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Larry G. Epstein Jiankang Zhang*

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Abstract

Evidence such as the Ellsberg Paradox shows that decision-makers do not assign probabilities to all events. It is intuitive that they may differ not only in the probabilities assigned to given events but also in the identity of the events to which they assign probabilities. This paper describes a theory of probability that is fully subjective in the sense that both the domain and the values of the probability measure are derived from preference. The key is a formal definition of 'subjectively unambiguous event.'

1. INTRODUCTION

1.1. Objectives

Savage's expected utility theory is typically referred to as providing a *subjective theory of probability*. That is because the probability measure underlies choice behavior. More precisely, it is derived from axioms on the preference ordering of uncertain prospects (acts defined on a state space S) and serves as a component in the representation of that preference. We begin by noting two critiques of the Savage model as a subjective theory of probability. Each claims that the Savage model delivers 'too much' to be completely satisfactory.

The first sense in which Savage delivers too much is that his axioms deliver not only the fact that preference is based on probabilities, but also the expected utility functional form. Because the use of probabilities seems more basic than any particular functional form, this aspect of the Savage theory is unattractive as a theory of probability. This critique is due to Machina and Schmeidler and it has been addressed by these authors in [16]. They

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describe axioms that deliver *probabilistically sophisticated* preferences. Roughly speaking, probabilistic sophistication entails a two-stage procedure for evaluating any act. First, the decision-maker uses a probability measure on the state space in order to translate the act into an induced distribution over outcomes (a lottery); and second, she uses a (not necessarily expected) utility function defined on lotteries to evaluate the induced lottery and produce a utility level for the act. Thus preference is based on probabilities, but in a way that does not impose superfluous functional form restrictions.

The second critique of the Savage theory that applies also to the Machina and Schmeidler extension is the one that motivates this paper. Both theories deliver too much in that they derive probabilities for *all* measurable events. Consequently, there is an important sense in which they fail to be subjective, as we now clarify. Both Savage and Machina-Schmeidler assume that the decision-maker contemplates all events, that is, all subsets of the state space are assumed admissible or measurable. Generalizations are possible whereby the class of measurable events can be taken to be an arbitrary exogenously specified σ-algebra Σ. The subjective nature of these theories is due to the fact that a probability measure p on Σ is derived from the decision-maker's preference ordering over the domain of Σ-measurable acts. However, the domain of the measure, either Σ or the power set, is *exogenous* to the model. In particular, and in stark contrast to the case for p, this domain does *not* depend on preference and it is *not* allowed to vary with the decision-maker, except in a trivial sense. The trivial exception is where the modeller *assumes* that two decision-makers a and b have different σ-algebras Σ_a and Σ_b, in which case the probability measures derived for a and b will have different domains. But the fact that a and b assign probabilities to different events is an assumption - it is not derived from the preferences or choice behavior of the two decision-makers. The σ-algebra that is appropriate for a or b is assumed to be given and derived from other considerations.

Exogeneity of the σ-algebra is not a limitation of a 'subjective theory' if it is believed that decision-makers assign probabilities to all events that are relevant to the context being modelled. In that case, the modeling context may dictate the appropriate specification for Σ, independently of preference. But choice behavior such as that exhibited in the Ellsberg Paradox and related evidence have demonstrated that many decision-makers do not assign probabilities to all events. In situations where some events are 'ambiguous', decision-makers may not assign probabilities to those events, though the likelihoods of 'unambiguous' events are represented in the standard probabilistic way. For example, in the case of the Ellsberg urn with balls of 3 possible colours, R, B and G, where the only objective information is that $R+B+G = 90$ and $R = 30$, events in the class

$$\mathcal{A} = \{\emptyset, \{R\}, \{B,G\}, \{R,B,G\}\} \tag{1.1}$$

are intuitively unambiguous. Most decision-makers would presumably assign them the obvious probabilities in deciding on how to rank bets based on the colour of a ball to be drawn at random. However, the use of probabilities for other events is inconsistent with the common 'ambiguity averse' preference ranking of such bets, namely a preference to bet on R (drawing a red ball) rather than on B and also a preference for betting on $\{B,G\}$ rather than on

$\{R, G\}$.

On the other hand, aversion to ambiguity is not universal. Some decision-makers are indifferent to ambiguity and behave in the non-paradoxical and fully probabilistic fashion. The lesson we take from this is that decision-makers may differ not only in the probabilities assigned to given events (an aspect not well illustrated by this example), but also in the identity of the events to which they assign probabilities. Thus a subjective theory of probability should make *both* the domain *and* the values of the probability measure subjective and based on preference.

The formulation of such a *fully subjective theory* of probability is our ultimate objective. Naturally, following the choice-theoretic tradition of Savage, it is preference rather than probability per se that is of prime importance. Thus the desired derivation of a probability measure is as one component of a preference representation that is to apply on an agent-specific subdomain of acts. We now clarify the nature of our contribution towards achieving a fully subjective theory.

1.2. Contribution

First our terminology must be made more precise. Expressions such as 'the event A is assigned a probability' are meaningless; a probability measure is defined primarily by additivity, a property that refers to a *class* (or collection) of events.[1] A meaningful statement is that '$\mathcal{B}$ is a class of events where p represents the decision-maker's likelihood relation'. Also meaningful, and our focus in this paper, is the stronger statement '$\mathcal{B}$ is a class of events such that preference is probabilistically sophisticated on the domain of all acts that are $\mathcal{B}$-measurable'. In general, there may be several such classes $\mathcal{B}$ and one might expect a fully subjective theory to derive them all from the given preference.

The accomplishment in this paper is more modest - we identify one particular class of events, denoted $\mathcal{A}$, whose elements are called (*subjectively*) *unambiguous* events. Then we show (Theorem 5.2) that the decision-maker is probabilistically sophisticated on the domain of unambiguous ($\mathcal{A}$-measurable) acts, given suitable axioms on preference. This representation result constitutes a contribution towards a fully subjective theory of probability, because both the domain $\mathcal{A}$ of the decision-maker's probability measure and the values assigned by the measure to events in $\mathcal{A}$ are derived from preference.

The definition of unambiguous events is the key to our model and constitutes a separate contribution. Thus some elaboration seems in order. Roughly speaking, a satisfactory formal definition must answer the question "which behavior or preference rankings reveal that the decision-maker views a given event as 'ambiguous'?" The Ellsberg urn with three colours illustrates our approach. The typical choices described earlier may be expressed in the form

$$R \succ_\ell B \quad \text{and} \quad R \cup G \prec_\ell B \cup G,$$

where $\succ_\ell$ may be read 'would rather bet on'. Thus the preference to bet on red rather

[1] Typically, that class is taken to be an algebra or σ-algebra. We adopt the more neutral term 'class' because a different mathematical structure will be relevant here, as explained shortly.

than blue is reversed if both winning events are expanded to include green. We view such a reversal as the behavioral manifestation of the intuitively ambiguous nature of the event G (drawing a green ball) and we use it as the basis for our definition of ambiguity in a general setting. In other words, we define ambiguity by a suitable lack of separability in preference and thus the collection $\mathcal{A}$ consists of those events that satisfy a suitable form of separability. The intuitive appeal of our definition, elaborated upon below, justifies singling out $\mathcal{A}$ amongst all the classes of events $\mathcal{B}$ where probabilistic sophistication prevails.

'Ambiguity' is a common word in informal discourse and is used in many different senses. Our definition attempts to capture one of these - imprecision regarding likelihood as illustrated by the Ellsberg Paradox. Alternatively, the word ambiguous is sometimes used when referring to an event that is in some sense not fully describable. To illustrate, consider the case where one of the states of the world is the catch-all 'none of the above' as in [14]. The decision-maker may not have a clear picture of the (sub-)contingencies underlying the catch-all state and thus it might be called ambiguous. However, if she can assign a precise probability to the catch-all state, because she can do so for the other states, then it would be subjectively unambiguous in our sense. More precisely, even without understanding the internal composition of the catch-all state, she might plausibly exhibit the behavior that we use to define 'unambiguous'. The behavior that corresponds intuitively to the noted incomplete understanding is the subject of the literature on 'missing states', in which [14] is the seminal article.

1.3. A By-Product

Typically, probability theory posits that any probability measure is defined on an algebra or σ-algebra, constructs that seem natural from a mathematical point of view. In a fully subjective theory, the domain $\mathcal{A}$ of the subjective probability measure, including its mathematical properties, are derived. This permits the appropriateness of the standard assumptions to be evaluated from a decision-theoretic point of view. This argument is due to Zhang [26], whose major finding in this regard we proceed to outline.

The major point is that $\mathcal{A}$ is typically not a σ-algebra or even an algebra. Moreover, at an intuitive level, while the class of unambiguous events is naturally taken to be closed with respect to complements and disjoint unions, it may not be closed with respect to intersections. This point may be illustrated by borrowing Zhang's example of an Ellsberg-type urn with 4 possible colours - R, B, G and W. Suppose that the only objective information is that the total number of balls is 100 and that $R + B = G + B = 50$. Then it is intuitive that the class of unambiguous events is

$$\mathcal{A} = \{S, \emptyset, \{B, R\}, \{B, G\}, \{G, W\}, \{R, W\}\}. \tag{1.2}$$

Observe that $\mathcal{A}$ fails to be an algebra, because while $\{B, R\}$ and $\{B, G\}$ are unambiguous, their intersection $\{B\}$ is not. As pointed out by Zhang, the appropriate mathematical structure for $\mathcal{A}$ is a λ-system (defined below), also sometimes called a Dynkin system.

For this paper, the fact that we cannot take $\mathcal{A}$ to be an algebra complicates the derivation of a probability measure on $\mathcal{A}$ and, in particular, prevents us from simply invoking existing results from [20], [9] and [16]. The arguments in these studies exploit the fact that the relevant class of events is closed with respect to intersections. We rely instead on a recent representation result in [27] for qualitative probabilities on λ-systems.

1.4. A More Applied Perspective

To this point, we have emphasized the importance for decision theory of the construction of a fully subjective theory of probability. However, there exists also more applied motivation for this paper.

The importance of the Ellsberg Paradox is due in part to the feeling that the phenomena of ambiguity and ambiguity aversion that it illustrates are likely important in many standard economic contexts and not merely in experimental settings. For example, it is intuitive that ambiguity may be important, perhaps as important as risk, for explaining behavior in asset markets. Clearly, formal investigation of this hypothesis requires a sound formal definition of ambiguity. We hope that our definition will serve this purpose. From this applied perspective, our representation result (Theorem 5.2) can be viewed as a form of confirmation that our definition is natural or appealing.

This seems an appropriate place to comment on the meaning of 'ambiguity aversion'. In the same way that risk aversion is defined by reference to riskless prospects or lotteries, any definition of ambiguity aversion requires the prior identification of unambiguous acts and events. In [6], one of us defined the notion of aversion to ambiguity (or uncertainty) beginning with an exogenously specified class of unambiguous events. It is straightforward to adapt the approach in [6] and to employ subjectively unambiguous events as the reference class used to define ambiguity aversion (see the concluding section for further details). In this way, this paper delivers also a definition of ambiguity aversion.[2]

See [4] for preliminary results regarding the effects of ambiguity aversion thus defined on asset returns in a representative-agent model. Other studies ([7] is one example) also claim to have derived effects of ambiguity aversion on asset returns, but, as argued in [6], such claims are suspect because they are not based on a satisfactory definition of ambiguity aversion.

1.5. Related Literature

Further comparison with the Machina-Schmeidler analysis seems worthwhile. If their axioms are imposed, then all events are subjectively unambiguous. Consequently, our Theorem 5.2 extends their main result by dropping the requirement that all events be unambiguous.[3] As a further consequence, only our model is consistent with Ellsberg-type behavior; events that are deemed ambiguous by the decision-maker are excluded from the domain $\mathcal{A}$ and are not necessarily assigned probabilities, removing the source of the paradox.

[2] Schmeidler [22] proposes a definition of ambiguity aversion for the case where preference is defined over two-stage Anscombe-Aumann acts (see [1]) rather than merely over Savage-style acts as here.

[3] A qualification is that our theorem delivers a countably additive probability measure, while theirs deals with the more general class of finitely additive subjective priors.

Several papers have studied ambiguity. Zhang [26] proposes a definition of 'subjectively unambiguous' that is closely related to the definition proposed here. A comparison is provided in Section 3.4.

The subjective nature of unambiguous events distinguishes our model from [10] and [6], where ambiguity is taken as a primitive. Here, in contrast, preference is the only primitive. The class of ambiguous events employed in [19] (and [13]) is 'subjective' in the sense that the authors require that preference satisfy Savage's Sure-Thing-Principle on the corresponding domain of acts. However, this requirement does not pin down a unique class of events, in contrast to the uniqueness of our class $\mathcal{A}$. Put another way, only our definition is *explicit and constructive.*

Ghirardato and Marinacci [11] define ambiguity in terms of preference in the case where the latter conforms to Choquet expected utility theory. The latter limitation is unattractive for the reason given in the next paragraph. If for the sake of further comparison one restricts attention to the Choquet expected utility framework, then their definition differs from ours. Rather than provide a detailed comparison of the two approaches (see, however, Section 3.6), we prefer to acknowledge that the intuitive appeal of any definition of 'subjectively unambiguous' is undoubtedly subjective, and to emphasize instead the 'practical' case for our definition - it delivers a representation result (Theorem 5.2). The latter is a major contribution because, as we have argued, it constitutes a first step towards a fully subjective theory of probability. (There is no parallel result in [11].) Keeping in mind the ultimate objective of deriving a probability measure on the class $\mathcal{A}$ of subjectively unambiguous events, it seems that a liberal definition of 'unambiguous' is advantageous. Our definition should be viewed from this perspective.

Finally, there exists an alternative response to evidence such as the Ellsberg Paradox that, one might argue, also delivers a fully subjective theory of probability. Schmeidler [22] and Gilboa and Schmeidler [12] respond to evidence that decision-makers' beliefs may not be representable by probabilities by proposing models in which beliefs are represented by more general mathematical constructs - 'non-additive probabilities' or capacities in the former paper and sets of priors in the latter. In the case of non-additive probabilities, one might proceed to identify a subdomain of events where additivity prevails as the class of subjectively unambiguous events. Similarly, in the case of a set of priors, a subdomain of events where all priors in the set agree provides a natural candidate as the class of subjectively unambiguous events. However, in either case, the resulting theory of subjective probability would deliver 'too much' in the sense of the first critique of the Savage model noted at the start of the paper. They would deliver specific models of preference *in addition to* a class of unambiguous events to which the decision-maker assigns probabilities. The use of probabilities seems more basic and is the sole focus of our model. Put another way, just as risk and risk aversion are meaningful notions even outside the expected utility framework (see [25] and [5], for example), it is desirable to have definitions of ambiguity and subsequently of ambiguity aversion that are not tied to any specific model of preference.

1.6. Outline

The paper proceeds as follows. Next we define λ-systems and some notation. Then Section 3 introduces our key definition of an unambiguous event. The remainder of the paper can be viewed as an attempt to justify the definition. In particular, the definition is examined in a number of Ellsberg-type settings and also within the context of some specific models of preference. Some preference axioms are described in Section 4 and our main result (Theorem 5.2) follows in Section 5. Section 6 concludes with suggestions for further research. Most proofs are relegated to appendices.

2. PRELIMINARIES

Let (S, Σ) be a measurable space where S is the set of states and Σ is a σ-algebra. All events in this paper are assumed to lie in Σ; we repeat this explicitly below only on occasion.

Say that a nonempty class of subsets $\mathcal{A} \subset \Sigma$ of S is a λ*–system* if

λ.1 $S \in \mathcal{A}$;

λ.2 $A \in \mathcal{A} \Longrightarrow A^c \in \mathcal{A}$; and

λ.3 $A_n \in \mathcal{A}$, $n = 1, 2, ...$ and $A_i \cap A_j = \emptyset$, $\forall i \neq j \Longrightarrow \cup_n A_n \in \mathcal{A}$.

This definition and terminology appear in [3, p. 36]. A λ–system $\mathcal{A}$ is closed with respect to complements and countable disjoint unions. The intuition for these properties is clear if we think of $\mathcal{A}$ as a class of events to which the decision-maker attaches probabilities. If she can assign a probability to event A, then the complementary probability is naturally assigned to A^c. Similarly, if she can assign probabilities to each of the disjoint events A and B, then the sum of these probabilities is naturally assigned to $A \cup B$. On the other hand, there is no such intuition supporting closure with respect to intersections, or equivalently, with respect to arbitrary unions. Lack of closure with respect to intersections differentiates λ–systems from algebras or σ–algebras. As illustrated by the example of an Ellsberg-type urn with 4 colours, λ–systems are more appropriate for modeling families of 'unambiguous' events.

We have the following equivalent definition [3, p. 43]:[4]

Lemma 2.1. *A nonempty class of subsets* $\mathcal{A} \subset \Sigma$ *of* S *is a* λ*–system if and only if*

$\lambda.1'$ $\emptyset, S \in \mathcal{A}$;

$\lambda.2'$ $A, B \in \mathcal{A}$ and $A \subseteq B \Longrightarrow B \backslash A \in \mathcal{A}$; and

$\lambda.3'$ $A_n \in \mathcal{A}$ and $A_n \subseteq A_{n+1}$, $n = 1, 2, ..., \Longrightarrow \cup_n A_n \in \mathcal{A}$.

[4]Collections of sets satisfying the following conditions are frequently called alternatively Dynkin systems or d-systems [24, p. 193].

Although $\mathcal{A}$ is not an algebra, a probability measure can still be defined on $\mathcal{A}$. Say that $p : \mathcal{A} \longmapsto [0, 1]$ is a (finitely additive) *probability measure* on $\mathcal{A}$ if:

P.1 $p(\emptyset) = 0$, $p(S) = 1$; and

P.2 $p(A \cup B) = p(A) + p(B)$, $\forall A, B \in \mathcal{A}$, $A \cap B = \emptyset$.

Countable additivity of p is defined in the usual way and will be stated explicitly where needed. Given a probability measure p on $\mathcal{A}$, call p *convex-ranged* if for all $A \in \mathcal{A}$ and $0 < r < 1$, there exists $B \subset A$, $B \in \mathcal{A}$, such that $p(B) = r\, p(A)$.[5]

As in Savage, we assume a set of outcomes $\mathcal{X}$. Prospects are modelled via (simple) acts, Σ-measurable maps from S to $\mathcal{X}$ having finite range. The set of acts is $\mathcal{F} = \{..., f, f', g, h, ...\}$. Given a λ-system $\mathcal{A}$, define $\mathcal{F}^{ua}$ by

$$\mathcal{F}^{ua} = \{f \in \mathcal{F} : f \text{ is } \mathcal{A}\text{-measurable}\}, \tag{2.1}$$

where f is $\mathcal{A}$-measurable if $\{s \in S : f(s) \in X\} \in \mathcal{A}$ for any subset X of $\mathcal{X}$. Thinking of $\mathcal{A}$ as the set of unambiguous events, $\mathcal{F}^{ua}$ is naturally termed the set of *unambiguous acts.*

3. UNAMBIGUOUS EVENTS

3.1. Definition

The primitives (S, Σ) and $\mathcal{X}$ are defined as above. The decision-maker has a preference order $\succeq$ on the set of acts $\mathcal{F}$. Unambiguous events are now defined from the perspective of $\succeq$.

Definition 3.1. *An event T is **unambiguous** if: (a) For all disjoint subevents A, B of T^c, acts h and outcomes x^*, x, z, $z' \in \mathcal{X}$,*

$$\begin{array}{c}\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in B \\ h(s) & \text{if } s \in T^c \backslash (A \cup B) \\ z & \text{if } s \in T \end{pmatrix} \succeq \begin{pmatrix} x & \text{if } s \in A \\ x^* & \text{if } s \in B \\ h(s) & \text{if } s \in T^c \backslash (A \cup B) \\ z & \text{if } s \in T \end{pmatrix} \Longrightarrow \\ \begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in B \\ h(s) & \text{if } s \in T^c \backslash (A \cup B) \\ z' & \text{if } s \in T \end{pmatrix} \succeq \begin{pmatrix} x & \text{if } s \in A \\ x^* & \text{if } s \in B \\ h(s) & \text{if } s \in T^c \backslash (A \cup B) \\ z' & \text{if } s \in T \end{pmatrix}; \end{array} \tag{3.1}$$

*and (b) The condition obtained if T is everywhere replaced by T^c in (a) is also satisfied. Otherwise, T is **ambiguous**.*

[5]When $\mathcal{A}$ is a σ-algebra and p is countably additive, this is equivalent to non-atomicity [18, pp. 142-3].

The set of unambiguous events is denoted $\mathcal{A}$. It is nonempty because $\emptyset$ and S are unambiguous. Observe that the defining invariance condition is required to be satisfied even if A or B is empty.

Turn to interpretation, assuming that $x^* \succ x$. The first two acts being compared yield identical outcomes if the true state lies in $(A \cup B)^c$. Thus the comparison is between 'bets conditional on $(A \cup B)$' with stakes x^* and x and the outcomes shown for $(A \cup B)^c$. The indicated ranking reveals that the decision-maker views A as conditionally more likely than B. Suppose now that the outcome on event T is changed from z to z'. If T is 'unambiguous', then this conditional likelihood ranking should not be affected because 'unambiguous' means or at least entails such separability or invariance. Constancy of acts on T is vital for this intuition. It is T in its entirety that is unambiguous and this does not imply anything about subsets.[6] This leads naturally to the restriction to acts that are constant on T. Finally, both T and T^c should satisfy such invariance because intuitively an event is unambiguous if and only if its complement is unambiguous.

For the exclusive purpose of the further interpretation offered in the next two paragraphs, suppose temporarily that Savage's axiom $P4$ is satisfied, that is, for all outcomes $x^* \succ x$ and for all events A and B, if

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in A^c \end{pmatrix} \succeq \begin{pmatrix} x^* & \text{if } s \in B \\ x & \text{if } s \in B^c \end{pmatrix},$$

then the same ranking obtains if the stakes of these bets are changed to y^* and y, for any $y^* \succ y$. This axiom delivers a (complete and transitive) likelihood relation $\succeq_\ell$ on events such that $A \succeq_\ell B$ if the decision-maker would rather bet on A than on B.

Given that $P4$ is satisfied, it is immediate that if T is unambiguous, then for all A and B such that $A \cup B = T^c$,

$$A \succeq_\ell B \iff A \cup T \succeq_\ell B \cup T. \tag{3.2}$$

This is shown by taking $z = x$ and $z' = x^*$ in (3.1). Another intuitive implication of the formal definition is that the likelihood relation $\succeq_\ell$ satisfies the following additivity property when restricted to $\mathcal{A}$: For all A, B and C in $\mathcal{A}$, with $A \cap C = B \cap C = \emptyset$,

$$B \succeq_\ell A \iff B \cup C \succeq_\ell A \cup C. \tag{3.3}$$

(This is proven as part of the proof of Theorem C.3.)

On the other hand, some readers may feel that our definition does not capture all intuitive aspects of 'unambiguous.' For example, if A and B are unambiguous, then it might be expected that (3.3) should be satisfied for all (not necessarily unambiguous) C disjoint from $A \cup B$; this is not the case given our definition. We do not claim to have captured the complete essence of 'unambiguous'. The argument that we offer for our definition is first

[6] This feature of our definition makes it clear that it has nothing to do with the sort of 'ambiguity' associated with incomplete understanding of the states comprising T , as discussed in Section 1.2 in connection with the literature on missing states.

that it captures some intuition and second that it helps to deliver a novel representation theorem whose importance was argued in the introduction. It remains to be seen whether alternative definitions will be similarly useful.

Before turning to examples, observe that null events are unambiguous, where, as in Savage, say that an event $T \subset S$ is *null* if for all acts f, g, $g' \in \mathcal{F}$,

$$\begin{pmatrix} f(s) & \text{if } s \in T^c \\ g(s) & \text{if } s \in T \end{pmatrix} \sim \begin{pmatrix} f(s) & \text{if } s \in T^c \\ g'(s) & \text{if } s \in T \end{pmatrix}.$$

Lemma 3.2. *If T is null, then T is unambiguous.*

Proof. Condition (3.1) is satisfied because changing the outcome on T from z to z' is a matter of indifference by the nullity of T. When T^c is substituted for T in (3.1), each weak preference ranking is necessarily indifference. ■

Finally, while we have emphasized the connection, albeit imperfect, between 'unambiguous' and 'having probability', the example of state-dependent-expected-utility shows that such a connection is not delivered by our definition of unambiguous without further restrictions. Let the utility function U be given by

$$U(f) = \int_S u_s(f)\, dp,$$

for a suitable collection $\{u_s\}$ of state-dependent vNM indices and some probability measure p. It follows immediately from the additive separability across states that all events are unambiguous. However, there is no meaningful sense in which U is based on probabilities. It is well-known that when the vNM index varies with the state, then the probability measure p is not unique (for example, the identical U is delivered by $v_s(\cdot) = a_s u_s(\cdot)$ and $dq = a_s^{-1}\, dp$) and its behavioral significance is unclear. We succeed later in delivering a unique probability measure for unambiguous events only by excluding such state-dependence. The preceding demonstrates that, in general, being unambiguous is more basic than "having probability", a distinction that will be illustrated further in Section 3.7.

3.2. Ellsberg Settings with a Single Urn

First, consider a state space consisting of only two points s_1 and s_2, corresponding to a single Ellsberg urn with balls having two possible colours. If $\emptyset \prec_\ell \{s_i\} \prec_\ell S$, $i = 1, 2$, then both singleton sets are necessarily unambiguous.[7] Support for this designation is provided by the fact that any such likelihood relation may be represented by a probability measure; take any probability measure p such that $p(s_1) < (\geq)\, 1/2$ if $\{s_1\} \prec_\ell (\succeq_\ell)\, \{s_2\}$.

[7]This presumes Savage's monotonicity axiom $P3$: For all non-null events E, acts f and outcomes x and y, $x \succeq y$ if and only if the act (x if E; $f(s)$ if $s \in E^c$) is weakly preferred to (y if E; $f(s)$ if $s \in E^c$).

Turn to the case of an urn with balls of 4 possible colours, as outlined in the introduction.[8] The state space is

$$S = \{B, R, G, W\},$$

where $S = 100$ and $B + R = B + G = 50$. The intuitive λ-system $\mathcal{A}$ of unambiguous events is described in (1.2). There is a natural probability measure p on $\mathcal{A}$ and it is reasonable to expect that a decision-maker would base choice on p when dealing with acts in $\mathcal{F}^{ua}$. In particular, she might conform to expected utility theory there, with utility function

$$U(f) = \int u(f)\,dp, \quad f \in \mathcal{F}^{ua},$$

for some vNM index u.

For the ranking of other acts, the following seems plausible: Though the decision-maker knows the probabilities of events in $\mathcal{A}$, no other information is available. Thus any measure in the set

$$P = \{m : \ m \text{ extends } p \text{ from } \mathcal{A} \text{ to } \Sigma\}, \tag{3.4}$$

is admissible. Suppose that this set of measures is used as described in the multiple-priors model [12]; that is, for some vNM index u, the utility of an act is computed by

$$U(f) = \min_{m \in P} \int_S u(f)\,dm. \tag{3.5}$$

Then the class of events that are subjectively unambiguous in the sense of our formal definition coincides with $\mathcal{A}$: Refer back to (3.4) and (3.5). For any $x^* \succ x$,

$$\begin{pmatrix} x^* & \text{if } s = W \\ x & \text{if } s = R \\ x & \text{if } s = G \\ z & \text{if } s = B \end{pmatrix} \sim \begin{pmatrix} x^* & \text{if } s = R \\ x & \text{if } s = W \\ x & \text{if } s = G \\ z & \text{if } s = B \end{pmatrix} \text{ when } z = x,$$

but indifference becomes '$\succ$' when z is set equal to x^*. This proves that both $\{B\}$ and its complement are ambiguous. Similarly for other singleton sets and their complements. To see that $\{R, G\}$ is ambiguous, observe that

$$\begin{pmatrix} x^* & \text{if } s \in \{B, W\} \\ z & \text{if } s \in \{R, G\} \end{pmatrix} \sim \begin{pmatrix} x & \text{if } s \in \{B, W\} \\ z & \text{if } s \in \{R, G\} \end{pmatrix} \text{ when } z = x,$$

but the first act is strictly preferred when $z = x^*$. This violates the defining property (3.1), taking $A = \{B, W\}$ and $B = \emptyset$. Finally, verify that each event identified in (1.2) as intuitively unambiguous is also unambiguous in the formal sense.

[8] The case of three colours may be discussed similarly. For a suitable specification of preference, $\mathcal{A}$ coincides with the intuitive class (1.1).

3.3. Global Probabilistic Sophistication

If "probabilities are assigned to all events", then all events should be unambiguous. Machina and Schmeidler give precise meaning to the expression in quotation marks. The precise definition of their class of probabilistically sophisticated preferences can be found in [16]. (See also Section 5 below, where we define the probabilistic sophistication of preference over any suitable domain of acts. The special case where that domain is the set of all acts $\mathcal{F}$ is the notion in [16]; it is occasionally distinguished here terminologically by the added adjective 'global'.) For the moment it suffices to note the following central axiom underlying probabilistic sophistication:

Axiom $P4^*$ (Strong Comparative Probability): For all disjoint events A and B, outcomes $x^* \succ x$ and $y^* \succ y$, and acts g and h,

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in B \\ g(s) & \text{if } s \notin A \cup B \end{pmatrix} \succeq \begin{pmatrix} x & \text{if } s \in A \\ x^* & \text{if } s \in B \\ g(s) & \text{if } s \notin A \cup B \end{pmatrix} \implies$$
$$\begin{pmatrix} y^* & \text{if } s \in A \\ y & \text{if } s \in B \\ h(s) & \text{if } s \notin A \cup B \end{pmatrix} \succeq \begin{pmatrix} y & \text{if } s \in A \\ y^* & \text{if } s \in B \\ h(s) & \text{if } s \notin A \cup B \end{pmatrix}. \tag{3.6}$$

It is immediate that if preference satisfies $P4^*$, then all events (in Σ) are unambiguous. The converse is also true if Savage's $P4$ is assumed.[9] We view these results as supportive of our definition. The former is noteworthy also because it ensures that our representation result Theorem 5.2 extends the main result in [16] (modulo the qualification mentioned in the introduction).

3.4. Linearly Unambiguous

The preceding is useful also for understanding why an alternative seemingly natural definition for 'unambiguous' was not adopted, thereby providing further perspective on our chosen definition. We motivated our definition in part by the suggestion that a necessary condition for an event to be unambiguous is that it be 'separable' from events in its complement. The following alternative definition embodies a stronger form of separability and therefore warrants some attention:

[9] We could obtain the unqualified equivalence between $P4^*$ and 'all events are unambiguous' if we strengthened our definition slightly to require invariance in (3.1) also if the outcomes x^* and x are replaced by y^* and y, where $y^* \succ y$. Our results below remain valid with this alternative definition. The distinction between the two alternative definitions of unambiguous seems to us to be a matter of taste.

Definition 3.3. *An event T is **linearly unambiguous** if: (a) For all acts f' and f and all outcomes z and z',*

$$\begin{pmatrix} f'(s) & \text{if } s \in T^c \\ z & \text{if } s \in T \end{pmatrix} \succeq \begin{pmatrix} f(s) & \text{if } s \in T^c \\ z & \text{if } s \in T \end{pmatrix} \implies \tag{3.7}$$

$$\begin{pmatrix} f'(s) & \text{if } s \in T^c \\ z' & \text{if } s \in T \end{pmatrix} \succeq \begin{pmatrix} f(s) & \text{if } s \in T^c \\ z' & \text{if } s \in T \end{pmatrix};$$

*and (b) The condition obtained if T is everywhere replaced by T^c in (a) is also satisfied. Otherwise, say that T is **linearly ambiguous**.*

It is apparent that if T is linearly unambiguous, then it is also unambiguous. The former is a stronger property because the indicated invariance is required for all acts f' and f and not just for the subclass of 'conditional binary acts' as in (3.1). The economic significance of this difference is described below and is similar to the discussion in [16, Section 4.2]. In any event, it is apparent that linear ambiguity embodies a stronger form of separability than does ambiguity. So why not use it as the key notion?

One answer is that (3.7) is too demanding to correspond to the intuitive notion of ambiguity. The invariance required by (3.7) may be violated because the decision-maker views outcomes in different states as complementary or substitutable for reasons that have nothing to do with ambiguity. For example, she may be probabilistically sophisticated, thus assigning probabilities to all events and translating any act into the induced lottery over outcomes, but then she might evaluate the lottery by a risk-preference utility functional that is not linear in probabilities (violating the Independence Axiom). Decision-makers who behave as in the Allais Paradox are of this sort. Many events would be linearly ambiguous for such decision-makers, though it seems intuitively that ambiguity has nothing to do with their preferences. In contrast, for such (probabilistically sophisticated) decision-makers all events are unambiguous, as we have just observed. Roughly speaking, our formal definition of ambiguity relates to behavior in the Ellsberg Paradox but not the Allais Paradox, while linear ambiguity confounds the two.[10]

A second (related) answer is that the existence of probabilistically sophisticated non-expected utility maximizers is at least a logical possibility if not an empirical fact. Such preferences are based on probabilities. A theory of subjective probability is richer if it includes them.

That is not to dispute the potential usefulness of the notion of linear ambiguity. Zhang [26], who originated the notion, shows that it can help to provide a fully subjective expected utility theory, in the sense of the introduction (see also Section 6). The linearity of the expected utility function explains our choice of terminology.

[10]Readers who attach little importance to the Allais Paradox may feel that such a view would justify using (3.7) in place of (3.1). We feel that it argues for imposing a form of the sure-thing principle on the subdomain of unambiguous acts, rather than for changing the meaning of unambiguous.

3.5. 'Unambiguous' and Updating

We observed in the introduction that one would expect the set of unambiguous events to be closed with respect to complements and disjoint unions. Closure with respect to complements is built into the definition of $\mathcal{A}$. For disjoint unions, it is readily shown that if T_1 and T_2 are disjoint and each satisfies (3.1), then so does $T_1 \cup T_2$. This does not prove that the union is in $\mathcal{A}$, because that requires also the appropriate form of (3.1) for $(T_1 \cup T_2)^c$; to prove this we employ a mild axiom as described in Lemma 5.1 below. For the next intuitive argument, we anticipate the fact that $\mathcal{A}$ will shortly be shown to be a λ-system.

Examination of updating is supportive of our definition. Imagine that prior to the eventual realization of the true state of the world, there is an intermediate stage at which the decision-maker learns that event T is true. At that point the relevant prospects are acts over T. Denote the set of such acts by $\mathcal{F}_T$ and by $\succeq_T$ the updated preference order over $\mathcal{F}_T$. We assume that updating takes the form prescribed by Machina [15]. Thus assume that $\succeq_T$ is defined by: For all f', $f \in \mathcal{F}_T$,

$$f' \succeq_T f \iff \begin{pmatrix} f'(s) & \text{if } s \in T \\ h(s) & \text{if } s \in T^c \end{pmatrix} \succeq \begin{pmatrix} f(s) & \text{if } s \in T \\ h(s) & \text{if } s \in T^c \end{pmatrix}. \tag{3.8}$$

Here h is the foregone unrealized alternative, an act over T^c. The updated order depends on h, but this dependence is suppressed in the notation.

We can now ask the following: Suppose that T and $R \subset T$ are unambiguous with respect to the initial order $\succeq$. Then is R necessarily unambiguous from the perspective of the updated order $\succeq_T$? (The latter expression is defined in the obvious way, using T as the new state space.) Intuition suggests a positive answer and that is the case for our definition.

Lemma 3.4. *Suppose that $\mathcal{A}$ is a λ-system (see Lemma 5.1). Let T and $R \subset T$ be in $\mathcal{A}$, that is, unambiguous with respect to the initial order $\succeq$. If preference is updated according to (3.8), then R is unambiguous with respect to $\succeq_T$.*

The proof is elementary. The added hypothesis that $\mathcal{A}$ is a λ-system is used only to show that $T \setminus R = (R \cup T^c)^c$ is also in $\mathcal{A}$.

3.6. Multiple-Priors and Choquet Expected Utility

Here we examine the nature of unambiguous events when preference is restricted to lie in the multiple-priors class [12] or in the Choquet expected utility (CEU) class [22]. In the context of these models, the functional forms for utility suggest definitions for 'unambiguous' that seem 'natural' on purely formal grounds. We compare these to our definition and find that, in general, they differ from our behaviorally based definition.

A multiple-priors preference order $\succeq$ is represented by the utility function

$$U(f) = \min_{m \in P} \int u(f)\, dm,$$

where $u : \mathcal{X} \longrightarrow \mathcal{R}^1$ and where P is a convex (and suitably closed) set of probability measures on (S, Σ). Say that all measures in P *agree* on an event T if $m(T) = m'(T)$ for all m and m' in P. One is tempted to identify unambiguous events with those events where all measures in P agree. This identification is justified in part.

Lemma 3.5. *The event T is unambiguous if all measures in the set P of priors agree on T.*

Proof. This follows from

$$U(f) = \min_{m \in P} \{u(x^*)\, m(A) + u(x)\, m(B) + \int_{T^c \setminus (A \cup B)} u(h) dm\} + q_T u(z),$$

where f is the first act appearing in (3.1) and q_T is the agreed probability of T. ■

Thus agreement on T of all measures in P is sufficient for T to be unambiguous. However, it is not necessary.[11] The reader may find this nonequivalence puzzling and perhaps even troubling. However, the class of events where all measures in P agree is an algebra and thus should not be expected to coincide with $\mathcal{A}$, which is only a λ-system. Second, we claim that the 'intuition' that suggests that unambiguous events are those where all measures agree is based on a necessarily superficial examination of the multiple-priors *functional form.* A serious study of ambiguity must be based on its *behavioral* manifestations. Naturally, our definition of ambiguity is expressed in terms of behavior. The behavioral characterization of the assumption that all measures in P agree on T has not been discussed in the literature, but it can be shown that agreement on T is equivalent to T being linearly unambiguous. We conclude in light of the discussion in Section 3.4 that measures in P may disagree on T for reasons that have nothing to do with ambiguity. There remains the question of characterizing 'subjectively unambiguous', in our sense, for the multiple priors model. Unfortunately, we have not progressed beyond the stated lemma.

Turn to the CEU model. Let preference be represented by U^{ceu}, where[12]

$$U^{ceu}(f) = \int_S u(f)\, d\nu.$$

Here $\nu : \Sigma \longrightarrow [0, 1]$ is a capacity and $u : \mathcal{X} \longrightarrow \mathcal{R}^1$. It is convenient to restrict attention to the special case where $u(\mathcal{X})$ has nonempty interior.

The CEU and multiple-priors models overlap. Say that ν is *convex* if

$$\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B),$$

[11]Take the capacity defined in (3.10) below with ϕ strictly convex. Then ν is convex and thus the Choquet expected utility function is also a member of the multiple-priors class with P equal to the core of ν. Apply (3.11).

[12]ν is a capacity if it maps Σ into $[0,1]$, $\nu(E') \geq \nu(E)$ whenever $E' \supset E$, $\nu(\emptyset) = 0$ and $\nu(S) = 1$. The indicated integral is a Choquet integral and equals $\Sigma_{i=1}^n u_i\, [\nu(\cup_{j=i}^n E_j) - \nu(\cup_{j=i+1}^n E_j)]$ if $E_i = \{s : u(f(s)) = u_i\}$ and $u_1 < ... < u_n$.

for all events A and B. The utility function U^{ceu} is also in the multiple-priors class if and only if ν is convex, in which case the relevant set of priors is $core(\nu)$. The core is defined as in co-operative game theory, that is, as the set of all finitely additive probability measures p on S satisfying

$$p(A) \geq \nu(A), \text{ for all } A \subset S.$$

A property of capacities that is weaker than convexity is exactness (see [21]): ν is *exact* if

$$\nu(A) = \min_{m \in core(\nu)} m(A), \text{ for all } A \subset S. \tag{3.9}$$

Two special cases of CEU preferences merit separate mention because they lie at opposite extremes in terms of the extent of subjective ambiguity that they reflect. The first has

$$\nu = \phi(p), \tag{3.10}$$

for some probability measure p and some strictly increasing $\phi : [0,1] \longrightarrow [0,1]$. The resulting preference order is probabilistically sophisticated (axiom $P4^*$ of Section 3.3 is readily verified) and thus all events in Σ are unambiguous in this case, that is, $\mathcal{A} = \Sigma$. Informal confirmation that this preference order does not reflect any concern with ambiguity is derived from the observation that it corresponds to the rank-dependent-expected-utility model that has been studied in the theory of preference over lotteries or (purely) risky prospects.

At the other extreme, suppose the capacity is given by

$$\nu(A) = \begin{cases} 1 & \text{if } A = S \\ 0 & \text{otherwise,} \end{cases}$$

which is often described as modeling *complete ignorance*. Justification for this name is provided by noting that the implied utility function U satisfies $U(f) = \inf_{s \in S} u(f(s))$; as well, the core of ν is the set of all probability measures on S. It is intuitive that complete ignorance should mean that all events other than $\emptyset$ and S be subjectively ambiguous. That is indeed the case - it is readily verified that $\mathcal{A} = \{\emptyset, S\}$ for this preference order.

For general capacities, $\mathcal{A}$ is intermediate between the above extremes and its characterization for a given ν is of interest. On purely formal (or mechanical) grounds, thinking of the capacity as analogous to a probability measure, albeit non-additive, each of the following classes of events seems plausible as a conjecture for how to characterize unambiguous events:

$$\mathcal{A}_0 = \{T \subset S : \nu(T + A) = \nu T + \nu A \text{ for all } A \subset T^c\},$$

$$\mathcal{A}_1 = \{T \subset S : \nu T + \nu T^c = 1\} \text{ and}$$

$$\mathcal{A}_2 = \cap_{m \in core(\nu)} \{A \subset S : mA = \nu A\}.$$

The first two classes may seem natural because they capture forms of additivity of the capacity. The class $\mathcal{A}_2$ consists of those events on which all measures in the core of ν agree, which also seems to reflect a lack of ambiguity, as discussed in the multiple-priors context.

In general, $\mathcal{A}_0 \subset \mathcal{A}_1$. When ν is exact, these three sets coincide.[13]

[13]This lemma appears in [11].

Lemma 3.6. *If ν is exact, then $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2$.*

However, none of the above classes coincides with the class $\mathcal{A}$ of subjectively unambiguous events. This is illustrated starkly by taking ν to be a distortion of a probability measure as in (3.10), with the distortion ϕ being strictly convex. Then[14]

$$\mathcal{A} = \Sigma \text{ and } \mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \{\emptyset, S\}. \tag{3.11}$$

Once again, we would argue that a sensible comparison of these alternative specifications of the class of unambiguous events must be based on behavioral foundations. For example, Zhang [26] shows that if $\mathcal{A}_0$ is closed with respect to complements, then it coincides with the class of linearly unambiguous events. We have already expressed our views on the relative merits of linear (un)ambiguity. Behavioral foundations for $\mathcal{A}_1$ are provided in [11]. Ghirardato and Marinacci provide a detailed discussion of the differences between their approach and ours. Under exactness, the above lemma shows that their definition coincides with Zhang's, namely, linear (un)ambiguity. Thus anyone convinced by our critique in Section 3.4 should be reluctant to accept the Ghirardato-Marinacci definition.

Some connections exist, however, between the alternative classes of events. The next result is immediate given Lemma 3.6 and the cited result from [26].

Lemma 3.7. *If $\mathcal{A}_0$ is closed with respect to complements, then $\mathcal{A}_0 \subset \mathcal{A}$. If ν is exact, then $\mathcal{A}_1 \subset \mathcal{A}$.*

Later (Corollary 5.4) we show that, under additional assumptions, all of the above classes coincide.

Finally, we provide a characterization of $\mathcal{A}$ (see Appendix A for a proof).[15]

Lemma 3.8. *T is unambiguous if and only if: For all pairwise disjoint events A, B, C and D, each disjoint from T,*

$$\nu(A \cup D) \geq \nu(B \cup D) \iff \nu(A \cup D \cup T) \geq \nu(B \cup D \cup T); \text{ and} \tag{3.12}$$

$$\text{if } (\nu(A \cup D) - \nu(B \cup D))\,(\nu(A \cup D \cup C) - \nu(B \cup D \cup C)) < 0, \text{ then} \tag{3.13}$$

$$\nu(A \cup D) - \nu(B \cup D) = \nu(A \cup D \cup T) - \nu(B \cup D \cup T) \text{ and} \tag{3.14}$$

$$\nu(A \cup D \cup C) - \nu(B \cup D \cup C) = \nu(A \cup D \cup C \cup T) - \nu(B \cup D \cup C \cup T); \tag{3.15}$$

and the above conditions are satisfied also by T^c.

[14]Under these assumptions, ν is a convex capacity, hence exact.

[15]The characterizing conditions on the capacity may appear ugly; they are much more complex than those used to define $\mathcal{A}_0$ and $\mathcal{A}_1$, for example. On the other hand, these conditions are well-founded, with behavioral counterpart given by (3.1), and the latter is highly interpretable and (in our view) intuitively appealing.

If the 'reversal' in (3.13) never occurs, then ν is (almost) a qualitative probability within T^c.[16] In that case, the ordinal condition (3.12) alone corresponds to 'T unambiguous'. However, when ν fails to be a qualitative probability within T^c, then 'T unambiguous' requires that the cardinal conditions in (3.14) and (3.15) obtain.

Observe that the above conditions do not involve u, which therefore has nothing to do with ambiguity.

3.7. A Two-Urn Example

We conclude our explication of the definition of 'subjectively unambiguous' with an example that illustrates a conceptual distinction between 'having probability' and 'being unambiguous'. In our model, unambiguous events have probabilities, but the reverse is false and should not be expected to hold. In addition, the example illustrates further how the condition defining 'subjectively unambiguous' may be checked in concrete settings. The result bolsters intuitive support for our definition.

A slight variation of Ellsberg's two-urn experiment illustrates the noted distinction.[17] Let S_2 be an urn containing 90 balls that are either red, blue or green in unknown proportions and S_1 an urn containing 30 balls of each colour. Typically, S_1 and S_2 are referred to as 'unambiguous' and 'ambiguous' urns respectively. This informal terminology is consistent with the view that decision-makers dislike ambiguity and with the typical preference for betting on drawing a ball having a specific colour from S_1 rather than from S_2. But such a ranking is also consistent with the use of a probability measure (presumably $(1/3, 1/3, 1/3)$) for the purpose of ranking bets *internal* to S_2. If we think of $S_1 \times S_2$ as the state space and $\mathcal{B}$ as the co-ordinate algebra $\{S_1 \times A_2 : A_2 \subset S_2\}$, then probabilistic sophistication on (acts measurable with respect to) $\mathcal{B}$ is commonly assumed, even though events in $\mathcal{B}$ are viewed as being ambiguous. Roughly, the existence of a probability measure that serves as the basis for the ranking of $\mathcal{B}$-measurable acts depends exclusively on the decision-maker's view of events *within* $\mathcal{B}$. On the other hand, whether or not events in $\mathcal{B}$ are subjectively ambiguous depends also on how they are viewed relative to events *outside* $\mathcal{B}$ (such as the comparison between drawing a red ball from S_2 as opposed to drawing it from S_1).

We proceed to examine in greater depth a generalization of this example. Let $S = S_1 \times S_2$, where $S_1 = S_2 = \Omega$ and Ω represents the possible states in each urn. We do not insist that Ω be finite. Let p be a probability measure on Ω. (Implicit is a σ-algebra on Ω such that the product σ-algebra equals Σ.) The decision-maker is told that p describes the distribution of states within the first urn S_1, but she is told less about the second urn S_2. For concreteness, take outcomes $\mathcal{X} \subset \mathcal{R}^1$.

As before $\mathcal{F}$ denotes the set of acts over S. Denote by $\mathcal{F}_i$ the set of acts over the state space S_i. As a first step in defining utility over $\mathcal{F}$, let $U_2 : \mathcal{F}_2 \longrightarrow \mathcal{R}^1$ be defined by

[16] The key defining property of a qualitative probability on T^c is ordinal additivity: For all events E, F and G, subsets of T^c such that G is disjoint from E and F, $\nu E \geq \nu F$ if and only if $\nu(E \cup G) \geq \nu(F \cup G)$.

[17] In Ellsberg's experiment, there were only 2 possible colours in each urn. We prefer to deal with 3 colours, for the reasons given in the footnote following Lemma 3.9.

$$U_2(f) = \int_{S_2} u(f)\, d\phi(p), \quad f \in \mathcal{F}_2, \tag{3.16}$$

where u is a continuous and strictly increasing vNM index, where $\phi : [0,1] \longrightarrow [0,1]$ is a strictly increasing and onto map and integration is in the sense of Choquet (see the preceding section). This defines the probabilistically sophisticated subclass of Choquet expected utility functions.

To define U on $\mathcal{F}$, observe that given any act f over S and $s_1 \in S_1$, the restriction $f(s_1, \cdot)$ can be viewed as an act over S_2, giving meaning to $U_2(f(s_1, \cdot))$. Thus U can be defined as follows: For each $f \in \mathcal{F}$,

$$U(f) = \int_{S_1} U_2(f(s_1, \cdot))\, dp(s_1) = \int_{S_1} \int_{S_2} u(f)\, d\phi(p(s_2)) dp(s_1). \tag{3.17}$$

In other words, uncertainty resolves in two stages. At the second stage and conditional on any s_1, U_2 is used to evaluate $f(s_1, \cdot)$. This evaluation can be viewed as producing the certainty equivalent outcome $u^{-1}(U_2(f(s_1, \cdot)))$, yielding the first stage act $s_1 \longmapsto u^{-1}(U_2(f(s_1, \cdot)))$, which is then evaluated using expected utility theory with vNM index u.[18]

This preference specification has a number of appealing features. First, preference to bet on $A_1 \times A_2$ over $B_1 \times B_2$ is independent of the stakes involved (Savage's P4), implying a complete and transitive likelihood relation $\succeq_\ell$ on all such rectangles. Moreover, $\succeq_\ell$ satisfies the following conditions for all events:

$$A_1 \times A_2 \succeq_\ell B_1 \times A_2 \iff A_1 \times A_2' \succeq_\ell B_1 \times A_2'$$

$$A_1 \times A_2 \succeq_\ell A_1 \times B_2 \iff A_1' \times A_2 \succeq_\ell A_1' \times B_2.$$

These equivalences reflect 'independence' between the two urns.

A second attractive feature of the preference specification is that it can explain the two-urn Ellsberg Paradox; indeed, it is often invoked for that purpose. To do so, suppose that

$$\phi(t) < t, \quad t \in (0,1). \tag{3.18}$$

This specialization implies a strict preference for betting on any event $E \subset \Omega$ when it is 'drawn' from S_1 rather than from S_2; that is, $E \times S_2 \succ_\ell S_1 \times E$.

Turn to subjectively unambiguous events and domains where preference is probabilistically sophisticated. There is at least one set of acts on which preference is probabilistically sophisticated, indeed expected utility. Define the co-ordinate σ-algebra

$$\mathcal{A}_1 = \{A_1 \times S_2 : A_1 \subset S_1\},$$

and identify $\mathcal{F}_1$ with the set of $\mathcal{A}_1$-measurable acts. (Define $\mathcal{A}_2$ similarly.) Then

$$U(f) = \int_{S_1} u(f)\, dp, \quad f \in \mathcal{F}_1.$$

[18]There is a clear parallel with the Anscombe-Aumann domain of two-stage acts that has played a large role in axiomatizations such as [22] and [12].

Acts in $\mathcal{F}_1$ are unambiguous in the sense of our formal definition and so the probabilistic sophistication exhibited on $\mathcal{F}_1$ is an implication of our later representation result, given that the axioms specified there are satisfied. Our definition requires some separability between an unambiguous event T and other events disjoint from T. Here observe that if $T = T_1 \times S_2$ and if f is an act yielding $g(s)$ if $s \notin T$ and $h(s)$ if $s \in T$, then $U(f) = \int_{T_1^c} U_2(g)\, dp + \int_{T_1} U_2(h)\, dp$. Such additive separability provides more than enough to imply that T is unambiguous.

On the other hand, because $U = U_2$ on $\mathcal{F}_2$, conclude that U is probabilistically sophisticated also on $\mathcal{F}_2$, or more precisely, on the set of $\mathcal{A}_2$-measurable acts. But, in general, events in $\mathcal{A}_2$ are ambiguous, as shown by the next result. Thus probabilistic sophistication prevails also on a subdomain of acts that are ambiguous.

Lemma 3.9. *In the context of the two-urn example defined by (3.17), suppose that p has full support and that either (i) Ω is finite and contains at least 4 elements, or (ii) $\Omega = [0,1]$. Suppose finally that*

$$\phi(t) \neq t \quad \text{for all } t \in (0,1). \tag{3.19}$$

Then each event of the form $T = S_1 \times T_2$ is ambiguous, where in case (i), T_2 is any non-empty proper subset of Ω and in case (ii), T_2 is any interval with $0 < p(T_2) < 1$.

The proof is straightforward, though tedious and lengthy (it is available upon request from the authors).[19] To show that a given T is ambiguous, one must provide disjoint events A, B and C, each disjoint from T, an act h and outcomes x^*, x, z and z' such that the invariance in (3.1) is violated when z is replaced by z'. Figure 1 illustrates the kinds of events that work for the T shown there ($B = B_1 \cup B_2$). To see intuitively why the noted invariance is violated, take the act h to be constant and equal to y on C (this suffices for the proof), suppose that B_1 is empty and let $x^* \succ y \succ x$. If $z = x^*$, then the best outcome x^* is attained on an event containing $[0,t_1] \times S_2$, and the latter has objective probability $p([0,t_1])$. This precision may lead to the preference for the conditional bet on A rather than on B (equal here to B_2) and hence to the first ranking shown in (3.1). However, if z is replaced by $z' = y$, the above perspective is changed and a reversal in ranking may occur.

The condition (3.19) is intuitive.[20] One possibility is (3.18), where the graph of ϕ lies everywhere below the 45° line, indicating a preference for betting on S_1, as explained earlier. It is noteworthy that a reversal of the strict inequality in (3.18) is also consistent with the condition in the lemma. In other words, our conclusion about the identity of ambiguous events is unchanged if there is a strict preference for betting on S_2, the intuitively ambiguous

[19] If Ω consists of two elements, as in Ellsberg's original two-urn example, then all events are unambiguous. There is insufficient scope for the separability required by our definition to have any bite. This is somewhat similar to the fact that in consumer demand theory, if there are only two goods (and if preference is suitably monotone), then each good is weakly separable. If Ω consists of three elements, the conclusion of the lemma is valid under slightly strengthened assumptions; for example, if the three elements have equal probability under p.

[20] In fact, the lemma is valid under the weaker assumption that $\phi(p(A)) + \phi(p(A^c)) \neq 1$ for all measurable A. We focus on (3.19) because it may be interpreted more simply.

urn. In that sense, ambiguity is logically distinct from the decision-maker's attitude towards ambiguity.

We offer one final observation regarding this example. In the above specification, preference conforms with expected utility on $\mathcal{F}_1$, which consists of unambiguous acts. But this is coincidence. For example, suppose that we reverse the 'order' of the component state spaces and defined U by

$$U(f) = \int_{S_2} \int_{S_1} u(f)\, dp(s_1) d\phi(p(s_2)).$$

Then U is an expected utility function on $\mathcal{F}_1$, but events in S_1 are in general ambiguous. In fact, for this preference order, neither urn is unambiguous. The change in designation of S_1 as a result of the change in preference reflects the subjective nature of ambiguity.

4. AXIOMS

Here we specify some axioms for the preference order $\succeq$. They will deliver not only the λ-system properties for $\mathcal{A}$ and a probability measure on these unambiguous events, but also the probabilistic sophistication of preference restricted to unambiguous acts. This representation result is the ultimate justification for our definition of 'unambiguous'. In particular, it confirms our rough intuition that unambiguous events are assigned probabilities, although probabilities may exist also on other collections of events.

The set $\mathcal{F}^{ua}$ of unambiguous acts is defined by (2.1). Also useful, for any given $A \in \mathcal{A}$, is the set of acts

$$\mathcal{F}_A^{ua} = \{f \in \mathcal{F} : f^{-1}(X) \cap A \in \mathcal{A} \text{ for all } X \subset \mathcal{X}\}.$$

Denote by $x \in \mathcal{X}$ both the outcome and the constant act producing the outcome x in every state. Preference statements like '$x \succeq y$' are therefore well-defined and have the obvious meaning.

Some of the axioms for $\succeq$ are slight variations of Savage's axioms, with names adapted from Machina and Schmeidler. Though the axioms are expressed in terms of $\mathcal{A}$, they constitute assumptions about $\succeq$ because $\mathcal{A}$ is derived from $\succeq$. A final remark is that the axioms relate primarily to $\succeq$ restricted to $\mathcal{F}^{ua}$.

Axiom 1. *(Monotonicity):* *For all outcomes x and y, non-null events $A \in \mathcal{A}$ and acts $g \in \mathcal{F}_{A^c}^{ua}$*

$$\begin{pmatrix} x & \text{if } s \in A \\ g(s) & \text{if } s \in A^c \end{pmatrix} \succeq \begin{pmatrix} y & \text{if } s \in A \\ g(s) & \text{if } s \in A^c \end{pmatrix} \Longleftrightarrow x \succeq y.$$

Axiom 2. *(Nondegeneracy):* *There exist outcomes x^* and x such that $x^* \succ x$.*

Axiom 3. *(**Weak Comparative Probability**): For all events A, $B \in \mathcal{A}$ and outcomes $x^* \succ x$ and $y^* \succ y$*

$$\begin{pmatrix} x^* & \textit{if } s \in A \\ x & \textit{if } s \in A^c \end{pmatrix} \succeq \begin{pmatrix} x^* & \textit{if } s \in B \\ x & \textit{if } s \in B^c \end{pmatrix} \iff$$

$$\begin{pmatrix} y^* & \textit{if } s \in A \\ y & \textit{if } s \in A^c \end{pmatrix} \succeq \begin{pmatrix} y^* & \textit{if } s \in B \\ y & \textit{if } s \in B^c \end{pmatrix}.$$

This is Savage's axiom $P4$ restricted to unambiguous events. As indicated earlier (Section 3.1), this axiom delivers the likelihood relation $\succeq_\ell$ on $\mathcal{A}$, where $A \succeq_\ell B$ if $\exists x^* \succ x$ such that

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in A^c \end{pmatrix} \succeq \begin{pmatrix} x^* & \text{if } s \in B \\ x & \text{if } s \in B^c \end{pmatrix}.$$

Another notable consequence of Weak Comparative Probability, that is specific to our setting, is that it immediately implies that the Machina-Schmeidler axiom $P4^*$ is satisfied on the domain $\mathcal{F}^{ua}$ of unambiguous acts, that is, the implication (3.6) is valid for all events A and B in $\mathcal{A}$ and all acts g and h in $\mathcal{F}^{ua}_{(A \cup B)^c}$. It might seem at first glance that this renders the remaining route to a suitable representation result routine, because given the key axiom $P4^*$, the remaining and relatively uncontentious Machina-Schmeidler axioms could be assumed and their result invoked. However, their arguments (suitably translated) rely on $\mathcal{A}$ being a σ-algebra, which is not generally the case when $\mathcal{A}$ is the set of unambiguous events.

The next axiom imposes suitable richness of the set of unambiguous events. It is clear from Savage's analysis that some richness is required to derive a probability measure on $\mathcal{A}$. Further, Savage's axiom $P6$ (suitably translated) is not adequate here because $\mathcal{A}$ is not a σ-algebra. However, the spirit of Savage's $P6$ is retained in the next axiom.

Axiom 4. ***(Small Unambiguous Event Continuity):*** *Let f, $g \in \mathcal{F}^{ua}$, $f \succ g$, with $f = (x_1, A_1; x_2, A_2; ...; x_n, A_n)$, $g = (y_1, B_1; y_2, B_2; ...; y_m, B_m)$, where each A_i and B_i lies in $\mathcal{A}$. Then for any x in $\mathcal{X}$, there exist two partitions $\{C_i\}_{i=1}^N$ and $\{D_j\}_{j=1}^M$ of S in $\mathcal{A}$ that refine $\{A_i\}_{i=1}^n$ and $\{B_j\}_{j=1}^m$ respectively, and satisfy:*

$$f \succ \begin{pmatrix} x & \textit{if } s \in D_k \\ g(s) & \textit{if } s \in D_k^c \end{pmatrix}, \textit{ for all } k \in \{1, ..., N\}; \tag{4.1}$$

and

$$\begin{pmatrix} x & \textit{if } s \in C_j \\ f(s) & \textit{if } s \in C_j^c \end{pmatrix} \succ g, \textit{ for all } j \in \{1, ..., M\}. \tag{4.2}$$

Very roughly, the axiom requires that unambiguous events can be decomposed into suitably 'small' unambiguous events. When $\mathcal{A}$ is closed with respect to intersections, as in the standard model [20] or [16] where it is taken to be the power set, then the axiom is implied by Savage's $P6$, given Axioms 1-3.[21]

The preceding axioms are largely familiar, at least when imposed on all of $\mathcal{F}$, rather than just on $\mathcal{F}^{ua}$ as here. The remaining two axioms are 'new' and are needed to accommodate the fact that $\mathcal{A}$ may not be a σ-algebra.

Say that a sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{F}^{ua}$ *converges in preference* to $f_\infty \in \mathcal{F}^{ua}$ if: For any two acts f_*, f^* in $\mathcal{F}^{ua}$ satisfying $f_* \prec f_\infty \prec f^*$, there exists an integer N such that

$$f_* \prec f_n \prec f^*, \text{ whenever } n \geq N.$$

Axiom 5. ***(Monotone Continuity):*** *For any $A \in \mathcal{A}$, outcomes $x^* \succ x$, act $h \in \mathcal{F}^{ua}_{A^c}$ and decreasing sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ with $A_1 \subseteq A$, define*

$$f_n = \begin{pmatrix} x^* & \text{if } s \in A_n \\ x & \text{if } s \in A \backslash A_n \\ h(s) & \text{if } s \in A^c \end{pmatrix} \text{ and } f_\infty = \begin{pmatrix} x^* & \text{if } s \in \cap_{n=1}^{\infty} A_n \\ x & \text{if } s \in A \backslash (\cap_{n=1}^{\infty} A_n) \\ h(s) & \text{if } s \in A^c \end{pmatrix}.$$

If $f_n \in \mathcal{F}^{ua}$ for all $n = 1, 2, ...$, then $\{f_n\}_{n=1}^{\infty}$ converges in preference to f_∞ and $f_\infty \in \mathcal{F}^{ua}$.

The name Monotone Continuity describes one aspect of the axiom, that requiring the indicated convergence in preference.[22] The second component of the axiom is the requirement that the limit f_∞ lie in $\mathcal{F}^{ua}$ whenever each f_n is unambiguous. This will serve in particular to ensure that $\mathcal{A}$ satisfies the 'countable' closure condition $\lambda.3$ or $\lambda.3'$ required by the definition of a λ-system.

It might be felt that given the correct definition of 'unambiguous', the derivation of a probability measure on $\mathcal{A}$ should be possible with little more than some richness requirements. The axioms stated thus far can arguably be interpreted as constituting such minimal requirements. However, they do not suffice and we need one final axiom. This may reflect the fact that only some aspects of 'unambiguous' are captured in our definition. In any event, the final axiom is intuitive and arguably weak. Its statement requires some preliminaries.

A finite partition with component events from $\mathcal{A}$ is denoted $\{A_i\}$. Henceforth all partitions have unambiguous components, even where not stated explicitly. Given such a partition, use the obvious abbreviated notation (x_i, A_i). For any permutation σ of $\{1, ..., n\}$, $\left(x_{\sigma(i)}, A_i\right)$ denotes the act obtained by permuting outcomes between the events. Say that the finite partition $\{A_i\}$ is a *uniform partition* if $A_i \sim_\ell A_j$ for all i and j and call $\{A_i\}$ *strongly uniform* if in addition it satisfies: For all outcomes $\{x_i\}$ and for all permutations σ,

$$\left(x_{\sigma(i)}, A_i\right) \sim (x_i, A_i). \tag{4.3}$$

[21] Savage's $P6$ applied to $\mathcal{F}^{ua}$ would require that, given x, if $f = (x_i, A_i)_{i=1}^n \succ g = (y_i, B_i)_{i=1}^n$, where every A_i and B_i lies in $\mathcal{A}$, then there exists a partition $\{G_i\}_{i=1}^N$ of S in $\mathcal{A}$ such that $f \succ (x, G_k;\ g, G_k^c)$ for all k. Given such a partition, and given that $\mathcal{A}$ is closed with respect to intersections, then the collection of events $D_{ij} = G_i \cap B_j$ satisfes the requirements in Axiom 4.

[22] A related axiom with the same name is used by Arrow [2, p. 48] to deliver the countable additivity of the subjective probability measure. Here as well, countable additivity will follow from Monotone Continuity, but as an unintentional by-product.

In particular, if $\{A_i\}_{i=1}^n$ is a strongly uniform partition, then for all index sets I and J, subsets of $\{1, 2, ..., n\}$,

$$\cup_{i \in I} A_i \sim_\ell \cup_{i \in J} A_i \text{ if } \mid I \mid = \mid J \mid .$$

Axiom 6. ***(Strong-Partition Neutrality):*** *For any two strongly uniform partitions $\{A_i\}_1^n$ and $\{B_i\}_1^n$, if $A_i \sim_\ell B_i$ for all i, then for all $\{x_i\}$,*

$$\begin{pmatrix} x_1 & \text{if } s \in A_1 \\ x_2 & \text{if } s \in A_2 \\ \cdots & \cdots \\ x_n & \text{if } s \in A_n \end{pmatrix} \sim \begin{pmatrix} x_1 & \text{if } s \in B_1 \\ x_2 & \text{if } s \in B_2 \\ \cdots & \cdots \\ x_n & \text{if } s \in B_n \end{pmatrix}. \tag{4.4}$$

The hypothesis that the A_i's and B_i's satisfy (4.3) expresses another sense in which these events are unambiguous. This makes the conclusion (4.4) natural and much weaker than if the indifference in (4.4) were required for all uniform partitions. The latter axiom would go a long way towards explicitly imposing probabilistic sophistication, an unattractive feature in the present exercise where the intention is that probabilistic sophistication on unambiguous acts should result primarily from the definition of 'unambiguous'. Axiom 6 is less vulnerable to such a criticism.

To support the claim that Strong-Partition Neutrality is a 'weak' axiom, observe that it is satisfied by *all* CEU orders, proving that it falls far short of imposing probabilistic sophistication. The reason is that if $\{A_i\}$ is a strongly uniform partition, then ν must be additive on the algebra generated by the partition.[23] Thus the indifference (4.4) is implied.

We turn next to the implications of these axioms.

5. PROBABILISTIC SOPHISTICATION ON UNAMBIGUOUS ACTS

Define "probabilistic sophistication on unambiguous acts $\mathcal{F}^{ua}$" by extending the definition of Machina and Schmeidler. For the convenience of the reader, the complete definition is stated here.

Some preliminary notions are required. Denote by $D(\mathcal{X})$ the set of probability distributions on $\mathcal{X}$ having finite support. A probability distribution $P = (x_1, p_1; ...; x_m, p_m)$ is said to *first-order stochastically dominate* $Q = (y_1, q_1; ...; y_n, q_n)$ with respect to the order $\succeq$ over the outcome set $\mathcal{X}$ if

$$\sum_{\{i: x_i \preceq x\}} p_i \leq \sum_{\{j: y_j \preceq x\}} q_j \qquad \text{for all } x \in \mathcal{X}.$$

Use the term *strict dominance* if the above holds with strict inequality for some $x \in \mathcal{X}$.

Given a real-valued function W defined on a mixture subspace $dom(W)$ of $D(\mathcal{X})$, say that W is *mixture continuous* if for any distributions P, Q and R in $dom(W)$, the sets

$$\{\lambda \in [0,\ 1] : W(\lambda P + (1-\lambda)Q) \geq W(R)\} \text{ and}$$
$$\{\lambda \in [0,\ 1] : W(\lambda P + (1-\lambda)Q) \leq W(R)\}$$

[23] Recall that we have defined a CEU order so that $u(\mathcal{X})$ has nonempty interior, where u is the vNM index.

are closed. Say that W is *monotonic* (with respect to stochastic dominance) if $W(P)(>) \geq W(Q)$ whenever P (strictly) stochastically dominates Q, P and Q in $dom(W)$.

Given a probability measure p on $\mathcal{A}$, denote by $P_{f,p} \in D(\mathcal{X})$ the distribution over outcomes induced by the act f. Define

$$D_p^{ua}(\mathcal{X}) = \{P_{f,p} : f \in \mathcal{F}^{ua}\}.$$

When p is convex-ranged, then $D_p^{ua}(\mathcal{X})$ is a mixture space.

We can finally state the desired definition. Say that $\succeq$ is *probabilistically sophisticated on* $\mathcal{F}^{ua}$ if there exists a convex-ranged probability measure p on $\mathcal{A}$ and a real-valued, mixture continuous and monotonic function W on $D_p^{ua}(\mathcal{X})$ such that $\succeq$ has utility function U of the form

$$U(f) = W(P_{f,p}). \tag{5.1}$$

Roughly speaking, the probability measure p is used to translate acts in $\mathcal{F}^{ua}$ into (purely risky) lotteries and these are evaluated by means of the risk preference functional W. No stand is taken on the functional form of W, apart from monotonicity and mixture continuity, thus capturing exclusively the decision-maker's reliance on probabilities for the evaluation of unambiguous acts. Subjective expected utility is merely one example, albeit an important one, in which W is an expected utility function on lotteries $D_p^{ua}(\mathcal{X})$ and thus U has the familiar form

$$U(f) = \int_S u(f)\, dp. \tag{5.2}$$

We remind the reader that because $\mathcal{A}$ and $\mathcal{F}^{ua}$ are derived from the given primitive preference relation $\succeq$ on $\mathcal{F}$, probabilistic sophistication so-defined is a property of $\succeq$ exclusively and does not rely on an exogenous specification of 'unambiguous acts.'[24]

Probabilistic sophistication with measure p implies that likelihood (or the ranking of unambiguous bets) is represented by p; that is,

$$A \succeq_\ell B \iff pA \geq pB, \quad \text{for all } A, B \in \mathcal{A}.$$

But (5.1) is much stronger, requiring that the ranking of all (not necessarily binary) unambiguous acts be based on p.

Turn to the implications of our axioms. A preliminary result (proven in Appendix B) is that they imply that $\mathcal{A}$ is a λ-system.

Lemma 5.1. *Under Axioms 2, 4 and 5, $\mathcal{A}$ is a λ-system. In particular, if T_1 and T_2 are disjoint unambiguous events, then $T_1 \cup T_2$ is unambiguous.*

The following is our main result:

Theorem 5.2. *Let $\succeq$ be a preference order on $\mathcal{F}$ and $\mathcal{A}$ the corresponding set of unambiguous events. Then the following two statements are equivalent:*

[24]If $\mathcal{A}$ and $\mathcal{F}^{ua}$ are replaced by Σ and $\mathcal{F}$ respectively, then one obtains the Machina-Schemidler definition of 'global' (i.e., on $\mathcal{F}$) probabilistic sophistication, with the minor difference that they take $\Sigma = 2^S$.

(a) $\succeq$ satisfies axioms 1-6.

(b) $\mathcal{A}$ is a λ-system and there exists a (unique) convex-ranged and countably additive probability measure p on $\mathcal{A}$ such that $\succeq$ is probabilistically sophisticated on $\mathcal{F}^{ua}$ with underlying measure p.

The important feature of the theorem is that both $\mathcal{A}$ and p are derived from preference. This feature distinguishes the theorem from the contributions of Savage and Machina-Schmeidler (who impose added axioms that imply $\mathcal{A} = \Sigma$), and render it a contribution towards a fully subjective theory of probability as discussed in the introduction.

We would like to clarify the sense in which we have established probabilistic sophistication on an endogenous domain. One way to accomplish this is as follows: Identify from preference some class of events $\mathcal{A}'$ and the corresponding set of measurable acts $\mathcal{F}'$. Then impose the Machina and Schmeidler axioms for preference $\succeq$ restricted to $\mathcal{F}'$, making the changes necessary to allow for the fact that $\mathcal{A}'$ may not be an algebra. Such an exercise would also generalize [16] and may be of some value. But this is decidedly *not* the nature of our contribution. For example, our representation result does *not* explicitly impose the key axiom $P4^*$ used in [16]. Admittedly, $P4^*$ on $\mathcal{F}^{ua}$ is implied by the definition of $\mathcal{F}^{ua}$ and by Weak Comparative Probability, but we view this as confirmation of the appropriateness of our definition of 'unambiguous.'

A noteworthy feature of our theorem is that is silent on the nature of preference on the domain of ambiguous acts. While at first glance this may seem like a weakness of our result, we feel to the contrary that it is a strength.[25] As argued in the introduction, a theory of probability should not deliver structure that is not germane to the use of probabilities. In this sense, restrictions on the ranking of ambiguous acts constitute excess baggage and it is a virtue of our representation theorem to have avoided them.

The bulk of the proof of Theorem 5.2 is found in Appendices C and D. The arguments in [20], [9] and [16] must be modified because only in the present setting is the relevant class of events $\mathcal{A}$ not closed with respect to intersections. A key step is to show (Appendix C) that our axioms for preference deliver the conditions for the implied likelihood relation that are used in [27] in order to obtain a representing probability measure. The proof of probabilistic sophistication is completed in Appendix D.

We conclude this section with two corollaries (proofs are provided in Appendix E). The first provides some reassurance that our definition of 'unambiguous' is appropriate. The second corollary elaborates on the performance of our theory within the framework of Schmeidler's Choquet expected utility model of preference.

Given the probability measure p on $\mathcal{A}$, one might approximate the likelihoods of ambiguous events by means of the inner and outer measures p_* and p^* defined as follows:[26] For each E in Σ,

$$p_*(E) = \sup\{p(A) : A \in \mathcal{A}, A \subset E\} \text{ and}$$

[25] We are indebted to Michelle Cohen and Mark Machina for helping to clarify our thinking on this point.
[26] See [23] and [8]. For an application to decision theory, see [26].

$$p^*(E) = \inf\{p(A) : A \in \mathcal{A},\, E \subset A\}.$$

These non-additive measures provide intuitive lower and upper bounds for the likelihood assessment of E; in particular, $p_*(\cdot) \leq p^*(\cdot)$.

Define

$$\overline{\mathcal{A}} = \{E : p_*(E) = p^*(E)\}.$$

Events in $\overline{\mathcal{A}}$ seem intuitively to be 'unambiguous'. The definition yields $\overline{\mathcal{A}} \supset \mathcal{A}$. Given our axioms, we can prove equality.

Corollary 5.3. *Let $\succeq$ be as in Theorem 5.2. Then $\overline{\mathcal{A}} = \mathcal{A}$.*

Next we apply our theorem to the CEU model. Recall the notation in Section 3.6. Say that a capacity ν is *convex-ranged on $\mathcal{A}$* if for every A in $\mathcal{A}$,

$$[0, \nu A] = \{\nu B : B \in \mathcal{A},\, B \subset A\}.$$

Say that ν is *continuous* if for all events in S,

$$\nu A_n \searrow \nu(\cap A_n) \text{ if } A_n \searrow \quad \text{and}$$

$$\nu A_n \nearrow \nu(\cup A_n) \text{ if } A_n \nearrow .$$

Corollary 5.4. *Let $\succeq$ be a CEU preference order with capacity ν and subjectively unambiguous events $\mathcal{A}$.*
(a) Suppose that ν is continuous and convex-ranged on $\mathcal{A}$. Then there exists a convex-ranged and countably additive probability measure p on $\mathcal{A}$ and a strictly increasing and onto map $\phi : [0,1] \longrightarrow [0,1]$ such that $\nu = \phi(p)$ on $\mathcal{A} = \overline{\mathcal{A}}$.
(b) Suppose in addition that $\nu(\mathcal{A}_0) = [0,1]$ and that $\mathcal{A}_0$ is closed with respect to complements. Then $\nu = p$ on $\mathcal{A}$.
(c) Suppose that ν is continuous and convex-ranged on $\mathcal{A}$, that ν is exact and that there exists an event A in $\mathcal{A}$ such that

$$\nu(A) + \nu(A^c) = 1 \text{ and } 0 < \nu(A) < 1. \tag{5.3}$$

Then $\nu = p$ on $\mathcal{A}$ and

$$\mathcal{A} = \mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2.$$

Under the conditions in part (a), the CEU order satisfies the axioms in Theorem 5.2, showing that the latter's scope extends beyond globally probabilistically sophisticated preferences. Part (b) gives conditions under which the CEU preference order is expected utility (and not merely probabilistically sophisticated) on the domain of unambiguous acts. Part (c) goes further and gives conditions under which $\mathcal{A}$ coincides with the families indicated and discussed earlier. Differences such as those described in (3.11) are eliminated, in part, through the assumption that $\nu(\mathcal{A}_1) \cap (0,1)$ is nonempty.[27]

[27] We owe this strong form of the result in (c) to Massimo Marinacci. Earlier versions of the paper assumed $\nu(\mathcal{A}_1) = [0,1]$ rather than merely (5.3).

6. FURTHER RESEARCH

Characterization of families of subjectively unambiguous events: Theorem 5.2 leaves open the question "is *any* suitably rich λ-system the family of subjectively unambiguous events for *some* preference order satisfying our axioms?" If not, what added properties are implied for such families?

We offer the following conjecture:

Conjecture 6.1. *Let $\mathcal{A}_0$ be any λ-system of events and let p be a convex-ranged and countably additive probability measure on $\mathcal{A}_0$. Then there exists a preference relation $\succeq$ on $\mathcal{F}$ such that: (i) Its class of subjectively unambiguous events, as defined in this paper, coincides with $\mathcal{A}_0$; and (ii) when restricted to the class of $\mathcal{A}_0$-measurable acts, $\succeq$ is probabilistically sophisticated with underlying probability measure p.*

Multiple domains of probabilistic sophistication: Ultimately, we would like to derive from preference *all* subdomains of acts where probabilistic sophistication prevails.

Degrees of ambiguity: We have classified events as being either unambiguous or ambiguous, but it seems natural to go further and to define a partial order 'more ambiguous' on events. It remains to be seen how this might be done.

Ambiguity aversion: As outlined in Section 1.4, we can adapt the approach in [6] and use subjectively unambiguous events as the reference class used to define ambiguity aversion. That leads to the following definition: Say that the preference order $\succeq$ on $\mathcal{F}$ is *ambiguity averse* if there exists another order $\succeq^{ps}$ on $\mathcal{F}$, that is probabilistically sophisticated there, such that

$$h \succeq^{ps} (\succ^{ps})\; f \quad \Longrightarrow \; h \succeq (\succ)\, f\,,$$

for all h in $\mathcal{F}^{ua}$ and f in $\mathcal{F}$. Here $\mathcal{F}^{ua}$ denotes the set of subjectively (for $\succeq$) unambiguous acts as defined in this paper. The interpretation begins with the view that any probabilistically sophisticated order $\succeq^{ps}$ is indifferent to ambiguity. Accordingly, if $\succeq^{ps}$ prefers the unambiguous act h to f, then so should the ambiguity averse $\succeq$, because $\succeq$ will discount f further due to its being ambiguous.

Further study of ambiguity aversion and its behavioral consequences are beyond the scope of this paper. We merely note that any CEU preference order satisfying the conditions of Corollary 5.4(c) is ambiguity averse according to the above definition. We also refer the reader to [4] for some implications in asset markets.

Axiomatization of expected utility on subdomain of linearly unambiguous acts: Because of the importance of the expected utility special case of probabilistic sophistication, one might be interested in a variation of Theorem 5.2 in which preference over unambiguous acts has the expected utility form. Such a representation result can be achieved as follows: Denote by $\mathcal{A}^* \subset \mathcal{A}$ the class of linearly unambiguous events and by $\mathcal{F}^{lua} \subset \mathcal{F}^{ua}$ the corresponding set of acts. Assume axioms 1 - 6, suitably reformulated by substituting $\mathcal{A}^*$ for $\mathcal{A}$ in the existing statements of these axioms. Then the proof of Theorem 5.2 may be routinely adapted to derive an expected utility representation on $\mathcal{F}^{lua}$. This is roughly the route followed in [26].[28]

A. APPENDIX: Choquet Expected Utility

Use '+' to denote disjoint union.

Proof of Lemma 3.8: Sufficiency may be proven by routine verification. We prove necessity.

Let T be unambiguous. If $x^* \succ x$, then $\nu(A+D) \geq \nu(B+D)$ iff

$$\begin{pmatrix} x^* & A+D \\ x & S\backslash(A+D) \end{pmatrix} \succeq \begin{pmatrix} x^* & B+D \\ x & S\backslash(B+D) \end{pmatrix} \iff$$

$$\begin{pmatrix} x^* & A+D \\ x & B \\ x & T^c\backslash(A+B+D) \\ x^* & T \end{pmatrix} \succeq \begin{pmatrix} x^* & B+D \\ x & A \\ x & T^c\backslash(A+B+D) \\ x^* & T \end{pmatrix} \iff$$

$$\begin{pmatrix} x^* & A+D+T \\ x & S\backslash(A+D+T) \end{pmatrix} \succeq \begin{pmatrix} x^* & B+D+T \\ x & S\backslash(B+D+T) \end{pmatrix} \iff$$

$$\nu(A+D+T) \geq \nu(B+D+T).$$

Suppose next that (3.13) is satisfied. In fact, suppose that

$$\nu(A+D) < \nu(B+D) \text{ and } \nu(A+D+C) > \nu(B+D+C). \tag{A.1}$$

(The other case is similar.)

The event T is unambiguous only if

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in B \\ y & \text{if } s \in C \\ h(s) & \text{if } s \in T^c\backslash(A+B+C) \\ z & \text{if } s \in T \end{pmatrix} \succeq \begin{pmatrix} x & \text{if } s \in A \\ x^* & \text{if } s \in B \\ y & \text{if } s \in C \\ h(s) & \text{if } s \in T^c\backslash(A+B+C) \\ z & \text{if } s \in T \end{pmatrix} \iff \tag{A.2}$$

a similar ranking obtains for the acts where z' replaces z. Suppose that h equals $\overline{y}$ on D and $\underline{y}$ on $T^c\backslash(A+B+C+D)$ and that

$$\overline{y} \succ x^* \succ y \succ x \succ \underline{y},\ z = y \text{ and } z' = x^*. \tag{A.3}$$

[28] In order to clarify the relation between this paper and [26], we point out that the latter does not contain a separate proof of the expected utility representation result. Rather it cites the arguments in this paper, suitably adapted, for a proof.

Then by the definition of Choquet integration, the above equivalence becomes

$$\begin{array}{l} u(x^*)\,\nu(A+D) \;+\; u(y)\,[\nu(C+T+A+D)-\nu(A+D)] \;+u(x)\,[1-\nu(C+T+A+D)] \quad \geq \\ u(x^*)\,\nu(B+D) \;+\; u(y)\,[\nu(C+T+B+D)-\nu(B+D)] \;+\; u(x)\,[1-\nu(C+T+B+D)] \end{array}$$

if and only if

$$\begin{array}{l} u(x^*)\,\nu(T+A+D) \;+\; u(y)\,[\nu(C+T+A+D)-\nu(T+A+D)]+ \\ u(x)\,[1-\nu(C+T+A+D)] \;\geq \\ u(x^*)\,\nu(T+B+D) \;+\; u(y)\,[\nu(C+T+B+D)-\nu(T+B+D)]+ \\ u(x)\,[1-\nu(C+T+B+D)]. \end{array}$$

Equivalently,

$$\begin{array}{l} [u(x^*)-u(y)]\;(\nu(A+D)-\nu(B+D))+ \\ [u(y)-u(x)]\,(\nu(C+T+A+D) \;-\; \nu(C+T+B+D)) \quad \geq 0 \end{array}$$

iff

$$\begin{array}{l} [u(x^*)-u(y)]\;\;(\nu(T+A+D)-\nu(T+B+D)) \;+ \\ [u(y)-u(x)]\;\;(\nu(C+T+A+D) \;-\; \nu(C+T+B+D)) \quad \geq 0, \end{array}$$

where this equivalence obtains for all outcomes.

By (A.1) and appropriate forms of (3.12), which has already been proven, conclude that $(\nu(A+D)-\nu(B+D))$ and $(\nu(T+A+D) - \nu(T+B+D))$ are both negative, while $(\nu(C+A+D) - \nu(C+B+D))$ and $(\nu(C+T+A+D) - \nu(C+T+B+D))$ are both positive. Because the range of u has nonempty interior, we can vary the above utility values sufficiently to conclude from the preceding equivalence that

$$\nu(T+A+D) \;-\; \nu(T+B+D) \;=\; \nu(A+D) \;-\; \nu(B+D). \tag{A.4}$$

Next apply a similar argument for the case

$$\overline{y} \succ x^* \succ y \succ x \succ \underline{y},\; z=x \text{ and } z'=x^*,$$

in place of (A.3). One obtains the equivalence

$$\begin{array}{l} [u(x^*)-u(y)]\;(\nu(A+D)-\nu(B+D))\;+ \\ [u(y)-u(x)]\,(\nu(C+A+D) \;-\; \nu(C+B+D)) \quad \geq 0 \end{array}$$

iff

$$\begin{array}{l} [u(x^*)-u(y)]\;\;(\nu(T+A+D)-\nu(T+B+D)) \;+ \\ [u(y)-u(x)]\;\;(\nu(C+T+A+D) \;-\; \nu(C+T+B+D)) \quad \geq 0, \end{array}$$

where this equivalence obtains for all outcomes. Apply (A.4) and conclude that

$$\nu(C+A+D)-\nu(C+B+D) \;=\; \nu(C+T+A+D)-\nu(C+T+B+D).$$

■

B. APPENDIX: λ-System

Proof of Lemma 5.1: Suppose that for some disjoint subsets A and B of $(T_1 \cup T_2)^c$, act h and outcomes x^*, x, z, $z' \in \mathcal{X}$, that

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in B \\ h & \text{if } s \in (T_1 \cup T_2)^c \backslash (A \cup B) \\ z & \text{if } s \in T_1 \cup T_2 \end{pmatrix} \succeq \begin{pmatrix} x & \text{if } s \in A \\ x^* & \text{if } s \in B \\ h & \text{if } s \in (T_1 \cup T_2)^c \backslash (A \cup B) \\ z & \text{if } s \in T_1 \cup T_2 \end{pmatrix}.$$

By (3.1) for T_2,

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in B \\ h & \text{if } s \in (T_1 \cup T_2)^c \backslash (A \cup B) \\ z & \text{if } s \in T_1 \\ z' & \text{if } s \in T_2 \end{pmatrix} \succeq \begin{pmatrix} x & \text{if } s \in A \\ x^* & \text{if } s \in B \\ h & \text{if } s \in (T_1 \cup T_2)^c \backslash (A \cup B) \\ z & \text{if } s \in T_1 \\ z' & \text{if } s \in T_2 \end{pmatrix},$$

which can be rewritten in the form

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in B \\ h & \text{if } s \in (T_1 \cup T_2)^c \backslash (A \cup B) \\ z' & \text{if } s \in T_2 \\ z & \text{if } s \in T_1 \end{pmatrix} \succeq \begin{pmatrix} x & \text{if } s \in A \\ x^* & \text{if } s \in B \\ h & \text{if } s \in (T_1 \cup T_2)^c \backslash (A \cup B) \\ z' & \text{if } s \in T_2 \\ z & \text{if } s \in T_1 \end{pmatrix}.$$

By (3.1) for T_1,

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in B \\ h & \text{if } s \in (T_1 \cup T_2)^c \backslash (A \cup B) \\ z' & \text{if } s \in T_2 \\ z' & \text{if } s \in T_1 \end{pmatrix} \succeq \begin{pmatrix} x & \text{if } s \in A \\ x^* & \text{if } s \in B \\ h & \text{if } s \in (T_1 \cup T_2)^c \backslash (A \cup B) \\ z' & \text{if } s \in T_2 \\ z' & \text{if } s \in T_1 \end{pmatrix}, \text{ or}$$

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in B \\ h & \text{if } s \in (T_1 \cup T_2)^c \backslash (A \cup B) \\ z' & \text{if } s \in T_2 \cup T_1 \end{pmatrix} \succeq \begin{pmatrix} x & \text{if } s \in A \\ x^* & \text{if } s \in B \\ h & \text{if } s \in (T_1 \cup T_2)^c \backslash (A \cup B) \\ z' & \text{if } s \in T_2 \cup T_1 \end{pmatrix}.$$

Therefore, $T_2 \cup T_1$ satisfies the appropriate form of (3.1).

It remains to prove that (3.1) is satisfied also by $(T_1 \cup T_2)^c$. By Small Unambiguous Event Continuity (Axiom 4) applied to the unambiguous events T_1 and T_2, there exists a partition $\{A_i\}_{i=1}^n$ of S in $\mathcal{A}$ such that $(T_1 \cup T_2)^c$ equals the finite disjoint union

$$(T_1 \cup T_2)^c = \bigcup_{A_i \subseteq (T_1 \cup T_2)^c} A_i.$$

Thus the first part of this proof establishes (3.1) for $(T_1 \cup T_2)^c$.

To complete the proof, it suffices to show that for any $\{A_n\}_{n=1}^{\infty}$, a decreasing sequence in $\mathcal{A}$, we have $\cap_{i=1}^{\infty} A \in \mathcal{A}$: By Nondegeneracy, there exist two outcomes $x^* \succ x$. Then

$$f_n = \begin{pmatrix} x^* & \text{if } s \in A_n \\ x & \text{if } s \in A_n^c \end{pmatrix} \in \mathcal{F}^{ua}, \text{ for all } n = 1, 2, ...$$

By Monotone Continuity (Axiom 5),

$$f_\infty = \begin{pmatrix} x^* & \text{if } s \in \cap_{n=1}^{\infty} A_n \\ x & \text{if } s \in (\cap_{n=1}^{\infty} A_n)^c \end{pmatrix} \in \mathcal{F}^{ua}.$$

Consequently, $\cap_{n=1}^{\infty} A_n \in \mathcal{A}$. ■

C. APPENDIX: Existence of Probability

The first step in proving Theorem 5.2 is to prove the existence of a probability measure representing $\succeq_\ell$. This appendix states a theorem (proven in [27]) that delivers such a probability measure given suitable properties for $\succeq_\ell$. The theorem extends [9, Theorem 14.2] to the present case of a λ-system of events. Next it is shown that these properties are implied by the axioms adopted for $\succeq$, as specified in Theorem 5.2.

For the following theorem, $\mathcal{A}$ denotes *any* λ-system and $\succeq_\ell$ is *any* binary relation on $\mathcal{A}$; that is, they are not necessarily derived from $\succeq$, though the subsequent application is to that case. Denote by

$$\mathcal{N}(\emptyset) = \{A \in \mathcal{A} : A \sim_\ell \emptyset\}.$$

Theorem C.1. *There is a unique finitely additive, convex-ranged probability measure p on $\mathcal{A}$ such that*

$$A \succeq_\ell B \Longleftrightarrow p(A) \geq p(B), \quad \forall\ A,\ B \in \mathcal{A}$$

if (and only if) $\succeq_\ell$ satisfies the following:

F1 $\emptyset \preceq_\ell A$, *for any* $A \in \mathcal{A}$

F2 $\emptyset \prec_\ell S$

F3 $\succeq_\ell$ *is a weak order*

F4 *If* $A, B, C \in \mathcal{A}$ *and* $A \cap C = B \cap C = \emptyset$, *then* $A \prec_\ell B \Longleftrightarrow A \cup C \prec_\ell B \cup C$.

F4' For any two uniform partitions $\{A_i\}_{i=1}^n$ *and* $\{B_i\}_{i=1}^n$ *of* S *in* $\mathcal{A}$, $\cup_{i \in I} A_i \sim_\ell \cup_{i \in J} B_i$, *if* $|I| = |J|$.

$F5$ (i) *If* $A \in \mathcal{A} \backslash \mathcal{N}(\emptyset)$*, then there is a finite partition* $\{A_1, A_2, ..., A_n\}$ *of* S *in* $\mathcal{A}$ *such that (1)* $A_i \subset A$ *or* $A_i \subset A^c$*,* $i = 1, 2, ..., n$*; (2)* $A_i \prec_\ell A$*,* $i = 1, 2, ..., n$*.*

(ii) *If* A*,* B*,* $C \in \mathcal{A} \backslash \mathcal{N}(\emptyset)$ *and* $A \cap C = \emptyset$*,* $A \prec_\ell B$*, then there is a finite partition* $\{C_1, C_2, ..., C_m\}$ *of* C *in* $\mathcal{A}$ *such that* $A \cup C_i \prec_\ell B$*,* $i = 1, 2, ..., m$*.*

$F6$ *If* $\{A_n\}$ *is a decreasing sequence in* $\mathcal{A}$ *and if* $A_* \prec_\ell \cap_0^\infty A_n \prec_\ell A^*$ *for some* A_* *and* A^* *in* $\mathcal{A}$*, then there exists* N *such that* $A_* \prec_\ell A_n \prec_\ell A^*$ *for all* $n \geq N$*.*

Axioms $F1$, $F2$, $F3$ and $F4$ are similar to those in [9, Theorem 14.2], while $F5$ strengthens the corresponding axiom there. The additional axioms $F4'$ and $F6$ are adopted here to compensate for the fact that $\mathcal{A}$ is not a σ-algebra.

For the remainder of the appendix, $\mathcal{A}$, $\succeq_\ell$ and $\succeq$ are as specified in Theorem 5.2 and the axioms stated in (a) are assumed. By Lemma 5.1, $\mathcal{A}$ is a λ-system. The objective now is to prove that conditions $F1 - F6$ are implied by the axioms given for $\succeq$. Proofs that are elementary are not provided.

Lemma C.2. *Let* $\{A_i\}$ *be a uniform partition of* S *in* $\mathcal{A}$*. Then for all outcomes* $\{x_i\}$ *and for all permutations* σ*,*

$$\left(x_{\sigma(i)}, A_i\right)_i \sim (x_i, A_i)_i \,. \tag{C.1}$$

In other words, every uniform partition is strongly uniform.

Proof. Without loss of generality, assume $x_1 \succ x_2$ and that

$$\begin{pmatrix} x_1 & \text{if } s \in A_1 \\ x_2 & \text{if } s \in A_2 \\ x_3 & \text{if } s \in A_3 \\ \cdots & \cdots \\ x_n & \text{if } s \in A_n \end{pmatrix} \succ \begin{pmatrix} x_2 & \text{if } s \in A_1 \\ x_1 & \text{if } s \in A_2 \\ x_3 & \text{if } s \in A_3 \\ \cdots & \cdots \\ x_n & \text{if } s \in A_n \end{pmatrix} ..$$

Since $\{A_i\}_{i=3}^n$ are unambiguous, the appropriate form of (3.1) implies

$$\begin{pmatrix} x_1 & \text{if } s \in A_1 \\ x_2 & \text{if } s \in A_1^c \end{pmatrix} \succ \begin{pmatrix} x_2 & \text{if } s \in A_2^c \\ x_1 & \text{if } s \in A_2 \end{pmatrix};$$

that is, $A_1 \succ_\ell A_2$, a contradiction. Similarly for the other cases. ∎

By showing that the axioms 1-6 for $\succeq$ imply properties $F1 - F6$ for $\succeq_\ell$, we prove the following:

Theorem C.3. *Let* $\succeq$ *be a preference order on* $\mathcal{F}$ *and denote by* $\mathcal{A}$ *the set of all unambiguous events. If* $\succeq$ *satisfies Axioms 1-6, then there exists a unique convex-ranged and countably additive probability measure on* $\mathcal{A}$ *such that*

$$A \succeq_\ell B \iff p(A) \geq p(B), \quad \text{for all } A, B \in \mathcal{A}.$$

Proof. Fix outcomes $x^* \succ x$. Properties $F1$ - $F3$ for $\succeq_\ell$ are immediate.
F4: (Note the role played here by the specific definition of $\mathcal{A}$; not any λ-system would do.) If $A \prec_\ell B$, then

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in A^c \end{pmatrix} \prec \begin{pmatrix} x^* & \text{if } s \in B \\ x & \text{if } s \in B^c \end{pmatrix}, \text{ or}$$

$$\begin{pmatrix} x^* & \text{if } s \in A \backslash B \\ x & \text{if } s \in B \backslash A \\ h & \text{if } s \in (A \cap B) \cup (C^c \backslash (A \cup B)) \\ x & \text{if } s \in C \end{pmatrix} \prec \begin{pmatrix} x & \text{if } s \in A \backslash B \\ x^* & \text{if } s \in B \backslash A \\ h & \text{if } s \in (A \cap B) \cup (C^c \backslash (A \cup B)) \\ x & \text{if } s \in C \end{pmatrix},$$

where

$$h = \begin{cases} x^* & \text{if } s \in A \cap B \\ x & \text{if } s \in C^c \backslash (A \cup B) \end{cases}.$$

Since C is unambiguous, deduce that

$$\begin{pmatrix} x^* & \text{if } s \in A \backslash B \\ x & \text{if } s \in B \backslash A \\ h & \text{if } s \in (A \cap B) \cup (C^c \backslash (A \cup B)) \\ x^* & \text{if } s \in C \end{pmatrix} \prec \begin{pmatrix} x^* & \text{if } s \in B \backslash A \\ x & \text{if } s \in A \backslash B \\ h & \text{if } s \in (A \cap B) \cup (C^c \backslash (A \cup B)) \\ x^* & \text{if } s \in C \end{pmatrix},$$

or $A \cup C \prec_\ell B \cup C$. Reverse the argument to prove the reverse implication.
F4$'$ follows from Lemma C.2 and Axiom 6.
F5 (i): Since $A \succ_\ell \emptyset$,

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in A^c \end{pmatrix} \succ x = \begin{pmatrix} x & \text{if } s \in A \\ x & \text{if } s \in A^c \end{pmatrix}.$$

By Small Unambiguous Event Continuity (Axiom 4), there is a partition $\{A_i\}_{i=1}^n$ of S in $\mathcal{A}$, refining $\{A, A^c\}$ and such that

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in A^c \end{pmatrix} \succ \begin{pmatrix} x^* & \text{if } s \in A_i \\ x & \text{if } s \in A_i^c \end{pmatrix}, \ i = 1, 2, ..., n.$$

That is, for each i, $A_i \subset A$ or $A_i \subset A^c$ and in addition, $A_i \prec_\ell A$.
F5 (ii): Let A, B and C be in $\mathcal{A} \backslash \mathcal{N}(\emptyset)$, $A \cap C = \emptyset$ and $A \prec_\ell B$. Then

$$f = \begin{pmatrix} x^* & \text{if } s \in B \\ x & \text{if } s \in B^c \end{pmatrix} \succ \begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in A^c \end{pmatrix} = \begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in C \\ x & \text{if } s \in A^c \backslash C \end{pmatrix} = g.$$

By Small Unambiguous Event Continuity, there exists a partition $\{C_1, ..., C_n\}$ of S in $\mathcal{A}$, refining $\{A, C, A^c \backslash C\}$ and such that

$$f = \begin{pmatrix} x^* & \text{if } s \in B \\ x & \text{if } s \in B^c \end{pmatrix} \succ \begin{pmatrix} x^* & \text{if } s \in C_i \\ g & \text{if } s \in C_i^c \end{pmatrix}, \ i = 1, 2, ..., n.$$

If $C_i \subset C$, then

$$\begin{pmatrix} x^* & \text{if } s \in B \\ x & \text{if } s \in B^c \end{pmatrix} \succ \begin{pmatrix} x^* & \text{if } s \in C_i \\ g & \text{if } s \in C_i^c \end{pmatrix} =$$

$$\begin{pmatrix} x^* & \text{if } s \in C_i \\ x^* & \text{if } s \in A \\ x & \text{if } s \in (A \cup C_i)^c \end{pmatrix} = \begin{pmatrix} x^* & \text{if } s \in A \cup C_i \\ x & \text{if } s \in (A \cup C_i)^c \end{pmatrix},$$

implying that $A \cup C_i \prec_\ell B$.
F6: Implied by Monotone Continuity. ■

D. APPENDIX: Proof of Main Result

Necessity of the axioms in Theorem 5.2: The necessity of Monotonicity, Nondegeneracy and Weak Comparative Probability is routine. Denote by $\succeq_D$ the order on $D_p^{ua}(\mathcal{X})$ represented by W.
Small Unambiguous Event Continuity: Let $f \succ g$ and x be as in the statement of the axiom. Denote by $P = (x_1, p_1; ...; x_n, p_n)$ and Q the probability distributions over outcomes induced by f and g respectively, and let $\underline{x}$ be a least preferred outcome in $\{x\} \cup \{x_1, x_2, ..., x_n\}$. Since $P \succ_D Q \succeq_D \delta_{\underline{x}}$, mixture continuity and monotonicity with respect to stochastic dominance ensure there exists some sufficiently large integer N such that $W\left((1-\frac{1}{N})P + \frac{1}{N}\delta_{\underline{x}}\right) > W(Q)$. Because p is convex-ranged, we can partition each set A_i into N equally probable events $\{A_{ij}\}_{j=1}^N$ in $\mathcal{A}$. Let $C_k = \cup_{i=1}^N A_{ik}$ for $k = 1, 2, ..., N$. Then $\{C_k\}_{k=1}^N$ is a partition of S in $\mathcal{A}$, $p(C_k) = 1/N$ and $p(A_i \backslash C_k) = (1 - 1/N)p_i$ for each i and k. Consequently, $[\underline{x}$ if $s \in C_k$; f if $s \notin C_k]$ induces the probability distribution $(1/N)\delta_{\underline{x}} + (1 - 1/N)P$ which is strictly preferred to Q. Combined with monotonicity with respect to first-order stochastic dominance, this yields $[x$ if $s \in C_k$; f if $s \notin C_k] \succeq [\underline{x}$ if $s \in C_k$; f if $s \notin C_k] \succ g$. Similarly for the other part of the axiom.
Monotone Continuity: Given a decreasing sequence $\{A_n\}_{n=1}^\infty$ in $\mathcal{A}$, $p(A_n) \searrow p(\cap_1^\infty A_i)$ by the countable additivity of p. The required convergence in preference is implied by mixture continuity of W. The limit f_∞ lies in $\mathcal{F}^{ua}$ because we are given that $\mathcal{A}$ is a λ-system.
Strong-Partition Neutrality: Immediate from (5.1).

Sufficiency of the axioms in Theorem 5.2: Let p be the measure provided by Theorem C.3.

Lemma D.1. *For unambiguous events A and B:*
(a) A is null iff $A \sim_\ell \emptyset$.
(b) If $A \sim_\ell B \sim_\ell \emptyset$ and $A \cap B = \emptyset$, then $A \cup B \sim_\ell \emptyset$.

Proof. (a) Fix $x \succ y$. Let A be null. Then

$$\begin{pmatrix} x & \text{if } s \in A \\ y & \text{if } s \in A^c \end{pmatrix} \sim \begin{pmatrix} y & \text{if } s \in A \\ y & \text{if } s \in A^c \end{pmatrix} = y = \begin{pmatrix} x & \text{if } s \in \emptyset \\ y & \text{if } s \in S \end{pmatrix},$$

implying that $A \sim_\ell \emptyset$. If A is not null, then by Monotonicity (Axiom 1),

$$\begin{pmatrix} x & \text{if } s \in A \\ y & \text{if } s \in A^c \end{pmatrix} \succ \begin{pmatrix} y & \text{if } s \in A \\ y & \text{if } s \in A^c \end{pmatrix} = \begin{pmatrix} x & \text{if } s \in \emptyset \\ y & \text{if } s \in S \end{pmatrix},$$

implying that $A \succ_\ell \emptyset$.

(b) Let $x \succ y$ and $A \cup B \succ_\ell \emptyset$, that is,

$$\begin{pmatrix} x & A \\ x & B \\ y & (A \cup B)^c \end{pmatrix} \succ y.$$

By (a), A and B are null and

$$y = \begin{pmatrix} y & A \\ y & B \\ y & (A \cup B)^c \end{pmatrix} \sim \begin{pmatrix} x & A \\ y & B \\ y & (A \cup B)^c \end{pmatrix} \sim \begin{pmatrix} x & A \\ x & B \\ y & (A \cup B)^c \end{pmatrix} \succ y.$$

This is a contradiction. ■

For each $f \in \mathcal{F}^{ua}$, define

$$P_f = \left(x_1,\ p(f^{-1}(x_1));\ \ldots;\ x_n,\ p(f^{-1}(x_n))\right).$$

Because p is fixed, it may be suppressed in the notation. Accordingly, write

$$P_f \in D^{ua}(\mathcal{X}) = \{P_f : f \in \mathcal{F}^{ua}\}.$$

Define the binary relation $\succeq_D$ on $D^{ua}(\mathcal{X})$ by

$$P \succeq_D Q \quad \text{if } \exists\ f \succeq g,\ P = P_f \text{ and } Q = P_g.$$

Lemma D.2. *If $P_f = P_g$, then $f \sim g$. Thus $\succeq_D$ is complete and transitive.*

Proof. We must prove that for any two partitions $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ of S in $\mathcal{A}$, if $A_i \sim_\ell B_i$, $i = 1, 2, \ldots, n$, then for all outcomes $\{x_i\}_{i=1}^n$,

$$\begin{pmatrix} x_1 & \text{if } s \in A_1 \\ x_2 & \text{if } s \in A_2 \\ \ldots & \ldots \\ x_n & \text{if } s \in A_n \end{pmatrix} \sim \begin{pmatrix} x_1 & \text{if } s \in B_1 \\ x_2 & \text{if } s \in B_2 \\ \ldots & \ldots \\ x_n & \text{if } s \in B_n \end{pmatrix}. \tag{D.1}$$

Case 1: $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ are uniform partitions of S in $\mathcal{A}$. The desired conclusion follows from Lemma C.2 and Axiom 6.
Case 2: All probabilities $\{p(A_i)\}_{i=1}^n$ and $\{p(B_i)\}_{i=1}^n$ are rational. Because p is convex-ranged, there exist $\{E_j\}_{j=1}^m$ and $\{C_j\}_{j=1}^m$, two uniform partitions of S in $\mathcal{A}$, such that

$$\begin{aligned} A_i &= \bigcup_{E_j \subseteq A_i} E_j,\ i = 1, 2, ..., n \text{ and} \\ B_i &= \bigcup_{C_j \subseteq B_i} C_j,\ i = 1, 2, ..., n. \end{aligned}$$

Now Case 1 may be applied.
Case 3: This is the general case where some of the probabilities $p(A_i)$ or $p(B_i)$ may be irrational. Suppose contrary to (D.1) that

$$f = \begin{pmatrix} x_1 & \text{if } s \in A_1 \\ x_2 & \text{if } s \in A_2 \\ ... & ... \\ x_n & \text{if } s \in A_n \end{pmatrix} \succ \begin{pmatrix} x_1 & \text{if } s \in B_1 \\ x_2 & \text{if } s \in B_2 \\ ... & ... \\ x_n & \text{if } s \in B_n \end{pmatrix} = g. \tag{D.2}$$

Without loss of generality, assume that $x_n \succ \cdots \succ x_2 \succ x_1$ and $p(A_1) = p(B_1)$ is irrational.

By the convex range of p over $\mathcal{A}$, there are rational numbers $r_m \nearrow_m$ and two increasing sequences $\{A_1^m\}_{m=1}^\infty$ and $\{B_1^m\}_{m=1}^\infty$ in $\mathcal{A}$ with $A_1^m \subset A_1$ and $B_1^m \subset B_1$, $m = 1, 2, ...$, such that $p(A_1^m) = p(B_1^m) = r_m \nearrow p(A_1) = p(B_1)$ as $m \longrightarrow \infty$. Accordingly,

$$p(A_1 \backslash A_1^m) = p(B_1 \backslash B_1^m) = p(A_1) - p(A_1^m) \searrow 0 \text{ as } m \longrightarrow \infty.$$

Thus, both $\{A_1 \backslash A_1^m\}_{m=1}^\infty$ and $\{B_1 \backslash B_1^m\}_{m=1}^\infty$ are decreasing sequences in $\mathcal{A}$ and

$$\cap_{m=1}^\infty (A_1 \backslash A_1^m) \sim_\ell \cap_{m=1}^\infty (B_1 \backslash B_1^m) \sim_\ell \emptyset. \tag{D.3}$$

Define

$$g_m = \begin{pmatrix} x_1 & \text{if } s \in B_1^m \\ x_2 & \text{if } s \in B_1 \backslash B_1^m \\ g & \text{if } s \in B_1^c \end{pmatrix},\ g_\infty = \begin{pmatrix} x_1 & \text{if } s \in \cap_{m=1}^\infty B_1^m \\ x_2 & \text{if } s \in B_1 \backslash (\cap_{m=1}^\infty B_1^m) \\ g & \text{if } s \in B_1^c \end{pmatrix}.$$

By Lemma D.1 and (D.3),

$$g_\infty \sim g \prec f.$$

By Monotone Continuity, g_m converges to g_∞ in preference as $m \longrightarrow \infty$. Conclude that there exists an integer N_1 such that

$$g_m \prec f \text{ whenever } m \geq N_1.$$

In particular,

$$g_{N_1} = \begin{pmatrix} x_1 & \text{if } s \in B_1^{N_1} \\ x_2 & \text{if } s \in B_1 \backslash B_1^{N_1} \\ g & \text{if } s \in B_1^c \end{pmatrix} \prec f.$$

By Monotonicity,

$$\begin{pmatrix} x_1 & \text{if } s \in A_1^{N_1} \\ x_2 & \text{if } s \in A_1 \backslash A_1^{N_1} \\ f & \text{if } s \in A_1^c \end{pmatrix} \succeq \begin{pmatrix} x_1 & \text{if } s \in A_1^{N_1} \\ x_1 & \text{if } s \in A_1 \backslash A_1^{N_1} \\ f & \text{if } s \in A_1^c \end{pmatrix} = f \succ \begin{pmatrix} x_1 & \text{if } s \in B_1^{N_1} \\ x_2 & \text{if } s \in B_1 \backslash B_1^{N_1} \\ g & \text{if } s \in B_1^c \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} x_1 & \text{if } s \in A_1^{N_1} \\ x_2 & \text{if } s \in A_2 \cup (A_1 \backslash A_1^{N_1}) \\ f & \text{if } s \in (A_1 \cup A_2)^c \end{pmatrix} \succ \begin{pmatrix} x_1 & \text{if } s \in B_1^{N_1} \\ x_2 & \text{if } s \in B_2 \cup (B_1 \backslash B_1^{N_1}) \\ g & \text{if } s \in (B_1 \cup B_2)^c \end{pmatrix}.$$

Note further that $A_2 \cup (A_1 \backslash A_1^{N_1}) \sim_\ell B_2 \cup (B_1 \backslash B_1^{N_1})$ since $p(A_2 \cup (A_1 \backslash A_1^{N_1})) = p(B_2 \cup (B_1 \backslash B_1^{N_1}))$. Thus a proof by induction establishes that

$$\begin{pmatrix} x_1 & \text{if } s \in A_1^{N_1} \\ x_2 & \text{if } s \in A_2^{N_2} \\ x_3 & \text{if } s \in A_3^{N_3} \\ \dots & \dots \\ x_n & \text{if } s \in A_n^{N_n} \end{pmatrix} \succ \begin{pmatrix} x_1 & \text{if } s \in B_1^{N_1} \\ x_2 & \text{if } s \in B_2^{N_2} \\ x_3 & \text{if } s \in B_3^{N_3} \\ \dots & \dots \\ x_n & \text{if } s \in B_n^{N_n} \end{pmatrix},$$

where $A_i^{N_i} \sim_\ell B_i^{N_i}$, $i = 1, 2, ..., n$ and every $p(A_i^{N_i}) = p(B_i^{N_i})$ is rational, contradicting Case 2. ■

The rest of the proof is similar to Steps 2-6 in the proof of [16, Theorem 2]. For example, in the proof of mixture continuity of $\succeq_D$ on $D^{ua}(\mathcal{X})$ (Step 3), Small Unambiguous Event Continuity may be used in place of Savage's $P6$ in order to overcome the lack of a σ-algebra structure for $\mathcal{A}$.

E. APPENDIX: Proofs of Corollaries

For the proof of Corollary 5.3, we need the following lemma:

Lemma E.1. *Let p_* be the inner measure induced from $(\mathcal{A}, p)$. Then for any event $E \in \Sigma$, there exists an increasing sequence $\{A_n\}$ in $\mathcal{A}$ with $A_n \subseteq A$ such that*

$$\lim_{n \to \infty} p(A_n) = p_*(A).$$

Proof. Claim: For any two events $A \subset B$ in $\mathcal{A}$, and any $r \in (p(A), p(B))$, there exists $C \in \mathcal{A}$ with $A \subset C \subset B$ such that $p(C) = r$. This is proven as follows: Because $p(B \backslash A) = p(B) - p(A) > 0$ and $0 < r - p(A) < p(B) - p(A) = p(B \backslash A)$, then by the convex range of p, there exists $D \in \mathcal{A}$, with $D \subset B \backslash A$ and $p(D) = r - p(A)$. Let $C = D \cup A$. Then $C \in \mathcal{A}$, $A \subset C \subset B$ and $p(C) = p(D) + p(A) = r$.

By the definition of p_*, there exist $\{B_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ with $B_n \subseteq A$, $n = 1, 2, ...$ such that

$$p(B_n) \geq p_*(A) - 1/n.$$

Without loss of generality, $p(B_n) < p(B_{n+1})$ for all n. Let $A_1 = B_1$. Since $p(B_2) > p(B_1)$, the claim implies that there exists $A_2 \in \mathcal{A}$ with $A_1 \subset A_2$ such that $p(A_2) = p(B_2)$. Proceed by induction to derive an increasing sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ such that $p(A_n) = p(B_n)$ for all n. ■

Proof of Corollary 5.3: Let p_* and p^* agree on E. By the above lemma, (and the corresponding result for p^*), there exist unambiguous events $\{A_n\}$ and $\{B_n\}$ such that

$$A_n \subset E \subset B_n,\ A_n \nearrow,\ B_n \searrow \text{ and } p(B_n \backslash A_n) < 1/n.$$

Because $\mathcal{A}$ is a λ-system, $A_\infty \equiv \cup A_n$ and $B_\infty \equiv \cap B_n$ are unambiguous. By countable additivity and Lemma D.1, $p(B_\infty \backslash A_\infty) = 0$ and $B_\infty \backslash A_\infty$ is null. Therefore, $E \backslash A_\infty$ is null and (by Lemma 3.2) unambiguous. Thus E, the disjoint union of A_∞ and $E \backslash A_\infty$, is unambiguous. ■

Proof of Corollary 5.4: (a) The assumptions on ν imply that $\succeq$ satisfies the axioms in Theorem 5.2. (Continuity implies Monotone Continuity for $\succeq$ and convex-ranged implies Small Unambiguous Event Continuity. To verify Monotonicity, apply the special nature of unambiguous events whereby they satisfy (3.1).) Therefore, there exists a convex-ranged and countably additive p representing the likelihood relation on $\mathcal{A}$ that is implicit in $\succeq$. Conclude that p must be ordinally equivalent to ν on $\mathcal{A}$.
(b) From (a) and Lemma 3.7, $\nu = \phi(p)$ on $\mathcal{A} \supset \mathcal{A}_0$, where p is convex-ranged on $\mathcal{A}$. Therefore, $\nu(\mathcal{A}_0) = \phi(p(\mathcal{A}_0)) = [0, 1]$. Because ϕ is (strictly) increasing and onto, conclude that $p(\mathcal{A}_0) = [0, 1]$. Now it is straightforward to prove that ϕ is the identity function. (For any two $x_1, x_2 \in [0, 1]$ with $x_1 + x_2 \leq 1$, there exist $A_1 \in \mathcal{A}_0$ and $A_2 \in \mathcal{A}$ such that $p(A_1) = x_1$, $p(A_2) = x_2$ and $A_1 \cap A_2 = \emptyset$. From the definition of $\mathcal{A}_0$, $\nu(A_1 \cup A_2) = \nu(A_1) + \nu(A_2)$, or

$$\begin{aligned}
\nu(A_1 \cup A_2) &= \phi(p(A_1 \cup A_2)) = \phi(p(A_1) + p(A_2)) \\
&= \phi(x_1 + x_2) = \nu(A_1) + \nu(A_2) \\
&= \phi(p(A_1)) + \phi(p(A_2)) \\
&= \phi(x_1) + \phi(x_2).
\end{aligned}$$

Since ϕ is continuous, ϕ is linear on [0, 1].)
(c) Let p be the measure provided by (a) and fix any measure q in *core*(ν). Continuity of ν implies that q is countably additive [21]. We show that A satisfies: For any $B \in \mathcal{A}$,

$$p(B) = p(A) \implies q(B) = q(A). \tag{E.1}$$

Then a recent result by Marinacci [17] allows us to conclude that $p = q$. Because this is true for any q in the core, conclude further that *core*$(\nu) = \{p\}$ and hence, because ν is exact, that

$$\nu = p \quad \text{on } \mathcal{A}.$$

From above, $\mathcal{A} \subset \mathcal{A}_2$, the class of events where all measures in the core agree. In addition, $\mathcal{A}_1 \subset \mathcal{A}$ by Lemma 3.7 and $\mathcal{A}_2 = \mathcal{A}_1 = \mathcal{A}_0$ by Lemma 3.6.

Thus it suffices to prove (E.1). Let $p(B) = p(A)$. Then

$$\nu(B) = \phi(p(B)) = \phi(p(A)) = \nu(A). \tag{E.2}$$

Similarly, $\nu(B^c) = \nu(A^c)$. From the hypothesis $\nu(A) + \nu(A^c) = 1$, deduce that

$$\nu(B) + \nu(B^c) = 1.$$

Because $q(\cdot) \geq \nu(\cdot)$, deduce further that $q(B) = \nu(B)$. Similarly, $q(A) = \nu(A)$. Finally, $q(B) = q(A)$ from (E.2). ■

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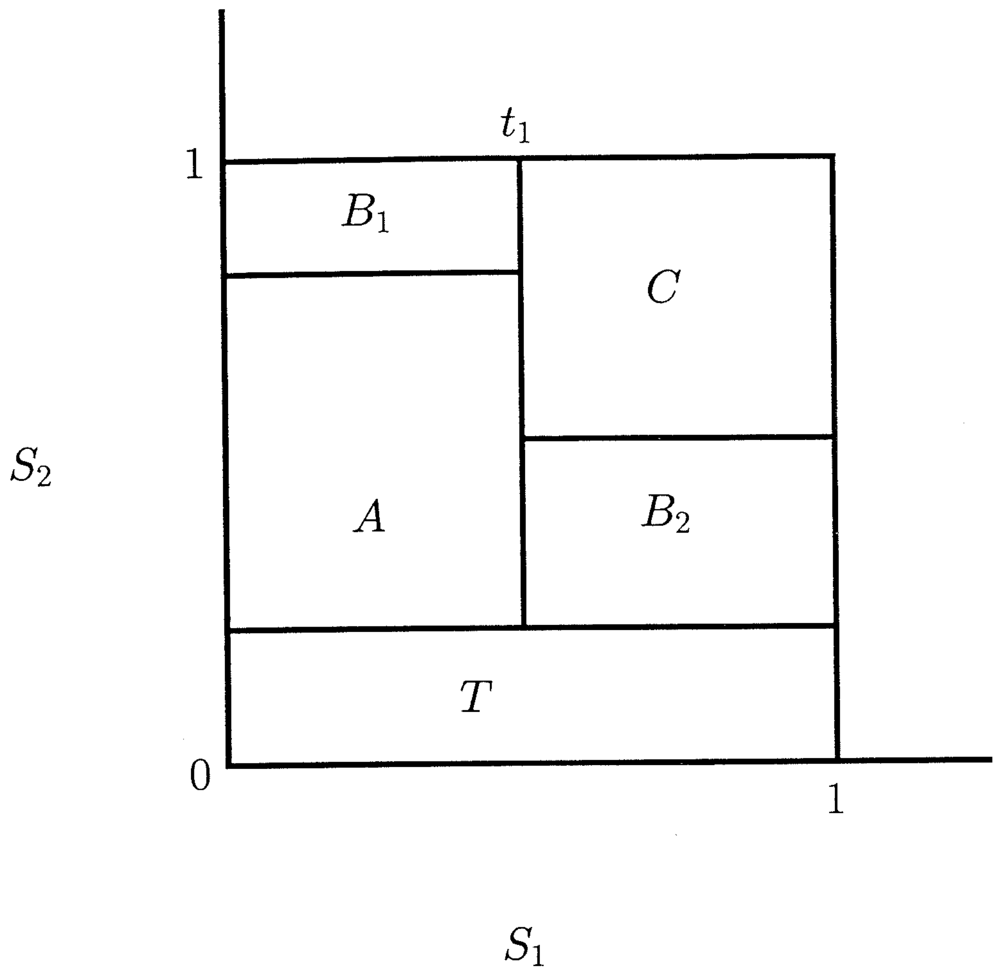

Figure 1: 2-Urn Example