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Larry G. Epstein and Jianjun Miao

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Larry G. Epstein and Jianjun Miao*

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Abstract

This paper describes a pure-exchange, continuous-time economy with two heterogeneous agents and complete markets. A novel feature of the economy is that agents perceive some security returns as ambiguous in the sense often attributed to Frank Knight. The equilibrium is described completely in closed-form. In particular, closed-form solutions are obtained for the equilibrium processes describing individual consumption, the interest rate, the market price of uncertainty, security prices and trading strategies. After identifying agents as countries, the model is applied to address the consumption home-bias and equity home-bias puzzles.

1. INTRODUCTION

This paper describes a pure-exchange, continuous-time economy with two agents and complete markets. A novel feature of the economy is that agents do not view all consumption processes or security returns as purely risky (probabilistic). Rather, they perceive some as ambiguous in the sense often attributed to Frank Knight.[1] Agents differ not only in endowments, but also in where they perceive

*Department of Economics, U. Rochester, Rochester, NY, 14627, lepn@troi.cc.rochester.edu, mias@troi.cc.rochester.edu. Epstein gratefully acknowledges the financial support of the NSF (grant SES-9972442). We have benefitted from comments by Gur Huberman and Martin Schneider. First version: March 2000.

[1]We deviate from Knight's terminology in using 'uncertainty' as a comprehensive term. In our terminology, every process or event is uncertain, and each is either risky or ambiguous (but not both).

ambiguity and in their aversion to ambiguity. The equilibrium is described completely in closed-form. In particular, closed-form solutions are obtained for the equilibrium processes describing individual consumption, the riskless rate, the market price of uncertainty, security prices and trading strategies.

1.1. Ambiguity

It is intuitive that in many situations a decision-maker may not have sufficient information or experience to assign precise probabilities to all future contingencies or states of the world. The Ellsberg Paradox provides an illustration; in particular, it describes intuitive behavior that is inconsistent with the reliance on any probability measure as a representation of beliefs. The lesson is that while a probability measure may adequately represent subjective 'mean likelihood', it cannot at the same time represent also the added dimension of beliefs emphasized by both Knight (confidence in likelihoods) and Keynes, who spoke of the 'weight' of supporting evidence [34]. We refer to this second dimension as ambiguity. The importance of the Ellsberg Paradox is that it is strongly suggestive of the importance of ambiguity also in nonexperimental settings and asset markets provide an obvious instance.

There are obvious questions concerning the incorporation of ambiguity in an empirically meaningful fashion. Most importantly, "*precisely* what is ambiguity?" One answer may be found in [24], which provides foundations for the modeling described in [10], which in turn is applied below. Chen and Epstein formulate intertemporal utility in a continuous-time setting that, given the foundations provided in [24], affords the Knightian distinction between risk and ambiguity. This is made possible through replacement of the single probability measure of the standard model with a set of probabilities or priors as proposed by Gilboa and Schmeidler [26] for an atemporal setting. Because utility is also recursive, we refer to it as *recursive multiple-priors utility.*[2]

Another natural concern is whether ambiguity should be expected to disappear because of learning. We do not model learning. Rather, we interpret our model as describing the steady state of an unmodeled learning process during which the individual has learned all she can about the environment. It is intuitive to us that even in the long run a decision-maker may *not* be completely confident that she knows *precisely* the probability law describing her environment. Agents in our economy take such imprecision into account in making consumption/savings and

[2]See [21] for an axiomatization.

portfolio decisions. (See Section 2.5 for further discussion of learning.)

1.2. Applications

Another motivation for exploring ambiguity is the well-known empirical failures of the risk-based model. One such failure is the "equity home bias" puzzle whereby individuals invest 'too little' in foreign securities [37]. Naturally, 'too little' is from the perspective of a model where securities are differentiated only via their risk characteristics. However, if foreign securities are more ambiguous than domestic ones, then the observed home bias may be optimal. The noted specification regarding ambiguity is the one we adopt, while interpreting the agents in our model as representative consumers in each of two countries.

Thus our model can be viewed as a formalization of the suggestion by French and Poterba that equity home bias may be due to differences in beliefs. They speculate [25, p. 225] that investors "may impute extra 'risk' to foreign investments because they know less about foreign markets, institutions and firms." They also cite evidence in [29] that "households behave as though unfamiliar gambles are riskier than familiar gambles, even when they assign identical probability distributions to the two gambles." The widespread tendency to invest in the familiar has been documented recently in [30], with the home country bias being just one instance. (See also [27].) We interpret the difference between the familiar and less familiar as the second dimension referred to by Knight and Keynes and we model it as a difference in ambiguity.

There exists survey evidence supporting systematic differences in returns expectations between domestic and foreign investors - investors tend to be more optimistic about domestic securities (see [42, 35, 43]). While these surveys do not address ambiguity and they elicit at best a single probability measure from each subject (rather than a set of priors), their findings are consistent with our model if we interpret the elicited measures as including an adjustment for ambiguity. (See the discussion in Section 4.4 for elaboration.) Thus we take these studies as providing further indirect support for our modeling approach.[3]

A second (related) application of our model is to cast light on the consump-

[3] Further supporting arguments may be found in [7, p. 1853] and the references cited therein. These arguments are offered to support the hypothesis of information asymmetry between domestic and foreign investors and thus ultimately to motivate a noisy rational expectations model where individuals have common single priors but observe different signals. However, they serve just as well to motivate a model with heterogeneous sets of priors.

tion behavior that is implied by efficient sharing of uncertainty.[4] In the standard model where individuals maximize additive expected utility functions, where utility indices may differ but the probability measure is common, efficiency implies that every individual's consumption level is a deterministic function of aggregate consumption. In an international setting, this implication contradicts evidence of a high correlation between country-specific consumption growth and country-specific output growth, leading to what Lewis [37, pp. 574-5] has termed the "consumption home bias." However, such correlation is consistent with efficiency in a world with ambiguity, at least as modeled here.

It may not be surprising to some that the assumption that foreign securities are more ambiguous than domestic ones leads to a bias towards domestic securities and to consumption growth that is sensitive to domestic shocks. However, it merits emphasis that these results are achieved as part of a dynamic general equilibrium and along with other (overidentifying) predictions that can be used to evaluate the model. These include: (i) positive correlation between security returns in the two countries and between returns and consumption growth rates within either country; and (ii) the country with the larger instantaneous mean growth rate of consumption has (under a suitable assumption on parameters) the higher instantaneous variance for consumption growth. A final prediction concerns an added piece of the equity home-bias puzzle whereby while foreign equity *holdings* by domestic residents are small, foreign equity *flows* are large and volatile [37, pp. 585-90].

1.3. Related Literature

While there are numerous papers dealing with existence and characterization of equilibrium in heterogeneous-agent economies (see [14], [16], [13] and [32], for example), there are fewer that derive qualitative or quantitative predictions (in a continuous-time setting). We discuss some of these here in order to place our model in perspective as a tool for understanding the workings of a heterogeneous-agent economy.

Dumas [17] and Wang [44] consider two-agent economies with complete markets; the former has linear production while Wang considers a pure exchange economy. Both authors assume expected additive utility maximization and per-

[4] The common terminology of efficient risk-sharing is not appropriate here because consumption processes typically involve elements of both risk and ambiguity. Hence, we refer throughout to the sharing of uncertainty.

mit some differences in utilities. Dumas relies completely on numerical techniques to analyse his model. Wang provides closed-form solutions for equilibrium consumption, interest rates and the market price of risk and a PDE determining security prices, but only by assuming a very special relation between the elasticity parameters of the felicities of the two agents [44].

Both cited models admit a representative agent. As a result, implications for aggregate variables and prices are similar to those delivered by representative agent models, given a suitable specification of utility for the representative agent. Further, the standard characterization of efficient consumption allocations (perfect correlation across all individuals with the aggregate) is valid for their models. In contrast, our model does not admit a representative agent and, as noted earlier, delivers a qualitatively different characterization of efficiency.

Both Dumas and Wang refer to the heterogeneity in utilities as modeling differences in risk aversion. However, because risk aversion and intertemporal substitutability are confounded in the standard utility specification, the interpretation of their results is problematic. For example, given the widely used isoelastic within-period felicity function ($u(c_t) = c_t^{\alpha}/\alpha$), a decrease in α increases risk aversion and simultaneously decreases the willingness to substitute across time. Therefore, such a model does not permit a clear understanding of which aspect of preference or which sort of heterogeneity (in risk aversion or in substitutability) is responsible for various properties of the equilibrium. A degree of disentangling is permitted by the recursive utility (or stochastic differential utility) model [12]. That model is applied in [18], where analytical solutions are provided for a specification in which there is heterogeneity in substitutability, but not in risk aversion.[5]

Our specification of utility also confounds risk aversion and substitution, but it disentangles these two aspects of preference from ambiguity aversion. Because we focus on ambiguity and heterogeneity in attitudes towards ambiguity, this degree of separation permits the interpretations suggested below. It is noteworthy that our model with heterogeneity with respect to ambiguity permits analytical solutions while models with heterogeneity in risk attitudes typically (with the exception of the special construction in [44]) require numerical analysis.

As discussed in [10] and below (Section 3.3), there is a limited observational equivalence between our model with ambiguity and a model in which individuals are expected utility maximizers with heterogeneous priors. Thus our model is

[5] See [40] for a discrete-time model that admits also differing risk attitudes but where numerical techniques must be adopted.

related to models with heterogeneous priors such as [11], [45] and [3], where closed-form solutions are also available. One difference is that these models assume incomplete information and Bayesian-rational inferences from observables, while, as mentioned earlier, we do not model learning. Rather we focus on a state where individuals have learned all they can about their environment and yet ambiguity persists. In Section 2.5, we argue that the persistence of ambiguity is at least plausible. While the reader may wish to reserve judgement on that aspect of our model, we emphasize at this point the particular appeal of incorporating ambiguity once one opens the door to heterogeneous beliefs, or homogeneous but wrong beliefs as in the discrete-time models [1] and [9]. In an environment in which there is disagreement about the probability of future states, an individual may question the reliability of her single prior and may wish to make decisions that are robust to errors in the prior. Such self-awareness and a desire for robust decisions lead naturally to consideration of sets of priors.[6]

1.4. Outline

The paper proceeds as follows: Recursive multiple-priors utility is described in the next section. The economy and equilibrium are described in Section 3. The nature of equilibrium and the model's application to the home bias puzzles in equities and consumption are discussed in Section 4. Proofs are relegated to an appendix.

2. RECURSIVE MULTIPLE-PRIORS UTILITY

In this section we outline a special case of recursive multiple-priors for a single individual that will be used later in the equilibrium model. The reader is referred to [10] for further details and for justification for asserted interpretations.

2.1. Consumption Processes

Time varies over $[0, T]$ and uncertainty is represented by a probability space $(\Omega, \mathcal{F}, P)$. Here, unlike in standard models, P represents neither the true objec-

[6] Models of 'robust decision-making' are described in [28] and [2] for discrete and continuous time settings respectively. As explained in [10], these models are similar in spirit but different in formal details from the recursive multiple-priors model. In particular, and in spite of their interpretation as modeling 'robust *decision-making*', this interpretation is lacking decision-theoretic foundations.

tive measure nor the subjective measure used by the individual being described. Its role is to define null events; there will be no disagreement or ambiguity about which events are null. Let $W = (W_t)$ be a standard d-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, P)$ and $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ the (augmented) filtration that it generates, representing the information available to the individual. Assume $\mathcal{F} = \mathcal{F}_T$ and that $\mathcal{F}_0$ is trivial. All processes in the sequel are progressively measurable with respect to $\mathbb{F}$ and all equalities involving random variables (processes) are understood to hold dP $a.s.$ ($dt \otimes dP$ $a.s.$).

There is a single consumption good at each instant. Consumption processes lie in $\mathcal{C}$, a subset of the set of positive progressively measurable processes that are also square integrable ($E_P \left[\int_0^T c_s^2 \, ds \right] < \infty$).

2.2. The Set of Priors

The first step in specifying a utility function on $\mathcal{C}$ is to specify the set of priors $\mathcal{P}$ on the state space $(\Omega, \mathcal{F}_T)$. Because all priors in $\mathcal{P}$ are taken to be mutually absolutely continuous with respect to P, they can be defined via their densities. These, in turn, may be defined by use of *density generators*, which is how we refer to any $\mathbb{R}^d$-valued process $\theta = (\theta_t)$ satisfying

$$\sup_t \mid \theta_t^i \mid \leq \kappa_i, \;\; i = 1, ..., d,$$

where $\kappa = (\kappa_1, ..., \kappa_d)^\top$ is a vector of non-negative parameters. Let Θ be the set of all such processes. Each density generator θ generates a P-martingale (z_t^θ) via the equation

$$dz_t^\theta \;=\; - \, z_t^\theta \, \theta_t \cdot dW_t, \quad z_0^\theta \;=\; 1, \tag{2.1}$$

or equivalently,

$$z_t^\theta \;=\; \exp\left\{ -\frac{1}{2} \int_0^t \parallel \theta_s \parallel^2 ds \;-\; \int_0^t \theta_s \cdot dW_s \right\}, \quad 0 \leq t \leq T. \tag{2.2}$$

Because $1 \;=\; z_0^\theta \;=\; E\left[z_T^\theta\right]$, z_T^θ is a P-density and thus determines a probability measure Q^θ on $(\Omega, \mathcal{F}_T)$ via

$$\frac{dQ^\theta}{dP} \;=\; z_T^\theta. \tag{2.3}$$

Finally, the set of priors is

$$\mathcal{P} = \{ Q^\theta : \theta \in \Theta \text{ and } Q^\theta \text{ is defined by (2.3)} \}. \tag{2.4}$$

When $\kappa = 0$, then $\mathcal{P}$ collapses to the single measure P as in a model without ambiguity. More generally, $\mathcal{P}$ is a nonsingleton that expands as any component of κ is increased. The natural interpretation is that ambiguity increases with κ, or alternatively, that ambiguity aversion increases with κ.[7]

For further clarification of our construction, think of a discrete-time event tree where nature determines motion through the tree and where $\mathcal{F}_T$ describes the set of terminal states or events. Fix a reference probability measure P on $\mathcal{F}_T$. At each time and state in the tree, the decision-maker's conditional beliefs about the state to be reached next period are represented by a set of densities with respect to the conditional measure induced by P. The set of densities determines a set of conditional probability measures over the state next period. Finally, the sets of conditional-one-step-ahead measures for all time-state pairs can be combined in the usual probability calculus way to deliver a set of measures on $\mathcal{F}_T$. (In this construction, admit all possible selections of a conditional measure at each time-state pair.) The definition (2.4) implements this construction in continuous-time; a density generator θ is a process that delivers the counterpart of the (logarithm of) a conditional-one-step-ahead density.

Note that by the Girsanov Theorem, $\left(W_t + \int_0^t \theta_s \, ds\right)$ is a Brownian motion relative to Q^θ. Thus the multiplicity of measures in $\mathcal{P}$ can be interpreted as modeling ambiguity about the drift of the driving process. The drift may be zero ($\theta = 0$) but another possibility according to $\mathcal{P}$ is that the 'true' measure is such that $(W_t^1 + \kappa_1 t, \, W_t^2 + \kappa_2 t)$ is a Brownian motion, corresponding to $\theta_t = (\kappa_1, \kappa_2)^\top$ for all t.

2.3. Utility

Define a utility process $(V_t(c))$ for each consumption process c in $\mathcal{C}$ as follows:

$$V_t(c) = \min_{Q \in \mathcal{P}} E_Q \left[\int_t^T e^{-\beta(s-t)} \, u(c_s) \, ds \,\middle|\, \mathcal{F}_t \right]. \tag{2.5}$$

[7] Given the subjective nature of ambiguity, there is an unavoidable confounding between the extent of ambiguity and the degree of aversion to it. For example, it is impossible to distinguish between the absence of ambiguity on the one hand and the presence of ambiguity combined with indifference to it on the other.

Under specified assumptions on $u(\cdot)$, the utility process is well-defined and it is recursive in the sense that[8]

$$V_t = \min_{Q \in \mathcal{P}} E_Q \left[\int_t^\tau e^{-\beta(s-t)} u(c_s) \, ds + e^{-\beta(\tau-t)} V_\tau \middle| \mathcal{F}_t \right], \; 0 \le t < \tau \le T.$$

Because utility also has the 'min-over priors' form of the multiple-priors model [26], we refer to $(V_t(\cdot))$ as the *recursive multiple-priors utility process.* Abbreviate $V_0(\cdot)$ by $V(\cdot)$ and refer to it as *recursive multiple-priors utility.*

Recursivity ensures dynamic consistency just as in the standard model with a single prior, which is obtained as the special case $\kappa = 0$. Recursivity follows from the fact that the utility process solves (uniquely) a backward stochastic differential equation (BSDE), that is, for each c, there exists a unique process $(V_t(c), \sigma_t(c))$ satisfying, for $0 \le t \le T$,

$$dV_t = [-u(c_t) \; + \; \beta V_t + \max_{\theta \in \Theta} \theta_t \cdot \sigma_t \,] \, dt + \sigma_t \cdot dW_t, \; V_T = 0. \tag{2.6}$$

Note that the volatility of utility $\sigma_t(c)$ is determined as part of the solution to the BSDE; it plays a key role in the sequel.

Additional conditions deliver a range of natural properties for utility. For example, if u is increasing and (strictly) concave, then so is each $V_t(\cdot)$.

The supergradients of utility are important for characterizing security prices and equilibrium more generally. A *supergradient* for V at the consumption process c is a process (π_t) satisfying

$$V(c') \; - \; V(c) \; \le \; E_P \left[\int_0^T \pi_t \, (c'_t - c_t) \, dt \right], \tag{2.7}$$

for all c' in $\mathcal{C}$. Because V $(= V_0)$ is a lower envelope of expected additive utility functions (2.5), the well-known structure of supergradients of the standard utility function immediately delivers supergradients for V [10, Section 4.7]. More precisely, define $\mathcal{P}_c$ to be the set of measures in $\mathcal{P}$ such that

$$V(c) \; = \; E_{Q^*} \left[\int_0^T e^{-\beta t} \, u(c_t) \, dt \right]. \tag{2.8}$$

[8] Sufficient conditions are that u be Borel measurable and that $E\left[\int_0^T u^2(c_t)\,dt\right] < \infty$ for all c in $\mathcal{C}$. Below, where we take u to be the log function, this square integrability condition determines the domain $\mathcal{C}$. Fortunately, the aggregate endowment process specified there lies in $\mathcal{C}$, implying that we can safely proceed in the equilibrium analysis under the assumption that for all intents and purposes the foundations for utility provided in [10, Theorem 2.3] apply to our model.

Then an appropriate envelope theorem delivers (assuming that u is differentiable) that any process $(\pi_t(c))$ of the following form is a supergradient for V at c:

$$\pi_t(c) = e^{-\beta t} u'(c_t) \left. \frac{dQ^*}{dP} \right|_{\mathcal{F}_t}, \quad Q^* \in \mathcal{P}_c. \tag{2.9}$$

However, $Q^* \in \mathcal{P}_c$ *if and only if* $Q^* = Q^{\theta^*}$ for some θ^* that solves (for every t) the instantaneous maximization appearing in (2.6), that is,[9]

$$\theta^* \in \Theta_c \equiv \{ (\theta_t) : \theta_t = \kappa \otimes sgn(\sigma_t) \text{ all } t \}. \tag{2.10}$$

Conclude, using (2.3), that each $(\pi_t(c))$ of the form

$$\pi_t(c) = e^{-\beta t} u'(c_t) z_t^{\theta^*}, \; \theta^* \in \Theta_c, \tag{2.11}$$

is a supergradient for V at c. Though there may be other supergradients at some processes c, and these would lead to different equilibria below, we restrict attention to equilibria corresponding to (2.11).

Note finally that the BSDE (2.6) may be simplified to

$$dV_t = [-u(c_t) + \beta V_t + \kappa \cdot \mid \sigma_t \mid] \, dt + \sigma_t \cdot dW_t, \quad V_T = 0, \tag{2.12}$$

where for any d-dimensional vector x, $\mid x \mid$ denotes the vector with i^{th} component $\mid x_i \mid$.

2.4. Example

We can compute utility explicitly for consumption processes c of the form

$$dc_t/c_t = \mu^c \, dt + s^c \cdot dW_t,$$

where μ^c and s^c are constant. Suppose that $u(c_t) = (c_t^\alpha - 1)/\alpha$, for $\alpha \leq 1$, where $\alpha = 0$ corresponds to the log specification. Then

$$V_t(c) = A_t \, (c_t^\alpha - 1) / \alpha - \frac{1}{\beta \rho} \frac{(\rho - \beta)}{\alpha} + e^{\beta(t-T)} \frac{1}{\beta \rho} \frac{\left[\rho - \beta e^{(\rho - \beta)(t-T)}\right]}{\alpha},$$

[9] For any d-dimensional vector x, $sgn(x)$ is the d-dimensional vector with i^{th} component equal to $sgn(x_i) = \mid x_i \mid / x_i$ if $x_i \neq 0$ and $= 0$ if $x_i = 0$. For any $y \in \mathbb{R}^d$, $y \otimes sgn(x)$ denotes the vector in $\mathbb{R}^d$ with i^{th} component $y_i sgn(x_i)$.

where

$$A_t = \rho^{-1}\left[1-\exp(\rho(t-T))\right] \quad \text{and}$$

$$(\rho - \beta)/\alpha = -\left(\mu^c - (1-\alpha)s^c \cdot s^c/2 - \kappa \cdot \mid s^c \mid\right).$$

The associated volatility is

$$\sigma_t = A_t c_t^{\alpha} s^c.$$

Evidently the utility of the given consumption process depends on the initial level of consumption and on the adjusted mean growth rate $\mu^c - (1-\alpha)s^c \cdot s^c/2 - \kappa \cdot \mid s^c \mid$, where the adjustment is both for risk (via the second term) and ambiguity (via the third term). We turn next to a discussion of ambiguity.

2.5. Ambiguity and Learning

The equilibrium model to follow deals with an economy in which the Brownian motion is 2-dimensional ($d = 2$) and for which, for each individual, at least one component of the ambiguity parameter κ is zero. Thus adopt these specializations also for the present discussion, taking $\kappa = (0, \overline{\kappa})^{\top}$.

All measures in $\mathcal{P}$ agree on events generated by the first component process $W^1 = (W_t^1)$ and thus we interpret these events as risky and consumption processes that are adapted to $\sigma(W_s^1 : s \leq t)$ are interpreted as involving only risk. Denote the set of such processes as $\mathcal{C}^{risk}$. However, there is disagreement within $\mathcal{P}$ about all other events. This leads, in particular, to the interpretation of consumption processes that are adapted to $\sigma(W_s^2 : s \leq t)$ as being ambiguous. An increase in $\overline{\kappa}$ has no effect on the ranking of risky processes. Therefore, we can express the behavioral significance of a change in $\overline{\kappa}$ in the following way: Let $\overline{\kappa}^* > \overline{\kappa}$ and let $V^*(\cdot)$ and $V(\cdot)$ be the corresponding utility functions. Then for all c in $\mathcal{C}$ and c^{risk} in $\mathcal{C}^{risk}$,

$$V(c^{risk}) \geq (>)\ V(c) \implies V^*(c^{risk}) \geq (>)\ V^*(c).$$

That is, whenever V rejects the ambiguous process c in favor of the risky and hence unambiguous c^{risk}, then so does V^*. In a natural sense, therefore, V^* is *more ambiguity averse* than V. We emphasize that the change from $\overline{\kappa}$ to $\overline{\kappa}^*$ does not affect attitudes towards risky consumption processes, supporting our earlier assertion that our model of preference delivers a degree of separation between attitudes towards risk and attitudes towards ambiguity.[10]

[10] The degree of risk aversion is modeled by the concavity of the felicity $u(\cdot)$; however, the latter will be held fixed in our equilibrium model where we adopt the logarithmic specification.

In a model with ambiguity, where individuals do not understand the driving process well enough to assign precise probabilities to events, the question noted in the introduction arises, namely, "why don't they learn?" We do not model learning; for example, the parameter κ is assumed constant and does not respond to data. We interpret our model as describing the steady state of an unmodeled learning process during which the individual has learned all she can about the environment. Moreover, we argue that ambiguity may very well persist in such a steady state, given an appropriate assumption about the individual's ex ante view.

Because the Ellsberg Paradox is the classic illustration of ambiguity, we recast the stochastic environment in terms of Ellsberg urns in order to consider briefly the question of learning. Thus suppose that W_t is real-valued ($d = 1$) and think of motion along the real line occurring in discrete-time. At each time t and at each position, there is an urn containing 100 balls, either red or black. No information is provided to the decision-maker about the color composition; she is told only that the total is 100 for each urn. A ball is drawn at random and motion is one unit to the right or left according as the color of the ball drawn is red or black. A different urn is used at each t but the decision-maker has no reason to believe that the color composition is different for different urns. Finally, because she cares about ambiguity, her ex ante view is expressible by a nonsingleton set of measures for each urn.

The way in which ambiguity changes over time depends both on the 'truth' and on how the urns are related according to the decision-maker's ex ante view. For simplicity, suppose that in fact there are 50 balls of each color in each urn, so that the true law of motion corresponds to a random walk. As for the subjective ex ante view, suppose first that she views the urns as being physically identical (having the identical color composition). Then observations of realized W_t's are viewed as though they were repeated samples from a single urn and it is intuitive that ambiguity will vanish and that the truth will be learned asymptotically.

At the other extreme, however, that the individual views the urns as 'independent', corresponding to a view of the data generating mechanism as changing through time in a way that she does not understand and cannot hope to learn. Then observed realizations correspond to draws from independent urns and intuition does *not* suggest that ambiguity would disappear asymptotically. Marinacci [38] proves a form of the LLN appropriate for this setting in which the connection between empirical frequencies and asymptotic beliefs is weakened *to a degree that depends on the extent of ambiguity in prior beliefs.* Asymptotically, the decision-maker believes that the limit frequency of any given color lies in an interval, where the interval collapses to a point if there is no ambiguity in prior beliefs but not more generally. This limiting situation is the one that we model.[11]

3. TWO-PERSON EQUILIBRIUM

3.1. The Economy

Information structure and preferences. The primitive probability space is $(\Omega, \mathcal{F}_T, P)$. Suppose the associated Brownian motion is 2-dimensional, $W_t = (W_t^1, W_t^2)$. We assume a population of two individuals. They have the common information structure represented by the augmented Brownian filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \le t \le T}$. In particular, the differing beliefs of the two individuals described below are *not* due to asymmetric information; they reflect differing prior views about the environment.[12] Note that the assumption that consumer i observes realizations of both W^i and W^j does not contradict the intuition described in the introduction whereby i is less familiar with securities that are driven primarily by W^j than with those driven by W^i. For example, Canadian sports fans have access to scores and satellite telecasts of soccer matches. However, typically they do not pay much attention to them with the result that many feel much more familiar with hockey and prefer to bet on hockey rather than on soccer matches.

There is a single perishable good (the numeraire), leading to the consumption set $\mathcal{C}$. Each individual has a recursive multiple-priors utility function on $\mathcal{C}$ and for $i = 1, 2$, c^i and $(V_t^i(\cdot))$ denote i's consumption and utility processes. Each utility function has the form (2.5) with common rate of time preference β and felicity function

$$u(c_t) = \log c_t. \tag{3.1}$$

Preferences differ, however, because individuals have different sets of priors, that is, different ignorance parameters κ^i. We assume that

$$\kappa^1 = (0, \kappa_1)^\top \quad \text{and} \quad \kappa^2 = (\kappa_2, 0)^\top. \tag{3.2}$$

[11] The general recursive multiple-priors model in [10] permits learning. This aspect of the model is being developed further in [22]; they model a decision-maker who views some aspects of her environment as learnable and others as not learnable. See [5] for the argument that economic time series may be generated by stochastic processes that could never be discovered from the data that they generate.

[12] See [39] for a discussion, in a Bayesian setting, of the merits of differing priors, rather than asymmetric information, as a basis for differing beliefs.

The interpretation is that i is more familiar with 'her own' component process W^i than with the other component W^j.[13] In extreme form this leads to no ambiguity for i about W^i, though W^j is ambiguous for her.[14] A concrete setting where this specification seems natural is where i is a representative consumer in country i in which W^i is the driving state process. Henceforth we adopt this interpretation and refer to individuals alternatively as countries.

We argued in Section 2.5 that the persistence of ambiguity is plausible given priors that reflect initial ambiguity, even given observability of realizations of the (domestic and) foreign state processes. At a theoretical level, this refutes the suggestion that ambiguity about foreign markets should be unimportant because of improved information about foreign security markets that is available in recent years. At a less formal level, many have claimed that it is not at all clear that investors could learn the true statistical model driving security returns even where one exists. For example, French and Poterba [25, p. 225] write that "the statistical uncertainties associated with estimating expected returns in equity markets makes it difficult for investors to learn that expected returns in domestic markets are not systematically higher than those abroad." In such an environment, where there may be less than complete confidence in estimated moments of expected returns, the investor may not treat these estimates as true in making portfolio decisions.[15] Rather she may be aware of the possibility that the estimates are wrong and thus seek to make robust decisions. As suggested in the introduction (Section 1.3), the multiple-priors model can be interpreted in these terms.

Securities markets. Investment opportunities are represented by a locally riskless bond earning the instantaneous interest rate r and by two securities, with respective (non-negative) dividend streams Y_t^1 and Y_t^2. Thus cumulative dividends are

[13] We write the second component of κ^1 as κ_1 to indicate that it is the ambiguity parameter for individual 1, though it relates to ambiguity about W^2. When referring to individuals i and j, it is understood that $i \neq j$.

[14] To describe i's set of priors $\mathcal{P}^i$ more formally, write $\mathcal{F}_T = \mathcal{F}_T^1 \otimes \mathcal{F}_T^2$, where $\mathcal{F}_t^i = \sigma\left(W_s^i : s \leq t\right)$, and denote by $P^{(1)}$ the first marginal of P. Then $\mathcal{P}^1$ consists of all products of the form $P^{(1)} \otimes Q$, where Q is a measure on $\mathcal{F}_T^2$ that is consistent with 1's ambiguity about W^2, as measured by κ_1; that is, Q is defined by the one-dimensional counterpart of (2.3). Similarly for $\mathcal{P}^2$.

[15] See [37] for a discussion of some of the literature on portfolio choice under estimation risk where it *is* assumed that estimates are treated as true.

described by

$$D_t^\top = \begin{cases} (0, \int_0^t Y_s^1 ds, \int_0^t Y_s^2 ds), & \text{if } 0 \leq t < T \\ (1, \int_0^T Y_s^1 ds, \int_0^T Y_s^2 ds) & \text{if } t = T. \end{cases} \tag{3.3}$$

Because some of our results do not require that we specify further the nature of the individual processes (Y_t^i), we defer further assumptions until they are needed (Section 4.4). In anticipation of the more detailed specification, the reader might think of (Y_t^i) being driven 'primarily' by the state process (W_t^i) associated with country i.

At each t, securities are traded in a competitive market at prices $S_t = (S_t^0, S_t^1, S_t^2)^\top$ denominated in units of consumption. In equilibrium, S is an Ito process so that the gain process $S + D$ is also an Ito process,

$$d(S_t + D_t) = \mu_t^G dt + s_t^G dW_t,$$

where μ_t^G is $\mathbb{R}^3$-valued and s_t^G is $\mathbb{R}^{3\times 2}$-valued. A *trading strategy* is an $\mathbb{R}^3$-valued process $\gamma = (\gamma_t)$, satisfying

$$\int_0^T |\gamma_t \cdot \mu_t^G| \, dt + \int_0^T \gamma_t^\top s_t^G (s_t^G)^\top \gamma_t \, dt < \infty.$$

This condition insures that the stochastic integral $\int \gamma_t \cdot d(S_t + D_t)$ is well defined. Note that $\gamma_t = \left(\gamma_{0,t}, \gamma_{1,t}, \gamma_{2,t}\right)^\top$, where $\gamma_{n,t}$ represents the number of shares of the bond ($n = 0$) and securities 1 and 2. The set of all trading strategies is denoted Γ.

Endowments and objectives. The aggregate endowment or output process (Y_t) is assumed to follow the geometric law

$$dY_t \,/\, Y_t \;=\; \mu^Y dt + s^Y \cdot dW_t, \tag{3.4}$$

where μ^Y and $s^Y = \left(s_1^Y, s_2^Y\right)^\top$ are constants. Aggregate dividends do not exhaust output. Rather we assume that

$$Y_t \;=\; Y_t^1 \,+\, Y_t^2 \,+\, \Phi_t,$$

where (Φ_t) is the part of aggregate output that is not traded.[16]

[16] The presence of a nontraded endowment complicates the model somewhat. We include it not for greater generality but primarily because, as explained in Section 4.4, it is unavoidable given the intuition we are trying to capture with our model and given our desire to obtain closed-form solutions.

Each country owns 1/2 of the nontraded endowment (Φ_t). Initial share holdings are given by

$$\gamma^1 = (0, \gamma^1_{1,0}, \gamma^1_{2,0})^\top \text{ and } \gamma^2 = (0, 1 - \gamma^1_{1,0}, 1 - \gamma^1_{2,0})^\top;$$

the assumption of zero initial bond holdings is made purely for simplicity.

Given a security price process S, individual i solves

$$\sup_{(e^i,\gamma^i)\in\mathcal{C}\times\Gamma} V^i(e^i) \tag{3.5}$$

subject to

$$S_t \cdot \gamma^i_t = S_0 \cdot \gamma^i_0 + \int_0^t \gamma^i_s \cdot d(S_s + D_s) - \int_0^t \left(e^i_s - \tfrac{1}{2}\Phi_s\right) ds, \;\; t \in [0,T], \tag{3.6}$$

and a credit constraint that is specified in Appendix A (see (A.15)).

The preceding defines the economy

$$\mathcal{E} \;=\; \Big((\Omega, \mathcal{F}, \mathbb{F}, \, P) \, , \, (W_t), \;\; \big(u, \beta, \kappa^i, \gamma^i_0\big)_{i=1,2} \, , \;\; (D_t), \, (Y_t) \Big) \, . \tag{3.7}$$

3.2. Equilibrium

We define two notions of equilibrium. An *Arrow-Debreu equilibrium* for the economy $\mathcal{E}$ is a tuple $((c^i)_{i=1,2}, p)$ where p is a non-negative real-valued (state) price process, c^i solves (for $i = 1, 2$)

$$\sup_{e^i\in\mathcal{C}} V^i(e^i) \quad \text{subject to}$$

$$E\left[\int_0^T p_s \left(e^i_s - \tfrac{1}{2}\Phi_s\right) ds\right] \le \gamma^i_{i,0} E\left[\int_0^T p_s Y^i_s \, ds\right] + \gamma^i_{j,0} E\left[\int_0^T p_s Y^j_s \, ds\right], \tag{3.8}$$

and where markets for contingent consumption clear, that is,

$$c^1 + c^2 \;=\; Y.$$

A *Radner equilibrium* for the economy $\mathcal{E}$ is a tuple $(\,(c^i, \gamma^i)_{i=1,2}, \, S)$ such that given the security price process S, (c^i, γ^i) solves problem (3.5) for $i = 1, 2$ and markets clear:

$$\gamma^1_t + \gamma^2_t \;=\; (0, 1, 1)^\top \text{ for all } t \text{ and } \; c^1 + c^2 \;=\; Y. \tag{3.9}$$

According to this definition, individuals make consumption and portfolio plans for the entire horizon at $t = 0$. Because of the sequential nature of markets, one may ask whether they have incentives to revise plans as time proceeds. However, the recursivity of utility, pointed out earlier, ensures that plans will be carried out.

The riskless rate and the bond price are related by

$$r_t \, dt = dS_t^{\mathbf{0}} \, / \, S_t^{\mathbf{0}}.$$

Let the returns process for risky securities be

$$dR_t^n = \frac{dS_t^n + Y_t^n dt}{S_t^n}, \quad n = 1, 2, \tag{3.10}$$

and write $R_t = (R_t^1, R_t^2)^\top$,

$$dR_t = b_t \, dt + s_t \, dW_t, \tag{3.11}$$

where b_t is $\mathbb{R}^2$-valued and each s_t is a 2×2 matrix. The equilibrium has complete markets (in the usual sense) if s_t is invertible. In that case, the state price process (p_t) satisfies

$$-dp_t \, / \, p_t = r_t \, dt + \eta_t \cdot dW_t, \quad p_0 = 1, \tag{3.12}$$

where $\eta_t \equiv s_t^{-1}(b_t - r_t \mathbf{1})$. To permit later use of the martingale approach, assume that r_t and η_t are uniformly bounded.[17] Typically, η_t is referred to as the market price of risk. We refer to it as the *market price of uncertainty* to reflect the fact that (W_t) and hence also security returns, embody both risk and ambiguity.

We establish existence of a complete markets equilibrium and characterize it 'almost' completely in closed form, under the assumption that[18]

$$0 \leq \kappa_1 < s_2^Y \quad \text{and} \quad 0 \leq \kappa_2 < s_1^Y. \tag{3.13}$$

Because these restrictions limit the ambiguity parameters to be 'small', they seem uncontentious. Moreover, they are crucial in delivering closed-form solutions. The derivation of even a limited analytical solution may seem surprising, (it was to

[17] As shown in Theorem 3.1, they are uniformly bounded in equilibrium.

[18] The positivity assumption for volatilities s_i^Y is without loss of generality because the negative of a P-Brownian motion is also a P-Brownian motion. In other words, the assumption and following theorem can be restated to apply to any parameters satisfying $0 \leq \kappa_i < \mid s_i^Y \mid$. The current statement is adopted for simplicity.

us), because supergradients for recursive multiple-priors utility depend on the volatility of the utility process (see (2.10) and (2.11)), about which one might expect typically to know very little. However, under (3.13), we show that the density generators that support equilibrium consumption processes are

$$\theta_t^{*1} = (0, \kappa_1)^\top \quad \text{and} \quad \theta_t^{*2} = (\kappa_2, 0)^\top \quad \text{for all } t, \tag{3.14}$$

which explicit expressions are the key to the availability of an analytical solution.

The description of equilibrium makes use of the process

$$\varsigma_t \equiv \exp\left\{ \frac{1}{2} \left((\kappa_1)^2 - (\kappa_2)^2 \right) t \; + \; \kappa_1 W_t^2 - \kappa_2 W_t^1 \right\} \tag{3.15}$$

and λ, the relative Pareto utility weight for country 2, which is given by (3.14), (2.2) and

$$\lambda \; = \; \frac{E\left[\int_0^T e^{-\beta t} z_t^{\theta^{*1}} \left(1 - \gamma_{1,0}^1 Y_t^1 / Y_t - \gamma_{2,0}^1 Y_t^2 / Y_t - \frac{1}{2} \Phi_t / Y_t \right) dt \right]}{E\left[\int_0^T e^{-\beta t} z_t^{\theta^{*2}} \left(\gamma_{1,0}^1 Y_t^1 / Y_t + \gamma_{2,0}^1 Y_t^2 / Y_t + \frac{1}{2} \Phi_t / Y_t \right) dt \right]}. \tag{3.16}$$

It is useful also to introduce the (shadow) price of the nontraded endowment given by

$$\overline{S}_t \; = \; \frac{1}{p_t} E\left[\int_t^T p_s \Phi_s \, ds \; \mid \; \mathcal{F}_t \right].$$

Write

$$d\overline{S}_t \; = \overline{\mu}_t \, dt \; + \; \overline{s}_t \cdot dW_t. \tag{3.17}$$

Define the (total) wealth process for i by

$$\overline{X}_t^i \; = \; S_t \cdot \gamma_t^i \; + \; \tfrac{1}{2} \, \overline{S}_t. \tag{3.18}$$

Theorem 3.1. *Assume (3.13) and define ς_t and λ as above.*

(i) There exists an Arrow-Debreu equilibrium $((c^i)_{i=1,2}, \, p)$ where

$$c_t^1 = \frac{1}{1 + \lambda \varsigma_t} Y_t, \quad c_t^2 = \frac{\lambda \varsigma_t}{1 + \lambda \varsigma_t} Y_t \; \text{ and} \tag{3.19}$$

$$p_t \; = \; \frac{e^{-\beta t} \, z_t^{\theta^{*i}}}{\left(c_t^i \, / \, c_0^i \right)}, \; \; i = 1 \; \text{ or } \; 2. \tag{3.20}$$

Here θ^{*i} and $z_t^{\theta^* i}$ are defined by (3.14) and (2.2).

(ii) Define prices of the two risky securities by

$$S_t^n = \frac{1}{p_t} E\left[\int_t^T p_\tau Y_\tau^n \, d\tau \middle| \mathcal{F}_t \right], \quad n = 1, 2, \tag{3.21}$$

and define the bond price by

$$S_t^0 = \frac{1}{p_t} E[p_T \mid \mathcal{F}_t]. \tag{3.22}$$

Let s_t be the returns volatility matrix as in (3.11). If s_t is invertible, then the Arrow-Debreu equilibrium $((c^i)_{i=1,2}, p)$ can be implemented by the Radner equilibrium $((c^i, \gamma^i)_{i=1,2}, S)$ described as follows:

(a) The interest rate r_t satisfies

$$r_t = \beta + \mu^Y - s^Y \cdot s^Y - \left[\kappa_2 s_1^Y - \frac{c_t^1}{Y_t} \left(\kappa_2 s_1^Y - \kappa_1 s_2^Y \right) \right], \tag{3.23}$$

the market price of uncertainty η_t is

$$\eta_t = s^Y + \begin{bmatrix} \kappa_2 c_t^2 / Y_t \\ \kappa_1 c_t^1 / Y_t \end{bmatrix} \tag{3.24}$$

and the state price process (p_t) satisfies (3.12).

(b) Excess returns for the two risky assets are

$$\begin{aligned} b_t^1 - r_t &= s_t^1 \cdot s^Y + \left(\kappa_2 \frac{c_t^2}{Y_t} s_t^{11} + \kappa_1 \frac{c_t^1}{Y_t} s_t^{12} \right), \\ b_t^2 - r_t &= s_t^2 \cdot s^Y + \left(\kappa_1 \frac{c_t^1}{Y_t} s_t^{22} + \kappa_2 \frac{c_t^2}{Y_t} s_t^{21} \right), \end{aligned} \tag{3.25}$$

where s_t^n is the n^{th} row of s_t, and s_t^{nm} is the (n, m) element of s_t.

(c) Wealth processes satisfy

$$\overline{X}_t^i = \beta^{-1} \left(1 - e^{-\beta(T-t)} \right) c_t^i. \tag{3.26}$$

(d) Trading strategies for the risky securities are given by[19]

$$\begin{bmatrix} S_t^1 \gamma_{1,t}^1 \\ S_t^2 \gamma_{2,t}^1 \end{bmatrix} = \overline{X}_t^1 (s_t s_t^\top)^{-1} (b_t - r_t \mathbf{1}) - \tfrac{1}{2} (s_t^\top)^{-1} \overline{s}_t + \overline{X}_t^1 \frac{\kappa_1}{\det(s_t)} \begin{bmatrix} s_t^{21} \\ -s_t^{11} \end{bmatrix} \tag{3.27}$$

[19] Trading strategies for the bond are described in the proof of the theorem.

$$\begin{bmatrix} S_t^1 \gamma_{1,t}^2 \\ S_t^2 \gamma_{2,t}^2 \end{bmatrix} = \overline{X}_t^2 (s_t s_t^\top)^{-1}(b_t - r_t \mathbf{1}) - \tfrac{1}{2}(s_t^\top)^{-1}\overline{s}_t + \overline{X}_t^2 \, \frac{\kappa_2}{\det(s_t)} \begin{bmatrix} -s_t^{22} \\ s_t^{12} \end{bmatrix},$$

where $\overline{s}_t$*, the volatility defined in (3.17), is given by*

$$\overline{s}_t \;=\; \tfrac{1-e^{-\beta(T-t)}}{\beta}\, Y_t \, s^Y \;-\; (s_t^1)^\top S_t^1 \;-\; (s_t^2)^\top S_t^2. \tag{3.28}$$

Given security prices as in (3.21), equations (3.10)-(3.11) determine the equilibrium drift and volatility of returns b_t and s_t. Thus the characterization of equilibrium provided by the theorem is complete, apart from the gap regarding the invertibility of s_t. For the particular specification of the individual processes (Y_t^i) described below (Section 4.4), we derive explicit solutions for S_t and confirm the invertibility of s_t, providing thereby a complete characterization of equilibrium. First, however, we discuss features of the equilibrium that are valid at the general level of the theorem.

Consider briefly equilibrium in the benchmark model $\kappa_1 = \kappa_2 = 0$. Because there is no ambiguity and (representative individuals in) both countries use the single and common probability measure P, equilibrium has the familiar form. For example, each country consumes a fixed proportion of the world output, implying equal growth rates of consumption. The riskless rate and market price of uncertainty are constant and depend in the familiar fashion on properties of the aggregate endowment process and excess returns for the risky securities are determined as in the representative-agent C-CAPM. Finally, each country's (value) portfolio of risky securities consists of two components, the mean-variance-efficient portfolio $\overline{X}_t^i (s_t s_t^\top)^{-1}(b_t - r_t \mathbf{1})$ and a component $-\frac{1}{2}(s_t^\top)^{-1}\overline{s}_t$ that hedges the risk due to the nontraded endowment.

3.3. Observational Equivalence

Before turning to a detailed discussion of equilibrium including, in particular, the role of ambiguity, consider an alternative interpretation of the equilibrium that does not involve ambiguity.

The supergradient (2.11) is identical to that for an expected additive utility maximizer who uses the single prior Q^* (see also (2.8)). It follows immediately, that our model's predictions can be generated alternatively by a model without ambiguity and in which beliefs are probabilistic, heterogeneous and (if P is the true measure), wrong. More precisely, the equilibrium described in the theorem is an equilibrium also for the economy in which the sets of priors $\mathcal{P}^1$ and $\mathcal{P}^2$ are replaced by the singletons Q^1 and Q^2 respectively, where these are defined by their densities

$$\frac{dQ^1}{dP} = \exp\left\{-\frac{1}{2}(\kappa_1)^2 T \;-\; \kappa_1 W_T^2\right\},$$

$$\frac{dQ^2}{dP} = \exp\left\{-\frac{1}{2}(\kappa_2)^2 T \;-\; \kappa_2 W_T^1\right\}.$$

Some insight into these measures and the difference between them is provided by noting (recall the end of Section 2.2) that $(W_t^1,\, W_t^2 + \kappa_1 t)$ is a Brownian motion under Q^1 and $(W_t^1 + \kappa_2 t,\, W_t^2)$ is a Brownian motion under Q^2.

We have already commented in Section 1.3 on the relative merits of our approach based on ambiguity. It seems to us to be: (i) less ad hoc than basing an explanation of behavior on a particular specification of heterogeneous and erroneous beliefs; and (ii) more coherent in that it models individuals as being aware of the possibility that any single probability measure that they consider could be wrong and seeking, therefore, to adopt robust decisions.

A final point that has not been made previously (see [10], however), is that the above observational equivalence is not complete. In the context of the equity premium puzzle, for example, one employs informally auxiliary 'data' regarding the degree of risk aversion. One aspect of the noted puzzle is that to explain historical averages of the excess return to equity one needs to assume a degree of risk aversion in excess of what seems plausible given introspection and/or casual observation. However, if only part of the excess return to equity is a premium for bearing risk, (with the remainder being a premium for bearing ambiguity), then only a smaller degree of risk aversion is required and consistency with other evidence may be possible. In this way, the (re)interpretation of security returns as involving ambiguity in addition to risk can matter for empirical performance.

4. THE NATURE OF EQUILIBRIUM

In the sequel, references to 'mean excess returns', covariances or other moments of distributions induced by stochastic processes are intended relative to the measure P. The reader may wish to think of P as being the true measure.

4.1. Which Country Faces More Ambiguity?

Naturally, our interpretation of the equilibrium described in the theorem centers on the presence of ambiguity. One aspect of the presence of ambiguity is the question "which country faces more ambiguity?" We will see that the answer influences several properties of equilibrium.[20]

It is important to understand how the above question differs from the issues addressed in Section 2.5. There we showed that given a preference order of the sort employed in our equilibrium model, where $\kappa = (0, \kappa_1)^\top$ (or $\kappa = (\kappa_2, 0)^\top$), an increase in κ_1 (or in κ_2) can be interpreted as an increase in ambiguity aversion. This justifies in part our interpretation of some expressions in the theorem as reflecting ambiguity aversion on the part of either 1 or of 2. In particular, such a change in κ_i models a hypothetical change in the single country i. However, it is not relevant to a comparisons of the two countries; for example, it does *not* justify interpreting $\kappa_2 > \kappa_1$ as "country 2 is more ambiguity averse than 1". The reason is that the two countries perceive ambiguity in different parts of the state space (1 views only W^2 as ambiguous, while 2 views only W^1 as ambiguous), while the two hypothetical versions of country i dealt with in Section 2.5 agree on the identity of the ambiguous events.

Our answer to the question posed at the start is that *country 2 faces more ambiguity than does country* 1 if

$$\kappa_2 s_1^Y - \kappa_1 s_2^Y > 0. \tag{4.1}$$

We use the aggregate output process (Y_t) to measure ambiguity. Thus an informal justification for the suggested interpretation of (4.1) is that it is true if s_1^Y is sufficiently large relative to s_2^Y and in that case, aggregate output is driven mostly by W^1, which is unambiguous for country 1 but ambiguous for 2.

For a more formal argument, let (Y_t^*) be the 'reference' process satisfying

$$dY_t^*/Y_t^* = \mu^* dt + (s_1^Y + s_2^Y)\, dW_t^1, \; Y_0^* = Y_0,$$

where the drift μ^* is chosen so that country 1 is indifferent between (Y_t^*) and the aggregate output process (Y_t). Because (Y_t^*) involves no ambiguity for 1, it serves as a 'risky equivalent' process for (Y_t) from the perspective of country 1 and $\mu^Y - \mu^*$ measures the 'cost' of ambiguity in (Y_t) for country 1. Because both processes are geometric, we can apply the illustrative calculation in Section 2.4 to compute (from the hypothesis that the two processes imply the same utility at time 0) that

$$\mu^Y - s^Y \cdot s^Y/2 - \kappa_1 s_2^Y = \mu^* - \left(s_1^Y + s_2^Y\right)^2/2.$$

[20] Put another way, the sign of $\kappa_2 s_1^Y - \kappa_1 s_2^Y$ affects the qualitative properties of equilibrium and our goal here is to suggest an interpretation for this sign.

Similarly for country 2 use the 'risky equivalent' process (Y_t^{**}), where

$$dY_t^{**}/Y_t^{**} = \mu^{**}dt + (s_1^Y + s_2^Y)\,dW_t^2,\ Y_0^{**} = Y_0;$$

and calculate that

$$\mu^Y - s^Y \cdot s^Y/2 - \kappa_2 s_1^Y = \mu^{**} - \left(s_1^Y + s_2^Y\right)^2/2.$$

Conclude that (4.1) is equivalent to $\mu^* > \mu^{**}$. Because the two reference risky processes (Y_t^*) and (Y_t^{**}) involve the same risk for both countries, (the measure P applies in both cases and the identical probability distributions are induced), we are justified in interpreting $\mu^* > \mu^{**}$, or equivalently,

$$\mu^Y - \mu^* < \mu^Y - \mu^{**},$$

as expressing that the cost of ambiguity for 1 is smaller than that for 2.

4.2. Consumption

Equation (3.19) makes explicit the implications of ambiguity for the equilibrium (or efficient) allocation of consumption. Individual consumption levels depend not only on the aggregate endowment but also on country-specific shocks W_t^1 and W_t^2. This dependence is readily understood as we now show.

Let θ^{*1} and θ^{*2} be the density generators given in (3.14) and $Q^{\theta^{*1}}$ and $Q^{\theta^{*2}}$ the corresponding measures as in (2.3). Given (2.8) and the discussion in Section 3.3, it is natural to refer to $Q^{\theta^{*1}}$ and $Q^{\theta^{*2}}$ as *ambiguity-adjusted probabilistic beliefs* of the two individuals. We noted above that $(W_t^1, W_t^2 + \kappa_1 t)$ is a Brownian motion under $Q^{\theta^{*1}}$. In particular, under $Q^{\theta^{*1}}$ the unconditional distribution for W_t^2 is $N(-\kappa_1 t, t)$, while it is $N(0,t)$ under P. This leftward shift as a result of country 1's ambiguity about W^2 is intuitive. Roughly, the assumption that aggregate output covaries with W^2 ($s_2^Y > 0$) implies that higher values of W_t^2 are better for both countries and particularly for 1.[21] As a multiple-priors decision-maker, country 1 evaluates prospects through the worst-case scenario. Thus she is led to attach relatively less weight (than under P) to good realizations of W_t^2, which explains the leftward shift and the related fact that the restricted density $\left.\frac{dQ^{\theta^{*1}}}{dP}\right|_{\mathcal{F}_t}$ is decreasing in W_t^2.

[21] In precise terms, the claim is that (under the parameter assumptions in the theorem) the equilibrium utility process for country 1 has positive volatility with respect to W_t^2. This is equivalent to (3.14), which is the key to the theorem.

The ambiguity adjusted probabilities affect consumption because, as in (2.9), i's marginal rate of substitution between time 0 and time t consumption is

$$MRS^i_{0,t} = \frac{e^{-\beta t} u'(c^i_t)}{u'(c^i_0)} \left. \frac{dQ^{\theta^{*i}}}{dP} \right|_{\mathcal{F}_t},$$

with the density term acting like a preference shock that redistributes weight away from states where W^j_t is large. Thus, if we define (ς_t) as in (3.15), or equivalently by[22]

$$\varsigma_t = \left. \frac{dQ^{\theta^{*2}} / dP}{dQ^{\theta^{*1}} / dP} \right|_{\mathcal{F}_t},$$

then (i) ς_t is increasing in W^2_t and decreasing in W^1_t; and (ii) a larger value for ς_t increases $MRS^2_{0,t}$ relative to $MRS^1_{0,t}$, inducing a shift in time t consumption towards individual 2. Given the log utility specification, the latter effect takes the precise form

$$\varsigma_t = \frac{c^2_t / c^2_0}{c^1_t / c^1_0},$$

that is, ς_t equals the relative average consumption growth rates of the two countries. Because ς_t measures the difference in ambiguity-adjusted beliefs (restricted to $\mathcal{F}_t$), we refer to (ς_t) as the *disagreement process* (of 2 relative to 1).[23]

The above simple intuition explains also other nonstandard features of equilibrium consumption processes. First, the presence of disagreement leads to the 'crossing' of individual consumption paths in some realizations; that is, even if $\lambda < 1$ and thus $c^1_0 > c^2_0$, country 2 consumes more than country 1 at times and states where ς_t is sufficiently large. Assuming that P is the true measure, then, conditional on $\mathcal{F}_\tau$, the (log) consumption ratio $\log (c^2_t / c^1_t)$ is normally distributed with mean $\left[\log (c^2_\tau / c^1_\tau) + \frac{1}{2} ((\kappa_1)^2 - (\kappa_2)^2) (t - \tau)\right]$ and variance $[((\kappa_1)^2 + (\kappa_2)^2) (t - \tau)]$; it is a P-martingale if $\kappa_1 = \kappa_2$.[24]

Unlike the case in the standard model, the consumption share c^i_t/Y_t of each country is stochastic. The behavior of these shares is readily deduced from Ito's

[22] Equivalence follows from (2.2) and (2.3).

[23] Note that this disagreement is endogenous and is determined as part of the equilibrium. This differentiates it from the disagreement process in Basak's model [3].

[24] Because each individual consumes a deterministic and common fraction of wealth in equilibrium (see (3.26)), the log wealth ratio has similar properties.

Lemma - $d\left(c_t^1/Y_t\right)$ is positively correlated with dY_t if and only if 1 faces less ambiguity than does 2 in the sense of (4.1). This is true, for example, if s_1^Y is sufficiently larger than s_2^Y. Then aggregate output is driven mostly by W^1, which situation is favorable for country 1 because her ambiguity concerns only the other process W^2 and country 1's consumption increases more than proportionately with total output. The mechanics underlying this effect stem from the following relation between output growth and the disagreement process:

$$cov_t\left(\frac{d\varsigma_t}{\varsigma_t}, \frac{dY_t}{Y_t}\right) = -\kappa_2 s_1^Y + \kappa_1 s_2^Y < 0.$$

In light of the connection described above between (ς_t) and marginal rates of substitution, the instantaneous change in the ratio $MRS_{0,t}^1/MRS_{0,t}^2$ covaries with dY_t; thus dY_t being positive leads to an increase in the share of consumption going to country 1.

Turn to instantaneous mean growth rates. Ito's Lemma applied to (3.19) shows that dc_t^i/c_t^i, $i = 1, 2$, have drifts

$$\begin{aligned} \mu_t^{c,1} &= \mu^Y + \frac{c_t^2}{Y_t}\left[\left(s_1^Y\kappa_2 - s_2^Y\kappa_1\right) + \left(\frac{c_t^2}{Y_t}(\kappa_2)^2 - \frac{c_t^1}{Y_t}(\kappa_1)^2\right)\right], \\ \mu_t^{c,2} &= \mu^Y - \frac{c_t^1}{Y_t}\left[\left(s_1^Y\kappa_2 - s_2^Y\kappa_1\right) + \left(\frac{c_t^2}{Y_t}(\kappa_2)^2 - \frac{c_t^1}{Y_t}(\kappa_1)^2\right)\right]. \end{aligned} \tag{4.2}$$

Evidently, mean growth rates differ from one another and from the rate for aggregate output, with one country growing faster and the other slower than aggregate output. To identify the faster growing country, assume for simplicity that

$$\kappa_1 = \kappa_2. \tag{4.3}$$

Then

$$\mu_t^{c,1} - \mu_t^{c,2} = \kappa_1\left(s_1^Y - s_2^Y\right) + (\kappa_1)^2\left(\frac{c_t^2 - c_t^1}{Y_t}\right). \tag{4.4}$$

Thus $s_1^Y > s_2^Y$ (which is here equivalent to (4.1), that is, 2 faces more ambiguity than 1) contributes to a larger mean growth rate in country 1 and this effect is larger the larger is the common degree of ambiguity aversion. The second component on the right is time varying and stabilizing in that it raises the relative mean growth rate of the country with lower consumption. The difference in mean growth rates is an increasing function of λ (for given realizations of W). Consequently, the noted difference increases if initial endowments are redistributed in favor of country 2.[25]

Second-order moments of consumption processes are also nonstandard. Once again, by Ito's Lemma, the volatilities of dc_t^i/c_t^i, $i = 1, 2$, are given by

$$\begin{aligned} s_t^{c,1} &= s^Y + \frac{c_t^2}{Y_t} \begin{bmatrix} \kappa_2 \\ -\kappa_1 \end{bmatrix}, \\ s_t^{c,2} &= s^Y + \frac{c_t^1}{Y_t} \begin{bmatrix} -\kappa_2 \\ \kappa_1 \end{bmatrix}. \end{aligned} \tag{4.5}$$

Thus, from (3.13), consumption growth rates in the two countries are positively correlated, as in the standard risk-based model. However, unlike the standard model, the country-specific growth rate $dc_t^1/c_t^1 - dY_t/Y_t$ is positively correlated with shocks in country 1, that is,

$$cov_t\left(dc_t^1/c_t^1 - dY_t/Y_t,\, dW_t^1\right) = \kappa_2\, s_1^Y\, c_t^2\,/\,Y_t > 0.$$

Such positive correlation is essentially what Lewis [37, p. 574] defines as *consumption home bias*. (See Section 4.4 for more on her definition and the predictions of our model.)

Assuming (4.3), then (4.5) implies that

$$\begin{aligned} s_t^{c,1} \cdot s_t^{c,1} &= s^Y \cdot s^Y + 2\,(\kappa_1)^2 \left(\frac{c_t^2}{Y_t}\right)^2 + 2\kappa_1 \left(\frac{c_t^2}{Y_t}\right) \left(s_1^Y - s_2^Y\right) \\ s_t^{c,2} \cdot s_t^{c,2} &= s^Y \cdot s^Y + 2\,(\kappa_1)^2 \left(\frac{c_t^1}{Y_t}\right)^2 - 2\kappa_1 \left(\frac{c_t^1}{Y_t}\right) \left(s_1^Y - s_2^Y\right) \end{aligned}$$

and hence that the difference in variances is

$$\begin{aligned} s_t^{c,1} \cdot s_t^{c,1} - s_t^{c,2} \cdot s_t^{c,2} &= 2\,(\kappa_1)^2 \left(\frac{c_t^2 - c_t^1}{Y_t}\right) + 2\kappa_1 \left(s_1^Y - s_2^Y\right) \\ &= 2\,\left(\mu_t^{c,1} - \mu_t^{c,2}\right). \end{aligned}$$

Though both means and variances are stochastic, the last equality implies that at all times and states, the country with higher mean growth rate also has

[25] It is straightforward to show that λ increases in response to such a redistribution of initial endowments. We use this fact frequently in the sequel.

the larger variance of consumption growth. The close connection between the difference in variances and the difference in mean growth rates implies also that factors underlying both are similar. For example, (i) $s_1^Y > s_2^Y$ contributes to a larger variance for consumption growth in country 1 relative to that in country 2 and (ii) a redistribution of initial endowments in favor of country 2 (that is, an increase in λ) increases the variance of consumption growth for country 1 relative to that for country 2. Consolidating with the previous discussion of mean growth rates and information about levels provided by (3.19), it follows that an initial redistribution towards country 2 results for that country in a higher initial level of consumption, and (in relative terms) a *lower* mean and variance for the rate of growth of consumption.

In terms of absolute (rather than relative) variance, consumption growth has a higher variance than aggregate output growth for at least one country, and for both countries if $s_1^Y = s_2^Y$.

4.3. Riskless Rate, Market Price of Uncertainty and Excess Returns

Equation (3.23) shows that like risk, ambiguity drives down the riskless rate; their effects are captured respectively by $s^Y \cdot s^Y$ (the variance of total output growth) and the last bracketed expression on the right. The riskless rate is stochastic and varies over time between the extremes $\beta + \mu^Y - s^Y \cdot s^Y - s_2^Y \kappa_1$ and $\beta + \mu^Y - s^Y \cdot s^Y - s_1^Y \kappa_2$, depending on the distribution of aggregate consumption. To interpret the latter dependence, assume (4.1). Then r_t is increasing in 1's share of total consumption. The reason for this dependence is that by (3.26), the noted consumption share serves as a proxy for 1's share of total wealth. Moreover, by (4.1) country 1 faces less ambiguity than does 2. Thus as the distribution of wealth shifts in favor of 1, the 'aggregate' ambiguity in the economy falls. Because ambiguity depresses the riskless rate, the latter is induced to rise.

Under (4.1), it is also the case that r_t is increasing as a function of 1's initial endowment (decreasing in λ). In the special case that 1 and 2 face the identical ambiguity ($\kappa_2 s_1^Y = \kappa_1 s_2^Y$), then r_t is constant and independent of the initial distribution.

Ambiguity acts to increase the market price of uncertainty, with the qualitative features of its effect being similar to those discussed for the riskless rate. The time variation of η_t is of particular interest. Refer to the component η_t^i as the domestic market price of uncertainty for country i. The significance of η_t^1, for example, is that it determines equilibrium excess returns for 'domestic securities' in country

1. That is, for a security whose return process (R_t^*) satisfies $dR_t^* = b_t^* dt + s_t^* dW_t^1$, its mean excess return equals

$$b_t^* - r_t = s_t^* \eta_t^1 \equiv s_t^* \left(s_1^Y + \kappa_2 - \kappa_2 c_t^1 / Y_t \right).$$

It is noteworthy that each domestic market price η_t^i is a decreasing function of c_t^i / Y_t. Campbell [8] argues that asset market data in a number of countries suggest that the (domestic) market price of uncertainty is negatively correlated with the level of domestic consumption. Our model delivers negative correlation, though with the share of aggregate consumption that occurs domestically. An immediate further implication is that the market price of uncertainty in country 1 is increasing in country 2's share of aggregate consumption. Finally, an increase in λ increases the domestic market price of uncertainty in country 1 and reduces that in country 2.

Turn to the excess returns (3.25). Rewrite them in vector form

$$b_t - r_t 1 = s_t s^Y + s_t \sum_{i=1}^{2} \frac{c_t^i}{Y_t} \theta_t^{*i},$$

where θ_t^{*i} satisfies (3.14) for each i. A corresponding decomposition of excess returns is derived in [10] in a representative agent model and they interpret the two components as premia for risk and ambiguity respectively. A similar interpretation applies here. The first risk premium term is the familiar instantaneous covariance of asset returns with the growth rate of aggregate consumption. The second component (which vanishes if each $\kappa_i = 0$) is a consumption-share weighted sum of individual ambiguity premia. If returns to the country i security are positively correlated with shocks in both countries ($s_t^{im} > 0$, for $m = 1, 2$), then the ambiguity premium for the security is positive. This is true in particular for each country given the specification of dividend processes described in the next section (see Corollary 4.1).

4.4. Country-Specific Securities and Home Bias

The properties of equilibrium discussed to this point depend on the hypothesis that aggregate output is geometric as in (3.4), but not on how that output is distributed between the dividend streams (Y_t^i) of the two traded securities and the nontraded endowment Φ. We turn now to properties that depend on the specification of (Y_t^i).

Henceforth assume that

$$Y_t^i / Y_t = v(W_t^i), \quad i = 1, 2, \tag{4.6}$$

where the 'share' function $v : \mathbb{R}^1 \longrightarrow (0, 1/2)$ is twice continuously differentiable with $v' > 0$. An immediate consequence is that

$$0 < Y_t^i, \; i = 1, 2, \text{ and } Y_t^1 + Y_t^2 < Y_t.$$

A second consequence is that, by Ito's Lemma,

$$\begin{aligned} dY_t^1/Y_t^1 &= a_t^1 \, dt + \left[s_1^Y + \frac{v'(W_t^1)}{v(W_t^1)} \right] dW_t^1 + s_2^Y \, dW_t^2 \\ dY_t^2/Y_t^2 &= a_t^2 \, dt + s_1^Y \, dW_t^1 + \left[s_2^Y + \frac{v'(W_t^2)}{v(W_t^2)} \right] dW_t^2 , \end{aligned} \tag{4.7}$$

for suitable drifts a_t^1 and a_t^2. Consequently, $dY_t^1/Y_t^1 - dY_t/Y_t$ is positively correlated with W_t^1 and uncorrelated with W_t^2. This justifies interpretation of Y_t^1 as the domestic security in country 1 - the idiosyncratic part of its growth rate is driven by domestic shocks. Moreover, because a similar statement applies to Y_t^2 and because the representative investor in country 1 views W_t^1 as unambiguous and W_t^2 as ambiguous, the foreign security is 'more ambiguous' for her.Thus the above specification of dividend streams is consistent with our guiding intuition, namely that foreign securities are more ambiguous than domestic securities.[26]

Given the specification for Y_t^i, we can elaborate on or reformulate the consumption home bias that is delivered by our model. From (4.5) and (4.7), it follows that

$$cov_t \left(dc_t^1/c_t^1 - dY_t/Y_t, \, dY_t^1/Y_t^1 - dY_t/Y_t \right) = \kappa_2 \frac{c_t^2}{Y_t} \frac{v'(W_t^1)}{v(W_t^1)} > 0.$$

In other words, there is positive correlation between country-specific consumption growth and country-specific output growth [37, p. 574].

[26] This discussion is admittedly informal. We do not yet have a well-founded formal definition of 'more ambiguous than'.

Conformity with the guiding intuition is the reason that we cannot specify dividends so that they exhaust total output and thus obviate the need for a nontraded endowment. For example, if we adopt (4.6) for country 1 and then define Y_t^2 as $Y_t - Y_t^1$, then Y_t^2/Y_t is driven by the shock in country 1.

Turn next to home bias in equities. Trading strategies for the two risky securities are given by (3.27). Suppose that our model is correct, including, in particular, regarding security prices and returns volatilities. Then, if one mistakenly adopts the standard model with no ambiguity, the first two expressions on the right side of (3.27) would be used to predict the components of the (value) portfolio of risky assets. The error that results is captured in the third term on the right which represents the effect of ambiguity. If volatilities satisfy

$$s_t^{ij} > 0, \; i, \, j = 1, \, 2, \; \text{ and } \; \det(s_t) > 0, \qquad (4.8)$$

then ambiguity induces country i to invest more in the domestic asset and less in the foreign asset. Thus from the perspective of a model that ignores ambiguity and focuses exclusively on the risk characteristics of securities, there is a seemingly irrational bias towards domestic securities. In this sense, if (4.8) is satisfied, our model can resolve the equity home bias puzzle, at least in qualitative terms.

Finally, (4.8) is valid, as shown in the following corollary of Theorem 3.1.

Corollary 4.1. *Let dividend processes be given by (4.6) and refer to the Arrow-Debreu equilibrium in Theorem 3.1. Then the returns volatility matrix s_t satisfies (4.8). In particular, the Radner equilibrium described in the Theorem exists.*

The positivity of returns volatilities has other noteworthy implications. In particular, it follows immediately that security returns in the two countries are positively correlated ($cov_t(dR_t^1, dR_t^2) > 0$) and, from (4.5), that returns are positively correlated with consumption growth in each country ($cov_t(dR_t^i, dc_t^i / c_t^i) > 0$).

Finally, with regard to home bias in equities, in the introduction we pointed to evidence that investors are more optimistic about domestic securities. Such a bias in expectations about mean returns can be identified in our model as follows: While the returns process for security 1 is given by (3.11), investors in the two countries view the driving processes W^1 and W^2 differently. In particular, in terms of the ambiguity adjusted probability measures (Section 4.2), 1 views $(W_t^1, \, W_t^2 + \kappa_1 t)$ as a Brownian motion while 2 views $(W_t^1 + \kappa_2 t, \, W_t^2)$ as a Brownian motion. Rewriting the returns process in terms of the Brownian driving process that is appropriate for each investor, leads to

$$\begin{aligned} dR_t^1 &= \left(b_t^1 - \kappa_1 s_t^{12}\right) dt + s_t^{11} \, dW_t^1 + s_t^{12} \, d(W_t^2 + \kappa_1 t), \\ dR_t^1 &= \left(b_t^1 - \kappa_2 s_t^{11}\right) dt + s_t^{11} \, d(W_t^1 + \kappa_2 t) + s_t^{12} \, dW_t^2. \end{aligned} \qquad (4.9)$$

Consequently, after adjusting for ambiguity, country 1 attaches a higher mean return to security 1 than does country 2 if and only if

$$\kappa_2 s_t^{11} > \kappa_1 s_t^{12}. \tag{4.10}$$

From the explicit expressions for the returns volatilities that are derived in Appendix B, conclude that (4.10) and hence the noted relative optimism are confirmed for our model in the symmetric case[27]

$$\kappa_2 s_1^Y = \kappa_1 s_2^Y.$$

We can interpret this prediction as being confirmed by the survey evidence regarding relative optimism about domestic securities cited in Section 1.2; to do so interpret elicited probability measures as including an adjustment for ambiguity.

It is noteworthy that differences in ambiguity-adjusted expectations are restricted to means. Agreement regarding volatilities is consistent with the well-known relative ease of estimating the variance-covariance matrix of returns.

4.5. A Further Parametrization and Trading Strategies

To study further the nature of trading strategies, we specialize (4.6) and assume:[28]

$$v(x) = \begin{cases} \frac{1}{4}\, e^x & \text{if } x \leq 0 \\ \frac{1}{2}\left(1 - \frac{1}{2}e^{-x}\right) & \text{if } x \geq 0. \end{cases}$$

It is readily computed that (with respect to the reference measure P),

$$E\left(Y_t^i \,/\, Y_t\right) = \tfrac{1}{4} \text{ and } \lim_{t \longrightarrow \infty} var\left(Y_t^i \,/\, Y_t\right) = \tfrac{1}{16}.$$

As explained following Theorem 3.1, security prices (3.21) are the key to the complete description of equilibrium. Since (3.20) provides an explicit characterization of state prices, closed-form solutions for all endogenous variables can be obtained if the dividend streams (Y_t^i) are specified so that the integration in (3.21) can be carried out analytically. The preceding specification for v permits such integration. For example, the formulae in Theorem 3.1 and (4.6) imply that S_t^1 can be written in the form

$$S_t^1 = c_t^1 \int_t^T e^{-\beta(\tau - t)}\, E_t\left[v(W_\tau^1)\right] d\tau + c_t^2 \int_t^T e^{-\beta(\tau - t)}\, E_t\left[\frac{z_\tau^{\theta^{*2}}}{z_t^{\theta^{*2}}} v(W_\tau^1)\right] d\tau$$

[27] See Section 4.1 for interpretation of this equality.

[28] Contrary to previous assumptions, v fails to be twice continuously differentiable though only at the origin. This does not affect preceding arguments, including (4.7), for example.

and the conditional expectations can be computed explicitly in terms of the standard univariate normal cdf. Therefore, $S_t^1 = h(t, W_t, Y_t)$ and $h(\cdot)$ is in closed-form up to the presence of some Riemann integrals.

However, the resulting expressions are lengthy and not easily interpreted and thus we have simulated our model numerically. For parameter values, we take[29]

$$\mu^Y = .0179,\ s_1^Y = s_2^Y = .0406,\ \beta = .02 \text{ and } T = 42.5\,.$$

To treat the two countries symmetrically, we assume that initial endowments are such that the relative utility weight λ equals 1 and that $\kappa_1 = \kappa_2$. Finally, the common value of the ambiguity parameter is specified to be .02.

To clarify the meaning and plausibility of the value .02, note that just as we derived (4.9), we can derive the corresponding 'ambiguity-adjusted' laws of motion for the aggregate endowment process. For country 1, for example, it is

$$dY_t/Y_t = (\mu^Y - s_2^Y \kappa_1)\ dt + s_1^Y dW_t^1 + s_2^Y\ (dW_t^2 + \kappa_1 t),$$

implying that the adjustment for ambiguity calls for lowering the mean to .0170, a reduction of only about 6%. From this perspective, a value of .02 for the κ_i's does not seem excessive.

As a benchmark, note that if $\kappa = 0$, then the equilibrium trading strategies are to buy and hold 1/2 share of each of the domestic and foreign securities. In contrast, Figure 1 describes the optimal holding $\gamma_{2,t}^1$ of the foreign security in one realization of the Brownian motion. There is a downward bias ($\gamma_{2,t}^1 < 1/2$) and continual retrading. To illustrate the latter, Figure 2 plots the corresponding turnover process $\mid d\gamma_{2,t}^1 \mid$.

5. CONCLUDING COMMENTS

We have extended the standard, log-utility, two-country general equilibrium model by incorporating a feature that seems to us to be intuitive, namely (greater) ambiguity about foreign securities. This extension moves predictions in the right direction in terms of helping to resolve the puzzles concerning home bias in consumption and equity. A more thorough (and quantitative) assessment of the model's usefulness for this purpose is left for future work. A multi-country extension would permit a fairer comparison with data.

[29] The values for μ^Y and s^Y are based on the discretized version of (3.4) and IFS quarterly consumption data (transformed into per capita terms) for the period 1957:1 to 1999:3.

On the other hand, we hope that the reader is already convinced of the broader potential usefulness of the model as a tool for exploring other questions concerning dynamic stochastic economies. The fact that the model admits a complete closed-form solution should make it useful in a variety of applications.

A. Appendix A

This appendix proves Theorem 3.1. The parametric restrictions

$$0 \leq \kappa_1 < s_2^Y \text{ and } 0 \leq \kappa_2 < s_1^Y \tag{A.1}$$

are adopted throughout.

Given consumption processes c^i, $i = 1, 2$, (σ_t^i) is the volatility associated with i's utility process $(V_t^i(c^i))$. Write

$$\sigma_t^1 = \left(\sigma_{1,t}^1, \sigma_{2,t}^1\right)^\top \text{ and } \sigma_t^2 = \left(\sigma_{1,t}^2, \sigma_{2,t}^2\right)^\top.$$

Thus $\sigma_{2,t}^1$ is that part of the volatility of 1's utility process that corresponds to W_t^2, the component that is ambiguous for 1.

Lemma A.1. *For the specific consumption processes c^1 and c^2 defined by (3.19), the volatilities of utility satisfy*

$$\sigma_{2,t}^1 > 0 \text{ and } \sigma_{1,t}^2 > 0. \tag{A.2}$$

Proof. Consider individual 1's utility process and show

$$\sigma_{2,t}^1 > 0; \tag{A.3}$$

the other inequality can be proven analogously.

Let Q be the measure defined as in (2.1)-(2.3) by taking

$$\theta_t = (0, \kappa_1)^\top \tag{A.4}$$

for all t. Define

$$V_t = E_Q\left[\int_t^T e^{-\beta(\tau - t)} \log c_\tau^1 \, d\tau \,\middle|\, \mathcal{F}_t\right]. \tag{A.5}$$

Then (V_t) is an Ito process and we can write

$$dV_t = \mu_t^V \, dt + \sigma_{1,t}^V dW_t^1 + \sigma_{2,t}^V dW_t^2.$$

Claim: $\sigma^V_{2,t} > 0$. If true, then we can conclude that $V_t = V^1_t(c^1)$ and hence that $\sigma^1_{2,t} = \sigma^V_{2,t} > 0$. The point, roughly, is that the positivity of the volatility $\sigma^V_{2,t}$ validates the specification (A.4) as the one that is consistent with the minimization over all priors in $\mathcal{P}$, as described in (2.10). In more formal terms, Girsanov's Theorem implies that (V_t) solves the BSDE:

$$dV_t = [-\log c^1_t + \beta V_t + \kappa_1 \mid \sigma^V_{2,t} \mid] \, dt + \sigma^V_t \cdot dW_t, \ V_T = 0.$$

But given that $\kappa = (0, \kappa_1)^\top$, this is the BSDE that defines, via an appropriate form of (2.12), the utility process $(V^1_t(c^1))$. By uniqueness of the solution, conclude that $V^1_t(c^1) = V_t$.

Turn to the proof of the claim. Given the explicit expression for V_t, a direct approach is possible. Substitute into (A.5) for c^1_τ using (3.19) to obtain $V_t = K_t - L_t$, where $K_t = E_Q\left[\int_t^T e^{-\beta(\tau-t)} \log Y_\tau \, d\tau \middle| \mathcal{F}_t\right]$ and $L_t = E_Q\left[\int_t^T e^{-\beta(\tau-t)} \log(1+\lambda\varsigma_\tau) \, d\tau \middle| \mathcal{F}_t\right]$. By Girsanov's Theorem, $(W^1_t, W^2_t + \kappa_1 t)$ is a martingale under Q. Thus we can compute K_t and, to a lesser degree, L_t. Because Y is a geometric process, compute that

$$K_t = a_t + d_t \left(s^Y_1 W^1_t + s^Y_2 W^2_t\right),$$

where a_t is deterministic and

$$d_t = \int_t^T e^{-\beta(\tau-t)} \, d\tau \, .$$

Write $L_t = H_t(W^1_t, W^2_t)$, where

$$H_t\left(w^1, w^2\right) = \int_t^T e^{-\beta(\tau-t)} \, E^{\tau-t} \log\left(1 + \lambda \, e^{\left(\frac{1}{2}((\kappa_1)^2 - (\kappa_2)^2)\tau + \kappa_1 x^2 - \kappa_2 x^1\right)}\right) d\tau$$

and the expectation $E^{\tau-t}$ refers to integration on the plane of points (x^1, x^2) with respect to the bivariate normal distribution $N(m_{\tau-t}, \Sigma_{\tau-t})$ with

$$m_{\tau-t} = \left(w^1, w^2 - \kappa_1(\tau - t)\right) \quad \text{and} \quad \Sigma_{\tau-t} = (\tau - t) \, I_{2\times 2}.$$

By Ito's Lemma and the preceding, it suffices to prove that (for all $(t, w^1, w^2) \in [0,T] \times \mathbb{R}^2$)

$$d_t s^Y_2 - \partial H_t(w^1, w^2) \,/\, \partial w^2 > 0. \tag{A.6}$$

Direct computation and reversing the order of differentiation and integration (by [6, p. 215]) yields

$$\frac{\partial}{\partial w^2} E^{\tau - t} \log \left(1 + \lambda\, e^{\left(\frac{1}{2}((\kappa_1)^2 - (\kappa_2)^2)\tau \,+\, \kappa_1 x^2 \,-\, \kappa_2 x^1\right)}\right) \;=$$

$$\frac{\partial}{\partial w^2} \left[\int_{\mathbb{R}^2} \log \left(1 + \lambda\, e^{\left(\frac{1}{2}((\kappa_1)^2 - (\kappa_2)^2)\tau \,+\, \kappa_1 (x^2 + w^2 - \kappa_1(\tau - t)) \,-\, \kappa_2 (x^1 - w^1)\right)}\right) dN(0, \Sigma_{\tau - t}) \right]$$

$< \kappa_1$, which leads to (A.6). ■

Lemma A.2. *For any given $\lambda > 0$, the consumption processes defined in (3.19) solve (uniquely)*[30]

$$\max \left\{ V^1(e^1) \,+\, \lambda\, V^2(e^2) : \, e^1,\, e^2 \in \mathcal{C},\, e^1 + e^2 \leq Y \right\}. \tag{A.7}$$

Proof. Clearly, c^1 and c^2 are feasible. Therefore, it suffices to verify that there exists a $\mathbb{R}^1_{++}$-valued shadow price process $\pi \,=\, (\pi_t)$ satisfying

$$\pi \,\in\, \partial V^1(c^1) \,\cap\, \lambda\, \partial V^2(c^2), \tag{A.8}$$

where $\partial V^i(c^i)$ denotes the set of supergradients for V^i at c^i. Given such a π, then

$$\begin{aligned} & V^1(e^1) \,+\, \lambda\, V^2(e^2) \,-\, V^1(c^1) \,-\, \lambda\, V^2(c^2) \\ \leq\;\; & E_P \left[\int_0^T \pi_t\, (e^1_t - c^1_t)\, dt \right] \,+\, \lambda\, E_P \left[\int_0^T (\pi_t / \lambda)\, (e^2_t - c^2_t)\, dt \right] \\ =\;\; & E_P \left[\int_0^T \pi_t\, (\Sigma_i (e^i_t - c^i_t))\, dt \right] \,=\, E_P \left[\int_0^T \pi_t\, (\Sigma_i e^i_t - Y_t)\, dt \right] \,\leq\, 0. \end{aligned}$$

To establish (A.8), recall (2.11) and thus that $\pi^i \,\in\, \partial V^i(c^i)$, where

$$\pi^i_t(c^i) \,=\, e^{-\beta t}\, z^{\theta^{*i}}_t / c^i_t,$$

and, following (2.10),

$$\theta^{*1} \,\in\, \Theta^1_c \,\equiv \{ \, (\theta_t) : \, \theta_t \,=\, (0, \kappa_1)^\top \,\otimes sgn(\sigma^1_t) \text{ all } t \}, \tag{A.9}$$

$$\theta^{*2} \,\in\, \Theta^2_c \,\equiv \{ \, (\theta_t) : \, \theta_t \,=\, (\kappa_2, 0)^\top \,\otimes sgn(\sigma^2_t) \text{ all } t \}.$$

[30] Because c^i denotes the equilibrium consumption process, we use e^i below to denote the generic process in $\mathcal{C}$.

By the positivity of volatilities in (A.2), (A.9) is equivalent to

$$\theta_t^{*1} = (0, \kappa_1) \quad \text{and} \quad \theta_t^{*2} = (\kappa_2, 0) \quad \text{for all } t.$$

Thus (A.8) is satisfied by π, where

$$\pi_t = e^{-\beta t}\, z_t^{\theta^{*1}} / c_t^1 = \lambda\, e^{-\beta t}\, z_t^{\theta^{*2}} / c_t^2. \quad \blacksquare \tag{A.10}$$

Turn to description of an equilibrium of the form $\left((c^i, \gamma^i)_{i=1,2}, S\right)$, where the c^i's are defined by (3.19) and where λ is defined by (3.16). Because these consumption processes are efficient, they can be implemented as part of an Arrow-Debreu equilibrium and subsequently also as part of a Radner equilibrium. To see this, let $p = \pi/\pi_0$, where π is defined in (A.10). Use p as a state price process; in particular, define security prices S^n, $n = 0, 1, 2$, by (3.22) and (3.21) and associated returns as in (3.11). Define person i's *financial wealth* by $X_t^i \equiv \gamma_t^i \cdot S_t$. Let

$$\psi_{n,t}^i \equiv S_t^n \gamma_{n,t}^i \,/\, X_t^i, \quad n = 1, 2, \tag{A.11}$$

and $\psi_t^i = \left(\psi_{1,t}^i, \psi_{2,t}^i\right)^\top$ denote portfolio shares. (If $X_t^i = 0$, let $\psi_{n,t}^i = 0$.) The proportion invested in the riskless asset is $1 - \psi_t^i \cdot \mathbf{1}$. Then i's initial total wealth is (letting i, $j = 1, 2$, $j \neq i$)

$$\begin{aligned} \overline{X}_0^i &= X_0^i + \frac{1}{2}\overline{S}_0 = \frac{1}{2}\gamma_0^i \cdot S_0 + \frac{1}{2}\overline{S}_0 \\ &= \gamma_{i,0}^i E\left[\int_0^T p_s Y_s^i \, ds\right] + \gamma_{j,0}^i E\left[\int_0^T p_s Y_s^j \, ds\right] + \frac{1}{2} E\left[\int_0^T p_s \Phi_s \, ds\right]. \end{aligned}$$

Therefore the static budget constraint (3.8) is equivalent to

$$E\left[\int_0^T p_s e_s^i \, ds\right] \leq \overline{X}_0^i, \quad e^i \in \mathcal{C}. \tag{A.12}$$

By the definition of λ, these constraints hold with equality if $e^i = c^i$. Finally, at c^i, each individual i satisfies the first-order conditions

$$e^{-\beta t} z_t^{\theta^{*i}} / c_t^i = \delta^i p_t, \tag{A.13}$$

for suitable multipliers $\delta^1 = 1$ and $\delta^2 = \lambda^{-1}$. Conclude that c^1 and c^2 are utility maximizing. Because they clear output markets, they constitute an Arrow-Debreu equilibrium allocation.

The following two Lemmas show that the static Arrow-Debreu equilibrium can be implemented by some security trading strategies γ^i, $i = 1, 2$, to form the Radner equilibrium described in Theorem 3.1.

First, notice that the budget constraint (3.6) is equivalent to the following familiar dynamic budget constraint:

$$dX_t^i = \left\{ \left[r_t + (\psi_t^i)^\top (b_t - r_t \mathbf{1}) \right] X_t^i \; - \; \left(e_t^i - \tfrac{1}{2}\Phi_t \right) \right\} \, dt + X_t^i (\psi_t^i)^\top \; s_t \, dW_t. \quad \text{(A.14)}$$

In order to rule out arbitrage opportunities [19], impose also the credit constraint

$$X_t^i \geq -\tfrac{1}{2}\,\overline{S}_t, \;\; t \in [0, T]. \quad \text{(A.15)}$$

Lemma A.3. *Let $S = (S^0, S^1, S^2)^\top$ be given by (3.21) and (3.22) and suppose that s_t is invertible. Then:*

(i) The state price process p satisfies (3.12).

(ii) If (e^i, ψ^i, X^i) satisfies the dynamic budget constraint (A.14) and the credit constraint (A.15), then e^i satisfies the static budget constraint (A.12).

(iii) Conversely, if e^i satisfies (A.12), there exist a portfolio share process ψ^i and financial wealth process X^i such that (e^i, ψ^i, X^i) satisfies (A.14) and (A.15). Moreover, if $e^i = c^i$, then ψ^i is unique up to equivalence and person i's financial wealth X_t^i is given by

$$X_t^i \; = \frac{1}{p_t} E\left[\int_t^T p_s (c_s^i - \tfrac{1}{2}\Phi_s) \, ds \middle| \, \mathcal{F}_t \right]. \quad \text{(A.16)}$$

Proof. (i) Since S is given by (3.21) and (3.22), the deflated gains process $(\int_0^t p_s dD_s + p_t S_t)$ is a 3-dimensional P-martingale and hence it has a zero drift. Then (3.12) follows from Ito's Lemma and the definitions of D and returns.

(ii) Adapt arguments from [31]. By (i), (A.14) and Ito's Lemma,

$$p_t X_t^i + \int_0^t p_\tau (e_\tau^i - \tfrac{1}{2}\Phi_\tau) \; d\tau = X_0^i + \int_0^t p_\tau X_\tau^i \left(s_\tau^\top \psi_\tau^i \; - \eta_\tau \right)^\top dW_\tau. \quad \text{(A.17)}$$

The left side of this equation is a local martingale. Because of the credit constraint (A.15), it is bounded below by a martingale and hence is a supermartingale. By the optional sampling theorem, therefore,

$$E\left[p_T X_T^i + \int_0^T p_t (e_t^i \; - \tfrac{1}{2}\Phi_t) \, dt \right] \leq X_0^i.$$

From $X_T^i \geq 0$, derive the static budget constraint (A.12).

(iii) Conversely, let e_t^i satisfy the static budget constraint (A.12), or equivalently,

$$E\left[\int_0^T p_t(e_t^i - \tfrac{1}{2}\Phi_t)\,dt\right] \leq X_0^i.$$

Introduce the P-martingale

$$H_t^i \equiv E\left[\int_0^T p_t(e_t^i - \tfrac{1}{2}\Phi_t)\,dt \,\middle|\, \mathcal{F}_t\right] - E\left[\int_0^T p_t(e_t^i - \tfrac{1}{2}\Phi_t)\,dt\right].$$

By the martingale representation theorem, H^i can be written as

$$H_t^i = \int_0^t (\phi_s^i)^\top dW_s,$$

for some progressively measurable $\mathbb{R}^2$-valued process ϕ^i with $\int_0^T \| \phi_s^i \|^2\, ds < \infty$, a.s. Let the financial wealth process (X_t^i) and portfolio share process (ψ_t^i) satisfy

$$X_t^i = \frac{1}{p_t}\left(X_0^i - \int_0^t p_s(e_s^i - \tfrac{1}{2}\Phi_s)\,ds + H_t^i\right) \text{ and } \tag{A.18}$$

$$\psi_t^i = (s_t^\top)^{-1}\left(\eta_t + \frac{\phi_t^i}{p_t X_t^i}\right). \tag{A.19}$$

Then

$$H_t^i = \int_0^t p_\tau X_\tau^i \left(s_\tau^\top \psi_\tau^i - \eta_\tau\right)^\top dW_\tau,$$

and, by (A.18),

$$\begin{aligned} p_t X_t^i &= X_0^i - \int_0^t p_\tau(e_\tau^i - \tfrac{1}{2}\Phi_\tau)\,d\tau + \int_0^t p_\tau X_\tau^i \left(s_\tau^\top \psi_\tau^i - \eta_\tau\right)^\top dW_\tau \\ &= X_0^i - E\left[\int_0^T p_t(e_t^i - \tfrac{1}{2}\Phi_t)\,dt\right] + E\left[\int_t^T p_s(e_s^i - \tfrac{1}{2}\Phi_s)\,ds \,\middle|\, \mathcal{F}_t\right]. \end{aligned}$$

From this one can verify that X_t^i satisfies the dynamic budget constraint (A.14) and the credit constraint (A.15).

If $e^i = c^i$, the static budget constraint (A.12) holds with equality. Consequently, (A.16) follows from the preceding equation.

Finally, consider the uniqueness of portfolio shares. By (A.16), $X_T^i = 0$ and M^i is a P-martingale, where

$$M_t^i \equiv p_t X_t^i + \int_0^t p_s (c_s^i - \tfrac{1}{2}\Phi_s)\, ds, \quad t \in [0,T]. \tag{A.20}$$

Suppose there are two such portfolios ψ^i and $\widehat{\psi}^i$ satisfying the stated properties. Let X^i and $\widehat{X}^i$ represent the corresponding financial wealth processes and (M_t^i) and $(\widehat{M_t^i})$ the corresponding P- martingales as in (A.20). By (A.17) and

$$M_T^i = \widehat{M_T^i} = \int_0^T B_s (c_s^i - \tfrac{1}{2}\Phi_s)\, ds,$$

the martingale

$$M_t^i - \widehat{M_t^i} = \int_0^t B_\tau X_\tau^i (\psi_\tau^i - \widehat{\psi}_\tau^i)^\top s_\tau \, dW_\tau, \quad t \in [0,T],$$

is identically zero. Thus the quadratic variation

$$< M^i - \widehat{M^i} >_t = \int_0^t (B_\tau X_\tau^i)^2 \parallel (\psi_\tau^i - \widehat{\psi}_\tau^i)^\top \, s_\tau \parallel^2 d\tau = 0, \;\; t \in [0,T].$$

Since s_t is invertible for all t, $\psi_t^i = \widehat{\psi}_t^i$ *a.s.* $dt \otimes dP$. ■

Lemma A.4. *Let the returns volatility matrix s_t be invertible. Then the Arrow-Debreu equilibrium $((c^i)_{i=1,2}, p)$ can be implemented to form a Radner equilibrium $((c^i, \gamma^i)_{i=1,2}, S)$, for some trading strategies $(\gamma^1, \gamma^2) \in \Gamma \times \Gamma$ and for security prices S^n, $n = 0$, 1, 2, given by (3.21)-(3.22).*

Proof. By the equivalence of the static and dynamic budget constraints proven in the preceding lemma, the two associated optimization problems are equivalent. Hence, we need only find trading strategies to clear all markets. Note that the static budget constraint (A.12) holds with equality in equilibrium.

Let $\gamma_{n,t}^i = X_t^i \psi_{n,t}^i / S_t^n, n = 1,2,$ and $\gamma_{0,t}^i = X_t^i (1 - \psi_{1,t}^i - \psi_{2,t}^i)/S_t^0$, where ψ_t^i is given by (A.19) and X_t^i is given by (A.16). Then $\gamma^i \in \Gamma$. By the preceding lemma, (c^i, ψ^i, X^i) satisfies the dynamic budget constraint (A.14). Stock markets clear if and only if

$$X_t^1 \psi_t^1 + X_t^2 \psi_t^2 = (S_t^1, S_t^2)^\top. \tag{A.21}$$

Sum financial wealth (A.16) over i and use pricing equation (3.21) and contingent consumption market clearing condition (3.9) to obtain

$$X_t^1 + X_t^2 = S_t^1 + S_t^2. \tag{A.22}$$

This equation and (A.21) imply that the bond market also clears. Therefore, we need only verify (A.21).

By Ito's Lemma, (3.12) and (A.14) to obtain (A.17). Sum (A.17) over i and apply Ito's Lemma and (A.22) to obtain

$$\begin{aligned} & d\left[p_t(S_t^1 + S_t^2)\right] \\ = & -p_t(Y_t^1 + Y_t^2)dt + p_t\left[X_t^1\left(\psi_t^1\right)^\top + X_t^2\left(\psi_t^2\right)^\top\right] s_t dW_t - p_t(S_t^1 + S_t^2)\,\eta_t \cdot dW_t. \end{aligned}$$

On the other hand, apply Ito's Lemma and use (3.10)-(3.12) to obtain

$$d\left[p_t(S_t^1 + S_t^2)\right] = a_t\,dt \ + p_t(S_t^1, S_t^2)s_t dW_t - p_t(S_t^1 + S_t^2)\,\eta_t \cdot dW_t,$$

for some process (a_t). Match the volatility terms in the above two expressions and apply invertibility of s_t to derive (A.21). ■

It remains to verify the security market conditions asserted in the theorem. Apply Ito's Lemma to the first-order conditions (A.13) and compare with (3.12) to derive

$$r_t = \beta + \mu_t^{c,i} - s_t^{c,i} \cdot s_t^{c,i} - s_t^{c,i} \cdot \theta_t^{*i} \quad \text{and} \tag{A.23}$$

$$\eta_t = s_t^{c,1} + \theta_t^{*1} = s_t^{c,2} + \theta_t^{*2}. \tag{A.24}$$

Substitute (4.5) and (3.14) into (A.24) to obtain (3.24).

Apply Ito's Lemma to the market clearing condition (3.9) and derive

$$\mu^Y = \sum_{i=1}^{2} \mu_t^{c,i} c_t^i / Y_t, \quad s^Y = \sum_{i=1}^{2} s_t^{c,i} c_t^i / Y_t.$$

Multiply c_t^i on each side of (A.23) and sum over i to obtain

$$r_t = \beta + \mu^Y - s^Y \eta_t.$$

Substitute expression (3.24) for η_t into the preceding to obtain (3.23).

By the definition of the market price of uncertainty process,

$$b_t - r_t \mathbf{1} = s_t \eta_t.$$

Substitute (3.24) for η_t into the preceding to obtain (3.25).

Finally, turn to parts (c)-(d). Equation (3.26) follows directly from the following Lemma.

Lemma A.5. *In equilibrium, consumption and total wealth are related by*

$$c_t^i = \frac{\beta}{1 - e^{-\beta(T-t)}} \overline{X}_t^i. \tag{A.25}$$

Proof. Use (A.16) and the definition of total wealth (3.18) to obtain

$$\overline{X}_t^i = \frac{1}{p_t} E\left[\int_t^T p_s c_s^i ds \middle| \mathcal{F}_t \right]. \tag{A.26}$$

Thus

$$\begin{aligned} \delta^i p_t \overline{X}_t^i &= E\left[\int_t^T \delta^i p_s c_s^i \, ds \middle| \mathcal{F}_t \right] \\ &= E\left[\int_t^T e^{-\beta s} z_s^{\theta^{*i}} \, ds \middle| \mathcal{F}_t \right] \\ &= \beta^{-1}(e^{-\beta t} - e^{-\beta T}) \, z_t^{\theta^{*i}}, \end{aligned}$$

where the second equality follows from the first-order conditions (A.13) and the third equality follows from the fact that $z_t^{\theta^{*i}}$ is a P-martingale. Apply (A.13) once more to derive (A.25). ■

Write

$$d\overline{X}_t^i / \overline{X}_t^i = \mu_t^{\overline{X},i} dt + s_t^{\overline{X},i} \cdot dW_t. \tag{A.27}$$

Thus $s_t^{\overline{X},i}$ is the volatility of i's total wealth process. From (A.25) and Ito's Lemma, deduce that

$$s_t^{\overline{X},i} = s_t^{c,i}. \tag{A.28}$$

From (A.19), the key to solve for portfolio shares and trading strategies is to solve for ϕ_t^i, the integrand in the martingale representation of H_t^i. Use (A.26) and the definition of $\overline{S}_t$ to rewrite H_t^i as

$$H_t^i = E\left[\int_0^T p_t c_t^i \, dt \middle| \mathcal{F}_t \right] - \frac{1}{2} E\left[\int_0^T p_t \Phi_t \, dt \middle| \mathcal{F}_t \right] - E\left[\int_0^T p_t (c_t^i - \tfrac{1}{2}\Phi_t) \, dt \right]$$

$$= p_t\overline{X}_t^i - \tfrac{1}{2}p_t\overline{S}_t + \int_0^t p_s c_s^i \, ds - \frac{1}{2}\int_0^t p_s \Phi_s \, ds - E\left[\int_0^T p_t(c_t^i - \tfrac{1}{2}\Phi_t)\, dt\right].$$

Apply Ito's Lemma to the above equation and use (3.12), (3.17) and (A.27) to obtain

$$p_t\overline{X}_t^i s_t^{\overline{X},i} - p_t\overline{X}_t^i\eta_t - \tfrac{1}{2}p_t\overline{s}_t + \tfrac{1}{2}p_t\overline{S}_t\eta_t = \phi_t^i.$$

Substitute this into (A.19) and use (A.28) and (3.18) to derive

$$\psi_t^i = \frac{1}{X_t^i}\left(s_t^\top\right)^{-1}\left(\overline{X}_t^i s_t^{c,i} - \tfrac{1}{2}\overline{s}_t\right).$$

Use (A.11), (A.24), (3.14) and $\eta_t = (s_t)^{-1}(b_t - r_t\mathbf{1})$ to substitute for $s_t^{c,i}$ in the above equation to obtain trading strategies (3.27) for the two risky securities. The trading strategy for bond is given by

$$\gamma_0^i = X_t^i(1 - \psi_t^i \cdot \mathbf{1}) \,/\, S_t^0.$$

By (3.18), (A.22), (A.25) and the market clearing condition (3.9),

$$\overline{X}_t^1 + \overline{X}_t^2 = X_t^1 + X_t^2 + \overline{S}_t = S_t^1 + S_t^2 + \overline{S}_t = \beta^{-1}\left(1 - e^{-\beta(T-t)}\right)Y_t.$$

Apply Ito's Lemma to this equation to obtain (3.28). ■

B. Appendix B

Proof of Corollary 4.1: Denote $E(\cdot|\mathcal{F}_t)$ by $E_t(\cdot)$. Substitute dividends processes (4.6) and the state price process (3.20) into pricing equations (3.21) to obtain

$$S_t^1 = c_t^1 K_t(W_t^1) + c_t^2 L_t(W_t^1) \quad \text{and} \tag{B.1}$$

$$S_t^2 = c_t^1 M_t(W_t^2) + c_t^2 N_t(W_t^2), \tag{B.2}$$

where

$$K_t(W_t^1) = \int_t^T e^{-\beta(\tau-t)} E_t\left[v(W_\tau^1)\right] d\tau,$$

$$L_t(W_t^1) = \int_t^T e^{-\beta(\tau-t)} E_t\left[\frac{z_\tau^{\theta^{*2}}}{z_t^{\theta^{*2}}} v(W_\tau^1)\right] d\tau,$$

$$M_t(W_t^2) = \int_t^T e^{-\beta(\tau-t)} E_t\left[\frac{z_\tau^{\theta^{*1}}}{z_t^{\theta^{*1}}} v(W_\tau^2)\right] d\tau,$$

$$N_t(W_t^2) = \int_t^T e^{-\beta(\tau-t)} E_t\left[v(W_\tau^2)\right] d\tau.$$

Since $v(\cdot)$ is increasing and positive, it is easy to show that $K_t(\cdot)$, $L_t(\cdot)$, $M_t(\cdot)$ and $N_t(\cdot)$ are all increasing and positive. (For example, to show that $E_t\left[\frac{z_\tau^{\theta^*2}}{z_t^{\theta^*2}} v(W_\tau^1)\right]$ is increasing, use the facts (i) $z_\tau^{\theta^*2}/z_t^{\theta^*2}$ depends only on the increment $(W_\tau^1 - W_t^1)$ and (ii) v is increasing.)

Apply Ito's Lemma to (B.1) and (B.2) to obtain all elements in the returns volatility matrix:

$$\begin{aligned} s_t^{11} &= (s_1^Y + \kappa_2 c_t^2/Y_t) K_t c_t^1/S_t^1 + (s_1^Y - \kappa_2 c_t^1/Y_t) L_t c_t^2/S_t^1 \\ &\quad + \left(c_t^1 K_t'(W_t^1) + c_t^2 L_t'(W_t^1)\right)/S_t^1, \end{aligned}$$

$$s_t^{12} = (s_2^Y - \kappa_1 c_t^2/Y_t) K_t c_t^1/S_t^1 + (s_2^Y + \kappa_1 c_t^1/Y_t) L_t c_t^2/S_t^1,$$
$$s_t^{21} = (s_1^Y + \kappa_2 c_t^2/Y_t) M_t c_t^1/S_t^2 + (s_1^Y - \kappa_2 c_t^1/Y_t) N_t c_t^2/S_t^2,$$

$$\begin{aligned} s_t^{22} &= (s_2^Y - \kappa_1 c_t^2/Y_t) M_t c_t^1/S_t^2 + (s_2^Y + \kappa_1 c_t^1/Y_t) N_t c_t^2/S_t^2 \\ &\quad + \left(c_t^1 M_t'(W_t^2) + c_t^2 N_t'(W_t^2)\right)/S_t^2, \end{aligned}$$

where prime denotes derivative. Given the assumption (A.1) on parameters, each term in above equations is positive and hence

$$s_t^{ij} > 0, \text{ for all } i,\ j = 1, 2.$$

For the determinant,

$$\begin{aligned} \det(s_t) &= s_t^{11} s_t^{22} - s_t^{12} s_t^{21} \\ &= a_t + \frac{c_t^1 c_t^2}{S_t^2 S_t^1}(s_1^Y \kappa_1 + s_2^Y \kappa_2)(K_t N_t - M_t L_t), \end{aligned}$$

where a_t is a positive process. We claim that $K_t > L_t$ and $N_t > M_t$. In fact,

$$\begin{aligned} K_t - L_t &= \int_t^T e^{-\beta(\tau-t)} E_t\left[v(W_\tau^1)\right] d\tau - \int_t^T e^{-\beta(\tau-t)} E_t\left[\frac{z_\tau^{\theta^*2}}{z_t^{\theta^*2}} v(W_\tau^1)\right] d\tau \\ &= \int_t^T e^{-\beta(\tau-t)} E_t\left[\left(1 - e^{-\frac{1}{2}(\kappa_2)^2(\tau-t) - \kappa_2(W_\tau^1 - W_t^1)}\right) v(W_\tau^1)\right] d\tau \\ &= \int_t^T e^{-\beta(\tau-t)} E_x\left[\left(1 - e^{-\frac{1}{2}(\kappa_2)^2(\tau-t) - \kappa_2 x}\right) v(x + W_t^1)\right] d\tau \ > \ 0, \end{aligned}$$

where E_x denotes expectation with respect to $N(0, \tau - t)$. To obtain the last inequality, use the fact that both $v(x+W_t^1)$ and $1-e^{-\frac{1}{2}(\kappa_2)^2(\tau-t)-\kappa_2 x}$ are increasing in x, which implies that

$$\begin{aligned} E_x\left[\left(1-e^{-\frac{1}{2}(\kappa_2)^2(\tau-t)-\kappa_2 x}\right)v(x+W_t^1)\right] &> \\ E_x\left[1-e^{-\frac{1}{2}(\kappa_2)^2(\tau-t)-\kappa_2 x}\right]\cdot E_x\left[v(x+W_t^1)\right] &= 0. \end{aligned}$$

Similarly, $N_t > M_t$. Therefore, $\det(s_t) > 0$. ■

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FIGURE 1: COUNTRY 1'S TRADING STRATEGY FOR FOREIGN ASSETS

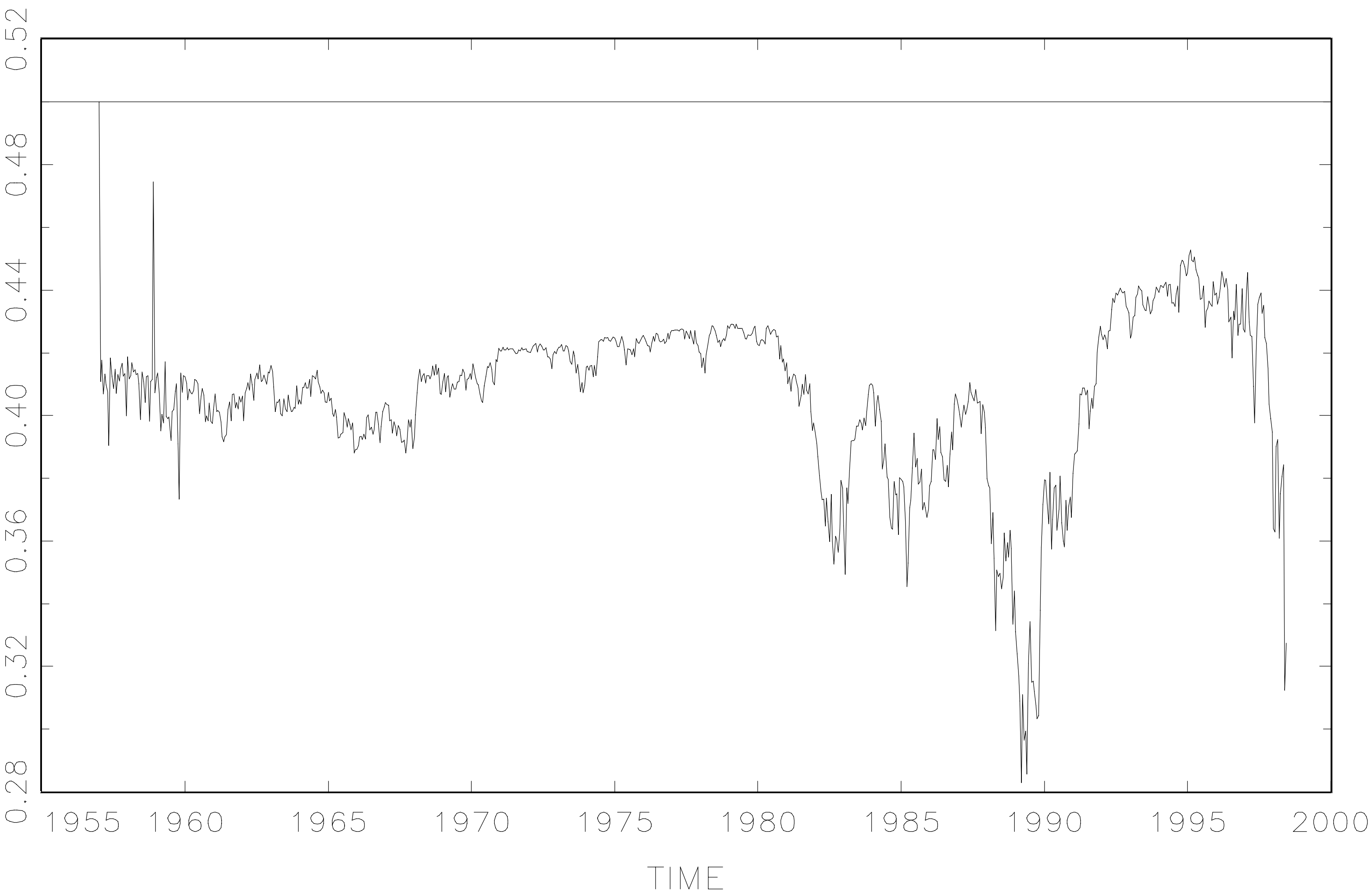

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FIGURE 2: TURNOVER OF COUNTRY 1'S DEMAND FOR FOREIGN ASSETS