-\delta) + (2x_j - \delta - 2m)(\delta) \\
&= \delta(-2x_i + 2x_j - 2\delta) \\
&> 2\delta(x_j - x_i - (x_j - x_i)) \\
&= 0.
\end{aligned}
$$

The inequality comes from the fact that $x_i < CE_i(c, E) \leq CE_j(c, E) < x_j$. This contradicts our choice of x.

(ii) A similar argument can be developed for the case when $E > \sum c_k/2$. □

and

Proof: (of Theorem 3): The first paragraph is identical to the first paragraph of the proof of Theorem 1.

(i) Suppose that $E \leq \sum_{k \in N} c_k/2$, so that $F(c, E) \leqq \frac{c}{2}$. Let $A(c, E) \equiv \{x \in \mathbb{R}^N_+ : 0 \leqq x \leqq \frac{c}{2}$ and $\sum x_i = E\}$. Since $A(c, E)$ is compact, there exists $x \in A(c, E)$, that maximizes the Lorenz ordering in $A(c, E)$. Suppose by contradiction that $x \neq CE(c, E)$.

Let $y^{max} = \max_j x_j$. Since $x \neq CE(c, E)$, there exists $i \in N$ such that $x_i < y^{max}$ and $x_i < c_i/2$. Let $\delta = \min\{\frac{c_i}{2}, y^{max}\} - x_i$. Let $N' = \{j \in N : x_j = y^{max}\}$.

Define $z \in A(c, E)$ as follows: $z_i = x_i + \frac{\delta}{2}$; for all $j \in N'$, $z_j = x_j - \frac{\delta}{2|N'|}$; and for all $k \notin N' \cup \{i\}$, $z_k = x_k$. It is straightforward to check that $z \in A(c, E)$ and that z Lorenz dominates x, contradicting our choice of x.

(ii) A similar argument can be developed for the case when $E > \sum c_k/2$. □

Remark.

In general, maximizing the Lorenz ordering is not the same thing as minimizing the difference between the largest and the smallest amounts any two claimants receive. To see this, compare the following two allocations: $x = (4, 9, 10)$ and $y = (5, 6, 12)$. Note that $\sum x_i = \sum y_i = 23$. $r(x) = 6$, $r(y) = 7$, but y Lorenz dominates x. However, in the two-person case, the two exercises coincide.